Dynamic Spectrum Management: Complexity and Duality

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Abstract

Consider a communication system whereby multiple users share a common frequency band and must choose their transmit power spectral densities dynamically in response to physical channel conditions. Due to co-channel interference, the achievable data rate of each user depends on not only the power spectral density of its own, but also those of others in the system. Given any channel condition and assuming Gaussian signaling, we consider the problem to jointly determine all users' power spectral densities so as to maximize a system-wide utility function (e.g., weighted sum-rate of all users), subject to individual power constraints. For the discretized version of this nonconvex problem, we characterize its computational complexity by establishing the NP-hardness under various practical settings, and identify subclasses of the problem that are solvable in polynomial time. Moreover, we consider the Lagrangian dual relaxation of this nonconvex problem. Using the Lyapunov theorem in functional analysis, we rigorously prove a result first discovered by Yu and Lui (2006) that there is a zero duality gap for the continuous (Lebesgue integral) formulation. Moreover, we show that the duality gap for the discrete formulation vanishes asymptotically as the size of discretization decreases to zero.

Keywords: Spectrum management, sum-rate maximization, complexity, duality

I. Introduction

In a multiuser communication system such as cognitive radio or Digital Subscribe Lines (DSL), interference mitigation is a major design and management objective. A standard approach to eliminate multiuser interference is to divide the available spectrum into multiple tones (or bands) and pre-assign them to the users on a non-overlapping basis (FDMA). Although such 'orthogonal channelization' approach is well-suited for high speed structured communication in which quality of service is a major

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concern, it can lead to high system overhead and low bandwidth utilization. This is because for such a system a frequency tone pre-assigned to a user can not be released to other users even if it is not needed when the user is idle, or is unusable due to poor channel conditions.

With the proliferation of various radio devices and services, multiple wireless systems sharing a common spectrum must coexist [9]. In such scenarios, a pre-engineered FDMA solution may no longer be feasible or desirable, and we are naturally led to a situation whereby users can dynamically adjust their transmit power spectral densities over the entire shared spectrum, potentially achieving significantly higher overall throughput. For such a multiuser system, each user's performance depends on not only the power allocation (across spectrum) of its own, but also those of other users in the system. To mitigate multiuser interference, proper spectrum management (i.e., power control) is needed for the maximization of the overall system performance. Spectrum management problem of this type also arises in a DSL system where multiple users communicate with a central office through separate telephone lines over a common spectrum. Due to electro-magnetic coupling, signals transmitted over different telephone wires bundled in close proximity may interfere with each other, resulting in significant signal distortion. In fact, such crosstalk is known to be the major source of signal distortion in a high speed DSL system [21]. Hence, for both wireless and wireline (DSL) applications, judicious management of spectrum among competing users can have a major impact on the overall system performance.

The dynamic spectrum management problem has recently become a topic of intensive research in the signal processing and digital communication community. From the optimization perspective, the problem can be formulated either as a noncooperative Nash game [5], [21] or as a cooperative utility maximization problem [4], [22]. Several algorithms were proposed to compute a Nash equilibrium solution (Iterative Waterfilling method (IWFA) [5], [21]) or globally optimal power allocations (Dual decomposition method [3], [11], [20]) for the cooperative game. Due to the problem's nonconvex nature, these algorithms either lack global convergence or may converge to a poor spectrum sharing strategy. Significant effort has been made to establish conditions which can ensure the existence and uniqueness of a Nash equilibrium solution as well as the convergence of IWFA [12], [15], [18], [21]. In an attempt to analyze the performance of the dual decomposition algorithms, Yu and Lui [20] studied the duality gap of the continuous sum-rate maximization problem and showed it to be zero in the frequency flat case. For the general frequency selective case, they used an intuitive but non-rigorous argument to suggest that the strong duality should still hold. Despite the aforementioned progress, a complete understanding of the problem's complexity status and a thorough duality analysis has not yet emerged. For example, the zero-duality gap result for the continuous formulation does not readily translate to asymptotic zero duality for the discrete spectrum management problem as the discretization becomes infinitely fine (see Section IV). The latter is key to study the performance of dual decomposition algorithms for practical OFDM based multiuser systems.

In this paper, we present a systematic study of the dynamic spectrum management problem, covering two key theoretical aspects: complexity and duality. Specifically, we determine the complexity status of the spectrum management problem under various practical settings as well as different choices of system utility functions, and identify subclasses which are polynomial time solvable. In so doing, we clearly delineate the set of computationally tractable problems within the general class of NP-hard spectrum management problems. Furthermore, we rigorously establish the zero-duality gap result of Yu and Lui [20] for the continuous formulation when the interference channels are frequency selective. The key steps in our analysis are to cast the continuous formulation under the Lebesgue integral framework and to use the Lyapunov theorem [13] from functional analysis. The latter theorem says that the integral of any set-valued function over a non-atomic measure space (in our case finite intervals) is convex, even if the individual values of the function are not convex. Finally, we show that the duality gap for the discretized spectrum management problem vanishes when the size of discretization approaches zero. This is the case even if the system utility is nonlinear (but nonetheless concave). The asymptotic zero duality result suggests that the Lagrangian dual decomposition approach [3], [11], [20] may be a viable way to reach approximate optimality for finely discretized spectrum management problems.

II. Problem Formulation

Consider a multi-user communication system consisting of K transmitter-receiver pairs sharing a common frequency band $f \in \Omega$. For simplicity, we will call each of such transmitter-receiver pair a "*user*". Upon normalization, we can assume Ω to be the unit interval in $\Re$, namely, $\Omega = [0,1]$. Each user k has a fixed transmit power budget which it can allocate across Ω so as to maximize its own utility. Let $s_k(f) : \Omega \mapsto [0, \infty)$ denote the power spectral density (or power allocation) function of user k. The transmit power budget of user k can be represented as

$$\int_\Omega s_k(f)df \le P_k,$$

where $P_k > 0$ is a given constant. Due to multi-user interference, user k's utility depends on not only its own allocation function $s_k(f)$, but also those of others $\{s_\ell(f) : \ell \neq k\}$. Let user k's utility function be denoted by

$$u_k(s_1, s_2, ..., s_K) = \int_\Omega R_k(s_1(f), \ldots, s_K(f))df,$$

where $R_k(\cdot) : \Omega \mapsto [0, +\infty)$ is a Lebesgue integrable, possibly non-concave function.

Due to the complex coupling between users' utility functions, it is generally impossible to maximize the utility functions u_1, u_2,..., u_K simultaneously. Instead, we seek to maximize a system-wide utility $H(u_1, ..., u_K)$ which carefully balances the interests of all users in the system. This leads to the following spectrum management problem:

$$
\begin{aligned}
\max \quad & H(u_1, \cdots, u_K) \\
\text{s.t.} \quad & u_1 = \int_\Omega R_1(s_1(f), \ldots, s_K(f))df \\
& \quad \vdots \\
& u_K = \int_\Omega R_K(s_1(f), \ldots, s_K(f))df \\
& \int_\Omega s_k(f)df \leq P_k,\ s_k(f) \geq 0 \text{ and Lebesgue measurable},\ k = 1, ..., K.
\end{aligned}
\tag{P_c}
$$

The subscript c in the notation "(P_c)" signifies the continuous domain of the formulation. The maximum value of (P_c) is called the *social optimum*.

There are four commonly used choices for the system utility function $H(u_1, \cdots, u_K)$:

i) Sum-rate utility: $H_1(u_1, ..., u_K) = \frac{1}{K}\sum_{k=1}^K u_k$;

ii) Proportional fairness utility: $H_2(u_1, ..., u_K) = \left(\prod_{k=1}^K u_k\right)^{1/K}$ (equivalent to maximizing $\sum_{k=1}^K \ln u_k$);

iii) Harmonic-rate utility: $H_3(u_1, ..., u_K) = K\left(\sum_{k=1}^K u_k^{-1}\right)^{-1}$ (equivalent to maximizing $-\ln(\sum_{k=1}^K u_k^{-1})$);

iv) Min-rate utility: $H_4(u_1, ..., u_K) = \min_{1\leq k\leq K} u_k$.

In general, these utility functions can be ordered

$$H_1 \geq H_2 \geq H_3 \geq H_4.$$

In terms of user fairness, the order is reversed.

The spectrum management problem (P_c) is in general nonconvex due to the nonconcavity of utility functions $u_1, u_2, ..., u_K$. Moreover, it is defined in continuous domain (infinite dimensional), with spectral density functions $s_1(f)$, $s_2(f)$,..., $s_K(f)$ as decision variables. As such, the spectrum management problem (P_c) is a difficult infinite dimensional nonlinear optimization problem.

