

GROMOV-WITTEN THEORY OF BANDED GERBES OVER SCHEMES

ELENA ANDREINI, YUNFENG JIANG, AND HSIAN-HUA TSENG

ABSTRACT. Let X be a smooth complex projective algebraic variety. Let \mathcal{G} be a G -banded gerbe with G a finite abelian group. We prove an exact formula expressing genus g orbifold Gromov-Witten invariants of \mathcal{G} in terms of those of X .

CONTENTS

1. Introduction	2
Conventions	2
Acknowledgments	3
2. Orbifold Gromov-Witten theory	3
2.1. Twisted stable maps	3
2.2. Virtual Fundamental Class	4
2.3. Gromov-Witten invariants and Descendant Potential	6
3. Twisted curves	7
3.1. The Picard group of prestable curves	7
3.2. The Picard group of twisted curves	8
3.3. The Brauer group	11
4. Rephrasing the moduli problem	11
5. The structure morphism p	18
6. Push-forward formula	21
6.1. Comparison of Obstruction Theories	21
6.2. Proof of the pushforward formula	23
7. Orbifold Gromov-Witten theory of banded gerbes	27
7.1. Orbifold Gromov-Witten invariants	27
7.2. Decomposition of Gromov Witten theory	28
Appendix A. A few useful facts about degree of morphisms	30
References	31

1. INTRODUCTION

This paper is a sequel to our paper [8]. Our goal is to study the Gromov-Witten theory of étale abelian banded gerbes over varieties. Let X be a smooth projective variety over \mathbb{C} and let G be a finite abelian group (viewed as a constant group scheme over X). Let $\epsilon : \mathcal{G} \rightarrow X$ be a G -banded gerbe over X . As explained in [8] our study of Gromov-Witten theory of \mathcal{G} is motivated by the so-called *decomposition conjecture* in physics [23]. In the case of a banded gerbe $\mathcal{G} \rightarrow X$ the decomposition conjecture states that the Gromov-Witten theory of \mathcal{G} is equivalent to the Gromov-Witten theory of the disjoint union of $|\hat{G}|$ copies of X , where $|\hat{G}|$ denotes the number of irreducible representations of G . See [37] for more discussions on the mathematical aspects of the decomposition conjecture.

In [8] we proved this conjecture for genus 0 Gromov-Witten theory of \mathcal{G} . Our approach consists of two steps. First, we studied the natural morphism¹ $p : \mathcal{K}_{0,n}(\mathcal{G}, \beta) \rightarrow \overline{\mathcal{M}}_{0,n}(X, \beta)$ between the moduli spaces of stable maps to \mathcal{G} and to X . We showed that this morphism can be factored as $\mathcal{K}_{0,n}(\mathcal{G}, \beta) \rightarrow P_n \rightarrow \overline{\mathcal{M}}_{0,n}(X, \beta)$, where $\mathcal{K}_{0,n}(\mathcal{G}, \beta) \rightarrow P_n$ is a G -gerbe and $P_n \rightarrow \overline{\mathcal{M}}_{0,n}(X, \beta)$ is obtained as a base-change of a functorial construction à la Matsuki-Olsson [28] applied to the stack $\mathfrak{M}_{0,n}$ of genus 0 prestable pointed curves². Such a construction produces a stack over $\mathfrak{M}_{0,n}$ which can be identified with an open substack of the stack of twisted curves $\mathfrak{M}_{0,n}^{tw}$. This result allows us to compare the push-forward $p_*[\mathcal{K}_{0,n}(\mathcal{G}, \beta)]^{vir}$ of the virtual fundamental class with the virtual fundamental class $[\overline{\mathcal{M}}_{0,n}(X, \beta)]^{vir}$. This in turn gives a comparison of genus 0 Gromov-Witten invariants of \mathcal{G} and X . In the second step we use some standard finite group theory to interpret our comparison result as a statement in agreement with the decomposition conjecture.

In this paper we study the *higher genus* Gromov-Witten invariants of \mathcal{G} following a similar strategy. Again we begin with studying the natural morphism

$$p : \mathcal{K}_{g,n}(\mathcal{G}, \beta) \rightarrow \overline{\mathcal{M}}_{g,n}(X, \beta)$$

between moduli spaces of stable maps. We first show that p can be similarly factored as

$$\mathcal{K}_{g,n}(\mathcal{G}, \beta) \rightarrow P_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}(X, \beta),$$

where $P_{g,n}$ is an open substack of $\mathfrak{M}_{g,n}^{tw} \times_{\mathfrak{M}_{g,n}} \overline{\mathcal{M}}_{g,n}(X, \beta)$. Even though, unlike in the genus 0 case, $\mathcal{K}_{g,n}(\mathcal{G}, \beta) \rightarrow P_{g,n}$ is not as simple as the structure morphism of a gerbe, we show that it is étale (see Proposition 5.1). It turns out that this property together with the properties of $P_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}(X, \beta)$ are sufficient to prove a comparison result between classes $p_*[\mathcal{K}_{g,n}(\mathcal{G}, \beta)]^{vir}$ and $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$, see Theorem 6.8. Using this we then draw conclusions on higher genus Gromov-Witten invariants by the method identical to the genus 0 case, see Theorem 7.3.

Conventions. Unless otherwise mentioned, we work over \mathbb{C} throughout this paper. By an *algebraic stack* we mean an algebraic stack over \mathbb{C} in the sense of [9]. By a *Deligne-Mumford stack* we mean an algebraic stack over \mathbb{C} in the sense of [16]. We assume

¹This natural morphism is obtained by composing stable maps to \mathcal{G} with the map $\omega : \mathcal{G} \rightarrow X$ and taking the relative coarse moduli space morphism.

²Given an algebraic stack X endowed with a locally free log structure the construction associates to it an algebraic stack \mathcal{X} defined as the category fibered in groupoids whose objects are morphisms to $f : T \rightarrow X$ and simple morphisms of locally free log structures $f^*\mathcal{M} \rightarrow \mathcal{M}'$.

moreover that all stacks (and schemes) are locally noetherian and locally of finite type. Following [21], logarithmic structures are considered on the étale site of schemes. For the extension of logarithmic structures to stacks, see [31]. Given a scheme (or a stack) X , a geometric point x of X , and an étale sheaf of sets \mathcal{F} on X , according to the standard notation we denote by $\mathcal{F}_{\bar{x}}$ the stalk of \mathcal{F} at x in the étale topology. A gerbe is an algebraic stack as in [27, Definition 3.15].

The main results in this paper are valid for banded G -gerbes over X with G a finite abelian group. For the sake of simplicity, in the main text we assume $G = \mu_r \subset \mathbb{C}^*$ is the cyclic group of r -th roots of unity. The case of general G requires only notational changes. In genus 0 this is spelled out in [8, Appendix A]. The higher genus case is similar and is left to the readers.

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2. ORBIFOLD GROMOV-WITTEN THEORY

In this section we recall some basic set-up and well known facts about orbifold Gromov-Witten theory that we will constantly use in what follows.

2.1. Twisted stable maps. We recall the definition of twisted curve here, see [2], [3], [6] for more details.

Definition 2.1 ([6], Definition 4.1.2). *A twisted nodal n -pointed curve over a scheme S is a morphism $\mathcal{C} \rightarrow S$ together with n closed substacks $\sigma_i \subset \mathcal{C}$ such that*

- \mathcal{C} is a tame Deligne-Mumford stack, proper over S , and étale locally is a nodal curve over S ;
- $\sigma_i \subset \mathcal{C}$ are disjoint closed substacks in the smooth locus of $\mathcal{C} \rightarrow S$;
- $\sigma_i \rightarrow S$ are étale gerbes;
- the map $\mathcal{C} \rightarrow C$ to the coarse moduli space C is an isomorphism away from marked points and nodes.

By definition the genus of a twisted curve $\mathcal{C} \rightarrow S$ is the genus of its coarse moduli space $C \rightarrow S$.

Throughout this paper we will always assume that twisted curves are *balanced*, i.e. at any twisted node, the local group acts on the two branches by opposite characters.

For more details on twisted curves that we will need in this paper we refer to [8], where the reader will find an introduction to the equivalence between twisted curves and log twisted curves introduced in [33].

Let S be a noetherian scheme and let \mathcal{X}/S be a proper Deligne-Mumford stack over S with projective coarse moduli space $X \rightarrow S$. We fix an ample invertible sheaf $\mathcal{O}_X(1)$ over X . Let $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ be the fibered category over S which to any S -scheme T associates the groupoid of the following data:

- A twisted n -pointed curve $(\mathcal{C}/T, \{\sigma_i\})$ over T ;

- A representable morphism $f : \mathcal{C} \rightarrow \mathcal{X}$ such that the induced morphism $\bar{f} : \mathcal{C} \rightarrow X$ between coarse moduli spaces is an n -pointed stable map of degree $\beta \in H_2^+(X, \mathbb{Z})$ (i.e. $\bar{f}_*[C] = \beta$).

According to [6, Theorem 1.4.1], the fibered category $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ is a Deligne-Mumford stack proper over S . As discussed in [3], there exists evaluation maps:

$$ev_i : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \rightarrow \bar{I}(\mathcal{X}), \quad 1 \leq i \leq n$$

taking values in the *rigidified inertia stack* $\bar{I}(\mathcal{X})$ of \mathcal{X} . This map is obtained as follows. The rigidified inertia stack $\bar{I}(\mathcal{X})$ may be defined as the stack of cyclotomic gerbes in \mathcal{X} , i.e. representable morphisms from cyclotomic gerbes to \mathcal{X} . The evaluation map ev_i is defined by associating to a twisted stable map $f : (\mathcal{C}/T, \{\sigma_i\}) \rightarrow \mathcal{X}$ its restriction to the i -th marked gerbe,

$$f|_{\sigma_i} : \sigma_i \rightarrow \mathcal{X},$$

which is an object of $\bar{I}(\mathcal{X})$. The rigidified inertia stack $\bar{I}(\mathcal{X})$ has an alternative description. Define the *inertia stack* of \mathcal{X} to be the fiber product over the diagonal:

$$I\mathcal{X} := \mathcal{X} \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X}.$$

By definition, objects of $I\mathcal{X}$ are pairs (x, g) where x is an object of \mathcal{X} and g is an element of the automorphism group of x . The rigidified inertia stack $\bar{I}(\mathcal{X})$ is obtained from $I\mathcal{X}$ by applying the rigidification procedure ([1], [4]). More details can be found in e.g. [3].

2.2. Virtual Fundamental Class. Let \mathcal{X} be a smooth proper DM stack with projective coarse moduli space. Let X be a smooth projective variety. In general both $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ and $\bar{M}_{g,n}(X, \beta)$ are neither smooth nor of pure dimension. Therefore they do not generally admit a fundamental cycle. In [11] the *virtual fundamental class* is introduced, which replaces the usual fundamental cycle. In order to construct it, an *obstruction theory* is needed.

Condition (†): We say that an object L^\bullet of $D(\mathcal{O}_{X_{\acute{e}t}})$ satisfies Condition (†) if

- (1) $h_i(L^\bullet) = 0$ for all $i > 0$,
- (2) $h_i(L^\bullet)$ is coherent, for $i = 0, 1$.

Definition 2.2. An obstruction theory is a morphism in $D(\mathcal{X})$ $\phi : E^\bullet \rightarrow L_{\mathcal{X}}^\bullet$ where $E^\bullet \in D(\mathcal{O}_{X_{\acute{e}t}})$ satisfies Condition (†) and $L_{\mathcal{X}}$ is Illusie's cotangent complex (for extension to DM stacks see [27] and to Artin stacks [32]). The morphism ϕ has to satisfy the following conditions:

- (1) $\phi^0 : h^0(E^\bullet) \rightarrow h^0(L_{\mathcal{X}}^\bullet)$ is an isomorphism;
- (2) $\phi^1 : h^1(E^\bullet) \rightarrow h^1(L_{\mathcal{X}}^\bullet)$ is surjective;

where h^i denotes the i -th complex cohomology and ϕ^i is the morphism induced by ϕ on cohomology sheaves.

Definition 2.3. An obstruction theory as in Definition 2.2 is called perfect if E^\bullet is locally quasi-isomorphic to a complex of vector bundles. It is called of perfect amplitude contained in $[-1, 0]$ if it is locally quasi-isomorphic to a complex of vector bundles $F^{-1} \rightarrow F^0$.

There is also a notion of *relative obstruction theory* for a morphism $\mathcal{X} \rightarrow \mathfrak{Y}$ from a DM stack to a smooth Artin stack of pure dimension. The definition is analogous to Definition 2.2 except that the absolute tangent complex $L_{\mathcal{X}}^\bullet$ is replaced by the relative

cotangent complex $L_{\mathfrak{X}/\mathfrak{Y}}^\bullet$. Given any separated DM stack \mathfrak{X} endowed with a perfect obstruction theory it is possible to associate to it a cycle in $A_*(\mathfrak{X})_{\mathbb{Q}}$ of the expected dimension called the *virtual fundamental class*. Such a cycle can be seen more conveniently as a bivariant class in $A^*(\mathfrak{X} \rightarrow S) \simeq A_*(\mathfrak{X})$, $S = \text{Spec } \mathbb{C}$. Analogously, in the relative case we get a class in $A_{\mathbb{Q}}^*(\mathfrak{X} \rightarrow \mathfrak{Y}) \simeq A_{\mathbb{Q}}^*(\mathfrak{X})$ (see [17, Ch. 17] for the theory of bivariant classes). The construction works as follows. Let us observe first that any DM type morphism of Artin stacks $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ defines a cone stack $\mathfrak{C}_{\mathfrak{X}/\mathfrak{Y}}$ called *intrinsic relative normal cone*. For any such a morphism one can find a diagram as follows [22]

$$(1) \quad \begin{array}{ccc} U^C & \longrightarrow & M \\ \downarrow \text{\acute{e}t} & & \downarrow \text{sm} \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

where the upper row is a closed immersion, the arrow on the left is étale and the arrow on the right is smooth. This is called in [11] a *local embedding*. Such a diagram defines a cone stack $[C_{U/M}/T_{M/\mathfrak{Y}}]$ which is shown to be in fact independent on the chosen local embedding of \mathfrak{X} . Moreover, the cone stacks constructed locally glue to yield a global cone stack.

Definition 2.4. *The intrinsic relative normal cone $\mathfrak{C}_{\mathfrak{X}/\mathfrak{Y}}$ is the unique cone stack such that $\mathfrak{C}_{\mathfrak{X}/\mathfrak{Y}}|_U \simeq [C_{U/M}/T_{M/\mathfrak{Y}}]$.*

Let $\mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism from a DM stack to a smooth Artin stack of pure dimension. Let E^\bullet be a perfect obstruction theory of amplitude contained in $[-1, 0]$ for \mathfrak{X} relative to \mathfrak{Y} . Let $E_\bullet := (E_{f_l}^\vee)^\bullet$, where the subscript f_l denotes the derived pullback to the big fppf site. The quasi-isomorphism class of such an obstruction theory determines a Picard stack $h^1/h^0((E_{f_l}^\bullet)^\vee) \simeq [E_1/E_0]$ (the stack theoretic quotient). Note that if E^\bullet admits a *global resolution*, namely it is globally isomorphic to a complex of vector bundles $F^{-1} \rightarrow F^0$, then the associated Picard stack is simply $[F_1/F_0]$. The conditions in Definition 2.2 ensure that there is a closed embedding $\mathfrak{C}_{\mathfrak{X}/\mathfrak{Y}} \hookrightarrow [E_1/E_0]$. If E^\bullet admits a global resolution by pulling-back $\mathfrak{C}_{\mathfrak{X}/\mathfrak{Y}}$ along $F_1 \rightarrow [F_1/F_0]$ we get a closed substack $C \hookrightarrow F_1$. The virtual fundamental class is obtained as the intersection of C with the zero section of F_1 .³

The stacks $\mathcal{K}_{g,n}(X, \beta)$ and $\overline{M}_{g,n}(X, \beta)$ admit perfect obstruction theories of amplitude contained in $[-1, 0]$ relative to $\mathfrak{M}_{g,n}^{tw}$ and $\mathfrak{M}_{g,n}$. In [10] it is shown that a perfect obstruction theory relative to $\mathfrak{M}_{g,n}$ for $\overline{M}_{g,n}(X, \beta)$ is given by

$$E^\bullet := R\pi_*(f^*\Omega_X \otimes \omega_\pi) \rightarrow L_{\overline{M}_{g,n}(X, \beta)/\mathfrak{M}_{g,n}},$$

where

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow \pi & & \\ \overline{M}_{g,n}(X, \beta) & & \end{array}$$

³In [22] intersection theory for Artin stacks was developed. Using such a theory it is possible to intersect $\mathfrak{C}_{\mathfrak{X}/\mathfrak{Y}}$ with the zero section of $[E_1/E_0]$ without needing to assume that the obstruction theory admits a global resolution.

is the universal stable map. Consider the universal twisted stable map to the gerbe \mathcal{G} :

$$\begin{array}{ccc} \mathfrak{c} & \xrightarrow{\tilde{f}} & \mathcal{G} \\ \downarrow \tilde{\pi} & & \\ \mathcal{K}_{g,n}(\mathcal{G}, \beta) & & \end{array}$$

According to [3] the moduli stack $\mathcal{K}_{g,n}(\mathcal{G}, \beta)$ has a perfect obstruction theory \tilde{E}^\bullet relative to $\mathfrak{M}_{g,n}^{tw}$ analogously constructed:

$$\tilde{E}^\bullet := R\tilde{\pi}_*(\tilde{f}^*\Omega_{\mathcal{G}} \otimes \omega_{\tilde{\pi}}) \rightarrow L_{\mathcal{K}_{g,n}(\mathcal{G}, \beta)/\mathfrak{M}_{g,n}^{tw}}.$$

The complex \tilde{E}^\bullet turns out to be quasi-isomorphic to the pullback of E^\bullet as an object in $\mathcal{D}_{Coh}(\mathcal{K}_{g,n}(\mathcal{G}, \beta))$.

