

On integrable system on S^2 with the second integral quartic in the momenta.

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Abstract

We consider integrable system on the sphere S^2 with an additional integral of fourth order in the momenta. At the special values of parameters this system coincides with the Kowalevski-Goryachev-Chaplygin system.

1 Introduction

Let us consider particle moving on the sphere $S^2 = \{x \in \mathbb{R}^3, |x| = a\}$. Entries of the vector x and angular momentum vector $J = p \times x$ are coordinates on the phase space T^*S^2 with the following Poisson brackets

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \quad \{x_i, x_j\} = 0, \quad (1)$$

where ε_{ijk} is the totally skew-symmetric tensor. The Casimir functions of the brackets (1)

$$A = \sum_{i=1}^3 x_i^2 = a^2, \quad B = \sum_{i=1}^3 x_i J_i = 0, \quad (2)$$

are in the involution with any function on T^*S^2 . The phase space T^*S^2 is four dimensional symplectic manifold. So, for the Liouville integrability of the corresponding equations of motion it is enough to find two functionally independent integrals of motion.

In this note we discuss an integrable system on T^*S^2 possessing integrals of second and fourth order in the momenta J_k . The corresponding Hamilton function has a natural form, i.e. it is a sum of a positive-definite kinetic energy and a potential. So, according to Maupertuis's principle, this natural integrable system on T^*S^2 immediately gives a family of integrable geodesic on S^2 . Integrals of the geodesic are also polynomials of the second and fourth degrees.

Remind that description of all the natural Hamiltonian systems on closed surfaces admitting integrals polynomial in momenta is a classical problem [1]. For the systems with polynomial in momenta integrals of degree one or two there exists a complete description and classification [2].

The Kowalevski top is an example of a conservative system on S^2 which possesses an integral of degree four in momenta [3]. Later Goryachev [4] and Chaplygin [5] found generalization of the Kowalevski system on S^2 . Recently, these results were extended in [6].

The main aim of this note is to consider some another generalization of the Kowalevski-Goryachev-Chaplygin system using the reflection equation theory [7].

2 Generic case

Following to [8] let us consider Lax matrix for the generalized Lagrange system

$$T(\lambda) = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} (\lambda)$$

with the following entries polynomial in the spectral parameter λ

$$\begin{aligned} A(\lambda) &= \lambda^2 - 2\lambda\alpha J_3 + \left(\alpha^2 - f(x_3)\right) J_3^2 - J_1^2 - J_2^2 - g(x_3), \\ B(\lambda) &= (x_1 + ix_2)m(x_3)\lambda + J_3(x_1 + ix_2)\ell(x_3) + (J_1 + iJ_2)n(x_3), \\ D(\lambda) &= -n(x_3)^2. \end{aligned} \quad (3)$$

Here α is an arbitrary numerical parameter, f, g, m, n and ℓ are some functions of x_3 and of the single non-trivial Casimir $a = \sqrt{x_1^2 + x_2^2 + x_3^2}$ (2).

The trace of the matrix $T(\lambda)$

$$\text{tr } T(\lambda) = A(\lambda) + D(\lambda) = \lambda^2 - \lambda H_L + K_L$$

gives rise integrals of motion in the involution for the generalized Lagrange system

$$H_L = 2\alpha J_3, \quad K_L = \left(\alpha^2 - f(x_3)\right) J_3^2 - J_1^2 - J_2^2 - g(x_3) - n(x_3)^2.$$

The corresponding equations of motion may be rewritten in the form of the Lax triad

$$\frac{d}{dt}T(\lambda) = \left[T(\lambda), M(\lambda) \right] + N(\lambda), \quad \text{tr } N(\lambda) = 0.$$

With algebraic point of view coefficients of the trace of $T(\lambda)$ give rise commutative subalgebra in the complete Poisson algebra generated by entries $T_{ij}(\lambda)$. All the generators of this subalgebra are linear polynomials on coefficients of entries $T_{ij}(\lambda)$, which are interpreted as integrals of motion for integrable system associated with matrix $T(\lambda)$.

Some special commutative subalgebras generated by quadratic polynomials on coefficients of $T_{ij}(\lambda)$ were considered in [8]. These subalgebras were associated with five integrable systems on S^2 with an additional integral of motion third order in the momenta.

According to [7, 9], we can try to construct another commutative subalgebra generated by quadratic polynomials on coefficients of $T_{ij}(\lambda)$, which are integrals of motion for another integrable system associated with the same matrix $T(\lambda)$.

Namely, using matrix $T(\lambda)$ (3) and standard machinery of the reflection equation theory [7, 9] we can construct another 2×2 matrix

$$L(\lambda) = \mathcal{K}_-(\lambda) T(\lambda) \mathcal{K}_+(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T^t(-\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (4)$$

with the following trace

$$\text{tr } L(\lambda) = -\lambda^6 - F_1 \lambda^4 - F_2 \lambda^2 - F_3. \quad (5)$$

Here the superscript t stands for matrix transposition and matrices $\mathcal{K}_\pm(\lambda)$ are numerical solutions of the reflection equation associated with the r -matrix of XXX type.