To facilitate numerical solution, we typically discretize the frequency band so that $\Omega = \{0, \frac{1}{N}, \frac{2}{N}, ..., 1\}$. In this way, the continuous formulation of the frequency management problem (P_c) can be discretized by replacing Lebesque measure with a discrete uniform measure on $[0, 1]$. In particular, user k's spectral density becomes

$$s_k^n \geq 0, \quad \frac{1}{N}\sum_{n=1}^N s_k^n \leq P_k$$

and the corresponding utility is

$$u_k = \frac{1}{N}\sum_{n=1}^{N} R_k(s_1^n, \ldots, s_K^n).$$

The corresponding social optimum is achieved by maximizing the total system utility $H(u_1, ..., u_K)$

$$\begin{aligned} \max \quad & H(u_1, \cdots, u_K) \\ \text{s.t.} \quad & u_1 = \frac{1}{N}\sum_{n=1}^{N} R_1(s_1^n, \ldots, s_K^n) \\ & \quad \vdots \\ & u_K = \frac{1}{N}\sum_{n=1}^{N} R_K(s_1^n, \ldots, s_K^n) \\ & \frac{1}{N}\sum_{n=1}^{N} s_k^n \le P_k,\ s_k^n \ge 0,\ k = 1, ..., K, \end{aligned} \qquad (P_d^N)$$

We use (P_d^N) to denote this discretized problem. Intuitively, $(P_d^N) \to (P_c)$ as $N \to \infty$, namely, as the discretization becomes infinitely fine, the discrete problem coincides with the continuous spectrum management problem. However, as we see in Section IV, this "limiting" argument can be problematic due to a mismatch between Riemann and Lebesgue integrals.

Rate maximization

Let x_k^n denote the transmitted complex Gaussian signal from user k (consisting of a transmitter and receiver pair) at tone n, and let $s_k^n := E|x_k^n|^2$ denote its power. For an AWGN channel, the received signal y_k^n is given by

$$y_k^n = \sum_{l=1}^{K} h_{l,k}^n x_l^n + z_k^n, \quad n \in \mathcal{N},\ k \in \mathcal{K},$$

where $z_k^n \sim CN(0, N_0)$ denotes the complex Gaussian channel noise with zero mean and variance N_0, and the complex scalars $\{h_{l,k}^n\}$ represent channel gain coefficients. In practice, $h_{l,k}^n$ can be determined by the distance between transmitter l and receiver k. The capacity region of this interference channel is still unknown. So it is reasonable (and natural) to treat the interference as white noise, especially if users do not have direct knowledge of the code/modulation schemes of other users in the system. In this way, we can write transmitter k's achievable data rate R_k^n at tone n [6] as

$$R_k^n(s_1^n, \ldots, s_K^n) = \ln\left(1 + \frac{|h_{k,k}^n|^2 s_k^n}{N_0 + \sum_{j\neq k} |h_{j,k}^n|^2 s_j^n}\right),$$

Upon normalizing the channel coefficients, we obtain

$$R_k^n(s_1^n, \ldots, s_K^n) \quad := \quad \ln\left(1 + \frac{s_k^n}{\sigma_k^n + \sum_{j\neq k} \alpha_{kj}^n s_j^n}\right), \qquad (1)$$

where $\sigma_k^n = N_0/|h_{k,k}^n|^2$ denotes the normalized background noise power, and $\alpha_{kj}^n = |h_{j,k}^n|^2/|h_{k,k}^n|^2$ is the normalized crosstalk coefficient from transmitter j to receiver k at tone n. Due to normalization, we have $\alpha_{kk}^n = 1$ for all k.

Notice that unlike the frequency flat case considered in [7], the channel coefficients $h_{j,k}^n$ vary according to tone index n due to frequency selectivity, resulting in a non-constant normalized noise power σ_k^n across tones. As it turns out, this crucial difference greatly complicates the spectrum management problem in the frequency selective case, making an otherwise convex optimization problem computationally intractable; see Section III.

For the continuous formulation, $R_k(\cdot)$ can represent the data rate achievable by user k at frequency f (in the sense of Shannon [6]):

$$R_k(s_1(f), \ldots, s_K(f)) = \ln\left(1 + \frac{s_k(f)}{\sigma_k(f) + \sum_{j\neq k} \alpha_{kj}(f) s_j(f)}\right), \tag{2}$$

where $\sigma_k(f) > 0$ signifies the noise power at user k on frequency f, and $\alpha_{kj}(f) > 0$ denotes the normalized path loss coefficient for the channel between user j and user k on frequency f. Clearly, the rate function $R_k(\cdot)$ and the utility function $u_k(\cdot)$ are both nonconcave. The spectrum management problem can be stated as

$$\begin{array}{ll} \max & H(u_1, \cdots, u_K) \\ \text{s.t.} & u_k = \displaystyle\int_\Omega \ln\left(1 + \frac{s_k(f)}{\sigma_k(f) + \sum_{j\neq k} \alpha_{kj}(f) s_j(f)}\right) df, \\ & \displaystyle\int_\Omega s_k(f) df \le P_k,\ s_k(f) \ge 0 \text{ and Lebesgue measurable},\ k = 1, ..., K. \end{array} \tag{P_f}$$

In practice (e.g., IEEE 802.11x standards), the available spectrum Ω is divided into multiple tones (or bands) and shared by the users. In this way and assuming $H(\cdot) = H_1(\cdot)$, the spectrum management problem (P_f) is discretized and becomes

$$\begin{array}{ll} \text{maximize} & \displaystyle\frac{1}{NK}\sum_{k=1}^{K}\sum_{n=1}^{N} \ln\left(1 + \frac{s_k^n}{\sigma_k^n + \sum_{j\neq k} \alpha_{kj}^n s_j^n}\right) \\ \text{subject to} & \displaystyle\frac{1}{N}\sum_{n=1}^{N} s_k^n \le P_k, \quad s_k^n \ge 0 \quad n = 1, 2, ..., N,\ k = 1, 2, ..., K. \end{array} \tag{P_1}$$

The main challenges of spectrum management are (i) nonconvexity, (ii) problem size ($N \ge 4000$, $K \ge 50$) and (iii) distributed optimization. A popular spectrum management approach is Frequency Division Multiple Access (FDMA) whereby the available tones (or bands) are shared by all the users on a non-overlapping basis. Such ‘orthogonal channelization’ approach is well-suited for high speed structured communication in which quality of service is a major concern. Mathematically, FDMA solutions can be described as

$$\mathcal{S} = \begin{cases} \left\{ \mathbf{s} \geq 0 \mid s_k^n s_j^n = 0, \ \forall\, k \neq j, \ \forall\, n \right\}, & \text{discrete,} \\ \\ \left\{ \mathbf{s}(f) \geq 0 \mid s_k(f) s_j(f) = 0, \ \forall\, k \neq j, \ \forall\, f \right\}, & \text{continuous.} \end{cases}$$

FDMA solutions are *not* necessarily vertex solutions.

III. Discrete Frequencies: Complexity Analysis

In this section, we investigate the complexity status of the spectrum management problem (P_f) under various practical settings as well as different choices of system utility functions. We provide a complete analysis on when the problem is NP-hard and also identify subclasses of the problem that are solvable in polynomial time. We will consider two separate cases: the case of many users and few tones (large K and fixed N), and the case of few users and many tones (fixed K and large N).

A. The case of many users and few tones

In this section, we fix N and analyze the complexity of the spectrum management problem (P_f) for large K and for various choices of system utility functions.

1) Maximization of sum-rate: Let us first consider the problem of maximizing the total system throughput or sum-rate. This corresponds to choosing a system utility function $H(u) = H_1(u) = \frac{1}{K}\sum_{k=1}^{K} u_k$. We show below that the resulting spectrum management problem is NP-hard for any fixed N.

The discrete sum-rate maximization problem is given by (P_1). We specialize (P_1) to the case $N = 1$:

$$\begin{array}{ll} \text{maximize} & H(s) = \displaystyle\sum_{k=1}^{K} \ln\left(1 + \frac{s_k}{\sigma_k + \sum_{j \neq k} \alpha_{kj} s_j}\right) \\ \text{subject to} & 0 \leq s_k \leq P_k,\ k = 1, ..., K. \end{array} \qquad (P_1)'$$

Notice that, for simplicity, we have dropped the constant factor $1/K$ from the objective function and removed the superscript n in our notations since $N = 1$. The resulting problem corresponds to the practical situation whereby multiple users share a single frequency band (say, a control channel), and wish to cooperate in order to maximize the sum-rate of all users.

Theorem 1: For the sum-rate utility function $H(u_1, ..., u_K) = \frac{1}{K}(u_1 + \cdots + u_K)$, the spectrum management problem (P_f) is strongly NP-hard for any fixed $N \geq 1$.

The proof is based on a polynomial time reduction from the maximum independent set problem. The details are relegated to Appendix A. Intuitively, if all cross talk coefficients are either 0 or ∞, then the optimal solution of (P_1) will have either $s_k = 0$ or $s_k = P_k$, for all users k. In this way, maximizing the sum-rate is equivalent to finding the largest subset of users which are mutually non-interfering, which is further equivalent to the maximum independent set problem in combinatorial optimization. The complexity analysis for (P_1) has an interesting consequence. It is well known that the maximum independent set problem is not only difficult to optimize, but also hard to approximate (cf. Trevisan [17]). In particular, for a K-node graph, there is a constant $c > 0$ such that if there is a polynomial-time K^{-c}-approximation algorithm[1] for the maximum independent set problem then P=NP. This inapproximability result, coupled with the polynomial transformation outlined in the preceding complexity analysis, implies that the sum-rate maximization problem cannot be well approximated even to within a factor of K^{-c} when the number of tones is 1.

2) Maximization of min-rate utility: We now study the complexity status of the spectrum management problem (P_f) when the system utility function is the minimum of all users' rates: $H(u) = \min_k u_k$. As we see next, unlike the sum-rate case considered earlier, the spectrum management problem becomes a convex optimization problem when $N = 1$.