2.3. Gromov-Witten invariants and Descendant Potential. In this paper we will work with cohomological GW-invariants. For us $[\mathcal{K}_{g,n}(\mathcal{X}, \beta)]^{vir}$ and $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$ will be classes in $H_*(\mathcal{K}_{g,n}(\mathcal{X}, \beta), \mathbb{Q})$ and in $H_*(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q})$. Let $\gamma_1, \dots, \gamma_n$ be homogeneous elements in $H^*(IX, \mathbb{Q})$. Then n -points genus g orbifold GW-invariants are defined⁴ as

$$(2) \quad \langle \gamma_1, \dots, \gamma_n \rangle_{g, \beta}^{\mathcal{X}} = \int_{[\mathcal{K}_{g,n}(\mathcal{X}, \beta)]^{vir}} \prod_{i=1}^n ev_i^*(\gamma_i).$$

The vector space $H^*(IX, \mathbb{Q})$ can be made into a graded vector space in the following way:

$$(3) \quad H^*(IX, \mathbb{Q}) = \bigoplus_{\Omega} H^{a-age(\Omega)}(\Omega, \mathbb{Q}),$$

where the direct sum is over the connected components $\Omega \subseteq IX$ and $age(-)$ is a locally constant function on the inertia stack defined in [3, Definition 7.1.1]. The genus g descendant potential is defined as follows. Let $\mathcal{L}_1, \dots, \mathcal{L}_n$ be the tautological line bundles over $\mathcal{K}_{g,n}(\mathcal{G}, \beta)$, defined as the line bundles whose fiber at a point is the cotangent space of the corresponding coarse moduli space curve. For any $i = 1, \dots, n$ let $\psi_i = c_1(\mathcal{L}_i)$. Note that with this definition the classes ψ_i are a pullback of the analogous classes in $H_*(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Q})$. Let $\alpha_1, \dots, \alpha_m$ be an additive basis of $H^*(X, \mathbb{Q})$. Let $t_{i,j}$, $i = 1, \dots, m$, $j \geq 0$ be supercommuting variables such that $\deg t_{i,j} = \deg \alpha_i$. We put $\gamma = \sum_{d=0}^{\infty} \sum_{i=1}^m t_{i,j} \tau_j \alpha_i$, where $\tau_j \alpha_i$ is defined as $\alpha_i \psi_*^j$ where $*$ = k if $\tau_j \alpha_i$ is inserted at the k -th place in (2). Then the genus g descendant potential is the series with coefficients in $\mathbb{C}[[t_{i,j}]]$ defined as follows

$$(4) \quad \mathcal{F}^g(\gamma, Q) = \sum_{n \geq 0} \sum_{\beta \in H_2(X, \mathbb{Z})} \frac{Q^\beta}{n!} \langle \gamma^n \rangle = \sum_{n \geq 0} \sum_{\beta \in H_2^+(X, \mathbb{Z})} \sum_{\substack{i_1, \dots, i_n \\ j_1, \dots, j_n}} \frac{Q^\beta}{n!} \prod_{k=1}^n t_{i_k, j_k} \langle \prod_{k=1}^n \alpha_{i_k} \psi_k^{j_k} \rangle_{\mathcal{X}, \beta}.$$

The genus g descendant potential is an exponential generating function for genus g descendant orbifold Gromov-Witten invariants.

⁴Here we ignore the subtlety involving IX and its rigidification.

3. TWISTED CURVES

In this section we present a few facts about prestable curves and twisted curves.

3.1. The Picard group of prestable curves. Let C be a genus g smooth curve. Let $\text{Pic } C^0$ denote the subgroup of degree 0 line bundles. The extension

$$(5) \quad 1 \rightarrow \text{Pic } C^0 \rightarrow \text{Pic } C \rightarrow \mathbb{Z} \rightarrow 1$$

is a semidirect product. Moreover $\text{Pic}^0 C$ is a divisible group and for any $r \in \mathbb{N}$ the kernel of the multiplication by r is a subgroup of rank r^{2g} [30]. Therefore, for any line bundle $L \in \text{Pic } C$ such that $\deg L$ is multiple of r , there are r^{2g} line bundles $N \in \text{Pic } C$ such that $N^{\otimes r} \simeq L$. If C is prestable, which means that it only admits ordinary double points as singularities, then $\text{Pic}^0 C$ is a semi-abelian variety (see e.g. [26]) as described in the following Lemma.

Lemma 3.1. *Let C be a genus g nodal prestable curve. Then there is an exact sequence*

$$(6) \quad 1 \rightarrow T \rightarrow \text{Pic } C \rightarrow \text{Pic } C^\nu \rightarrow 1,$$

where T is an algebraic torus and C^ν denotes the normalization of C . Therefore $\text{Pic } C$ is a semi-abelian variety.

Remark 3.2. *The algebraic torus T in the Lemma above is isomorphic to \mathbb{G}_m^s where s is the number of double points. It can be thought of as parametrizing descent data for line bundles. Indeed it is a well known fact (see e.g. [20]) that the category of line bundles over C is equivalent to the category of line bundles over C^ν endowed with an identification of the fibers over the preimages of the nodes. Isomorphisms of 1-dimensional k -vector spaces are given by $\mathbb{G}_{m,k}$.*

In the following it will be convenient to make use of *dual graphs* (introduced in [12], Definition 1.1 and Definition 1.5). Given a prestable curve the associated dual graph encodes its topological type.

Definition 3.3. *A graph τ is a quadruple $(F_\tau, V_\tau, j_\tau, \partial_\tau)$, where F_τ and V_τ are finite sets, $\partial_\tau : F_\tau \rightarrow V_\tau$ is a map and $j_\tau : F_\tau \rightarrow F_\tau$ an involution. We call F_τ the set of flags, V_τ the set of vertices, $S_\tau = \{f F_\tau | j_\tau f = f\}$ the set of tails and $E_\tau = \{\{f_1, f_2\} \subset F_\tau | f_1 = j_\tau f_2\}$ the set of edges of τ . For $v \in V_\tau$ let $F_\tau(v) = \partial_\tau^{-1}(v)$ and $|v| = \#F_\tau(v)$, the valence of v .*

Definition 3.4. *A modular graph is a graph τ endowed with a map $g_\tau : V_\tau \rightarrow \mathbb{Z}_{\geq 0}$; $v \mapsto g(v)$. The number $g(v)$ is called the genus of v .*

Notation 3.5. *Let τ be a modular graph. We denote by $b_1(\tau)$ the first Betti number of τ defined as $b_1(\tau) = 1 - |V_\tau| + |E_\tau|$*

The following lemma characterizes the torsion subgroups of the Picard group of a prestable curve.

Lemma 3.6. *Let C be a nodal prestable curve of dual graph τ . Let $r \in \mathbb{N}$. Let $\text{Pic } C[r]$ be the r -torsion part of the Picard group. Then*

$$(7) \quad |\text{Pic } C[r]| = 2g(\tau) - b_1(\tau),$$

where $b_1(\tau) = 1 - \#V_\tau + \#E_\tau$ is the first Betti number of the graph τ .

3.2. The Picard group of twisted curves. As for ordinary prestable curves in order to describe the Picard group of a twisted curve along with its torsion subgroups we will make use of dual graphs encoding the topological type of the curve and the isotropy groups of its special points. In [7] we introduced *gerby modular graphs* generalizing Definition 3.4. They will allow us to suitably label strata and irreducible components of stacks parametrizing twisted curves and twisted stable maps and will make much easier our notation in Section 6.2.

Notation 3.7. Let \mathcal{U} be a finite set and $\mathfrak{o} : \mathcal{U} \rightarrow \mathbb{N}$, $\mathfrak{a} : \mathcal{U} \rightarrow \mathbb{Q}$ two set maps.

The following example is important.

Example 3.8. Let \mathcal{X} be a smooth proper Deligne-Mumford stack. A triple $(\mathcal{U}, \mathfrak{o}, \mathfrak{a})$ is obtained as follows. Let $\mathcal{U} = \mathcal{U}(\mathcal{X})$ be the set of connected components of the rigidified cyclotomic inertia stack of \mathcal{X} . Let \mathfrak{o} be the set map $\mathcal{U}(\mathcal{X}) \rightarrow \mathbb{N}$ taking $U \in \mathcal{U}(\mathcal{X})$ to the integer r such that $U \subseteq \bar{I}_{\mu_r}(\mathcal{X})$. Let $\mathfrak{a} : \mathcal{U}(\mathcal{X}) \rightarrow \mathbb{Q}$ be the set map taking U to its age.

Definition 3.9 (Gerby dual graph). Let $(\mathcal{U}, \mathfrak{o}, \mathfrak{a})$ be as in Notation 3.7. A gerby modular graph $\tilde{\tau}$ associated to $(\mathcal{U}, \mathfrak{o}, \mathfrak{a})$ is the data of an underlying modular graph τ with a map $\mathfrak{g} : F_{\tilde{\tau}} \rightarrow \mathcal{U}$ such that $\mathfrak{g}(f) = \mathfrak{g}(f')$ whenever the flags f, f' form an edge $\{f, f'\} \in E_{\tilde{\tau}}$. We define $\gamma := \mathfrak{o} \circ \mathfrak{g}$.

Let A be a semigroup with indecomposable zero. A gerby A -graph is a gerby modular graph $\tilde{\tau}$ whose underlying modular graph τ is endowed with an A -structure (i.e. a map $V_{\tau} \rightarrow A$).

Notation 3.10. Let $\tilde{\tau}$ be a gerby dual graph over τ . We establish the following conventional notations,

- For any edge $e_j \in E_{\tilde{\tau}}$ we put $r_j := \gamma(e_j)$. If \mathcal{C} is a twisted curve with gerby dual graph $\tilde{\tau}$, we denote by Γ_{e_j} the automorphisms group of a twisted node of \mathcal{C} corresponding to e_j . Such a group is cyclic of order r_j ;
- for any flag $p_i \in F_{\tilde{\tau}}/E_{\tilde{\tau}}$ we put $b_i = \gamma(p_i)$. If \mathcal{C} is a twisted curve with gerby dual graph $\tilde{\tau}$, We denote by Γ_{p_i} the automorphisms group of a stacky point of \mathcal{C} corresponding to p_i . Such a group is cyclic of order b_i ;

Lemma 3.11. Let \mathcal{C} be a twisted curve over $\text{Spec } \mathbb{C}$ of gerby dual graph $\tilde{\tau}$. Let $\nu : \mathcal{C}^{\nu} \rightarrow \mathcal{C}$ be its normalization. Then there is an exact sequence

$$(8) \quad 1 \rightarrow \prod_{i=1}^s \mathbb{C}^* \rightarrow \text{Pic } \mathcal{C} \rightarrow \text{Pic } \mathcal{C}^{\nu} \rightarrow \prod_{i=1}^s \text{Pic } B\Gamma_{e_i} \rightarrow 1$$

where $s = |E_{\tilde{\tau}}|$ and $B\Gamma_{e_i}$ are the gerbes corresponding to twisted nodes.

PROOF. It follows from the normalization exact sequence

$$(9) \quad 1 \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow \nu_* \mathcal{O}_{\mathcal{C}^{\nu}} \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{B\Gamma_{e_i}} \rightarrow 1$$

by taking the invertible elements and by computing the long exact cohomology sequence. \square

Lemma 3.12. Let \mathcal{C} be a twisted curve over $\text{Spec } \mathbb{C}$ with coarse moduli space C . Let $\tilde{\tau}$ be the dual graph of \mathcal{C} . With reference to Notation 3.10, there is an exact sequence

$$(10) \quad 1 \rightarrow \text{Pic } C \rightarrow \text{Pic } \mathcal{C} \rightarrow \left(\bigoplus_{e_j \in E_{\tilde{\tau}}} \mathbb{Z}/r_j \mathbb{Z} \oplus \bigoplus_{p_i \in F_{\tilde{\tau}}/E_{\tilde{\tau}}} \mathbb{Z}/b_i \mathbb{Z} \right) \rightarrow 1.$$

PROOF. Consider the exact sequence of complexes

$$(11) \quad 1 \rightarrow \pi_* \mathbb{G}_m \rightarrow R\pi_* \mathbb{G}_m \rightarrow \frac{R\pi_* \mathbb{G}_m}{\pi_* \mathbb{G}_m} \rightarrow 1.$$

The Hypercohomology long exact sequence yields

$$(12) \quad 1 \rightarrow H^1(C, \mathbb{G}_m) \xrightarrow{\pi^*} H^1(\mathcal{C}, \mathbb{G}_m) \xrightarrow{res} H^0(C, R^1\pi_* \mathbb{G}_m) \rightarrow 1,$$

where the arrow on the right is surjective because of Tsen's theorem (cf. Section 3.3). By a result proven in [1, Proposition A.0.1] there is a canonical isomorphism

$$(13) \quad H^0(C, R^1\pi_* \mathbb{G}_m) \simeq \prod_{e_j \in E_{\tilde{\tau}}} H^1(\Gamma_{e_j}, \mathbb{G}_m) \times \prod_{p_i \in F_{\tilde{\tau}}/E_{\tilde{\tau}}} H^1(\Gamma_{p_i}, \mathbb{G}_m),$$

where the groups above are group cohomology for the trivial action of Γ_{e_j} , Γ_{p_i} on \mathbb{G}_m . It is a standard result that there are isomorphisms $H^1(\Gamma_{e_j}, \mathbb{G}_m) \simeq \prod_j \mathbb{Z}/r_j \mathbb{Z}$, $H^1(\Gamma_{p_i}, \mathbb{G}_m) \simeq \prod_i \mathbb{Z}/b_i \mathbb{Z}$. The map *res* in (12) is given by restricting line bundles to the twisted points of \mathcal{C} . \square

Remark 3.13. *Let \mathcal{C} be a twisted curve with gerby dual graph $\tilde{\tau}$. Let $\#E_{\tilde{\tau}} = s$, $\#(F_{\tilde{\tau}} \setminus E_{\tilde{\tau}}) = n$. There is a commutative diagram*

$$(14) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \text{Pic } C & \longrightarrow & \text{Pic } \mathcal{C} & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}/b_i \oplus \bigoplus_{k=1}^s \mathbb{Z}/r_k \longrightarrow 1 \\ & & \downarrow & & \downarrow \nu^* & & \downarrow \mathbb{I} \oplus \Delta^{-1} \\ 1 & \longrightarrow & \text{Pic } C^\nu & \longrightarrow & \text{Pic } \mathcal{C}^\nu & \longrightarrow & \bigoplus_{i=1}^n \mathbb{Z}/b_i \oplus \bigoplus_{k=1}^s \mathbb{Z}/r_k^{\oplus 2} \longrightarrow 1 \\ & & & & \downarrow & \swarrow & \\ & & & & \bigoplus_{k=1}^s \mathbb{Z}/r_k & & \\ & & & & \downarrow & & \\ & & & & 1 & & \end{array}$$

where the map $\mathbb{I} \oplus \Delta^{-1}$ is the identity of the first factor and the anti-diagonal on the second factor. The diagonal arrow is its cokernel. This corresponds to the fact that a line bundle which is non trivial when restricted to a twisted node over \mathcal{C} pulls-back over \mathcal{C}^ν to a line bundle involving opposite powers of the tautological line bundles associated to the two preimages of the node. This is a consequence of the fact that nodes of twisted curves are balanced. We observe that any two line bundles in $\text{Pic } \mathcal{C}$ pulling back to isomorphic line bundles in $\text{Pic } \mathcal{C}^\nu$ differ by a degree zero line bundle because the normalization map is of degree 1.

The following lemma gives a characterization of torsion subgroups for unmarked twisted curves.