Let us begin with the following partial numerical solutions the reflection equation

$$\mathcal{K}_+(\lambda) = \begin{pmatrix} b_1 \lambda + b_0 & \lambda \\ 0 & -b_1 \lambda + b_0 \end{pmatrix}, \quad \mathcal{K}_-(\lambda) = \mathcal{K}_+^t(\lambda), \quad (6)$$

which depend on two arbitrary parameters b_0 and b_1 only.

Substituting $T(\lambda)$ (3) and \mathcal{K}_\pm (6) into $L(\lambda)$ (4) one gets that function F_3 in (5) depends of variables x_3 and J_3 only. So, if we want to consider integrable system differed from the generalized Lagrange system we have to put $F_3 = \text{const}$. It leads to the following expressions of the functions $f(x_3)$ and $g(x_3)$

$$f(x_3) = \alpha^2 - \frac{2\ell(x_3)x_3}{n(x_3)} + \frac{\ell(x_3)^2(a^2 - x_3^2)}{n(x_3)^2}, \quad g(x_3) = \frac{d}{n(x_3)^2}, \quad (7)$$

where d is arbitrary numerical parameter.

Theorem 1 *If functions $f(x_3)$ and $g(x_3)$ are given by (7) then the third coefficient F_3 in (5) is a constant*

$$F_3 = 2b_0^2 d,$$

*while two remaining coefficients F_1 and F_2 are in the involution on T^*S^2 if and only if*

$$n(x_3, a) = c_1 \sin \left(\alpha \arctan \left(\frac{x_3}{\sqrt{a^2 - x_3^2}} \right) \right) + c_2 \cos \left(\alpha \arctan \left(\frac{x_3}{\sqrt{a^2 - x_3^2}} \right) \right).$$

Here α, c_1, c_2 are arbitrary parameters and all another functions in (3) are equal to

$$\alpha = 0, \quad m = 0, \quad \ell = n \frac{\sqrt{x_3^2 - a^2} - x_3 (\ln(x_3 + \sqrt{x_3^2 - a^2}) + c_3)}{(x_3^2 - a^2)(\ln(x_3 + \sqrt{x_3^2 - a^2}) + c_3)},$$

$$\alpha \neq 0 \quad m = -\frac{n'}{\alpha}, \quad \ell = \frac{(\alpha^2 n - x_3 n')n}{(x_3^2 - a^2)n'},$$

where $n' = \frac{\partial n(x_3, a)}{\partial x_3}$.

The proof is straightforward.

So, two functions F_1 and F_2 are in the involution $\{F_1, F_2\} = 0$ on the phase space T^*S^2 . Moreover, direct calculation yields that they are functionally independent functions on T^*S^2 . It means that these functions F_1 and F_2 define an integrable system on the sphere.

Integrals of motion F_1 and F_2 are quadratic and quartic polynomials in the momenta. For instance, at $\alpha \neq 0$ the corresponding Hamilton function is equal to

$$\begin{aligned}
 H = \frac{F_1}{2} = & J_1^2 + J_2^2 + \left(2\alpha^2 + (a^2 - x_3^2) \frac{\ell^2}{n^2} - 2x_3 \frac{\ell}{n} \right) J_3^2 + 2b_0 \frac{n'}{\alpha} x_1 \\
 & + 2b_1 (J_1 n + J_3 x_1 (\ell - 2n')) + b_1^2 \left(n^2 + \frac{(a^2 - x_3^2) n'^2}{\alpha^2} \right) + \frac{d}{n^2}.
 \end{aligned} \tag{8}$$

For brevity we do not present the second integral of motion F_2 explicitly. This function F_2 may be restored from the definitions (3-5) and conditions of the Theorem 1.

If we consider more generic solutions of the reflection equations

$$\mathcal{K}_+(\lambda) = \begin{pmatrix} b_1 \lambda + b_0 & \lambda \\ 0 & -b_1 \lambda + b_0 \end{pmatrix}, \quad \mathcal{K}_-(\lambda) = \begin{pmatrix} d_1 \lambda + d_0 & 0 \\ \lambda & -d_1 \lambda + d_0 \end{pmatrix},$$

which depend of four parameters, one gets the same integrals of motion up to rescaling of x and rotations

$$x \rightarrow bUx, \quad J \rightarrow UJ, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{pmatrix}, \tag{9}$$

where b and ϕ are the suitable parameters.

Up to such transformations integrals of motion F_1 and F_2 depend of five numerical parameters $\alpha, b_0, b_1, c_1/c_2$ and d . Remind, that another two parametric family of integrable systems on the sphere with fourth order integral of motion was studied in [6]. However, ansatz for the Hamilton function proposed in [6] looks like more restrictive than Hamiltonians (8). The relations between these systems will be studied in the forthcoming publications.