Let $N = 1$ and consider the min-rate utility function $H(u) = \min_k u_k$. The corresponding spectrum management problem (P_f) becomes (after dropping the superscript n)

$$\begin{array}{ll} \text{maximize} & \min\limits_{1\le k\le K} \ln\left(1 + \dfrac{s_k}{\sigma_k + \sum_{j\neq k}\alpha_{kj}s_j}\right) \\ \text{subject to} & 0 \le s_k \le P_k,\ k = 1, ..., K, \end{array}$$

which, by the monotonicity of $\ln(\cdot)$ function, is equivalent to

$$\begin{array}{ll} \text{maximize} & \min\limits_{1\le k\le K} \dfrac{s_k}{\sigma_k + \sum\limits_{j\neq k}\alpha_{kj}s_j} \\ \text{subject to} & 0 \le s_k \le P_k,\ k = 1, ..., K. \end{array}$$

This is known as a generalized fractional linear programming problem and can be solved by parametric linear programming. In particular, introducing an auxiliary variable τ, we obtain the following equivalent formulation

$$\begin{array}{ll} \text{maximize} & \tau \\ \text{subject to} & s_k \ge \tau(\sigma_k + \sum_{j\neq k}\alpha_{kj}s_j) \\ & \tau \ge 0,\ 0 \le s_k \le P_k,\ k = 1, ..., K, \end{array}$$

[1]For a maximization problem $\max_{f\in\Omega} H(f)$ with $H(f)$ nonnegative, we say $\hat{f}$ is an δ-approximate solution if $H(\hat{f}) \ge \delta \max_{f\in\Omega} H(f)$.

which can be easily solved using a binary search on τ. This shows that the case of $N = 1$ is polynomial time solvable. However, if $N > 2$ then the problem remains NP-hard.

Theorem 2: For the min-rate system utility function $H(u) = \min_k u_k$, the spectrum management problem (P_f) is polynomial time solvable (in fact equivalent to a parametric linear program) when $N = 1$, and is strongly NP-hard when $N \geq 3$.

The proof of Theorem 2 consists of a polynomial time reduction from the 3-colorability problem, i.e., the problem to determine if the nodes of a given graph can be assigned one of the three colors so that no two adjacent nodes are colored the same. The 3-colorability problem is known to be NP-hard. Intuitively, for any given graph, we can think of nodes in the graph as users, and colors as frequency tones. If we set the normalized crosstalk coefficients $\alpha^n_{\ell k}$ to be either a (sufficiently large) constant or zero, depending on if nodes ℓ and k are adjacent or not. In this way, the min-rate optimal solution for the corresponding spectrum management problem will try to assign frequency tones to users so that each tone is utilized by at least one user in a non-interfering manner. This then leads to a solution to the coloring problem. We provide details of the analysis in Appendix B.

There is still a missing case of $N = 2$ that is not covered by Theorem 2. While we have not been able to characterize its complexity, we can reduce it to a simple optimization problem involving a single variable.

Theorem 3: For the min-rate utility function $H(u) = \min_k u_k$ and $N = 2$, the spectrum management problem (P_f) is equivalent to a single-variable optimization problem.

Proof. When $N = 2$, the maximization of min-rate utility becomes

$$\boxed{\begin{array}{ll} \text{maximize} & \displaystyle\min_{1\leq k\leq K} \left(\ln\left(1 + \frac{s_k^1}{\sigma_k^1 + \sum_{j\neq k} \alpha_{kj}^1 s_j^1}\right) + \ln\left(1 + \frac{s_k^2}{\sigma_k^2 + \sum_{j\neq k} \alpha_{kj}^2 s_j^2}\right)\right) \\ \text{subject to} & s_k^1 + s_k^2 \leq P_k,\ s_k^1, s_k^2 \geq 0,\ k = 1, ..., K. \end{array}}$$

By a bisection search on the objective function, this maximization problem can always be broken down to a series of feasibility problems:

$$(F_\alpha) \left\{ \begin{array}{l} \ln\left(1 + \dfrac{s_k^1}{\sigma_k^1 + \sum_{j\neq k} \alpha_{kj}^1 s_j^1}\right) + \ln\left(1 + \dfrac{s_k^1}{\sigma_k^1 + \sum_{j\neq k} \alpha_{kj}^1 s_j^1}\right) \geq \ln \alpha \\ s_k^1 + s_k^2 \leq P_k,\ \ s_k^1,\ \ s_k^2 \geq 0,\ k = 1, ..., K. \end{array} \right.$$

We claim that for a fixed $\alpha > 0$, then it takes polynomial-time to check if (F_α) is feasible or not. In fact, (F_α) is equivalent to the existence of $\beta > 0$ such that

$$\begin{cases} 1 + \dfrac{s_k^1}{\sigma_k^1 + \sum_{j\neq k} \alpha_{kj}^1 s_j^1} \geq \dfrac{\alpha}{\beta} \\ 1 + \dfrac{s_k^2}{\sigma_k^2 + \sum_{j\neq k} \alpha_{kj}^2 s_j^1} \geq \beta \\ s_k^1 + s_k^2 \leq P_k,\ s_k^1, s_k^2 \geq 0,\ k = 1, ..., K, \end{cases}$$

which amounts to checking the feasibility of the system

$$\begin{cases} \min\limits_{1\leq k\leq K} \dfrac{\sigma_k^2 + s_k^2 + \sum_{j\neq k} \alpha_{kj}^2 s_j^2}{\sigma_k^2 + \sum_{j\neq k} \alpha_{kj}^2 s_j^2} \geq \beta \geq \max\limits_{1\leq k\leq K} \dfrac{\alpha(\sigma_k^1 + \sum_{j\neq k} \alpha_{kj}^1 s_j^1)}{\sigma_k^1 + s_k^1 + \sum_{j\neq k} \alpha_{kj}^1 s_j^1} \\ s_k^1 + s_k^2 \leq P_k,\ s_k^1, s_k^2 \geq 0,\ k = 1, ..., K, \end{cases}$$

and this leads to solving the following parameterized convex optimization problem:

$$\begin{array}{lll} (Q_\beta) & \text{minimize} & \max\limits_{1\leq k\leq K} \dfrac{\sigma_k^1 + \sum_{j\neq k} \alpha_{kj}^1 s_j^1}{\sigma_k^1 + s_k^1 + \sum_{j\neq k} \alpha_{kj}^1 s_j^1} \\ & \text{subject to} & \min\limits_{1\leq k\leq K} \dfrac{\sigma_k^2 + s_k^2 + \sum_{j\neq k} \alpha_{kj}^2 s_j^2}{\sigma_k^2 + \sum_{j\neq k} \alpha_{kj}^2 s_j^2} \geq \beta \\ & & s_k^1 + s_k^2 \leq P_k,\ s_k^1, s_k^2 \geq 0,\ k = 1, ..., K. \end{array}$$

Denote the optimal value of (Q_β) to be $v(\beta)$. Maximizing α is equivalent to finding the maximum of $\beta/v(\beta)$ for $\beta > 0$, i.e., $\max_{0<\beta} \beta/v(\beta)$. ■

3) Maximization of harmonic-rate utility: When the system utility function is given by $H(u) = H_3(u) = K(u_1^{-1} + \cdots + u_K^{-1})^{-1}$, the spectrum management problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \sum\limits_{k=1}^{K} \left(\sum\limits_{n=1}^{N} \ln \left(1 + \dfrac{s_k^n}{\sigma_k^n + \sum_{j\neq k} \alpha_{kj}^n s_j^n} \right) \right)^{-1} \\ \text{subject to} & \sum\limits_{n=1}^{N} s_k^n \leq P_k,\ k = 1, ..., K, \\ & s_k^n \geq 0,\ k = 1, ..., K;\ n = 1, ..., N. \end{array} \qquad (P_3)$$

Similar to the min-rate utility case, we have the following complexity characterization result.

Theorem 4: For the harmonic-rate utility function $H(u) = K\left(\sum_{k=1}^K u_k^{-1}\right)^{-1}$, the spectrum management problem (P_3) is a convex optimization problem (thus polynomially solvable) when $N = 1$, and is strongly NP-hard when $N \geq 3$.

Proof. We only consider the case of $N = 1$. The NP-hardness proof for $N \geq 3$ can be found in Appendix

C. When $N = 1$, problem (P_3) becomes (after dropping the superscript n)

$$\begin{aligned} &\text{minimize} \quad \sum_{k=1}^{K} \left(\ln \left(1 + \frac{s_k}{\sigma_k + \sum_{j \neq k} \alpha_{kj} s_j} \right) \right)^{-1} \\ &\text{subject to} \quad 0 \leq s_k \leq P_k,\ k = 1, ..., K. \end{aligned}$$

Introducing new variables t_k, we can rewrite the above problem as

$$\begin{aligned} &\text{minimize} \quad \sum_{k=1}^{K} t_k \\ &\text{subject to} \quad 1/t_k \leq \ln \left(1 + \frac{s_k}{\sigma_k + \sum_{j \neq k} \alpha_{kj} s_j} \right) \\ &\qquad\qquad\quad 0 \leq s_k \leq P_k,\ t_k \geq 0,\ k = 1, ..., K. \end{aligned}$$

By a nonlinear variable transformation, $s_k := \exp(y_k)$, (P_3) can be equivalently turned into

$$\begin{aligned} &\text{minimize} \quad \textstyle\sum_{k=1}^{K} t_k \\ &\text{subject to} \quad \ln \left(\sigma_k \exp(-y_k) + \sum_{j \neq k} \alpha_{kj} \exp(y_j - y_k) \right) + \ln \left(\exp\left(\frac{1}{t_k}\right) - 1 \right) \leq 0, \qquad (P_3)' \\ &\qquad\qquad\quad y_k \leq \ln P_k,\ t_k \geq 0,\ k = 1, ..., K. \end{aligned}$$

It is well known that $\ln \left(\sigma_k \exp(-y_k) + \sum_{j \neq k} \alpha_{kj} \exp(y_j - y_k) \right)$ is a convex function in y. Moreover, it can checked easily that $\ln(\exp(1/t) - 1)$ is convex over $t > 0$. Consequently, $(P_3)'$ is a convex program, hence solvable in polynomial-time in terms of the dimension and the required solution precision. ■

We remark that the complexity status of (P_3) remains unknown when $N = 2$.