Lemma 3.14. *Let \mathcal{C} be an unmarked twisted curve with gerby dual graph $\tilde{\tau}$. With reference to Notation 3.10 we define $l_e := \text{g.c.d.}(r_e, r) \forall e$. Then there is a non split short exact sequence*

$$(15) \quad 1 \rightarrow \mathbb{Z}/r \mathbb{Z}^{2g(\tilde{\tau}) - b_1(\tilde{\tau})} \rightarrow \text{Pic } \mathcal{C}[r] \rightarrow \bigoplus_{e \in E_{\tilde{\tau}}}^{n.s.} \mathbb{Z}/l_e \mathbb{Z} \rightarrow 1,$$

where the direct sum $\bigoplus_{e \in E_{\tilde{\tau}}}^{n.s.}$ is taken over the non separating nodes.

PROOF. Note that $\mathbb{Z}/r\mathbb{Z}^{2g(\bar{\tau})-b_1(\bar{\tau})} = \mathbb{Z}/r\mathbb{Z}^{2g(\tau)-b_1(\tau)} \simeq \text{Pic } C[r]$. We will see that line bundles which are non-trivial when restricted to non-separating nodes contribute to $\text{Pic } \mathcal{C}[r] \simeq H^1(\mathcal{C}, \mu_r)$, where we use the equivalence between the category of r -torsion line bundles and the category of μ_r -principal bundles. Consider the exact sequence of complexes

$$(16) \quad 1 \rightarrow \pi_*\mu_r \rightarrow R\pi_*\mu_r \rightarrow \frac{R\pi_*\mu_r}{\pi_*\mu_r} \rightarrow 1.$$

By taking the Hypercohomology long exact sequence we get the short exact sequence

$$(17) \quad 1 \rightarrow H^1(C, \mu_r) \rightarrow H^1(\mathcal{C}, \mu_r) \rightarrow \text{Ker}(H^0(C, R^1\pi_*\mu_r) \xrightarrow{\delta} H^2(C, \mu_r)) \rightarrow 1.$$

The morphism $H^1(\mathcal{C}, \mu_r) \rightarrow H^0(C, R^1\pi_*\mu_r)$ can be seen as the restriction map, which takes an r -torsion line bundle to its restriction to the stacky points. The boundary morphism δ is explicitly described in [14]. Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$ be a class in $H^0(C, R^1\pi_*\mu_r) \simeq \prod_{e \in E_{\bar{\tau}}} H^1(B\Gamma_e, \mu_r) \simeq \text{Pic}(\prod_{e \in E_{\bar{\tau}}} B\Gamma_e)$. Let $\mathcal{L}_{\vec{\alpha}}$ be any line bundle over \mathcal{C} which restricts to a line bundle of class $\vec{\alpha}$ over $\prod_e B\Gamma_e$. Then $\delta(\vec{\alpha})$ is the isomorphism class of the gerbe over C induced by the r -th root of $\mathcal{L}_{\vec{\alpha}}^{\otimes r}$. Note that there is a commutative diagram (cfr. [14, Lemma 3.2.19])

$$(18) \quad \begin{array}{ccc} H^1(C^\nu, R^1\pi_*^\nu\mu_r) & \longrightarrow & H^2(C^\nu, \mu_r) \\ \uparrow \nu^* & & \uparrow \wr \\ H^1(C, R^1\pi_*\mu_r) & \longrightarrow & H^2(C, \mu_r), \end{array}$$

where C^ν denotes the normalization of C . Using the notations in Remark 3.13, the above diagram becomes

$$(19) \quad \begin{array}{ccc} \bigoplus_{k=1}^s \mathbb{Z}/r_k^{\oplus 2} & \longrightarrow & \bigoplus_{v \in \tau} (\mathbb{Z}/r)_v \\ \uparrow \Delta^{-1} & & \sim \updownarrow \\ \bigoplus_{k=1}^s \mathbb{Z}/r_k & \longrightarrow & \bigoplus_{v \in \tau} (\mathbb{Z}/r)_v \end{array}$$

where for any v , $(\mathbb{Z}/r)_v = \mathbb{Z}/r$. The maps $\bigoplus_{k=1}^s \mathbb{Z}/r_k^{\oplus 2} \rightarrow \bigoplus_{v \in \tau} (\mathbb{Z}/r)_v$ take an element (m_1, \dots, m_{2s}) in the image of Δ^{-1} to $\sum_{e_k \in E_v} (r/r_k)m_k$. Choose a set of line bundles whose images in $\text{Pic } \mathcal{C}/\text{Pic } C$ generate it. Assume moreover that for any such line bundle there exists a node e in \mathcal{C} such that the restriction of this line bundle to the complement of e in \mathcal{C} is trivial. The class of each line bundle of the above set is $(0, \dots, \alpha, \dots, 0)$ with α in the k -th position. If e_k is a non separating node, then α is sent to zero, because the preimages of the node in the normalization belong to the same irreducible component. If e_k is separating, its preimages in \mathcal{C}^ν belong to two different irreducible components, hence the image of α is $(\mathfrak{r}r_k\alpha, -\mathfrak{r}r_k\alpha) \in (\mathbb{Z}/r)_v \oplus (\mathbb{Z}/r)_{v'}$ where v, v' are the vertices associated to the irreducible components of \mathcal{C}^ν containing the preimages of the node. Therefore $\text{Ker}(\delta)$ is generated by the subset of line bundles whose restriction to non-separating nodes is non trivial. The claim follows from the fact that for any node e , $H^1(B\Gamma_e, \mu_r) \simeq \mathbb{Z}/l_e$. \square

For any twisted curve \mathcal{C} we will fix a set of (isomorphisms classes of) line bundles in $\text{Pic } \mathcal{C}$ lifting the standard basis of $\text{Pic } \mathcal{C}/\text{Pic } C$. We start from the right factor group in (13) in Lemma 3.12. Recall that any smooth twisted curve is obtained as stack over its coarse moduli space by the construction known as *taking roots of line bundles* as shown in [3], [13]. This construction provides for any marked twisted point p_i a *tautological line*

bundle whose fiber over p_i is the standard representation of $\Gamma_{p_i} \simeq \mu_{b_i}$. We will denote such line bundles by \mathcal{T}_i for any marked point p_i . If p_i is twisted, \mathcal{T}_i has non-trivial image in $\text{Pic } \mathcal{C} / \text{Pic } C$ and it is our choice for the lift of its image. We come to the second factor in (13). Unlike twisted marked points, for twisted nodes there is no universal construction providing a canonical choice for line bundles whose restriction to a twisted node is non trivial. However, due to Lemma 3.14 we can choose a set of line bundles, that we will denote by $\mathcal{Q}_1, \dots, \mathcal{Q}_s$, $s = |E_{\bar{\tau}}|$, satisfying the following properties

Property 3.15.

- (1) for any $e_i \in E_{\bar{\tau}}$, $i = 1, \dots, s$, $\mathcal{Q}|_{e_i}$ is the standard representation of $\Gamma_{e_i} \simeq \mu_{r_i}$;
- (2) for any $i = 1, \dots, s$, $\mathcal{Q}_i^{r_i} \simeq \mathcal{O}_{\mathcal{C}}$, $r_i = \gamma(e_i)$;
- (3) for any $i = 1, \dots, s$ $\mathcal{Q}_i|_{\mathcal{C} \setminus e_i} \simeq \mathcal{O}_{\mathcal{C} \setminus e_i}$.

We remark that any two line bundles $\mathcal{Q}_i, \mathcal{Q}'_i$ satisfying the above conditions for some i differ by an r_i -torsion line bundle.

3.3. The Brauer group. The Brauer group of a smooth algebraic curve over $\text{Spec } \mathbb{C}$ is trivial. This is a consequence of Tsen's theorem as explained e.g. in [29, III §2 Example 2.22 Case (d)]. In fact by the same argument the Brauer group of a prestable curve is also trivial. It is not hard to see that the same result also holds for the Brauer group of a twisted curve. Consider sequence (17) in Lemma 3.14. The result follows by taking the long exact cohomology sequence and by the result in [1, Proposition A.0.1]. Indeed for Γ any cyclic group $H^2(\Gamma, \mathbb{G}_m) = 0$. See also [34].

Lemma 3.16. *Let $\mathcal{C}_A \rightarrow \text{Spec } A$ be a twisted curve over an artinian ring. Then $H^2(\mathcal{C}_A, \mathcal{O}_{\mathcal{C}_A}^*) = 0$.*

PROOF. Let $I \subset A$ be the ideal of the closed point 0. Let \mathcal{C}_0 denote the pullback of the curve to the closed point. By flatness $I \cdot \mathcal{O}_{\mathcal{C}_A}$ is the ideal sheaf of \mathcal{C}^0 . The result follows by taking the long exact cohomology sequence of the short exact sequence

$$(20) \quad 1 \rightarrow 1 + I \cdot \mathcal{O}_{\mathcal{C}_A} \rightarrow \mathcal{O}_{\mathcal{C}_A}^* \rightarrow \mathcal{O}_{\mathcal{C}_0}^* \rightarrow 1,$$

and from $H^2(\mathcal{C}_A, \mathcal{O}_{\mathcal{C}_A}) = 0$. \square

Lemma 3.17. *Let $\pi : \mathcal{C}_S \rightarrow S$ be a twisted curve over a scheme S . Let $\mathcal{G} \rightarrow \mathcal{C}$ be a μ_r -banded gerbe of class $\alpha \in H^2(C, \mu_r)$. Then étale locally over S , \mathcal{G} is isomorphic to a gerbe of roots of a line bundle.*

PROOF. As explained e.g. in [25], \mathcal{G} is a root gerbe if and only if it admits a line bundle whose fibers carry an action of μ_r given by multiplication by $\chi(\alpha)$ where χ denotes the natural inclusion of the sheaf of r -th roots of unity in $\mathcal{O}_{\mathcal{C}}$. By standard limit arguments, such kind of sheaf exists over a geometric fiber $\mathcal{G}_{\bar{s}}$ of π and extends to an étale neighborhood of \bar{s} . \square

4. REPHRASING THE MODULI PROBLEM

Recall that for the purpose of computing GW invariants, we can pretend that there are evaluation morphisms taking values on the usual inertia stack (rather than on the rigidified inertia stack)

$$ev_i : \mathcal{K}_{g,m}(\mathcal{G}, \beta) \rightarrow I\mathcal{G} \simeq \mathcal{G} \times_X G,$$

where the last canonical isomorphism is due to the fact tha \mathcal{G} is a G -banded gerbe. Given an n -tuple $\vec{g} \in G^{\times n}$ we define

$$(21) \quad \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}} := \cap_{i=1}^n ev_i^{-1}(I\mathcal{G}_{g_i}),$$

where $I\mathcal{G}_{g_i} := \mathcal{G} \times \{g_i\}$. The substack $\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}}$ is either empty or an open and closed component of $\mathcal{K}_{g,n}(\mathcal{G}, \beta)$. It turns out that $\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}} \neq \emptyset$ if and only if \vec{g} is β -admissible in the sense of Definition 4.5. By definition $\mathcal{K}_{g,n}(\mathcal{G}, \beta) = \coprod_{\vec{g}} \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}}$.

Let $\mathcal{G} \rightarrow X$ be a μ_r -gerbe. The moduli problem of twisted stable maps to \mathcal{G} relative to the moduli problem of stable maps to X is equivalent to the moduli problem of twisted stable maps to gerbes induced by pullback over the prestable curves admitting stable maps to X . This fact follows easily from the universal property of the fiber product in the strict 2-category of algebraic stacks. The datum of a twisted stable map $[\tilde{f} : \mathcal{C} \rightarrow \mathcal{G}]$ is equivalent to the outer part of the following diagram:

$$(22) \quad \begin{array}{ccccc} \mathcal{C} & & \xrightarrow{\tilde{f}} & & \mathcal{G} \\ & \searrow h & & \searrow & \downarrow \\ & \mathcal{G}_C & \longrightarrow & \mathcal{G} & \\ & \downarrow & & \square & \downarrow \\ \mathcal{C} & & \xrightarrow{f} & & X \end{array}$$

By the universal property the morphism \tilde{f} induces a unique representable morphism $h : \mathcal{C} \rightarrow \mathcal{G}_C$, where \mathcal{G}_C is the pull-back gerbe.

In order to benefit from the above reformulation of the moduli problem we need to characterize the gerbes obtained by pullback to the prestable curves admitting stable maps to X . Note that for a family of prestable curve $\pi : C \rightarrow S$, the Brauer group of a geometric fiber is isomorphic to the stalk of the constructible sheaf $R^2\pi_*\mu_r$ at the corresponding geometric point (see e.g. [29], VI 2 Corollaries 2.3 and 2.5). In particular the Brauer group is not constant over S . The class of the pull-back gerbe \mathcal{G}_C in $H^2(C, \mu_r)$ induces a global section of $R^2\pi_*\mu_r$ whose evaluation at any geometric point \bar{s} of S is the class of the gerbe induced by pull-back over $C_{\bar{s}}$. By applying the relative trace map to the section of $R^2\pi_*\mu_r$ defined by $[\mathcal{G}_C] \in H^2(C, \mu_r)$, we get a function over S with values in \mathbb{Z}/r . A priori such a function is only locally constant. In fact we will prove it is constant. The element of \mathbb{Z}/r obtained by evaluating this function has the following geometric meaning. The gerbe $\mathcal{G}_{C_{\bar{s}}}$ induced over any geometric curve $C_{\bar{s}}$ is isomorphic to the gerbe of r -th root of some line bundle. Any two line bundles over $C_{\bar{s}}$ of the same degree induce isomorphic root gerbes. Therefore we can associate to the gerbe $\mathcal{G}_{C_{\bar{s}}}$ the degree of any line bundle inducing (a gerbe isomorphic to) it. If $C_{\bar{s}}$ is reducible, we consider the total degree of such a line bundle. Moreover, we can consider componentwise the degree mod r , since the r -th root gerbe of a line bundle whose degree is a multiple of r is trivial. In Lemma 4.1 and Proposition 4.3 we describe the restriction map of the gerbe \mathcal{G} along stable maps to X . It turns out that once the curve class $\beta = f_*[C]$ is fixed, the total degree of any line bundle inducing a gerbe isomorphic to the pullback of \mathcal{G} to the geometric fibers of the domain curve is constant over $\overline{M}_{g,n}(X, \beta)$ and only depends on β and on the class of \mathcal{G} .

Lemma 4.1. *Let $f : C \rightarrow X$ be an object of $\overline{M}_{g,n}(X, \beta)(\mathbb{C})$. Let $\mathcal{G} \rightarrow X$ be a gerbe of class $\alpha \in H^2(X, \mu_r)$. Then*

$$f^*\mathcal{G} \simeq \sqrt[r]{L/C}$$

for any $L \in \text{Pic } C$ with $\deg L = \alpha^{an} \cap \overline{\beta}$, where α^{an} is the image of α in $H^2(X^{an}, \mathbb{Z}/r)$ and $\overline{\beta}$ is the image of β in $H_2(X^{an}, \mathbb{Z}/r)$. Here (and henceforth) $\sqrt[r]{L/C}$ denotes the stack of r -th roots of the line bundle L .