3 Special cases

At $\alpha = 0$ the Hamiltonian reads as

$$\begin{aligned}
 H_0 = \frac{F_1}{2} &= J_1^2 + J_2^2 + \left(\frac{x_3^2}{x_3^2 - a^2} - \frac{1}{(\ln(x_3 + \sqrt{x_3^2 - a^2}) + c_3)^2} \right) J_3^2 \\
 &+ 2b_1c_2 \left(J_1 - \frac{(\ln(x_3 + \sqrt{x_3^2 - a^2}) + c_3)x_3 - \sqrt{x_3^2 - a^2}}{(x_3^2 - a^2)(\ln(x_3 + \sqrt{x_3^2 - a^2}) + c_3)} x_1 J_3 \right).
 \end{aligned} \tag{10}$$

It define new integrable system on the sphere, which depend of two parameters b_1c_2 and c_3 only.

If $\alpha = 1$ or $\alpha = 2$ one gets

$$n = c_1x_3 + c_2\sqrt{a^2 - x_3^2} \quad \text{and} \quad n(x_3) = c_1x_3\sqrt{a^2 - x_3^2} + c_2(a^2 - 2x_3^2),$$

respectively. Even in these particular cases the corresponding Hamiltonian H (8) remains a huge function. We will present it imposing some additional restrictions only.

At $\alpha = 1$ and $c_2 = 0$ the Hamiltonian (8) is equal to

$$H_1 = \frac{F_1}{2} \Big|_{c_2=0} = J_1^2 + J_2^2 + 2J_3^2 + 2b_1c_1(J_1x_3 - 2J_3x_1) + 2b_0c_1x_1 + \frac{d}{c_1^2x_3^2} - b_1^2c_1^2a^2. \tag{11}$$

After canonical transformation

$$J_1 \rightarrow J_1 - c_1b_1x_3, \quad J_2 \rightarrow J_2, \quad J_3 \rightarrow J_3 + c_1b_1x_1, \quad x_k \rightarrow x_k$$

this Hamiltonian H_1 (11) reads as

$$H_1 = J_1^2 + J_2^2 + 2J_3^2 + 2c_1b_0x_1 + c_1^2b_1^2(x_1^2 - x_2^2) + \frac{d}{c_1^2x_3^2}. \tag{12}$$

It is the Hamilton function for the Kowalewski-Chaplygin-Goryashev top. Canonical transformations (9) allow us to rewrite the Hamilton function (12) for the Kowalevski-Goryachev-Chaplygin system in the standard form [10]. The corresponding Lax matrix $L(\lambda)$ (4) was constructed in [10]. The separation of variables associated with this matrix is discussed in [11].

At $\alpha = 1$ and $c_1 = 0$ the corresponding Hamilton function (8) is equal to

$$\begin{aligned}
 \tilde{H}_1 = \frac{F_1}{2} \Big|_{c_1=0} &= J_1^2 + J_2^2 + \frac{2x_3^4 - a^4}{x_3^2(x_3^2 - a^2)} J_3^2 + 2c_2b_0 \frac{x_1x_3}{\sqrt{x_3^2 - a^2}} + \frac{d}{c_2^2(x_3^2 - a^2)} \\
 &+ 2c_2b_1 \left(\sqrt{x_3^2 - a^2} J_1 - \frac{2x_3^2 + a^2}{x_3\sqrt{x_3^2 - a^2}} x_1 J_3 \right) - b_1^2c_2^2a^2.
 \end{aligned} \tag{13}$$

At $\alpha = 2$ and $c_1 = 0$ it has the form

$$\begin{aligned} \tilde{H}_2 = \frac{F_1}{2} \Big|_{c_1=0} &= J_1^2 + J_2^2 + \left(5 + \frac{a^2}{x_3^2}\right) J_3^2 - 4b_0c_2x_3x_1 + \frac{d}{c_2^2(2x_3^2 - a^2)^2} \\ &- 2c_2b_1 \left((2x_3^2 - a^2)J_1 - \frac{6x_3^2 + a^2}{x_3} x_1J_3 \right) + b_1^2c_2^2a^4. \end{aligned}$$

These Hamiltonians define new integrable systems on the sphere, which depend of three arbitrary parameters only.

4 Summary

Using the Lax matrix for the generalized Lagrange system and the standard construction of the commutative subalgebras from the reflection equation theory we construct new integrable system on the sphere. The corresponding Hamilton function is given by (10, 8) while the second integral is fourth order polynomial in the momenta.

These integrals depend of five numerical parameters $\alpha, b_0, b_1, c_1/c_2$ and d up to canonical transformations. At the special values of parameters we recover the Kowalevski-Goryachev-Chaplygin system.

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