4) Maximization of proportional fairness utility: Consider the system utility function $H(u) = H_2(u) = \left(\prod_{k=1}^{K} u_k \right)^{1/K}$. Notice that maximizing $(\prod_k u_k)^{1/K}$ (the geometric mean of users' data rates) is equivalent to maximizing $\sum_k \ln u_k$. Thus, the spectrum management problem (P_f) becomes

$$\begin{aligned} &\text{maximize} \quad \sum_{k=1}^{K} \ln \left(\sum_{n=1}^{N} \ln \left(1 + \frac{s_k^n}{\sigma_k^n + \sum_{j \neq k} \alpha_{kj}^n s_j^n} \right) \right) \qquad (P_4) \\ &\text{subject to} \quad \sum_{n=1}^{N} s_k^n \leq P_k,\ s_k^n \geq 0,\ k = 1, ..., K,\ n = 1, ..., N. \end{aligned}$$

Below is our complexity characterization result for the proportional fairness utility maximization problem.

Theorem 5: For the proportional fairness utility function $H(u) = \left(\prod_{k=1}^{K} u_k \right)^{1/K}$, the spectrum management problem (P_f) is convex when $N = 1$ and is NP-hard for $N \geq 3$.

Proof. For $N = 1$, the corresponding spectrum management problem becomes

$$\begin{array}{ll} \text{maximize} & \sum_{k=1}^{K} \ln\ln\left(1 + \frac{s_k}{\sigma_k + \sum_{j\neq k}\alpha_{kj}s_j}\right) \\ \text{subject to} & 0 \le s_k \le P_k,\ k = 1, ..., K. \end{array} \qquad (P_4)'$$

Similar to the proof of Theorem 4, let us introduce some auxiliary variables t_k and the constraints

$$\exp(t_k) \le \ln\left(1 + \frac{s_k}{\sigma_k + \sum_{j\neq k}\alpha_{kj}s_j}\right),$$

which can be equivalently written as

$$\frac{\sigma_k + \sum_{j\neq k}\alpha_{kj}s_j}{s_k} \le \frac{1}{\exp(\exp(t_k)) - 1}.$$

By the variable transformation $s_j := \exp(y_j)$, $j = 1, ..., K$, we rewrite the above inequality as

$$\ln\left(\sigma_k \exp(-y_k) + \sum_{j\neq k}\alpha_{kj}\exp(y_j - y_k)\right) + \ln\left(\exp(\exp(t_k)) - 1\right) \le 0. \qquad (3)$$

It can be checked that the function $h_2(t) := \ln\left(\exp(\exp(t)) - 1\right)$ is convex. This shows that constraints of the type in (3) are convex. Since $(P_4)'$ can be equivalently written as

$$\begin{array}{ll} \text{maximize} & \sum_{k=1}^{K} t_k \\ \text{subject to} & \ln\left(\sigma_k \exp(-y_k) + \sum_{j\neq k}\alpha_{kj}\exp(y_j - y_k)\right) + \ln\left(\exp(\exp(t_k)) - 1\right) \le 0, \\ & y_k \le \ln P_k,\ k = 1, ..., K, \end{array}$$

it is therefore a convex optimization problem.

The NP-hardness proof is similar to the harmonic rate case (consisting of a reduction from 3-colorability problem), and is given in Appendix D. ■

The complexity status remains unknown for the two-tone case with proportional fairness criterion.

B. The case of many tones and few users

So far we have analyzed the computational complexity of the spectrum management problem for various choices of system utility functions when the number of users K is large while the number of tones N is small and fixed. In what follows, we consider the other case when the number of users K is small and fixed while the number of frequency tones N grows to infinity. For the sum-rate maximization problem (corresponding to the system utility function $H(u) = H_1(u) = \sum_{n=1}^{N} u_n$), the recent work of

[8] shows that the resulting spectrum management problem is NP-hard even when there are only two users in the system. This NP-hardness result effectively shatters any hope to efficiently compute the exact optimal spectrum sharing strategy. The combinatorial growth of possible frequency assignments simply renders the problem intractable. Below we generalize this result to other system utility functions.

Theorem 6: The two-user spectrum management problem (P_d^N) is NP-hard when the system utility function $H(u)$ is given by the following choices: $H_1(u) = \sum_{n=1}^N u_n$ (sum-rate), $H_2(u) = \sum_{n=1}^N \ln u_n$ (proportional fairness), $H_3(u) = (\sum_{n=1}^N u_n^{-1})^{-1}$ (harmonic-rate), or $H_4(u) = \min_n u_n$ (min-rate).

The proof of this complexity characterization relies on a reduction from the so called equipartition problem: given an even number of integers, determine if they can be partitioned into two subsets of same size such that the sum of integers in the two subsets are equal. To see how the equipartition problem is related to the spectrum management problem, let us imagine a situation whereby two users with the same noise power spectrum are to share a common set of frequency tones. Assume that the crosstalk coefficients are sufficiently strong on every tone. It follows from [8] that the optimal sum-rate spectrum sharing strategy must be FDMA. To maximize the system utility (defined by any of the mentioned utility functions) among all FDMA strategies, the two users should partition the tones in a way that best balances the total noise power across the frequency tones assigned to the two users. In this way, deciding on which user should get exactly which subset of tones becomes essentially the aforementioned equipartition problem. The details of the proof are relegated to Appendix E.

For a single user system ($K = 1$), there is no multiuser interference, so the optimal spectrum management problem becomes convex, as long as $H(u)$ is concave. In fact, in this case, all four system utility functions H_1, H_2, H_3 and H_4 coincide, and the optimal solution can be found via the well-known waterfilling algorithm in polynomial time. Below is a summary of the complexity status of the discrete spectrum management problem (P_f).

Problem Class \ Utility Function	Sum-Rate H_1 FDMA Soln	Sum-Rate H_1 (arithmetic mean)	Proportional Fairness H_2 (geometric mean)	Harmonic mean H_3	Min-Rate H_4
K=1, N arbitrary	Convex Opt (Waterfilling)	Convex Opt (Waterfilling)	Convex Opt (Waterfilling)	Convex Opt (Waterfilling)	Convex Opt (Waterfilling)
K≥2 and fixed, N arbitrary	NP-hard	NP-hard	NP-hard	NP-hard	NP-hard
N>2 and fixed, K arbitrary	Strongly NP-hard	Strongly NP-hard	NP-hard	Strongly NP-hard	Strongly NP-hard
N=1, K arbitrary	Strongly NP-hard	Strongly NP-hard	Convex Opt	Convex Opt	LP

In the above table, the column 'Sum-Rate H1 (FDMA solution)' represents the optimization model where the objective is to maximize the total rates, while the users' power spectrums are *constrained* to have no overlap. The second column 'Sum-Rate H1 (arithmetic mean)' represents the model with the same objective but the FDMA constraints are removed.

IV. Duality

The discrete rate-maximization problems considered in Section III are mostly NP-hard. This motivates us to consider efficient algorithms which can find high quality approximate solutions for the rate-maximization problem in polynomial time. One natural approach is to consider the dual formulation and apply Lagrangian relaxation.