PROOF. Since we work over \mathbb{C} we have a canonical isomorphism $(\mathbb{Z}/r)_X \xrightarrow{\sim} (\mu_r)_X$. Therefore from now on we will identify μ_r and its tensor powers with \mathbb{Z}/r . Moreover, because \mathbb{Z}/r is a finite group, for any $i \geq 0$ there are isomorphisms between étale and complex cohomology ([29, III §3.3])

$$(23) \quad \begin{array}{ccc} H^i(X, \mathbb{Z}/r) & \rightarrow & H^i(X^{an}, \mathbb{Z}/r) \\ \delta & \mapsto & \delta^{an} \end{array}$$

Since f is a proper map of proper schemes over \mathbb{C} , a restriction map for the étale cohomology $f^* : H^i(X, \mathbb{Z}/r) \rightarrow H^i(C, \mathbb{Z}/r)$ is defined. A pushforward map $f_* : H^i(C, \mathbb{Z}/r) \rightarrow H^{i-2c}(X, \mathbb{Z}/r)$, where $c := \dim X - \dim C$, is also defined by duality. A class $\eta \in H^i(C, \mathbb{Z}/r)$ is taken to the unique class $f_*\eta$ such that the condition

$$\text{Tr}_C(f^*\xi \cap \eta) = \text{Tr}_X(\xi \cap f_*\eta).$$

is satisfied for any $\xi \in H^{2-i}(X, \mathbb{Z}/r)$. The operation $\text{Tr}(f^*(-) \cap 1)$, where 1 is the generator of $H^0(C, \mathbb{Z}/r)$, defines an element f_*1 of the dual space of $H^2(X, \mathbb{Z}/r)$. Such an element is the same for any map of class β . Indeed, by the isomorphism (23) we get

$$(24) \quad \text{Tr}_C(f^*\alpha \cap 1) = \text{Tr}_{C^{an}}(f_{an}^* \alpha^{an} \cap [\overline{C}]^{P.D.}) = \text{Tr}_{X^{an}}(\alpha^{an} \cap \overline{\beta}^{P.D.}),$$

where $[\overline{C}]$ and $\overline{\beta}$ are the images of the fundamental class $[C]$, resp. of β , in $H_2(C^{an}, \mathbb{Z}/r)$, resp. in $H_2(X^{an}, \mathbb{Z}/r)$, and P.D. means the Poincaré dual. \square

Remark 4.2. *Note that when there is $L \in \text{Pic } X$ such that $\mathcal{G} \simeq \sqrt[r]{L/X}$, by the functoriality of the r -th root construction, the restriction map takes $\alpha \in H^2(X, \mu_r)$ to the class of the gerbe root of a line bundle of degree $c_1(L) \cap \beta$ modulo r . Here β is seen as usual as an element of $\text{Hom}_{\mathbb{Z}}(\text{Pic } X, \mathbb{Z})$.*

Proposition 4.3. *Consider the universal diagram*

$$(25) \quad \begin{array}{ccc} \mathfrak{C} & \xrightarrow{\mathfrak{f}} & X \\ \pi \downarrow & & \\ \overline{M}_{g,n}(X, \beta) & & \end{array}$$

where \mathfrak{C} is the universal curve and \mathfrak{f} is the universal stable map. Let $e : H^2(\mathfrak{C}, (\mu_r)_{\mathfrak{C}}) \rightarrow H^0(\overline{M}_{g,n}(X, \beta), R^2\pi_*\mu_r)$ be the edge map obtained from the Leray's spectral sequence for the sheaf $(\mu_r)_{\mathfrak{C}}$ and the morphism π . Then the map

$$\text{Tr}(e(\mathfrak{f}^*\sqcup)) : H^2(X, \mu_r) \rightarrow (\mathbb{Z}/r)_{\overline{M}_{g,n}(X, \beta)}$$

determines a constant global section of $(\mathbb{Z}/r)_{\overline{M}_{g,n}(X, \beta)}$. The evaluation of such a section at a geometric point \overline{p} gives the total degree of any line bundle whose r -th root gerbe is isomorphic to the gerbe $\mathcal{G}_{C_{\overline{p}}}$ obtained by pulling back \mathcal{G} to $C_{\overline{p}}$. Moreover such a degree depends linearly on β modulo r .

PROOF. Let $(\mu_r)_\mathfrak{C}$ be the sheaf of r -th roots of unity over \mathfrak{C} . The higher direct image sheaves $R^i\pi_*(\mu_r)_\mathfrak{C}$ are constructible sheaves. From the low degree exact sequence associated to the Leray's spectral sequence for the sheaf $(\mu_r)_\mathfrak{C}$ and the morphism π , we get the following morphism of cohomology groups

$$(26) \quad H^2(\overline{M}_{g,n}(X, \beta), \mu_r) \xrightarrow{H^2(\pi^*)} H^2(C, \mu_r) \xrightarrow{e} H^0(\overline{M}_{g,n}(X, \beta), R^2\pi_*(\mu_r)).$$

Note that we denoted by μ_r both the sheaf $(\mu_r)_\mathfrak{C}$ and $\pi_*(\mu_r)_\mathfrak{C} \simeq (\mu_r)_{\overline{M}_{g,n}(X, \beta)}$, where the last isomorphism holds because families of prestable curves are geometrically connected. Let $\alpha = [\mathcal{G}] \in H^2(X, \mu_r)$. The pullback $\mathfrak{f}^*\alpha \in H^2(\mathfrak{C}, (\mu_r)_\mathfrak{C})$ induces via the edge map e a global section of the constructible sheaf $R^2\pi_*(\mu_r)_\mathfrak{C}$. Let $\overline{p} : \text{Spec } \mathbb{C} \rightarrow \overline{M}_{g,n}(X, \beta)$ be a geometric point. Let $C_{\overline{p}}$ be the pre-stable curve over \overline{p} and $\iota_{\overline{p}} : C_{\overline{p}} \rightarrow \mathfrak{C}$ the inclusion. Let $\alpha_{\overline{p}}$ be the image of $\mathfrak{f}^*\alpha$ along the map $H^2(\iota_{\overline{p}}^*) : H^2(\mathfrak{C}, (\mu_r)_\mathfrak{C}) \rightarrow H^2(C_{\overline{p}}, (\mu_r)_{C_{\overline{p}}})$. By the universal property of the quotient (obviously $H^2(\iota_{\overline{p}}^*) \circ H^2(\pi^*) = 0$) it follows from sequence (26) that there is a factorization of $H^2(\iota_{\overline{p}}^*)$ through the edge map e as follows

$$(27) \quad \begin{array}{ccc} H^2(\mathfrak{C}, (\mu_r)_\mathfrak{C}) & \xrightarrow{e} & H^0(\overline{M}_{g,n}(X, \beta), R^2\pi_*(\mu_r)_\mathfrak{C}) \\ \downarrow H^2(\iota_{\overline{p}}^*) & & \downarrow H^0(\overline{p}^*) \\ H^2(C_{\overline{p}}, (\mu_r)_{C_{\overline{p}}}) & \xrightarrow{\sim} H^0(\overline{p}, (R^2\pi_*(\mu_r)_{C_{\overline{p}}})) \xleftarrow{\sim} & H^0(\overline{p}, \overline{p}^*(R^2\pi_*(\mu_r)_\mathfrak{C})) \end{array}$$

where the isomorphism on the right of the last row holds by proper base change theorem. Therefore the evaluation of $e(\mathfrak{f}^*\alpha)$ at a geometric point \overline{p} gives $\alpha_{\overline{p}}$ in $\overline{p}^*R^2\pi_*(\mu_r)_\mathfrak{C} \simeq H^2(C_{\overline{p}}, (\mu_r)_{C_{\overline{p}}})$. Since the universal curve is representable relative to its base, there is a trace map constructed as in the case of schemes:

$$(28) \quad R^2\pi_*(\mu_r)_\mathfrak{C} \xleftarrow{\sim} R^2\pi_!(\mu_r)_{\mathfrak{C}^{sm}} \xrightarrow{\text{Tr}} \mathbb{Z}/r,$$

where $\mathfrak{C}^{sm} \subseteq \mathfrak{C}$ is the smooth locus of the universal curve. The natural morphism $R^2\pi_!(\mu_r)_{\mathfrak{C}^{sm}} \rightarrow R^2\pi_*(\mu_r)_\mathfrak{C}$ in (28) is an isomorphism because $\mathfrak{C} \setminus \mathfrak{C}^{sm}$ is finite over the base. A priori $\text{Tr}(e(\mathfrak{f}^*\alpha))$ is only locally constant. However, by Lemma 4.1 we conclude that it is in fact constant, because its evaluation depends only on α and β . \square

Notation 4.4. *In what follows for the sake of simplicity we will denote by $\alpha \cap \beta$ the element $\alpha^{an} \cap \overline{\beta} \in \mathbb{Z}/r$ computed in Lemma 4.1. This element characterizes the gerbe \mathcal{G}_C pulled back to a prestable curve C of class β . In fact it is the total degree of any line bundle over C whose r -th root is isomorphic to \mathcal{G}_C .*

Definition 4.5. *Given an identification $G \simeq \mu_r$, a β -admissible vector is an n -tuple $\vec{g} = (\exp(\frac{2\pi\sqrt{-1}m_1}{b_1}), \dots, \exp(\frac{2\pi\sqrt{-1}m_n}{b_n})) \in G^{\times n}$, with $b_i | r$ and $(m_i, b_i) = 1$ for $i = 1, \dots, n$, and such that*

$$\prod_{i=1}^n \exp(\frac{2\pi\sqrt{-1}m_i}{b_i}) = \exp(\frac{2\pi\sqrt{-1}k}{r}),$$

where $k = \alpha \cap \beta \pmod{r}$.

Lemma 4.6. *Let C be a prestable curve of dual graph τ . Let $\mathcal{G} \simeq \sqrt[r]{L/C}$ for some $L \in \text{Pic } C$. Let \mathfrak{C} be a twisted curve of gerby graph $\tilde{\tau}$ over C admitting a representable morphism to \mathcal{G} . By the universal property of root gerbes, there exists a pair (N, ϕ) such that $\phi : N^{\otimes r} \simeq L$. Let $n = |F_{\tilde{\tau}} \setminus E_{\tilde{\tau}}|$. Then*

$$(29) \quad N \simeq \otimes_{i=1}^n \mathcal{J}_i^{m_i} \otimes M,$$

where $(m_i, b_i) = 1$, the restriction of M to the marked points is trivial and for any $e \in E_{\tilde{\tau}} M|_e$ is a faithful representation of $\Gamma_e \simeq \mu_{r_e}$.

PROOF. Let us consider the normalization $\nu : \mathcal{C}^\nu \rightarrow \mathcal{C}$. This is a representable morphism. Let $l = 1, \dots, m$ be an index labeling the irreducible components of \mathcal{C} . For any l there are induced morphisms $\mathcal{C}_l^\nu \rightarrow \mathcal{C}_l$, with \mathcal{C}_l^ν a smooth twisted curve. Let $n_l := |F_{\tilde{\tau}(v_l)} \setminus E_{\tilde{\tau}(v_l)}|$, $k_l := |E_{\tilde{\tau}(v_l)}^{n.l.}|$, $s_l := |E_{\tilde{\tau}(v_l)}^{loop}|$, where $\tilde{\tau}(v_l)$ is the gerby graph of the irreducible component \mathcal{C}_l . Then

$$(30) \quad \mathcal{N}|_{\mathcal{C}_l^\nu} \simeq \otimes_{i=1}^{n_l} \mathcal{T}_i^{m_i} \otimes \otimes_{j=1}^{k_l} \mathcal{T}_j^{m_j} \otimes \otimes_{t=1}^{s_l} \mathcal{T}_t^{m_t} \otimes M'$$

where M' is pulled-back from the coarse moduli space C_l^ν . Note that there is a bijection between marked points and separating nodes of \mathcal{C}_l and of its normalization. On the contrary, for any non-separating node of \mathcal{C}_l there are two marked points in its preimage in \mathcal{C}_l^ν . To study representability of $f_l^\nu : \mathcal{C}_l^\nu \rightarrow \mathcal{G}$ by [6, Lemma 4.4.3], it suffices to study the homomorphism

$$(31) \quad \text{Aut}(\sigma_i) \rightarrow \text{Aut}(f_l^\nu(\sigma_i)),$$

induced by f_l^ν on stack points. Here σ_i can be either a stack point mapping to a smooth point or to a node in \mathcal{C}_l . By σ_i we mean precisely a morphism $\tilde{h}_i : \text{Spec } K \rightarrow \mathcal{C}$ from an algebraically closed field K to \mathcal{C}_l^ν with image in the special locus. By the root construction description of \mathcal{C}_l^ν (see e.g. [13, Example 2.7] and [3, Section 4.2]), the stack point σ_i is equivalent to the data $(h_i, \mathcal{N}_i, \tau_i, \phi_i)$, where $h_i : \text{Spec } K \rightarrow C$ with image p_i , \mathcal{N}_i is a line bundle over $\text{Spec } K$, $\phi_i : \mathcal{N}_i^{\otimes r_i} \xrightarrow{\sim} h_i^* \mathcal{O}(p_i)$, τ_i is a section of \mathcal{N}_i such that $\phi_i(\tau_i^{r_i}) = h_i^* s_i$, hence $\tau_i = 0$. The image $f_l^\nu(\sigma_i)$ is given by $\tilde{h}_i^* \nu^* \mathcal{N}$ and $\tilde{h}_i^* \psi : \tilde{h}_i^* \nu^* \mathcal{N}^{\otimes r} \simeq \tilde{h}_i^* \pi^* f^* L$. Note that $\tilde{h}_i^* \mathcal{T}_i$ is naturally isomorphic to \mathcal{N}_i . An automorphism $\epsilon \in \text{Aut}(\sigma_i) \simeq \mu_{r_i}$ is mapped to $\epsilon^{m_i} \in \text{Aut}(f_l^\nu(\sigma_i)) \simeq \mu_r$ since $\mathcal{N} = \pi^* M' \otimes \otimes_{i=1}^n \mathcal{T}_i^{m_i}$. We conclude by observing that for any σ_i in \mathcal{C}_l^ν , the restriction of the normalization map is an isomorphism with its image. Hence in particular for any e in $E_{\tilde{\tau}} M|_e$ is isomorphic to $\mathcal{T}_i^{m_i}$ over σ_i for some σ_i in \mathcal{C}^ν . Therefore $M|_e$ is a faithful representation of $\Gamma_e \simeq \text{Aut}(\sigma_i)$ because $\mathcal{T}_i^{m_i}$ is. \square

Lemma 4.7. *Let $[f : (\mathcal{C}, \{\sigma_i\}) \rightarrow \mathcal{G}] \in \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}}(\mathbb{C})$ with $\mathcal{G} \simeq \sqrt[r]{L/X}$ for some L in $\text{Pic } X$. Then \tilde{f} is equivalent to a line bundle*

$$(32) \quad \mathcal{N} \simeq \otimes_{i=1}^n \mathcal{T}_i^{m_i} \otimes M$$

where M is as in Lemma 4.6 and for $i = 1, \dots, n$ m_i are determined by \vec{g} .

PROOF. By definition the morphism $\tilde{f}|_{\sigma_i} : B\mu_{r_i} \simeq \sigma_i \rightarrow \mathcal{G}$ is equivalent to an injective homomorphism

$$\mu_{r_i} \hookrightarrow \mu_r, \quad \exp(2\pi\sqrt{-1}/r_i) \mapsto g_i.$$

The argument in the proof of Lemma 4.6, applied to the irreducible component of \mathcal{C} containing σ_i , shows that we may write

$$(33) \quad g_i = \exp(2\pi\sqrt{-1} \frac{m_i}{b_i}), \quad \text{with } 0 \leq m_i < b_i, \text{ and } (m_i, b_i) = 1.$$

Furthermore, if $\mathcal{L}^{1/r}$ is the universal r -th root of \mathcal{L} over \mathcal{G} , then $\tilde{f}|_{\sigma_i}^* \mathcal{L}^{1/r}$ is the μ_{b_i} -representation on which the standard generator $\exp(2\pi\sqrt{-1}/b_i) \in \mu_{b_i}$ acts by multiplication by $\exp(2\pi\sqrt{-1}m_i/b_i)$. In other words

$$(34) \quad \text{age}_{\sigma_i}(\tilde{f}^* \mathcal{L}^{1/r}) = \frac{m_i}{b_i}.$$

□

Lemma 4.8. *Let $[\tilde{f} : \mathcal{C} \rightarrow \mathcal{G}]$ be an object of $\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}}(\mathbb{C})$ with underlying map between coarse spaces being $[f : C \rightarrow X]$. Then the orders of the isotropy groups of marked points and separating nodes are determined by \vec{g} .*

PROOF. The proof proceeds by induction on the number of irreducible components of \mathcal{C} . If \mathcal{C} is irreducible the claim follows from Lemma 4.7. Let us prove the induction step. Let $\mathcal{C}_1 \subset \mathcal{C}$ be an irreducible component containing only a separating node e . Let $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$. Set $\beta_1 = f_*[C_1]$. Then $\tilde{f}|_{\mathcal{C}_1}$ is equivalent to a line bundle \mathcal{N}_1 of degree k_1/r , where $k_1 = c_1(L) \cap \beta_1$. By Riemann-Roch for twisted curves

$$(35) \quad \frac{k_1}{r} - \sum_{i=1}^n \text{age}_{p_i}(\mathcal{N}_1) = \text{age}_e(\mathcal{N}_1) + N = \frac{k_1}{r} - \sum_{i=1}^{n_1} \frac{m_i}{b_i} = \frac{m_e}{b_e} + N$$

where $N \in \mathbb{N}$. Note that the preimage of e along the closed immersion $i : \mathcal{C}_1 \hookrightarrow \mathcal{C}$ is a smooth point, that we denote again by e . By taking the fractional part of (35) we get

$$(36) \quad \left\langle \frac{k_1}{r} - \sum_{i=1}^{n_1} \frac{m_i}{b_i} \right\rangle = \frac{m_e}{b_e} \quad (m_e, b_e) = 1.$$

We conclude by observing that the order of the node e is given by b_e . □

Remark 4.9. *We observe that unlike separating nodes, \vec{g} does not determine the order of non-separating nodes. In Lemma 4.8 we used Riemann-Roch componentwise in order to determine the order of the inertia group of separating nodes. That works because the preimage of a separating node along the inclusion of an irreducible components is a smooth point. Of course this is false for a non-separating nodes. Riemann-Roch formula for twisted curve does not take into account the ages of line bundles at nodes. This is a consequence of the fact that twisted nodes are assumed to be balanced.*

Definition 4.10. *Let τ be a dual graph. Let \vec{g} be an admissible vector. We say that a dual graph $\tilde{\tau}$ over τ is \vec{g} -compatible if the following conditions are satisfied:*

- i) for all $p_i \in F_{\tilde{\tau}}/E_{\tilde{\tau}}$, we have $\gamma(p_i) = b_i$ where b_i is determined by \vec{g} as in Definition 4.5.
- ii) for all non-looping $e_j \in E_{\tilde{\tau}}$, we have $\gamma(e_j) = s_j$ where s_j is determined by \vec{g} as in the proof of Lemma 4.8.