Consider the discrete rate-maximization problem (P_d^N):

$$\begin{array}{ll} \max & H(u_1, \cdots, u_K) \\ \text{s.t.} & u_k \le \frac{1}{N}\sum_{n=1}^{N} \ln\left(1 + \frac{s_k^n}{\sigma_k^n + \sum_{j\neq k}\alpha_{kj}^n s_j^n}\right) \\ & \frac{1}{N}\sum_{n=1}^{N} s_k^n \le P_k,\ s_k^n \ge 0,\ k = 1, ..., K. \end{array} \qquad (P_d^N)$$

The corresponding Lagrangian function is given by

$$\begin{aligned} L_N(s,u;\lambda,\mu) &= H(u_1,\cdots,u_K) + \sum_{k=1}^{K}\lambda_k\left[\frac{1}{N}\sum_{n=1}^{N}\ln\left(1+\frac{s_k^n}{\sigma_k^n+\sum_{j\neq k}\alpha_{kj}^n s_j^n}\right) - u_k\right] \\ &\quad + \sum_{k=1}^{K}\mu_k\left[P_k - \frac{1}{N}\sum_{n=1}^{N}s_k^n\right] \\ &= H(u_1,\cdots,u_K) - \sum_{k=1}^{K}u_k\lambda_k + \sum_{k=1}^{K}P_k\mu_k \\ &\quad + \frac{1}{N}\sum_{n=1}^{N}\sum_{k=1}^{K}\left[\lambda_k\ln\left(1+\frac{s_k^n}{\sigma_k^n+\sum_{j\neq k}\alpha_{kj}^n s_j^n}\right) - \mu_k s_k^n\right], \end{aligned}$$

and the dual function is given by

$$\begin{array}{rcll} g_N(\lambda,\mu) & := & \sup & L_N(s,u;\lambda,\mu) \\ & & \text{s.t.} & u \in \Re^K,\ s_k^n \ge 0,\ k=1,...,K. \end{array}$$

Thus, the Lagrangian dual of (P_d^N) can be written as

$$\begin{array}{ll} \min & H^*(\lambda) + P^{\mathrm{T}}\mu + \bar{g}_N(\lambda,\mu) \\ \text{s.t.} & \lambda,\mu \in \Re_+^K \end{array} \qquad (D_d^N)$$

where H^* is the convex conjugate dual function of H defined by

$$H^*(\lambda) := \sup_{t \in \Re^K} \left(H(t) - \lambda^{\mathrm{T}} t \right) \tag{4}$$

and

$$\begin{aligned} \bar{g}_N(\lambda, \mu) \quad := \quad \max \quad & \frac{1}{N} \sum_{n=1}^{N} \sum_{k=1}^{K} \left[\lambda_k \ln \left(1 + \frac{s_k^n}{\sigma_k^n + \sum_{j \neq k} \alpha_{kj}^n s_j^n} \right) - \mu_k s_k^n \right] \\ \text{s.t.} \quad & s_k^n \geq 0, \; n = 1, ..., N; \; k = 1, ..., K. \end{aligned}$$

The conjugate function H^* can be computed explicitly for various choices of system utility functions. It is relatively easy to verify the following.

1) For weighted sum-rate function $H(u) = w^{\mathrm{T}} u$, the conjugate function $H^*(\lambda) = 0$ for $\lambda = w$ and $H^*(\lambda) = \infty$ when $\lambda \neq w$.
2) For the proportional fairness system utility function $H(u) = \sum_{k=1}^{K} \ln u_k$, the conjugate function $H^*(\lambda) = K - \sum_{k=1}^{K} \ln \lambda_k$.
3) For the min-rate utility function $H(u) = \min_{1 \leq k \leq K} u_k$, the conjugate function $H^*(\lambda) = 0$ if $\sum_{k=1}^{K} \lambda_k \geq 1$, and $H^*(\lambda) = +\infty$ otherwise.
4) For the harmonic-rate utility function $H(u) = (\sum_{k=1}^{K} u_k^{-1})^{-1}$, the conjugate function $H^*(\lambda) = 0$ if $\sum_{k=1}^{K} \sqrt{\lambda_k} \geq 1$, and $H^*(\lambda) = +\infty$ otherwise.

Unlike the original problem (P_d^N), its dual (D_d^N) is convex which is potentially easy to solve. Let P_N^* and D_N^* denote their respective optimal values. It follows from weak duality that $P_N^* \leq D_N^*$. It is known from Section III that (P_d^N) is typically NP-hard, suggesting that the primal (P_d^N) and the dual (D_d^N) are in general not equivalent. Indeed, the following example suggests that the duality gap $D_N^* - P_N^*$ is typically positive.

Example: Consider a case with two users sharing one frequency. The corresponding sum-rate maximization problem is given by

$$\begin{aligned} \text{maximize} \quad & \ln \left(1 + \tfrac{s_1}{1+s_2} \right) + \ln \left(1 + \tfrac{s_2}{1+s_1} \right) \\ \text{subject to} \quad & 0 \leq s_1 \leq P_1, \quad 0 \leq s_2 \leq P_2, \end{aligned} \tag{5}$$

where $P_i > 0$ is the power budget for user i, ($i = 1, 2$). Let the optimal value of the above problem be $v(P_1, P_2)$. We show below that $v(P_1, P_2)$ is not concave function of $P_1, P_2 > 0$, which implies that the duality gap is positive. [This concavity property of $v(P_1, P_2)$ was called time-sharing property in [20].]

Note that the objective is actually equivalent to maximizing $\frac{(1+s_1+s_2)^2}{(1+s_1)(1+s_2)}$, which is a quasi-convex function on its domain (see [1]). Therefore, its maximum value is attained at the vertices. In other words,

$$v(P_1, P_2) = \max\{\ln(1 + P_1), \ln(1 + P_2), 2\ln(1 + P_1 + P_2) - \ln(1 + P_1) - \ln(1 + P_2)\}. \tag{6}$$

So if $P_1 = P_2 = 2$, then $P_N^* = v(2,2) = \ln 3$ which is attained by $(s_1^*, s_2^*) = (2,0)$ or $(0,2)$ (FDMA solution). However, the corresponding dual formulation of (5) can be written as

$$\begin{aligned} \text{minimize} \quad & P_1\mu_1 + P_2\mu_2 + \bar{g}_1(\mu) \\ \text{subject to} \quad & \mu = (\mu_1,\, u_2) \geq 0, \end{aligned} \tag{7}$$

where

$$\bar{g}_1(\mu) = \max_{s_1, s_2 \geq 0} \ln\left(1 + \frac{s_1}{1+s_2}\right) + \ln\left(1 + \frac{s_2}{1+s_1}\right) - \mu_1 s_1 - \mu_2 s_2.$$

Notice that $\bar{g}_1(\mu)$ is a convex function that is symmetric with respect to μ_1 and μ_2. Thus, if $P_1 = P_2 = 2$, then the dual objective function in (7) is symmetric with respect to μ_1 and μ_2. It follows that the optimal dual solution must have $\mu_1 = \mu_2$. Direct computation of $\bar{g}_1(\mu)$ (via KKT condition) shows that, when $\mu_1 = \mu_2 = 1/5$, the associated primal power levels must be either $(s_1, s_2) = (4,0)$ or $(0,4)$. Moreover, the subdifferential of $\bar{g}_1$ at $(1/5, 1/5)$ is equal to $\partial\bar{g}_1(\frac{1}{5}, 15) = \text{convex hull}\{(0,-4), (-4,0)\}$. Plugging these values into (7), we see that the dual optimality condition $(0,0) \in (P_1, P_2) + \partial\bar{g}_1(\mu_1, \mu_2)$ is satisfied at $\mu_1 = \mu_2 = 1/5$. Therefore, we obtain the optimal dual solutions $\mu_1^* = \mu_2^* = 1/5$, and the minimum dual objective value $D_N^* = \ln 5$ which is strictly larger than the primal objective value $P_N^* = \ln 3$.

When $H(u) = w^{\mathrm{T}}u$ (weighted sum-rate), a so called *time-sharing property* was introduced in [20] which can ensure zero duality gap $D_N^* - P_N^* = 0$. This property essentially requires the region of achievable rates $\{(u_1, \cdots, u_K) \mid \sum_{n=1}^N s_k^n \leq P_k,\, s_k^n \geq 0,\, 1 \leq k \leq K\}$ to be convex, where each u_k is defined in (P_d^N). Mathematically, the time-sharing property is equivalent to the concavity of the mapping $v(P_1, P_2)$. Unfortunately, time-sharing property does not hold in general. For instance, consider the preceding example where $v(P_1, P_2)$ is given by (6). Simple calculation shows that

$$2\ln(1 + P_1 + P_2) - \ln(1 + P_1) - \ln(1 + P_2) \geq \ln(1 + P_1) \text{ and } P_1, P_2 > 0$$

if and only if $P_2 + 1 - P_1^2 \geq 0$. Similarly,

$$2\ln(1 + P_1 + P_2) - \ln(1 + P_1) - \ln(1 + P_2) \geq \ln(1 + P_2) \text{ and } P_1, P_2 > 0$$

if and only if $P_1 + 1 - P_2^2 \geq 0$. Let $\Omega := \{(P_1, P_2) \in \Re^2_{++} \mid P_2 + 1 - P_1^2 \geq 0,\, P_1 + 1 - P_2^2 \geq 0\}$. We have $v(P_1, P_2) = 2\ln(1 + P_1 + P_2) - \ln(1 + P_1) - \ln(1 + P_2)$ for $(P_1, P_2) \in \Omega$. Now we see that $v(P_1, P_2)$ cannot be concave in Ω, since, for instance its Hessian at an interior point of Ω, $(1,1)$, is given by

$$\nabla^2 v(P_1, P_2)|_{(P_1,P_2)=(1,1)} = \begin{pmatrix} \frac{1}{36} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{1}{36} \end{pmatrix},$$

which is clearly not negative semidefinite.

A. Asymptotic Strong Duality

When the duality gap $D_N^* - P_N^*$ is nonzero, the dual problem (D_d^N) is not equivalent to the primal problem (P_d^N). Nonetheless, (D_d^N) may still provide a close approximation of (P_d^N). This motivates us to find an upper bound on the duality gap between (P_d^N) and (D_d^N). Notice that, for the continuous spectrum management problem (corresponding to $N = \infty$), the duality gap $D_\infty^* - P_\infty^* = 0$, as was established in [20] in the context of sum-rate maximization. This suggests that the duality gap $D_N^* - P_N^*$ for finite N should vanish asymptotically as $N \to \infty$ for general system utility functions. In this section, we show that this asymptotic strong duality result indeed holds true for general system utility functions, thanks to a hidden convexity resulting from the frequency set Ω being an interval (rather than a finite discrete set). The key in our analysis is the so-called Lyapunov theorem for vector measures [13].