The definition of β -admissible vector is useful to characterize twisted stable maps to arbitrary banded gerbes. Let \mathcal{G} be a μ_r -banded gerbe. Let $\alpha \in H^2(X, \mu_r)$ be the class of \mathcal{G} . Let $[f : C \rightarrow X]$ be a geometric point in $\overline{M}_{g,n}(X, \beta)$. Let k be a lift to \mathbb{Z} of $\alpha \cap \beta$. Let us choose a degree k line bundle L over C and an isomorphism θ such that $\theta : f^*\mathcal{G} \xrightarrow{\sim} \sqrt[r]{L/C}$. Let us consider the diagram

$$(37) \quad \begin{array}{ccccc} & & \tilde{f} & & \\ & & \curvearrowright & & \\ \mathcal{C} & \xrightarrow{h} & \mathcal{G}_C & \xrightarrow{g} & \mathcal{G} \\ & \searrow^{(\mathcal{N}, \phi)} \alpha \Rightarrow & \nearrow^{\theta^{-1}} & \square & \downarrow \\ & \sqrt[r]{L/C} & & & X \\ & & C & \xrightarrow{f} & \\ & & & & \end{array}$$

We claim that the twisted stable map $\tilde{f} : \mathcal{C} \rightarrow \mathcal{G}$ determines a β -admissible vector \vec{g} which can be computed as the β -admissible vector of (\mathcal{N}, ϕ) . Indeed it turns out that \vec{g} is well defined in the sense that it does not depend on the choice of L and θ . First we know that (\mathcal{N}, ϕ) determines some \vec{g} as in Lemma 4.11. Let $\{p_i\}$ be the set of marked points in \mathcal{C} . Let us consider the restrictions $h|_{p_i}$ and $\tilde{f}|_{p_i}$. They define (up to the choice of a section) objects of the cyclotomic inertia stacks $I_\mu \mathcal{G}_C$ and $I_\mu \mathcal{G}$. We recall that over \mathbb{C} the cyclotomic inertia stack is canonically isomorphic to the usual inertia stack. By definition of banded gerbe, there are canonical isomorphisms $I\mathcal{G} \simeq \mathcal{G} \times_X (\mu_r)_X$ and $I\mathcal{G}_C \simeq \mathcal{G}_C \times_C (\mu_r)_C$, where $(\mu_r)_C \simeq f^{-1}(\mu_r)_X$. Indeed \mathcal{G}_C is canonically banded by the restriction of $(\mu_r)_X$ to C ([18, IV 2 Corollary 2.2.4]). There is a 2-commutative diagram

$$(38) \quad \begin{array}{ccc} I\sqrt[r]{L/C} & \xrightarrow{\sim} & \sqrt[r]{L/C} \times_C (\mu_r)_C \\ \downarrow & \Rightarrow & \downarrow (\theta^{-1}, id) \\ I\mathcal{G}_C & \xrightarrow{\sim} & \mathcal{G}_C \times_C (\mu_r)_C \\ \downarrow & \Rightarrow & \downarrow (g, f_*) \\ I\mathcal{G} & \xrightarrow{\sim} & \mathcal{G} \times_X (\mu_r)_X, \end{array}$$

where the arrows on the left are the morphisms between the inertia stacks induced by θ^{-1} and g while f_* denotes the morphism of schemes corresponding to the morphism of sheaves $(\mu_r)_X \rightarrow f_*(\mu_r)_C$. The restriction $h|_{p_i}$ is an object of $I\sqrt[r]{L/C}$, therefore by (38) and by the projection $\mathcal{G}_C \times_C (\mu_r)_C \rightarrow (\mu_r)_C$ it defines an object of μ_r . By the 1-commutativity of the upper part of diagram (37), we see that $\tilde{f}|_{p_i}$ is associated to the same element of μ_r . Once we choose θ and L we get a bijection between the possible representable morphisms $\mathcal{C} \rightarrow \mathcal{G}_C$ and the representable morphisms $\mathcal{C} \rightarrow \sqrt[r]{L/C}$. Since θ is an isomorphism of banded gerbes, which is the same as to say that the upper square of diagram (38) commutes, any two morphisms (\mathcal{N}, ϕ) , h corresponding to each other via θ define the same n -tuple of elements of μ_r . Therefore we can associate to a twisted stable map $[\tilde{f} : \mathcal{C} \rightarrow \mathcal{G}]$ over $[f : C \rightarrow X]$ the n -tuple of elements of μ_r defined by the morphism $\mathcal{C} \rightarrow \sqrt[r]{L/C}$ associated to \tilde{f} by any choice of θ and L . Such an n -tuple by definition is a β -admissible vector. We will sometimes call a twisted stable map with β -admissible vector \vec{g} a twisted stable map of *contact type* \vec{g} .

The above discussion allows to state the following Lemma.

Lemma 4.11. *Let $\mathcal{G} \simeq \mathcal{G}$ be a μ_r -banded gerbe. Let $[f : C \rightarrow X]$ be an object of $\overline{M}_{g,n}(X, \beta)(\mathbb{C})$. Let $\mathcal{C} \rightarrow C$ be a twisted curve over C such that the order of all of its special points divides r . Then a representable morphism $\mathcal{C} \rightarrow f^*\mathcal{G}$ lifting f is equivalent to a pair (\mathcal{N}, ϕ) , where $\mathcal{N} \in \text{Pic } \mathcal{C}$ and $\phi : \mathcal{N}^r \rightarrow f^*\mathcal{L}$ satisfy the following conditions*

- (1) $\phi : \mathcal{N}^r \xrightarrow{\sim} f^*\mathcal{L}$;
- (2) $\mathcal{N} \simeq \otimes_{i=1}^n \mathcal{T}_i^{m_i} \otimes \mathcal{M}$, where \mathcal{T}_i are the tautological line bundles associated to the marked points and $\mathcal{M} \in \text{Pic } \mathcal{C}$ such that, for any marked point p_i , $\mathcal{M}|_{p_i} \simeq \mathcal{O}_{p_i}$;
- (3) $\vec{g} \in \mu_r^{\times n}$, defined as the n -tuple (g_1, \dots, g_n) such that $g_i = \exp(\frac{2\pi\sqrt{-1}m_i}{b_i})$, is β -admissible;
- (4) for any twisted node $e \in \mathcal{C}$, $\mathcal{M}|_e$ is a vector bundle carrying a faithful representation of the isotropy group of e .

Moreover if $[f : \mathcal{C} \rightarrow \mathcal{G}]$ is in $\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}}$, then the order of the inertia groups of marked points and of separating nodes is determined. In other words, the dual graph $\tilde{\tau}$ of \mathcal{C} is \vec{g} -compatible over the dual graph τ of its coarse moduli space C .

PROOF. It follows from Lemmas 4.6, 4.8 and from the discussion starting on page 16 \square

Proposition 4.12. *Let $\mathcal{G} \rightarrow X$ be a μ_r -banded gerbe. Let $[f : (C, p_1, \dots, p_n) \rightarrow X]$ be an object of $\overline{M}_{g,n}(X, \beta)(\mathbb{C})$. Let τ be the dual graph of C . Let $\vec{g} = (g_1, \dots, g_n) \in \mu_r^{\times n}$ a β -admissible vector. Let $\tilde{\tau}$ be a \vec{g} -compatible gerby graph such that for any $e \in E_{\tilde{\tau}}^{\text{loop}}$ $\gamma(e)|r$. Let \mathcal{C} be the twisted curve over C with dual gerby graph $\tilde{\tau}$. Then \mathcal{C} admits twisted stable maps to \mathcal{G} with associated β -admissible vector \vec{g} lifting $[f : C \rightarrow X]$. Moreover, the number of non isomorphic twisted stable maps is*

$$(39) \quad N(\tilde{\tau}) = r^{2g-b_1(\tau)} \times \prod_{l=1}^{|E_{\tilde{\tau}}|} \phi(\gamma(e_l))$$

where $\phi(\)$ denotes the Euler totient function.

PROOF. We sketch the proof, details are left to the reader. We fix a line bundle L in $\text{Pic } C$ such that $f^*\mathcal{G} \simeq \sqrt[r]{L/C}$. Let \mathcal{C} be a twisted curve as above. Then a twisted stable map $\mathcal{C} \rightarrow \mathcal{G}$ lifting f corresponds to a pair (\mathcal{N}, ϕ) such that $\phi : \mathcal{N}^{\otimes r} \simeq L$. By Lemma 4.11 we know that the admissible vector \vec{g} fixes the restriction of \mathcal{N} to $\bigoplus_{p_i \in F_{\tilde{\tau}}/E_{\tilde{\tau}}} B\mu_{b_i} \bigoplus \bigoplus_{e_j \in E_{\tilde{\tau}}^{\text{nl}}} B\mu_{r_j}$. By definition of β -admissible vector and by discussion on page 11 it is possible to lift this image to a line bundle \mathcal{N}' which is an r -th root of L and it is trivial when restricted to the complement of marked points and separating nodes. In order to get a line bundle \mathcal{N} corresponding to a representable morphism (cfr Lemma 4.6) we need to tensor \mathcal{N}' with an r -torsion line bundle whose restriction to non-separating nodes yields a faithful representation of their inertia groups and which is trivial elsewhere. Such a line bundle can be obtained as a tensor product of line bundles \mathcal{Q}_l satisfying Property 3.15. The condition on the faithfulness of the representations of the inertia groups of the nodes is equivalent to the requirement for each $e_l \in E_{\tilde{\tau}}^{\text{loop}}$ the associated line bundle \mathcal{Q}_l appears raised to a tensor power coprime with $\gamma(e_l)$ (see again Lemma 4.6). Moreover, \mathcal{N} can be further tensored by r -torsion line bundles pulled-back from C to obtain non-isomorphic r -th roots of L . The number of non-isomorphic (\mathcal{N}, ϕ) yielding non-isomorphic twisted stable maps to \mathcal{G} is computed as

$$(40) \quad N(\tilde{\tau}) = r^{2g-b_1(\tau)} \times \prod_{l=1}^{|E_{\tilde{\tau}}|} \phi(\gamma(e_l)),$$

where $\phi(-)$ denotes the Euler totient function, whose evaluation on $n \in \mathbb{N}$ gives the number of integers less than n and coprime with n . \square

5. THE STRUCTURE MORPHISM p

As sketched in the introduction, in order to compare orbifold Gromov-Witten invariants of \mathcal{G} with Gromov-Witten invariants of X we need to prove a push-forward formula

for the virtual fundamental class of $\mathcal{K}_{g,n}(\mathcal{G}, \beta)$. We consider the following diagram:

$$(41) \quad \begin{array}{ccccc} & & p & & \\ & \swarrow & \text{---} & \searrow & \\ \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\bar{g}} & \xrightarrow{t} & P_{g,n} & \xrightarrow{q'} & \overline{M}_{g,n}(X, \beta) \\ & \searrow & \downarrow & \square & \downarrow \\ & & \mathfrak{M}_{g,n}^{tw} & \xrightarrow{q} & \mathfrak{M}_{g,n} \end{array}$$

Here the morphism q maps a twisted curve to the underlying prestable curve. The right vertical arrow is the forgetful morphism taking a stable map $[f : C \rightarrow X]$ to the prestable curve C . The stack $P_{g,n}$ is defined as the fiber product. There is a natural morphism $p : \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\bar{g}} \rightarrow \overline{M}_{g,n}(X, \beta)$ associating to a twisted stable map $[\tilde{f} : \mathcal{C} \rightarrow \mathcal{G}]$ the underlying stable map $[f : C \rightarrow X]$ between the coarse moduli spaces. Just like the case of $\overline{M}_{g,n}(X, \beta)$ there is a natural forgetful morphism $\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\bar{g}} \rightarrow \mathfrak{M}_{g,n}^{tw}$ taking a twisted stable map $[\tilde{f} : \mathcal{C} \rightarrow \mathcal{G}]$ to the twisted curve \mathcal{C} . The square defined by the forgetful morphisms and the morphisms q and q' is commutative. Therefore by the universal property of the fiber product there is an induced morphism t .

We will prove that the pushforward map

$$p_* : H_*(\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\bar{g}}, \mathbb{Q}) \rightarrow H_*(\overline{M}_{g,n}(X, \beta), \mathbb{Q})$$

takes $[\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\bar{g}}]^{vir}$ to a multiple of $[\overline{M}_{g,n}(X, \beta)]^{vir}$, see Theorem 6.8 below. In order to prove this push-forward formula we will need the following Proposition, which says that the morphism t in (41) is étale.

Proposition 5.1. *The morphism t in diagram (41) is étale.*

PROOF. Since all our stacks are locally noetherian locally of finite type, to prove that a morphism is étale we can use the infinitesimal lifting criterion and check it over square zero extensions of Artinian local rings ([19, §17]). Let

$$(42) \quad 1 \rightarrow I \rightarrow B \rightarrow A \rightarrow 1$$

be a square zero extension of local Artinian rings. Assume that we have the following 2-commutative diagram

$$(43) \quad \begin{array}{ccc} \text{Spec } A & \longrightarrow & \mathcal{K}_{g,n}(\mathcal{G}, \beta) \\ \downarrow & \Rightarrow & \downarrow \\ \text{Spec } B & \longrightarrow & P_{g,n} \end{array}$$

This is the datum of an object $(\mathcal{C}_B, f_B : \mathcal{C}_B \rightarrow X)$ of $P_{g,n}(B)$, an object $(\mathcal{C}_A, \tilde{f}_A : \mathcal{C}_A \rightarrow \mathcal{G})$ of $\mathcal{K}_{g,n}(\mathcal{G}, \beta)(A)$ and a pair of isomorphisms $\varphi \in \text{Mor } \overline{M}_{g,n}(X, \beta)(A)$ and

$\psi \in \text{Mor } \mathfrak{M}_{g,n}(A)$ making the following diagram commutative

$$(44) \quad \begin{array}{ccc} \mathcal{C}_A & \xrightarrow[\sim]{\psi} & \mathcal{C}_B|_A \\ \downarrow & & \downarrow \\ C_A & \xrightarrow[\sim]{\varphi} & C_B|_A \\ f_A \downarrow & & \downarrow f_B|_A \\ X & \xrightarrow{id} & X. \end{array}$$

Here f_A denotes the morphism induced by \tilde{f}_A by passing to the coarse moduli spaces. The morphism t is étale if and only if there exists a unique (up to isomorphism) arrow $\text{Spec } B \rightarrow \mathcal{K}_{g,n}(\mathcal{G}, \beta)$ making the following diagram 2-commute

$$(45) \quad \begin{array}{ccc} \text{Spec } A & \longrightarrow & \mathcal{K}_{g,n}(\mathcal{G}, \beta) \\ \downarrow & \nearrow & \downarrow \\ \text{Spec } B & \longrightarrow & P_{g,n}. \end{array}$$