Definition: A measure is *non-atomic* if every set of non-zero measure has a subset with strictly less non-zero measure.

The standard Lebesgue measure is non-atomic, while the uniform measure on a finite set is atomic. The following is a convenient form of Lyapunov Theorem due to Blackwell [2].

Lemma 1: Let v be a non-atomic measure on a Borel field $\mathcal{B}$ generated from subsets of a space Ω. Let $g_i(x(\cdot), \cdot)$ be compatible with $\mathcal{B}$-measurable function $x(\cdot)$ (i.e., if $x(\cdot)$ is $\mathcal{B}$-measurable then $g_i(x(\cdot), \cdot)$ is $\mathcal{B}$-measurable), $i = 1, ..., m$. Then,

$$\left\{ \left(\begin{array}{c} \int_\Omega g_1(x(\cdot), \cdot) dv \\ \vdots \\ \int_\Omega g_m(x(\cdot), \cdot) dv \end{array} \right) \middle| \; x \text{ is } \mathcal{B}\text{-measurable} \right\}$$

is a convex set.

It is important to notice that there is no assumption on the convexity of g_i functions or the set Ω. The convexity of the image of the integral mapping is due to the non-atomic property of measure v. We now use Lemma 1 to argue the asymptotic strong duality for the continuous formulation of the spectrum management problem.

Suppose that the system utility function H is monotonically nondecreasing componentwise and jointly concave. We can equivalently rewrite (P_c) as

$$\begin{array}{ll} \max & H(u_1,\cdots,u_K) \\ \text{s.t.} & u_1 \le \int_{f\in\Omega} \ln\left(1+\frac{s_1(f)}{\sigma_1(f)+\sum_{j\neq 1}\alpha_{1j}(f)s_j(f)}\right)df \\ & \vdots \\ & u_K \le \int_{f\in\Omega} \ln\left(1+\frac{s_K(f)}{\sigma_K(f)+\sum_{j\neq K}\alpha_{Kj}(f)s_j(f)}\right)df \\ & \int_{f\in\Omega} s_k(f)df \le P_k,\ k=1,...,K, \\ & s_k(f)\ge 0,\ f\in\Omega,\ s_k(\cdot) \text{ is Lebesgue measurable};\ k=1,...,K. \end{array} \qquad (P_c)$$

The following theorem follows from Lemma 1.

Theorem 7: Let $v(P)$ be the optimal value of (P_c) (also known as the perturbation function of (P_c)). Suppose that H is monotonically increasing componentwise and jointly concave. Then $v(P)$ is a concave function in P.

Proof. Let P^1 and P^2 be two parameter vectors. Let $s^1(f)$ and $s^2(f)$ be optimal solutions for (P_c) with parameters P^1 and P^2 respectively. Let

$$t_k^i = \int_{f\in\Omega} \ln\left(1+\frac{s_k^i(f)}{\sigma_k(f)+\sum_{j\neq k}\alpha_{kj}(f)s_j^i(f)}\right)df,\ \ i=1,2;\ k=1,...,K.$$

Then, by Lemma 1, there exist nonnegative Lebesgue measurable functions $\{\bar{s}_k(f)\}$ such that

$$\int_{f\in\Omega} \ln\left(1+\frac{\bar{s}_k(f)}{\sigma_k(f)+\sum_{j\neq k}\alpha_{kj}(f)\bar{s}_j(f)}\right)df = (t^1+t^2)/2$$

and

$$\int_{f\in\Omega} \bar{s}_k(f)df = \left(\int_{f\in\Omega} s_k^1(f)df + \int_{f\in\Omega} s_k^2(f)df\right)\Big/\, 2 \le (P_k^1+P_k^2)/2.$$

Therefore, the optimal value of (P_c) with parameter $(P^1+P^2)/2$ satisfies

$$\begin{aligned} v((P^1+P^2)/2) &\ge H((t_1^1+t_1^2)/2,\cdots,(t_K^1+t_K^2)/2) \\ &\ge (H(t_1^1,\cdots,t_K^1)+H(t_1^2,\cdots,t_K^2))/2 = (v(P^1)+v(P^2))/2. \end{aligned}$$

where the second inequality is due to concavity of H. ■

A consequence of Theorem 7 is that the Lagrangian dual problem admits no duality gap with the original problem. Specifically, define the following Lagrangian function

$$\begin{aligned} L(s,t;\lambda,\mu) &= H(t_1,\cdots,t_K) + \sum_{k=1}^{K} \lambda_k \left[\int_{f\in\Omega} \ln\left(1 + \frac{s_k(f)}{\sigma_k(f) + \sum_{j\neq k}\alpha_{kj}(f)s_j(f)}\right) df - t_k \right] \\ &\quad + \sum_{k=1}^{K} \mu_k \left[P_k - \int_{f\in\Omega} s_k(f) df \right] \\ &= H(t_1,\cdots,t_K) - \sum_{k=1}^{K} t_k\lambda_k + \sum_{k=1}^{K} P_k\mu_k \\ &\quad + \int_{f\in\Omega} \sum_{k=1}^{K} \left[\lambda_k \ln\left(1 + \frac{s_k(f)}{\sigma_k(f) + \sum_{j\neq k}\alpha_{kj}(f)s_j(f)}\right) - \mu_k s_k(f) \right] df. \end{aligned} \tag{8}$$

Let

$$\begin{aligned} g(\lambda,\mu) \quad := \quad & \sup \quad L(s,t;\lambda,\mu) \\ & \text{s.t.} \quad t \in \Re^K \\ & \qquad\; s_k(f) \geq 0,\ f \in \Omega,\ s_k(\cdot) \text{ is Lebesgue measurable};\ k = 1,...,K. \end{aligned}$$

Since t is separated from s, we may simplify the expression for $g(\lambda,\mu)$ by using the conjugate dual function of H (cf. (4)) which is convex. We have

$$g(\lambda,\mu) = H^*(\lambda) + p^{\mathrm{T}}\mu + \bar{g}(\lambda,\mu),$$

where

$$\begin{aligned} \bar{g}(\lambda,\mu) \quad := \quad & \max \quad \int_{f\in\Omega} \sum_{k=1}^{K} \left[\lambda_k \ln\left(1 + \frac{s_k(f)}{\sigma_k(f) + \sum_{j\neq k}\alpha_{kj}(f)s_j(f)}\right) - \mu_k s_k(f) \right] df \\ & \text{s.t.} \quad s_k(f) \geq 0,\ f \in \Omega,\ s_k(\cdot) \text{ is Lebesgue measurable};\ k = 1,...,K. \end{aligned}$$

Clearly, $g(\lambda,\mu)$ is convex jointly in λ and μ. The Lagrangian dual problem of (P_c) is defined as

$$\boxed{\begin{array}{ll} \text{minimize} & g(\lambda,\mu) \\ \text{subject to} & \lambda,\mu \in \Re^K_+. \end{array} \qquad (D_c)}$$

Due to Theorem 7, the perturbation function $v(P)$ is concave. By a well known result in convex analysis (Section 34 of [14]), this immediately implies that the duality gap is zero; see also Theorem 1 in Yu and Lui [20].

Corollary 1: Suppose that the system utility function $H(u_1,...,u_K)$ is jointly concave in $(u_1,u_2,...,u_K)$ and is nondecreasing in each u_k. Then, the optimal values of (P_c) and (D_c) are equal; i.e., the strong duality relation holds.

Since the concavity and monotonicity assumptions in Corollary 1 are satisfied by the min-rate, harmonic-rate, proportional fairness rate and sum-rate functions, it follows that the duality gap between (P_c) and

(D_c) is zero for all of these choices of system utility functions. By a "continuity" argument, this should imply that the duality gap between the discrete primal-dual pair (P_d^N) and (D_d^N) should vanish when $N \to \infty$. This is what we establish in the next theorem.

Theorem 8: Suppose the system utility function $H(u_1, u_2, ..., u_K)$ is jointly concave and continuous in $(u_1, ..., u_K)$, and is monotonically non-decreasing in each argument. Moreover, assume each user's utility function is given by $u_i = R_i(s_1(f), s_2(f), \ldots, s_K(f))$, with R_i nonnegative and Lebesgue measurable, where $s_j(f)$ is the nonnegative and Lebesgue integrable power spectral density function of user j. Let P_N^* and D_N^* denote the optimal values of (P_d^N) and (D_d^N) respectively. Then the duality gap $P_N^* - D_N^*$ vanishes asymptotically in the sense that

$$\liminf_{N \to \infty} (P_N^* - D_N^*) = 0.$$

In light of Corollary 1, we only need to show that the optimal values of (P_d^N) and (D_d^N) converge respectively to those of (P_c) and (D_c) respectively, as $N \to \infty$. The main difficulty with the proof is that the continuous formulations (P_c) and (D_c) involve Lebesgue integrals while the discrete formulations (P_d^N) and (D_d^N) involve Riemann sum of Lebesgue integrals. It is well known that we cannot in general approximate the value of a Lebesgue integral by a Riemann integral. The mismatch of integrals arise because Lyapunov theorem works only for Lebesgue integrals while in spectral management applications we are confined to Riemann sum type of discrete formulations. Fortunately, for the optimization problems considered here, the mismatch can be resolved. We leave the details of the proof to Appendix F.

V. Discussions

For a communication system in which users must share a common bandwidth, dynamic spectrum management (DSM) offers a great potential to significantly improve total system performance and spectral efficiency. This paper considers the computational challenges associated with DSM. If the potential benefits of DSM are to be realized, these challenges must be properly addressed. The complexity results of this paper suggest that for a given channel condition, computing the optimal spectrum sharing strategy is generally difficult, unless either the number of users in the system or the number of shared frequency tones is small (1 or 2). Even for a moderately sized problem (with $10 \sim 20$ users and $1000 \sim 2000$ frequency tones), finding the globally optimal spectrum sharing strategy can be computationally prohibitive. Consequently, our goal for DSM should be more realistic. The most that we can hope for is to be able to efficiently determine an approximately optimal spectrum sharing strategy with provably good quality.

One efficient approach to find high quality approximately optimal spectrum sharing strategies is through Lagrangian relaxation. This is because the dual formulation of the spectrum management problem is always convex and is amenable to distributed implementation. The duality analysis in this paper shows that the duality gap vanishes as the size of discretization decreases to zero, suggesting that the optimal spectrum management problem is asymptotically convex. The main reason for the vanishing duality gap is a hidden convexity associated with the continuous formulation due to the Lyapunov theorem in functional analysis. The asymptotic strong duality suggests that it may be possible to devise a polynomial time approximation scheme for the continuous spectrum management problem (P_c). That is, it may be possible to find an ϵ-optimal spectrum sharing strategy for (P_c) in time that is polynomial in K and $1/\epsilon$, where $\epsilon > 0$. However, to achieve this goal, it will be necessary to develop a strengthened duality analysis which explicitly bounds the size of duality gap for any finite discretization. We plan to address these and other related issues in a forthcoming paper.

A number of extensions to the current work are possible. For example, rather than maximizing a system-wide utility function as in the formulation (P_c), a telecom system operator may wish to minimize the total transmission power while ensuring a given data rate for each user. This leads to the following QoS (quality of service) constrained optimization:

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{K} \int_{f\in\Omega} s_k(f)df \\ \text{subject to} \quad & \int_{f\in\Omega} \ln\left(1 + \frac{s_k(f)}{\sigma_k(f) + \sum_{j\neq k} \alpha_{kj}(f)s_j(f)}\right) df \geq r_k, \\ & s_k(f) \geq 0, \ \text{Lebesgue integrable}, \ k = 1, 2, ..., K, \end{aligned}$$

where r_k is the required data rate for user k. The corresponding discretized version becomes

$$\begin{aligned} \text{minimize} \quad & \frac{1}{N} \sum_{k=1}^{K} \sum_{n=1}^{N} s_k^n \\ \text{subject to} \quad & \frac{1}{N} \sum_{n=1}^{N} \ln\left(1 + \frac{s_k^n}{\sigma_k^n + \sum_{j\neq k} \alpha_{kj}^n s_j^n}\right) \geq r_k, \ s_k^n \geq 0, \ k = 1, 2, ..., K. \end{aligned}$$

For the one-tone case ($N = 1$), the discrete formulation is simply a linear program (solvable in polynomial time). Also, if only one user is present ($K = 1$), then the problem is solved by the iterative water-filling procedure. For other general cases ($K \geq 2$ or $N \geq 3$), the proof techniques of Section III can be easily adapted to show the NP-hardness of the above QoS constrained problem. Moreover, Lyapunov theorem can again be applied to the above pair of continuous-discrete formulations and the asymptotic strong duality still holds.

Our work can also be extended to other resource management problems in multiuser communication such as transmission time management. In the latter case, we only need to change "f, $s(f)$, $\int \cdot\, df$, etc" to "t, $s(t)$, $\int \cdot\, dt$, ..." respectively. Management of hybrid resources such as time-frequency sharing can also be treated similarly.

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APPENDIX

Appendix A Proof of Theorem 1

We now present a polynomial transformation of the maximum independent set problem on a graph to $(P_1)'$. Since the former is NP-hard, this will imply the NP-hardness of $(P_1)'$ and (P_1). Suppose that $G = (V, E)$ is an undirected graph. An independent set of G is a subset $S \subseteq V$ such that no two nodes in S are connected: for any $v_i, v_j \in S$, $(v_i, v_j) \notin E$. To find an independent set with a given size is NP-hard.

Consider a connected graph with K vertices, i.e. $|V| = K$. For each $v_i \in V$, let

$$\alpha_{ij} = \begin{cases} MK^2, & \text{if } v_j \text{ is adjacent to } v_i; \\ 0, & \text{otherwise,} \end{cases}$$

where M is any positive number greater than K, and $\sigma_k = M$, $P_k = 1$, $k = 1, ..., K$. In this way, the feasible set becomes a Cartesian product of probability simplices. We claim that G has a maximum independent set of size $|I|$ if and only if the corresponding $(P_1)'$ has an optimal value v^* satisfying

$$|I| \ln \left(1 + \frac{1}{M}\right) \le v^* < (|I| + 1) \ln \left(1 + \frac{1}{M}\right).$$

If G has a maximum independent set I, then by letting

$$s_k = \begin{cases} 1, & \text{if } v_k \in I, \\ 0, & \text{otherwise,} \end{cases}$$

we have a solution for $(P_1)'$ with an objective value equal to $|I| \ln \left(1 + \frac{1}{M}\right)$.

On the other hand, suppose that one has an optimal solution s^* for $(P_1)'$ with optimal value v^*. By a direct computation, we have

$$\frac{\partial^2 H}{\partial s_k^2} = -\frac{1}{(s_k + M + MK^2 \sum_{(v_k, v_j) \in E} s_j)^2} + \sum_{i:(v_i, v_k) \in E} \frac{1}{(K^{-2} + \sum_{(v_i, v_j) \in E} s_j)^2}$$

Since G is connected, so the second sum is not vacuous and the definition of M ensures that $\frac{\partial^2 H}{\partial s_k^2} > 0$ for all feasible vector s. Thus, the objective function $H(s)$ is convex with respect to every component of s (though not jointly convex in s). Since the maximum of a convex function over a polytope is always attainable at a vertex, it follows that we can assume that s^* is a 0-1 vector. Let

$$S := \{v_k \mid s_k^* = 1,\, 1 \le k \le K\}.$$

Let I be a maximum independent set contained in S. Then, it follows from the property $M > K$ that

$$\begin{aligned} v^* &= \sum_{v_k \in S} \ln\left(1 + \frac{1}{M + MK^2|\{v_j \in S \mid (v_j, v_k) \in E\}|}\right) \\ &< |I| \ln\left(1 + \frac{1}{M}\right) + \sum_{v_k \in S} \ln\left(1 + \frac{1}{M + MK^2}\right) \\ &\le |I| \ln\left(1 + \frac{1}{M}\right) + K \ln\left(1 + \frac{1}{M + MK^2}\right) \\ &\le |I| \ln\left(1 + \frac{1}{M}\right) + \ln\left(1 + \frac{1}{M}\right) = (|I| + 1) \ln\left(1 + \frac{1}{M}\right) \end{aligned}$$

where the strict inequality is due to the connectedness of G which implies that $|\{v_j \in S \mid (v_j, v_k) \in E\}| \ge 1$. Thus, we have

$$|I| \ln\left(1 + \frac{1}{M}\right) \le v^* < (|I| + 1) \ln\left(1 + \frac{1}{M}\right),$$

establishing our claim.

In case $N \ge 2$, we consider N copies of graph G, called it $G^N = (V^N, E^N)$, defined as follows: $(v_k^i, v_k^j) \in E^N$ for any $1 \le i \ne j \le N$, and $(v_k^i, v_l^j) \in E^N$ iff $(v_k, v_l) \in E$, where $1 \le i, j \le N$, $1 \le k \ne l \le K$. Then, an independent set in G corresponds to an independent set in G^N, and vice versa. Hence, (P_1) is in general strongly NP-hard for any fixed integer $N \ge 1$.

Appendix B Proof of Theorem 2

The $N = 1$ case has been treated earlier. We only need to prove the problem is strongly NP-hard for $N = 3$. The general case of $N > 3$ by setting $\alpha_{kj}^n = 0$ and $\sigma_k^n = M$, for all $n > 3$ and all k, j, where

$$M \ge \max_k \{\sigma_k^1, \sigma_k^2, \sigma_k^3\} + \max_{k,j} \{\alpha_{kj}^1, \alpha_{kj}^2, \alpha_{kj}^3\} \left(\sum_k P_k\right)$$

is a constant. With this choice, it can be checked that all tones numbered $N > 3$ are too noisy to be used by any user in the system. In this case, the general $N > 3$ case is reduced to a three tone case.

When $N=3$, the spectrum management problem (P_f) becomes

$$\begin{array}{ll} \text{maximize} & \min_{1\le k\le K} \sum_{n=1}^{3} \ln\left(1+\frac{s_k^n}{\sigma_k^n+\sum_{j\neq k}\alpha_{kj}^n s_j^n}\right) \\ \text{subject to} & s_k^1+s_k^2+s_k^3\le P_k,\ s_k^1,s_k^2,s_k^3\ge 0,\ k=1,...,K. \end{array} \qquad (P_2)$$

To prove the strong NP-hardness, we construct a polynomial transformation from the so called vertex 3-coloring problem to (P_2). Given a connected graph $G=(V,E)$ with K vertices, (i.e. $|V|=K$), the 3-coloring problem requires the determination of whether or not there is a partition $V=V^1\cup V^2\cup V^3$ (mutually exclusive) such that V^1,V^2,V^3 are all independent sets of the graph. For each graph G, we define a corresponding spectrum management problem (P_2) as follows: for $n=1,2,3,$, let

$$\alpha_{ij}^n = \begin{cases} 3, & \text{if } v_j \text{ is adjacent to } v_i;, \\ 0, & \text{otherwise.} \end{cases}$$

Also, set $\sigma_k^n=1$ and $P_k=1$, for all n,k.