Let us start by observing that the pullback of \mathcal{G} to \mathcal{C}_B is a root gerbe by Lemma 3.16. Choose a line bundle L_B over C_B and an isomorphism ξ_B such that $\xi_B : f_B^* \mathcal{G} \xrightarrow{\sim} \sqrt[r]{L_B/C_B}$. Let $\iota : C_B|_A \hookrightarrow C_B$ denote the inclusion. There is a composite isomorphism

$$(46) \quad f_A^* \mathcal{G} \xrightarrow{\sim} \varphi^* \iota^* f_B^* \mathcal{G} \xrightarrow[\varphi^* \iota^* \xi_B]{\sim} \sqrt[r]{\varphi^* \iota^* L_B/C_A},$$

where the first arrow is canonical. Define $L_A := \varphi^* \iota^* L_B$. According to this definition and isomorphism (46), \tilde{f}_A corresponds to a pair (\mathcal{N}_A, ϕ_A) where \mathcal{N}_A is a line bundle over \mathcal{C}_A and ϕ_A is an isomorphism such that $\phi_A : \mathcal{N}_A^r \xrightarrow{\sim} L_A$. To give an extension of \tilde{f}_A to B is equivalent to giving an extension (\mathcal{N}_B, ϕ_B) of the pair (\mathcal{N}_A, ϕ_A) and an isomorphism ϕ_B lifting ϕ_A such that $\phi_B : \mathcal{N}_B^{\otimes r} \simeq L_B$. By flatness we have an exact sequence

$$(47) \quad 1 \rightarrow J \rightarrow \mathcal{O}_{\mathcal{C}_B} \rightarrow \mathcal{O}_{\mathcal{C}_A} \rightarrow 1,$$

where $J = I \otimes_B \mathcal{O}_{\mathcal{C}_B}$ is a square zero ideal in $\mathcal{O}_{\mathcal{C}_B}$. By restricting to the subsheaves of invertible elements and by taking the long exact cohomology sequence we get

$$(48) \quad 1 \rightarrow H^1(\mathcal{C}_A, \mathcal{O}_{\mathcal{C}_A}) \otimes J \rightarrow \text{Pic } \mathcal{C}_B \rightarrow \text{Pic } \mathcal{C}_A \rightarrow 1,$$

where the last arrow is surjective because of dimensional reasons. Exactness on the left is due to the surjection $H^0(\mathcal{O}_B^*) \rightarrow H^0(\mathcal{O}_A^*)$. Choose a line bundle \mathcal{N}_B lifting \mathcal{N}_A . Let $\mathcal{S}_B = \mathcal{N}_B^{\otimes r} \otimes L_B^\vee$. Then $\mathcal{S}_B|_A \simeq \mathcal{O}$, hence \mathcal{S}_B belongs to the subgroup $H^1(\mathcal{C}_A, \mathcal{O}_{\mathcal{C}_A}) \otimes J$. Since J is a square zero ideal, it is not hard to see that $H^1(\mathcal{C}_A, \mathcal{O}_{\mathcal{C}_A}) \otimes J$ is divisible (e.g. by working with cocycles). We can therefore assume that $\mathcal{N}_B^{\otimes r} \simeq L_B$. We need to show that any two line bundles $\mathcal{N}'_B, \mathcal{N}''_B$ lifting \mathcal{N}_A and satisfying the above condition are isomorphic. Indeed by assumption $\mathcal{N}'_B \otimes \mathcal{N}''_B^\vee$ is an r -torsion line bundle in $H^1(\mathcal{C}_A, \mathcal{O}_{\mathcal{C}_A}) \otimes J$. Again by using that J is square zero one can see that this group does not contain torsion. Moreover any two choices of (\mathcal{N}_B, ϕ_B) lifting (\mathcal{N}_A, ϕ_A) and such that $\phi_B : \mathcal{N}_B^{\otimes r} \xrightarrow{\sim} L_B$ define two isomorphic morphisms to $\sqrt[r]{L_B/C_B}$. \square

Let $\mathcal{K} := \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}}$ and $P := P_{g,n}$. We observe that the relative inertia of the morphism $t : \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}} \rightarrow P_{g,n}$, defined as

$$I(\mathcal{K}/P) = \mathcal{K} \times_{\mathcal{K} \times_P \mathcal{K}} \mathcal{K},$$

contains as a subsheaf the étale sheaf $(\mu_r)_{\mathcal{K}}$. Indeed, the automorphisms group of an object $[f : \mathcal{C}_T \rightarrow \mathcal{G}]$ over T leaving \mathcal{C}_T fixed is $f^*(\mu_r)_{\mathcal{G}} \simeq (\mu_r)_{\mathcal{C}_T}$ because \mathcal{G} is a μ_r -banded gerbe. Twisted curves are geometrically connected, therefore $\Gamma(\mathcal{C}_T, \mu_r) = \Gamma(T, \mu_r)$. Whenever the the inertia of a finitely presented stack \mathcal{X} contains an étale subgroup stack G there exists a construction called *rigidification* [4], which yields a canonical morphism $\mathcal{X} \rightarrow \mathcal{X} \!/\!/ G$ such that any morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ whose relative inertia $I(\mathcal{X}/\mathcal{Y})$ contains G factors through $\mathcal{X} \rightarrow \mathcal{X} \!/\!/ G$. Moreover $\mathcal{X} \rightarrow \mathcal{X} \!/\!/ G$ is an étale gerbe. In our case the rigidification of \mathcal{K} along μ_r is isomorphic to the relative coarse moduli space [5] for the morphism t . The relative coarse moduli space construction is recalled in Appendix A. By the above arguments we get the following

Lemma 5.2. *The morphism $t : \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}} \rightarrow P_{g,n}$ factors through*

$$(49) \quad \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}} \rightarrow \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}} \!/\!/ \mu_r \rightarrow P_{g,n},$$

where the first arrow is a μ_r -banded gerbe and the second arrow is representable and étale.

PROOF. The second arrow is representable because the rigidification coincides in this case with the relative coarse moduli space. The second arrow is also étale because the first arrow is étale and surjective, and being étale for a morphism is a property étale local on the source. \square

6. PUSH-FORWARD FORMULA

Consider again diagram (41). In this Section we prove the pushforward formula, Theorem 6.8, which states that the pushforward along p of $[\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}}]^{vir}$ is a multiple of $[\overline{M}_{g,n}(X, \beta)]^{vir}$ in $H_*(\overline{M}_{g,n}(X, \beta), \mathbb{Q})$. We will show that the multiplicative factor is given by r^{2g-1} . The first step in the proof is to show that the natural perfect relative obstruction theory for $\mathcal{K}_{g,n}(\mathcal{G}, \beta)$ is quasi-isomorphic to the pull-back of the natural perfect relative obstruction theory for $\overline{M}_{g,n}(X, \beta)$. We refer the reader to Section 2.2 for definitions and notations.

6.1. Comparison of Obstruction Theories.

Lemma 6.1. *Let $s : \mathcal{K}_{g,n}(\mathcal{G}, \beta) \rightarrow \overline{M}_{g,n}(X, \beta)$ be the natural morphism. There is a natural isomorphism of objects in $\mathcal{D}_{Coh}(\mathcal{K}_{g,n}(\mathcal{G}, \beta))$*

$$s^* E^\bullet \xrightarrow{\sim} \widetilde{E}^\bullet.$$

PROOF. We will prove the statement for $\widetilde{E}^{\vee \bullet} = R\widetilde{\pi}_* \widetilde{f}^* T_{\mathcal{G}}$ and $E^{\vee \bullet} = R\pi_* f^* T_X$. Consider the complex $Ls^* R\pi_* f^* T_X$ in $\mathcal{D}_{coh}(\mathcal{K}_{g,n}(\mathcal{G}, \beta))$. It suffices to show that $Ls^* R\pi_* f^* T_X \simeq$

$R\tilde{\pi}_* \tilde{f}^* T_{\mathcal{G}}$. For this we consider the diagram

$$\begin{array}{ccccc}
\mathcal{C} & \xrightarrow{\tilde{f}} & \mathcal{G} & & \\
\rho \searrow & & \epsilon \searrow & & \\
& & \mathcal{G} & \xrightarrow{\epsilon} & X \\
\tilde{\pi} \searrow & & \downarrow & & \\
& & C & \xrightarrow{f} & X \\
& & \downarrow \pi & & \\
& & \mathcal{K}_{g,n}(\mathcal{G}, \beta) & \xrightarrow{s} & \overline{M}_{g,n}(X, \beta)
\end{array}$$

Observe that $\epsilon^* T_X \simeq T_{\mathcal{G}}$. Also we have $R\rho_* L\rho^* \simeq Id$ because the map ρ is the relative coarse moduli space for the map $\tilde{\pi}$. The arrow s is flat, because it is the composition of a flat morphism with an étale morphism. The arrow π is flat because it is the structure morphism of the universal curve. Moreover the square in the above diagram is cartesian, hence we calculate (using [27, Proposition 13.1.9])

$$\begin{aligned}
Ls^* R\pi_* f^* T_X &\simeq R\pi'_* Lp'^* f^* T_X \\
&\simeq R\pi'_* R\rho_* L\rho^* Lp'^* f^* T_X \\
&\simeq R\pi'_* R\rho_* \tilde{f}^* \epsilon^* T_X \\
&\simeq R\pi'_* R\rho_* \tilde{f}^* T_{\mathcal{G}} \\
&\simeq R\tilde{\pi}_* \tilde{f}^* T_{\mathcal{G}}.
\end{aligned}$$

Since s^* is exact, we write s^* for Ls^* . \square

Lemma 6.2. *The diagram of morphisms in $\mathcal{D}_{coh}(\mathcal{K}_{g,n}(\mathcal{G}, \beta))$*

$$(50) \quad \begin{array}{ccc}
s^* E^\bullet & \longrightarrow & s^* L_{\overline{M}_{g,n}(X, \beta)/\mathfrak{M}_{g,n}}^\bullet \\
\downarrow & & \downarrow \\
\tilde{E}^\bullet & \longrightarrow & L_{\mathcal{K}_{g,n}(\mathcal{G}, \beta)/\mathfrak{M}_{g,n}^{tw}}^\bullet
\end{array}$$

is commutative.

PROOF. The obstruction theories above are determined by adjunction from the morphisms obtained by composing the arrows in the following diagram

$$(51) \quad \begin{array}{ccc}
s^* f^* L_X^\bullet & \longrightarrow & \tilde{f}^* L_{\mathcal{G}} \\
\downarrow & & \downarrow \\
s^* L_C & \longrightarrow & L_{\mathcal{C}}^\bullet \\
\downarrow & & \downarrow \\
s^* L_{C/C'} & \longrightarrow & L_{\mathcal{C}/\mathcal{C}'}^\bullet \\
\uparrow & & \uparrow \\
s^* \pi^* L_{\overline{M}_{g,n}(X, \beta)/\mathfrak{M}_{g,n}}^\bullet & \longrightarrow & \tilde{\pi}^* L_{\mathcal{K}_{g,n}(\mathcal{G}, \beta)/\mathfrak{M}_{g,n}^{tw}}^\bullet
\end{array}$$

Each arrow in diagram (51) is commutative since every arrow is part of a transitivity exact sequence involving cotangent complexes. Commutativity follows from functorial properties of transitivity exact sequences (cfr. [24, 2.1.5]). \square

Corollary 6.3. *The induced morphism $s^*E^\bullet \rightarrow L_{\mathcal{K}_{g,n}(\mathcal{G},\beta)/\mathfrak{M}_{g,n}^{tw}}$ coincides with the orbifold GW obstruction theory $\tilde{E}^\bullet \rightarrow L_{\mathcal{K}_{g,n}(\mathcal{G},\beta)/\mathfrak{M}_{g,n}^{tw}}$.*

Corollary 6.4. *For any β -admissible vector \vec{g} , the induced morphism $s^*E^\bullet \rightarrow L_{\mathcal{K}_{g,n}(\mathcal{G},\beta)^{\vec{g}}/\mathfrak{M}_{g,n}^{tw}}$ coincides with the orbifold GW obstruction theory $\tilde{E}^\bullet \rightarrow L_{\mathcal{K}_{g,n}(\mathcal{G},\beta)^{\vec{g}}/\mathfrak{M}_{g,n}^{tw}}$.*

PROOF. $\mathcal{K}_{g,n}(\mathcal{G},\beta)^{\vec{g}} \hookrightarrow \mathcal{K}_{g,n}(\mathcal{G},\beta)$ is an open immersion. \square

6.2. Proof of the pushforward formula. By Corollary 6.4 the natural perfect relative obstruction theory \tilde{E}^\bullet over $\mathcal{K}_{g,n}(\mathcal{G},\beta)^{\vec{g}}$ is quasi-isomorphic to the pull-back of the natural perfect relative obstruction theory E^\bullet over $\overline{M}_{g,n}(X,\beta)$. We know that such obstruction theories admit global resolutions. We choose a length 2 complex of vector bundles F^\bullet quasi isomorphic to E^\bullet . Its pullback to $\mathcal{K}_{g,n}(\mathcal{G},\beta)^{\vec{g}}$, that we denote by \tilde{F}^\bullet , is a global resolution of \tilde{E}^\bullet . We denote the dual complexes by \tilde{F}_\bullet and F_\bullet . We have a cartesian diagram (put $\mathcal{K} = \mathcal{K}_{g,n}(\mathcal{G},\beta)^{\vec{g}}$, $\overline{M} = \overline{M}_{g,n}(X,\beta)$)

$$(52) \quad \begin{array}{ccc} \tilde{C} & \xrightarrow{p''} & C \\ \downarrow & \square & \downarrow \\ \tilde{F}_1 & \xrightarrow{p'} & F_1 \\ \tilde{\pi} \downarrow & \square & \downarrow \pi \\ \mathcal{K} & \xrightarrow{p} & \overline{M} \end{array}$$

where $\tilde{C} = \tilde{F}_1 \times_{[\tilde{F}_1/\tilde{F}_0]} \mathfrak{C}_{\mathcal{K}/\mathfrak{M}^{tw}}$ and $C = F_1 \times_{[F_1/F_0]} \mathfrak{C}_{\overline{M}/\mathfrak{M}}$. We observe that \tilde{F}_1 and F_1 are vector bundles. Hence the intersections with the zero sections provide isomorphisms $0_{\mathcal{K}}^! : A_*(\tilde{F}_1)_{\mathbb{Q}} \xrightarrow{\sim} A_*(\mathcal{K})_{\mathbb{Q}}$ and $0_{\overline{M}}^! : A_*(F_1)_{\mathbb{Q}} \xrightarrow{\sim} A_*(\overline{M})_{\mathbb{Q}}$. By compatibility of Gysin morphisms with proper pushforward we get that for any $\alpha \in A_*(\tilde{F}_1)_{\mathbb{Q}}$

$$(53) \quad p_* 0_{\mathcal{K}}^! \alpha = 0_{\overline{M}}^! p'_* \alpha.$$

We argue in the spirit of [15] that $p_*[\mathcal{K}]^{vir} = d[\overline{M}]^{vir}$ for some $d \in \mathbb{Q}$ if and only if

$$(54) \quad p'_*[\tilde{C}] = d[C] \in A_*(F_1)_{\mathbb{Q}}.$$

We will show that \tilde{C} is the pullback of C and that

$$(55) \quad p'_*[\tilde{C}] = p'_*p'^*[C] = r^{2g-1}[C] \in A_*(F_1)_{\mathbb{Q}}.$$

Proposition 6.5. *Consider diagram (41). The intrinsic normal cone $\mathfrak{C}_{\mathcal{K}/\mathfrak{M}^{tw}}$ is isomorphic to the pullback of the relative intrinsic normal cone $\mathfrak{C}_{\overline{M}/\mathfrak{M}}$, where $\mathcal{K} = \mathcal{K}_{g,n}(\mathcal{G},\beta)^{\vec{g}}$, $\overline{M} = \overline{M}_{g,n}(X,\beta)$, $\mathfrak{M}^{tw} = \mathfrak{M}_{g,n}^{tw}$ and $\mathfrak{M} = \mathfrak{M}_{g,n}$.*

PROOF. Note first that the relative intrinsic normal cone $\mathfrak{C}_{P_{g,n}/\mathfrak{M}^{tw}}$ is isomorphic to the pullback of $\mathfrak{C}_{\overline{M}/\mathfrak{M}}$ because q is flat. This follows from [11, Proposition 7.2]. We claim that $\mathfrak{C}_{\mathcal{K}/\mathfrak{M}^{tw}}$ is isomorphic to $t^*\mathfrak{C}_{P_{g,n}/\mathfrak{M}^{tw}}$. By the functorial properties of the intrinsic normal cone there is a natural morphism

$$(56) \quad \mathfrak{C}_{\mathcal{K}/\mathfrak{M}^{tw}} \rightarrow t^*\mathfrak{C}_{P_{g,n}/\mathfrak{M}^{tw}}.$$

Being an isomorphism is a local property. We will show that such a morphism is an isomorphism in local charts. Any local embedding of \mathcal{K} over $\mathfrak{M}_{g,n}^{tw}$ is also a local embedding for $P_{g,n}$ over $\mathfrak{M}_{g,n}^{tw}$ because t is étale. Let (U, M) such a local embedding. Restrict the natural morphism (56) to U . By the local description of the intrinsic relative normal cone we get

$$(57) \quad \begin{array}{ccc} \mathfrak{C}_{\mathcal{K}/\mathfrak{M}^{tw}}|_U & \longrightarrow & t^* \mathfrak{C}_{P_{g,n}/\mathfrak{M}^{tw}}|_U \\ \downarrow \sim & & \downarrow \sim \\ [C_U/T_M/\mathfrak{M}^{tw}] & \xrightarrow{\sim} & [C_U/T_M/\mathfrak{M}^{tw}]. \end{array}$$

□

In order to prove the equality in (55) we will show that p , hence p' is of pure degree. We will use the facts recalled in Appendix A. We make a few preliminary remarks. The morphism $t : \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}} \rightarrow P_{g,n}^{\vec{g}}$ is étale by Proposition 5.1. The morphism $g : \mathfrak{M}_{g,n}^{tw} \rightarrow \mathfrak{M}_{g,n}$ is flat by [33]. Moreover $p : \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}} \rightarrow \overline{M}_{g,n}(X, \beta)$ is proper because it is a morphism between proper stacks and is quasi finite because e.g. it is a composition of two quasi finite morphisms. Indeed, the image of $\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}}$ in $\mathfrak{M}_{g,n}^{tw}$ factors through the open substack where the orders of isotropy groups of points in the boundary divisors divide r . Such an open substack is of finite type and quasi-finite over $\mathfrak{M}_{g,n}$. Therefore q' is also quasi-finite. The morphism t is quasi-finite because it is étale. Finally, $p = q' \circ t$, hence p is quasi-finite.

Proposition 6.6. *With the notations in diagram (52)*

$$(58) \quad p'_*[\tilde{C}] = r^{2g-1}[C].$$

PROOF. Let us denote by C^r the reduced substack of an irreducible component of C . Let $\tilde{C}' = p^{-1}(C^r)$ and let \tilde{C}^r be the reduced substack of \tilde{C} and \tilde{C}' . Let $m(C)$, $m(\tilde{C}')$ denote the geometric multiplicities. Equation (58) can be written as

$$(59) \quad \begin{aligned} p'_* p'^*[C] &= m(C)m(\tilde{C}')p'_*[\tilde{C}^r] = m(C)m(\tilde{C}') \deg(\tilde{C}^r/C^r)[C^r] \\ &= \deg(\tilde{C}'/C^r)[C]. \end{aligned}$$

We compute the degree of \tilde{C}' over C^r by using the characterization given in Appendix A. Assume that C^r factors through an integral substack $\overline{M}(\tau)^r$ of $\overline{M}_{g,n}(X, \beta)$ such that its generic geometric points correspond to stable maps with domain curve of dual graph τ . Let $\mathcal{K}(\tau)$ be the preimage of $\overline{M}(\tau)^r$ along p . The dimension of the ring $K(\mathcal{K})$ over $K(\overline{M}(\tau)^r)$, $[K(\mathcal{K}) : K(\overline{M}(\tau)^r)]$, is equal to the number of points weighted with multiplicity in a generic geometric fiber of $\overline{p} : \overline{\mathcal{K}}(\tau) \rightarrow \overline{M}(\tau)^{red}$, which is the relative coarse moduli space of $p : \mathcal{K}(\tau) \rightarrow \overline{M}(\tau)^{red}$ (see Appendix A). Let $E_{\tau}^{loop} \subset E_{\tau}$ be the subset of looping edges. Let $P(\tau) = P_{g,n} \times_{\overline{M}_{g,n}(X, \beta)} \overline{M}(\tau)^{red}$. Let $\overline{x} : \text{Spec } \mathbb{C} \rightarrow \overline{M}(\tau)^{red}$ be a generic geometric point. Let $P(\tau)(\overline{x})$ be the fiber of $P(\tau) \rightarrow \overline{M}(\tau)^{red}$ over \overline{x} . Then

$$(60) \quad \mathcal{K}(\tau) \times_{\overline{M}(\tau)^{red}} \text{Spec } \mathbb{C} \simeq \mathcal{K}(\tau) \times_{P(\tau)} P(\tau)(\overline{x}).$$

Let $A(\tau)$ be the set of all \vec{g} -compatible gerby graphs over τ . For any $\tilde{\tau}$ in $A(\tau)$ and for any $i = 1, \dots, |E_{\tilde{\tau}}|$ let $s_i(\tilde{\tau}) = \gamma(e_i)$, $e_i \in E_{\tilde{\tau}}$. Then

$$(61) \quad P(\tau)(\overline{x}) = \coprod_{\tilde{\tau} \in A(\tau)} [\text{Spec } \mathbb{C}[x_1, \dots, x_r]/(x_i^{s_i(\tilde{\tau})})/\mu(\tilde{\tau})] := \coprod_{\tilde{\tau} \in A(\tau)} W(\tilde{\tau}),$$

where $\mu(\tilde{\tau}) = \mu_{s_1(\tilde{\tau})} \times \dots \times \mu_{s_r(\tilde{\tau})}$ and $(\epsilon_1, \dots, \epsilon_r)$ in $\mu(\tilde{\tau})$ acts on (x_1, \dots, x_r) taking it to $(\epsilon_1 x_1, \dots, \epsilon_r x_r)$. Let $J(\tau) \subset A(\tau)$ be the set of \vec{g} -compatible gerby graphs $\tilde{\tau}$ over τ such that for any $e_i \in E_{\tilde{\tau}}$, $\gamma(e_i)$ divides r . Let $I(\tilde{\tau})$ be the number of non-isomorphic line bundles over a twisted curve of gerby dual graph $\tilde{\tau}$. For any $\tilde{\tau} \in J(\tau)$ there is a cartesian diagram

$$(62) \quad \begin{array}{ccccc} \coprod_{l(\tilde{\tau})} (B\mu_r)_{l(\tilde{\tau})} & \longrightarrow & \coprod_{l(\tilde{\tau})} Z(\tilde{\tau})_{l(\tilde{\tau})} & \longrightarrow & \mathcal{K}(\tau)(\bar{x}) \\ \downarrow & & \downarrow & & \downarrow \\ & & \coprod_{l(\tilde{\tau})} W(\tilde{\tau})_{l(\tilde{\tau})} & \longrightarrow & \mathcal{K}(\tau) \not\!/\!/\mu_r(\bar{x}) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } \mathbb{C} & \longrightarrow & W(\tilde{\tau}) & \longrightarrow & P(\tau)(\bar{x}) \end{array}$$

where $1 \leq l(\tilde{\tau}) \leq I(\tilde{\tau})$, $(B\mu_r)_{l(\tilde{\tau})} = (B\mu_r)$, $Z(\tilde{\tau})_{l(\tilde{\tau})} = Z(\tilde{\tau})$ and $W(\tilde{\tau})_{l(\tilde{\tau})} = W(\tilde{\tau})$. By Lemma 5.2 the upper vertical arrows are the structure morphisms of a μ_r -gerbe and the lower vertical arrows are étale and representable. In our case, as can be checked by using groupoid presentations⁵, the lower central arrow is a trivial cover. The upper left corner is a disjoint union of $I(\tilde{\tau})$ trivial gerbes. We recall that $I(\tilde{\tau})$ is the number of non isomorphic r -torsion line bundles over a twisted curve of dual graph $\tilde{\tau}$ associated to a fixed \vec{g} and with fibers carrying faithful representations of the isotropy groups of the nodes. Connected components of $\mathcal{K}(\tau)(\bar{x}) \times_{P(\tau)(\bar{x})} W(\tilde{\tau})$ are labeled by the corresponding connected component in $\coprod_{l(\tilde{\tau})} (B\mu_r)_{l(\tilde{\tau})}$. Note that for any $\tilde{\tau}$ the (relative) coarse moduli space of $W(\tilde{\tau})$ (over $\text{Spec } \mathbb{C}$) is $\text{Spec } \mathbb{C}$ and for any $\tilde{\tau}$ and any $l(\tilde{\tau})$ the same is true for $Z(\tilde{\tau})_{l(\tilde{\tau})}$. Therefore the following diagram is cartesian

$$(63) \quad \begin{array}{ccc} \coprod_{\tilde{\tau} \in J(\tau)} \coprod_{l(\tilde{\tau})} Z(\tilde{\tau})_{l(\tilde{\tau})} & \longrightarrow & \mathcal{K}(\tau) \\ \pi_{\bar{x}} \downarrow & \square & \downarrow \pi \\ \coprod_{\tilde{\tau} \in J(\tau)} \coprod_{l(\tilde{\tau})} (\text{Spec } \mathbb{C})_{l(\tilde{\tau})} & \longrightarrow & \overline{\mathcal{K}}(\tau) \\ \bar{p}_{\bar{x}} \downarrow & \square & \downarrow \bar{p} \\ \text{Spec } \mathbb{C} & \xrightarrow{\bar{x}} & \overline{M}(\tau)^r \end{array}$$

Here $\bar{p} : \overline{\mathcal{K}} \rightarrow \overline{M}(\tau)^r$ is the relative coarse moduli space for p and $\bar{p}_{\bar{x}}$ is the relative coarse moduli space of $\bar{p}_x \circ \pi_{\bar{x}}$ because the formation of the relative coarse moduli space commutes with arbitrary base change for tame stacks. The number of points in the fiber of $\bar{p}_{\bar{x}}$ is computed in Lemma 6.7 below. It is equal to r^{2g} and it does not depend on τ . According to the degree formula in Appendix A, in order to compute the degree of p we need to multiply by the number $\delta(\mathcal{K}_{g,n}(\mathcal{G}, \beta)) / \delta(\overline{M}_{g,n}(X, \beta))$ in case there are non trivial generic stabilizers. Note that for any $T \rightarrow \mathcal{K}_{g,n}(\mathcal{G}, \beta)$ corresponding to an object $[\mathcal{C} \rightarrow \mathcal{G}]$ over $[C \rightarrow X]$, there is a surjection of sheaves of groups $\underline{Aut}_T(\mathcal{C} \rightarrow \mathcal{G}) \rightarrow \underline{Aut}_T(C \rightarrow X)$. Hence $\delta(\mathcal{K}_{g,n}(\mathcal{G}, \beta)) / \delta(\overline{M}_{g,n}(X, \beta))$ is equal to the degree of the relative inertia $I(\mathcal{K}/\overline{M})$. This is in turn equal to $\deg I(\mathcal{K}/\overline{\mathcal{K}}) = r$, because $\mathcal{K} \rightarrow \overline{\mathcal{K}}$ is a μ_r -gerbe. By putting all together we get

$$(64) \quad \deg p = r^{2g-1}.$$

⁵Artinian local rings with residue field \mathbb{C} admit only trivial étale covers

□

Lemma 6.7. *Consider the morphism $\bar{p} : \overline{\mathcal{K}}(\tau) \rightarrow \overline{\mathcal{M}}(\tau)$ defined in Proposition 6.6. Let $\bar{x} : \text{Spec } \mathbb{C} \rightarrow \overline{\mathcal{M}}(\tau)^r$ be a generic geometric point. Then the number of geometric points of $\overline{\mathcal{K}}(\tau)$ over \bar{x} is equal to r^{2g} .*

PROOF. We first observe that geometric points of $\overline{\mathcal{K}}(\tau)$ are in bijection with geometric points of $\mathcal{K}(\tau)$. Therefore we have to count the number of (non isomorphic) twisted curves \mathcal{C} over C with \vec{g} -compatible gerby dual graph over τ admitting twisted stable maps to \mathcal{G} . Moreover, for any such \mathcal{C} we have to count the number of non-isomorphic twisted stable maps to \mathcal{G} . Note first that a twisted curve over $\text{Spec } \mathbb{C}$ is determined by its coarse moduli space and by the order of the isotropy groups of its special points⁶. Let us start by computing the number of \vec{g} -compatible gerby graphs $\tilde{\tau}$ over τ such that the corresponding \mathcal{C} admits a twisted stable map to \mathcal{G} . By Proposition 4.12 we know that any twisted curve with \vec{g} -compatible $\tilde{\tau}$ such that for any $e_i \in E_{\tilde{\tau}}^{loop}$ $\gamma(e_i)$ divides r admits twisted stable maps to \mathcal{G} . Choose a line bundle L such that $f^*\mathcal{G} \simeq \sqrt{L/C}$. We fix a set of line bundles satisfying the properties listed in Property 3.15 and we use the same notations we used there. After making this choice, the admissible vector \vec{g} determines a line bundle \mathcal{S} over \mathcal{C} such that $\deg L \otimes \mathcal{S}^{-r}$ is multiple of r . Explicitly, $\mathcal{S} \simeq \otimes_{i=1}^n \mathcal{J}_i^{m_i} \otimes \otimes_{j \in E_{\tilde{\tau}}} \mathcal{Q}_j^{t_j}$ where for all $e_i \in F_{\tilde{\tau}} \setminus E_{\tilde{\tau}}$, $(m_i, \gamma(e_i)) = 1$ and for all $e_j \in E_{\tilde{\tau}} \setminus E_{\tilde{\tau}}^{loop}$, $(\gamma(e_j), t_j) = 1$. Any line bundle defined as

$$(65) \quad \mathcal{N}_{\vec{s}} = \mathcal{S} \otimes \mathcal{R}_{\vec{s}} = \mathcal{S} \otimes \otimes_{e_l \in E_{\tilde{\tau}}^{loop}} \mathcal{Q}_l^{s_l}, \quad (s_l, \gamma(e_l)) = 1 \quad \forall e_l \in E_{\tilde{\tau}}^{loop}$$

corresponds to a twisted stable map $\mathcal{C} \rightarrow \sqrt{L/C} \simeq f^*\mathcal{G}$. Indeed by definition $\mathcal{R}_{\vec{s}}^{\otimes r} \simeq \mathcal{O}_{\mathcal{C}}$. The condition $(s_l, \gamma(e_l)) = 1$ as usual implies the representability of the corresponding morphism. By tensoring \mathcal{S} with different possible choices of line bundles of the form $\mathcal{R}_{\vec{s}}$ we get different isomorphism classes of twisted stable maps. The number of choices of $\mathcal{R}_{\vec{s}}$ is computed by

$$(66) \quad N(r_1, \dots, r_k) = \prod_{l=1}^k \phi(r_l),$$

where $k = |E_{\tilde{\tau}}^{loop}|$, $r_l = \gamma(e_l)$ for any $e_l \in E_{\tilde{\tau}}^{loop}$ and $\phi(-)$ is the Euler totient function⁷. Moreover, by tensoring any $\mathcal{N}_{\vec{s}}$ with different possible choices of r -torsion line bundles in $\text{Pic } C$ we get $r^{2g-b_1(\tau)}$ non isomorphic twisted stable maps. Finally, in order to get the total number of geometric points over \bar{x} we have to sum over all $\tilde{\tau} \in J(\tilde{\tau})$ with $J(\tilde{\tau})$ defined in Proposition 6.6. This is the same as summing the numbers $N(r_1, \dots, r_n)$ over all possible n -tuples (r_1, \dots, r_n) of integers dividing r . Let $|E_{\tilde{\tau}}^{loop}| = k$. By the properties of Euler totient function we get

$$(67) \quad \tilde{N} = \sum_{r_1, \dots, r_k | r} N(r_1, \dots, r_k) = \sum_{r_1, \dots, r_k | r} \prod_{l=1}^k \phi(r_l) = r^k = r^{b_1(\tau)}.$$

⁶This can be understood e.g. by considering the characterization of twisted curves in terms of log-twisted curves given in [33] and by observing that the only isomorphism class of locally free log structure over $\text{Spec } \mathbb{C}$ corresponds to a monoid of the form \mathbb{N}^r , where $r \in \mathbb{N}$ is equal to the number of nodes of the coarse moduli space C

⁷For any $n \in \mathbb{N}$, the Euler totient function $\phi(n)$ is defined to be the number of integers less than n and coprime with n . The following property holds: $\sum_{d|n} \phi(d) = n$.

Finally, we get the total number of points over \bar{x} as

$$(68) \quad N_{tot} = \tilde{N} \times r^{2g-b_1(\tau)} = r^{2g}.$$

Note that this number does not depend on the dual graph τ . \square

Combining the above discussions, we get the desired pushforward formula.

Theorem 6.8. *Let \vec{g} be a β -admissible vector. Let $p : \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}} \rightarrow \overline{M}_{g,n}(X, \beta)$ be the morphism defined in diagram (41). Then*

$$(69) \quad p_*[\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}}]^{vir} = r^{2g-1}[\overline{M}_{g,n}(X, \beta)]^{vir} \in H^*(\overline{M}_{g,n}(X, \beta), \mathbb{Q}).$$

PROOF. It follows from arguments at the beginning of Section 6.2 and from Proposition 6.6. \square

7. ORBIFOLD GROMOV-WITTEN THEORY OF BANDED GERBES

In this Section we examine the Gromov-Witten invariants of the gerbe \mathcal{G} using results in previous sections. In particular in Theorem 7.3 we prove the decomposition conjecture for \mathcal{G} .

7.1. Orbifold Gromov-Witten invariants. Let

$$\epsilon : \mathcal{G} \rightarrow X$$

be a G -banded gerbe with G a finite abelian group over X . Let $\alpha \in H^2(X, G)$ be the isomorphism class of \mathcal{G} (as a G -banded gerbe). Since the gerbe is G -banded there is a canonical isomorphism

$$(70) \quad I\mathcal{G} = G \times_X \mathcal{G} \simeq \coprod_{g \in G} \mathcal{G}_g,$$

where \mathcal{G}_g is a root gerbe isomorphic to \mathcal{G} . Let $\epsilon_g : \mathcal{G}_g \rightarrow X$ be the induced morphism. On each component there is an isomorphism between the rational cohomology groups

$$\epsilon_g^* : H^*(X, \mathbb{Q}) \xrightarrow{\simeq} H^*(\mathcal{G}_g, \mathbb{Q}).$$

To simplify notation, from now on we assume $G = \mu_r$ for some $r \in \mathbb{N}$. The discussion for the general G requires only notational changes. The genus 0 case is spelled out explicitly in [8, Appendix A]. The higher genus case is similar.

Let $\vec{g} = (g_1, \dots, g_n)$ be a β -admissible vector. There are evaluation maps

$$ev_i : \mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}} \rightarrow \bar{I}(\mathcal{G})_{g_i},$$

where $\bar{I}(\mathcal{G})_{g_i}$ is a component of the *rigidified inertia stack* $\bar{I}(\mathcal{G}) = \cup_{g \in \mu_r} \bar{I}(\mathcal{G})_g$. Although the evaluation maps ev_i do not take values in $I\mathcal{G}$, as explained in [3], Section 6.1.3, one can still define a pull-back map at cohomology level,

$$ev_i^* : H^*(\mathcal{G}_{g_i}, \mathbb{Q}) \rightarrow H^*(\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}}, \mathbb{Q}).$$

Given $\delta_i \in H^*(\mathcal{G}_{g_i}, \mathbb{Q})$ for $1 \leq i \leq n$ and integers $k_i \geq 0, 1 \leq i \leq n$, one can define descendant orbifold Gromov-Witten invariants

$$\langle \delta_1 \bar{\psi}_1^{k_1}, \dots, \delta_n \bar{\psi}_n^{k_n} \rangle_{g,n,\beta}^{\mathcal{G}} := \int_{[\mathcal{K}_{g,n}(\mathcal{G}, \beta)^{\vec{g}}]^{vir}} \prod_{i=1}^n ev_i^*(\delta_i) \bar{\psi}_i^{k_i},$$

where $\bar{\psi}_i$ are the pullback of the first Chern classes of the tautological line bundles over $\overline{M}_{g,n}(X, \beta)$ (which by abuse of notation we also denote by $\bar{\psi}_i$).

For classes $\delta_i \in H^*(\mathcal{G}_i, \mathbb{Q})$, set $\bar{\delta}_i = (\epsilon_{g_i}^*)^{-1}(\delta_i)$. Descendant Gromov-Witten invariants $\langle \bar{\delta}_1 \bar{\psi}_1^{k_1}, \dots, \bar{\delta}_n \bar{\psi}_n^{k_n} \rangle_{0,n,\beta}^X$ of X are similarly defined. Theorem 6.8 implies the following comparison result.

Theorem 7.1.

$$\langle \delta_1 \bar{\psi}_1^{k_1}, \dots, \delta_n \bar{\psi}_n^{k_n} \rangle_{g,n,\beta}^{\mathcal{G}} = r^{2g-1} \langle \bar{\delta}_1 \bar{\psi}_1^{k_1}, \dots, \bar{\delta}_n \bar{\psi}_n^{k_n} \rangle_{g,n,\beta}^X.$$

Moreover, if \vec{g} is not admissible, then the Gromov-Witten invariants of \mathcal{G} vanish.

PROOF. Denote by $\bar{e}v_i : \overline{M}_{g,n}(X, \beta) \rightarrow X$ the i -th evaluation map. Using the definition of ev_i^* one can check that $ev_i^*(\delta_i) = p^* \bar{e}v_i^*(\bar{\delta}_i)$. Note also that $p^* \bar{\psi}_i = \bar{\psi}_i$. Thus using Theorem 6.8 we have

$$\begin{aligned} \langle \delta_1 \bar{\psi}_1^{k_1}, \dots, \delta_n \bar{\psi}_n^{k_n} \rangle_{g,n,\beta}^{\mathcal{G}} &= \int_{[\mathcal{X}_{g,n}(\mathcal{G}, \beta)]^{\vec{g}} \text{vir}} \prod_{i=1}^n ev_i^*(\delta_i) \bar{\psi}_i^{k_i} \\ &= \int_{[\mathcal{X}_{g,n}(\mathcal{G}, \beta)]^{\vec{g}} \text{vir}} \prod_{i=1}^n p^* \bar{e}v_i^*(\bar{\delta}_i) \bar{\psi}_i^{k_i} \\ &= \int_{[\mathcal{X}_{g,n}(\mathcal{G}, \beta)]^{\vec{g}} \text{vir}} \prod_{i=1}^n p^*(\bar{e}v_i^*(\bar{\delta}_i) \bar{\psi}_i^{k_i}) \\ &= r^{2g-1} \int_{[\overline{M}_{g,n}(X, \beta)]^{\text{vir}}} \prod_{i=1}^n \bar{e}v_i^*(\bar{\delta}_i) \bar{\psi}_i^{k_i} \\ &= r^{2g-1} \langle \bar{\delta}_1 \bar{\psi}_1^{k_1}, \dots, \bar{\delta}_n \bar{\psi}_n^{k_n} \rangle_{g,n,\beta}^X. \end{aligned}$$

□

7.2. Decomposition of Gromov Witten theory. In the following we use complex numbers \mathbb{C} as coefficients for the cohomology. For $\bar{\alpha} \in H^*(X, \mathbb{C})$ and an irreducible representation ρ of G , we define

$$\bar{\alpha}_\rho := \frac{1}{r} \sum_{g \in G} \chi_\rho(g^{-1}) \epsilon_g^*(\bar{\alpha}),$$

where χ_ρ is the character of ρ . The map $(\bar{\alpha}, \rho) \mapsto \bar{\alpha}_\rho$ clearly defines an additive isomorphism

$$(71) \quad \bigoplus_{[\rho] \in \widehat{G}} H^*(X)_{[\rho]} \simeq H^*(I\mathcal{G}, \mathbb{C}),$$

where \widehat{G} is the set of isomorphism classes of irreducible representations of G , and for $[\rho] \in \widehat{G}$ we define $H^*(X)_{[\rho]} := H^*(X, \mathbb{C})$.

Theorem 7.1 together with orthogonality relations of characters of G implies the following

Theorem 7.2. *Given $\bar{\alpha}_1, \dots, \bar{\alpha}_n \in H^*(X, \mathbb{Q})$ and integers $k_1, \dots, k_n \geq 0$, we have*

$$\begin{aligned} &\langle \bar{\alpha}_{1\rho_1} \bar{\psi}_1^{k_1}, \dots, \bar{\alpha}_{n\rho_n} \bar{\psi}_n^{k_n} \rangle_{g,n,\beta}^{\mathcal{G}} \\ &= \begin{cases} r^{2g-2} \langle \bar{\alpha}_1 \bar{\psi}_1^{k_1}, \dots, \bar{\alpha}_n \bar{\psi}_n^{k_n} \rangle_{g,n,\beta}^X \chi_\rho(\exp(\frac{-2\pi\sqrt{-1}(\alpha \cap \beta)}{r})) & \text{if } \rho_1 = \rho_2 = \dots = \rho_n =: \rho, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(Recall that $\alpha \in H^2(X, \mu_r)$ is the class of the gerbe $\mathcal{G} \rightarrow X$.)

PROOF. By our definition we have

$$\begin{aligned} & \langle \bar{\alpha}_{1\rho_1} \bar{\psi}_1^{k_1}, \dots, \bar{\alpha}_{n\rho_n} \bar{\psi}_n^{k_n} \rangle_{g,n,\beta}^{\mathcal{G}} \\ &= r^{2g-1} \sum_{g_1, \dots, g_n \in \mu_r} \prod_{i=1}^n \chi_{\rho_i}(g_i^{-1}) \langle \prod_{i=1}^n \epsilon_{g_i}^*(\bar{\alpha}_i) \bar{\psi}_i^{k_i} \rangle_{g,n,\beta}^{\mathcal{G}}. \end{aligned}$$

The term associated to $\vec{g} := (g_1, \dots, g_n)$ in the above sum vanishes unless \vec{g} is an admissible vector. This implies that $\prod_{i=1}^n g_i = \exp(\frac{2\pi\sqrt{-1}(\alpha \cap \beta)}{r})$. We rewrite this as $g_n^{-1} = \exp(\frac{-2\pi\sqrt{-1}(\alpha \cap \beta)}{r}) \prod_{i=1}^{n-1} g_i$. Substitute this into above equation and use Theorem 7.1 to get

$$\begin{aligned} & \langle \bar{\alpha}_{1\rho_1} \bar{\psi}_1^{k_1}, \dots, \bar{\alpha}_{n\rho_n} \bar{\psi}_n^{k_n} \rangle_{g,n,\beta}^{\mathcal{G}} \\ &= r^{2g-1} \sum_{g_1, \dots, g_{n-1} \in \mu_r} \chi_{\rho_n}(\exp(\frac{-2\pi\sqrt{-1}(\alpha \cap \beta)}{r})) \left(\prod_{i=1}^{n-1} \chi_{\rho_i}(g_i^{-1}) \chi_{\rho_n}(g_i) \right) \frac{1}{r} \langle \prod_{i=1}^n \bar{\alpha}_i \bar{\psi}_i^{k_i} \rangle_{g,n,\beta}^X. \end{aligned}$$

Applying the orthogonality condition

$$\frac{1}{r} \sum_{g \in \mu_r} \chi_{\rho}(g^{-1}) \chi_{\rho'}(g) = \delta_{\rho, \rho'},$$

we find

$$\begin{aligned} & \langle \bar{\alpha}_{1\rho_1} \bar{\psi}_1^{k_1}, \dots, \bar{\alpha}_{n\rho_n} \bar{\psi}_n^{k_n} \rangle_{g,n,\beta}^{\mathcal{G}} \\ &= \frac{1}{r} \chi_{\rho_n}(\exp(\frac{-2\pi\sqrt{-1}(\alpha \cap \beta)}{r})) \left(\prod_{i=1}^{n-1} \delta_{\rho_i, \rho_n} \right) r^{2g-1} \langle \prod_{i=1}^n \bar{\alpha}_i \bar{\psi}_i^{k_i} \rangle_{g,n,\beta}^X. \end{aligned}$$

The result follows. \square

We now reformulate this in terms of generating functions. Let

$$\{\bar{\phi}_i \mid 1 \leq i \leq \text{rank} H^*(X, \mathbb{C})\} \subset H^*(X, \mathbb{C})$$

be an additive basis. According to the discussion above, the set

$$\{\bar{\phi}_{i\rho} \mid 1 \leq i \leq \text{rank} H^*(X, \mathbb{C}), [\rho] \in \widehat{G}\}$$

is an additive basis of $H^*(\mathcal{I}\mathcal{G}, \mathbb{C})$. Recall that the genus g descendant potential of \mathcal{G} is defined to be

$$(72) \quad \begin{aligned} & \mathcal{F}_{\mathcal{G}}^g(\{t_{i\rho, j}\}_{1 \leq i \leq \text{rank} H^*(X, \mathbb{C}), \rho \in \widehat{G}, j \geq 0}; Q) := \\ & \sum_{\substack{n \geq 0, \beta \in H_2(X, \mathbb{Z}) \\ i_1, \dots, i_n; \rho_1, \dots, \rho_n; j_1, \dots, j_n}} \frac{Q^\beta}{n!} \prod_{k=1}^n t_{i_k \rho_k, j_k} \langle \prod_{k=1}^n \bar{\phi}_{i_k \rho_k} \bar{\psi}_k^{j_k} \rangle_{g,n,\beta}^{\mathcal{G}}. \end{aligned}$$

The descendant potential $\mathcal{F}_{\mathcal{G}}^g$ is a formal power series in variables $t_{i\rho, j}$, $1 \leq i \leq \text{rank} H^*(X, \mathbb{C})$, $\rho \in \widehat{G}$, $j \geq 0$ with coefficients in the Novikov ring $\mathbb{C}[[\overline{NE}(X)]]$, where $\overline{NE}(X)$ is the Mori cone of the coarse moduli space of \mathcal{G} . Here Q^β are formal variables labeled by classes $\beta \in \overline{NE}(X)$. See e.g. [38] for more discussion on descendant potentials for orbifold Gromov-Witten theory.

Similarly the genus g descendant potential of X is defined to be

$$(73) \quad \mathcal{F}_X^g(\{t_{i, j}\}_{1 \leq i \leq \text{rank} H^*(X, \mathbb{C}), j \geq 0}; Q) := \sum_{\substack{n \geq 0, \beta \in H_2(X, \mathbb{Z}) \\ i_1, \dots, i_n; j_1, \dots, j_n}} \frac{Q^\beta}{n!} \prod_{k=1}^n t_{i_k, j_k} \langle \prod_{k=1}^n \bar{\phi}_{i_k} \bar{\psi}_k^{j_k} \rangle_{0,n,\beta}^X.$$

The descendant potential \mathcal{F}_X^g is a formal power series in variables $t_{i,j}$, $1 \leq i \leq \text{rank} H^*(X, \mathbb{C})$, $j \geq 0$ with coefficients in $\mathbb{C}[[\overline{NE}(X)]]$ and Q^β is (again) a formal variable. Theorem 7.2 may be restated as follows.

Theorem 7.3.

$$\mathcal{F}_g^g(\{t_{i\rho,j}\}_{1 \leq i \leq \text{rank} H^*(X, \mathbb{C}), \rho \in \widehat{G}, j \geq 0}; Q) = r^{2g-2} \sum_{[\rho] \in \widehat{G}} \mathcal{F}_X^g(\{t_{i\rho,j}\}_{1 \leq i \leq \text{rank} H^*(X, \mathbb{C}), j \geq 0}; Q_\rho),$$

where Q_ρ is defined by the following rule:

$$Q_\rho^\beta := Q^\beta \chi_\rho \left(\exp \left(\frac{-2\pi\sqrt{-1}(\alpha \cap \beta)}{r} \right) \right),$$

and χ_ρ is the character associated to the representation ρ .

Theorem 7.3 confirms the decomposition conjecture for genus g Gromov-Witten theory of \mathcal{G} .

APPENDIX A. A FEW USEFUL FACTS ABOUT DEGREE OF MORPHISMS

We want to characterize the degree of proper and quasi-finite morphisms of algebraic stacks in terms of number of points in the generic geometric fibers. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a separated quasi-finite dominant morphism between Deligne-Mumford stacks with \mathcal{Y} integral. Then according to [39] its degree is given by

$$(74) \quad \deg(\mathcal{X}/\mathcal{Y}) = \frac{\delta(\mathcal{Y})}{\delta(\mathcal{X})} [K(\mathcal{X}) : K(\mathcal{Y})],$$

where $\delta\mathcal{X} := \deg(I\mathcal{X}/\mathcal{X})$, $\delta\mathcal{Y} := \deg(I\mathcal{Y}/\mathcal{Y})$. If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is representable, then the degree of the field extension $[K(\mathcal{X}) : K(\mathcal{Y})]$ is equal to the number of points counted with multiplicity in the generic geometric fiber. A similar characterization can be given in the general case for a representable morphism $\bar{f} : \bar{\mathcal{X}} \rightarrow \mathcal{Y}$ canonically determined by f as described below. Let \mathcal{X} and \mathcal{Y} be algebraic stacks. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism locally of finite presentation such that the relative inertia

$$\text{Ker}(I(\mathcal{X}) \rightarrow f^*I(\mathcal{Y})) \simeq \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$$

is finite. Then in [5] it is proved that there exists a factorization of f

$$(75) \quad \mathcal{X} \xrightarrow{\pi} \bar{\mathcal{X}} \xrightarrow{\bar{f}} \mathcal{Y},$$

called the *relative coarse moduli space*, such that

- (1) \bar{f} is representable;
- (2) π is proper and quasi-finite;
- (3) $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\bar{\mathcal{X}}}$;
- (4) for any $\mathcal{X}' \xrightarrow{\pi'} \mathcal{X}' \xrightarrow{f'} \mathcal{Y}$ with f' representable there is a unique morphism $h : \mathcal{X}' \rightarrow \bar{\mathcal{X}}$ such that $\pi' = h \circ \pi$ and $\bar{f} = f' \circ h$.

Moreover the formation of the relative coarse moduli space commutes with flat base change and, if f is tame, with arbitrary base change.

We observe that $\pi_*K(\mathcal{X}) = K(\bar{\mathcal{X}})$ if \mathcal{X} is reduced. Indeed $K(\mathcal{X})$ is the ring of morphisms $\mathcal{X} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ defined over an open dense substack. For any such a morphism there is a factorization $\mathcal{X} \rightarrow \bar{\mathcal{X}} \rightarrow X \rightarrow \mathbb{A}_{\mathbb{C}}^1$, where $\bar{\mathcal{X}}$ is the absolute coarse moduli space of X .

Since we work over the complex numbers and $\bar{f} : \bar{\mathcal{X}} \rightarrow \mathcal{Y}$ is a representable morphism $[K(\bar{\mathcal{X}}) : K(\mathcal{Y})]$ can be computed as the number of points (weighted by the geometric multiplicity) in a generic geometric fiber.

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INTERNATIONAL SCHOOL FOR ADVANCED STUDIES, VIA BEIRUT 2-4, 34100 TRIESTE, ITALY

E-mail address: `andreini.elena@gmail.com`

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, SOUTH KENSINGTON CAMPUS, LONDON SW7 2AZ, UNITED KINGDOM

E-mail address: `y.jiang@imperial.ac.uk`

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, 100 MATH TOWER, 231 WEST 18TH AVE., COLUMBUS, OH 43210, USA

E-mail address: `hhtseng@math.ohio-state.edu`