We claim that graph G is 3-colorable if and only if (P_2) has an optimal value greater or equal than $\ln 2$. If a 3-coloring solution $V=V^1\cup V^2\cup V^3$ exists, then we may let

$$s_k^1=\begin{cases} 1, & \text{if } v_k\in V^1, \\ 0, & \text{otherwise,}\end{cases} \quad \text{and } s_k^2=\begin{cases} 1, & \text{if } v_k\in V^2, \\ 0, & \text{otherwise,}\end{cases} \quad \text{and } s_k^3=\begin{cases} 1, & \text{if } v_k\in V^3, \\ 0, & \text{otherwise,}\end{cases}$$

with $k=1,...,K$. This achieves an objective value $\ln 2$ for (P_2).

Now suppose that we have a solution for (P_2) with an objective value at least $\ln 2$. Let

$$V^n := \left\{ v_k \;\middle|\; \ln\left(1+\frac{s_k^n}{\sigma_k^n+\sum_{j\neq k}\alpha_{kj}^n s_j^n}\right)\ge \frac{1}{3}\ln 2,\ 1\le k\le K \right\},\ n=1,2,3.$$

Clearly, we have $V=V^1\cup V^2\cup V^3$. We claim that each V^n must be an independent set. To see this, suppose the contrary so that there are two nodes $v_k,v_j\in V^n$ (for some $n=1,2,3$) which are adjacent in G. Then the above definition of V^n implies that

$$s_k^n\ge(2^{1/3}-1)(1+3s_j^n) \quad \text{and} \quad s_j^n\ge(2^{1/3}-1)(1+3s_k^n)$$

where we have used the definitions of σ_i^n and α_{ij}^n. Combining these two inequalities yields

$$s_k^n\ge(2^{1/3}-1)(1+3(2^{1/3}-1)(1+3s_k^n))$$

which implies that $s_k^n<0$. This is a contradiction, so the nodes v_j and v_k cannot be adjacent.

Notice that the sets V^1, V^2, V^3 may be overlapping. In this case, we can redefine the sets as $V^1:=V^1$, $V^2:=V^2\backslash V^1$, $V^3:=V^3\backslash(V^1\cup V^2)$. In this way, $V=V^1\cup V^2\cup V^3$ forms a partition and gives a 3-color solution for G. Since the above transformation involves only numbers that are at most polynomial in

K (in fact constant in K), this establishes the strong NP-hardness of the original spectrum management problem.

Appendix C Proof of Theorem 4

The proof of strongly NP-hardness for the case $N \geq 3$ is similar to the min-rate case considered in Theorem 2, and we only provide an outline below. Consider a graph $G = (V, E)$ with $|V| = K$. We need to show that the graph is 3-colorable if and only if the optimal value of the (P_f) is at least $K/\ln 2$. Let

$$\mu = \frac{\ln 2}{3K} \text{ and } \alpha_{ij} = \begin{cases} 1/\mu^2, & \text{if } v_j \text{ is adjacent to } v_i; \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

and $\sigma_k^n = 1$ for $n = 1, 2, 3$, and $k = 1, 2, ..., K$. We claim that G is 3-colorable if and only if the following problem:

$$\begin{aligned} &\text{maximize} && \left(\sum_{k=1}^{K} \left(\sum_{n=1}^{3} \ln \left(1 + \frac{s_k^n}{1 + \sum_{j \neq k} \alpha_{kj} s_j^n} \right) \right)^{-1} \right)^{-1} \\ &\text{subject to} && \sum_{n=1}^{3} s_k^n \leq 1,\ s_k^n \geq 0,\ k = 1, ..., K,\ n = 1, 2, 3, \end{aligned} \tag{10}$$

has an optimal value at least $\frac{\ln 2}{K}$.

"$\Longrightarrow$": If the graph is indeed 3-colorable, then there is a partition of the vertices, say $V = V^1 \cup V^2 \cup V^3$, such that V^n is an independent set, $n = 1, 2, 3$. Let

$$s_k^1 = \begin{cases} 1, & \text{if } v_k \in V^1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } s_k^2 = \begin{cases} 1, & \text{if } v_k \in V^2, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } s_k^3 = \begin{cases} 1, & \text{if } v_k \in V^3, \\ 0, & \text{otherwise,} \end{cases}$$

with $k = 1, ..., K$. Clearly this is a feasible solution of the above problem, whose objective value equals $\frac{\ln 2}{K}$.

"$\Longleftarrow$": Let f^* be the optimal value of (10) with $f^* \geq \frac{\ln 2}{K}$. Let us define

$$V^n := \left\{ v_k \,\middle|\, \ln \left(1 + \frac{s_k^n}{1 + \sum_{j \neq k} \alpha_{kj} s_j^n} \right) \geq \mu,\ 1 \leq k \leq K \right\},\ n = 1, 2, 3. \tag{11}$$

Since $\{s_k^n \mid n = 1, 2, 3; k = 1, ..., K\}$ is an optimal solution for (10) with optimal value f^*, for each given k, $1 \leq k \leq K$, it follows that

$$\sum_{n=1}^{3} \ln \left(1 + \frac{s_k^n}{1 + \sum_{j \neq k} \alpha_{kj} s_j^n} \right) \geq f^* \geq \frac{\ln 2}{K} = 3\mu.$$

By the above inequality and (11), we have $\cup_{n=1}^{3} V^n = V$. What remains to be seen is that each V^n forms an independent set. For this purpose, take any two vertices $v_k, v_l \in V^n$, and we wish to show that $(v_k, v_l) \notin E$. First, we note, due to $v_k \in V^n$, that

$$s_k^n \geq \frac{s_k^n}{1+\sum_{j\neq k}\alpha_{kj}s_j^n} \geq \exp(\mu) - 1 \geq \mu,$$

and similarly, $s_l^n \geq \mu$, since $v_l \in V^n$. Suppose by contradiction that $(v_k, v_l) \notin E$ and so $\alpha_{kj} = 1/\mu^2$. Then,

$$\frac{\mu}{1+\mu} = \frac{1}{1+\frac{1}{\mu^2}\mu} \geq \frac{1}{1+\frac{1}{\mu^2}s_l^n} \geq \frac{s_k^n}{1+\sum_{j\neq k}\alpha_{kj}s_j^n} \geq \mu,$$

which is clearly a contradiction.

Finally, we notice that the sets V^1, V^2, V^3 may be overlapping. In this case, we can redefine the sets as $V^1 := V^1$, $V^2 := V^2 \backslash V^1$, $V^3 := V^3 \backslash (V^1 \cup V^2)$. In this way, $V = V^1 \cup V^2 \cup V^3$ forms a partition and gives a 3-color solution for G. Since the polynomial transformation outlined above involves only numbers that are polynomial in K, we conclude the original spectrum management problem is strongly NP-hard.

Appendix D Proof of Theorem 5

For the case $N \geq 3$, we use a polynomial time reduction (similar to the one used in Theorem 2) to transform the 3-colorability problem to (P_4). Consider a graph $G = (V, E)$ with $|V| = K$. Let

$$\lambda = \left(\frac{1}{3}\right)^K \ln 2 \quad \text{and} \quad \alpha_{ij} = \begin{cases} 1/\lambda^2, & \text{if } v_j \text{ is adjacent to } v_i; \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

and $\sigma_k = 1,\ k = 1, ..., K$.

We claim that the graph is 3-colorable if and only if the following problem:

$$\begin{array}{ll} \text{maximize} & \sum_{k=1}^{K} \ln\left(\sum_{n=1}^{3} \ln\left(1+\frac{s_k^n}{1+\sum_{j\neq k}\alpha_{kj}s_j^n}\right)\right) \\ \text{subject to} & \sum_{n=1}^{3} s_k^n \leq 1,\ k = 1, ..., K, \\ & s_k^n \geq 0,\ k = 1, ..., K;\ n = 1, 2, 3, \end{array} \tag{13}$$

has an optimal value at least $K \ln\ln 2$.

"$\Longrightarrow$": If the graph is indeed 3-colorable, then there is a partition of the vertices, say $V = V^1 \cup V^2 \cup V^3$, such that V^n is an independent set, $n = 1, 2, 3$. Let

$$s_k^1 = \begin{cases} 1, & \text{if } v_k \in V^1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } s_k^2 = \begin{cases} 1, & \text{if } v_k \in V^2, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } s_k^3 = \begin{cases} 1, & \text{if } v_k \in V^3, \\ 0, & \text{otherwise,} \end{cases}$$

with $k = 1, ..., K$. Clearly this is a feasible solution of (13), whose objective value equals $K \ln \ln 2$.

"$\Longleftarrow$": Let f^* be the optimal value of the above problem with $f^* \geq K \ln \ln 2$. Let us define

$$V^n := \left\{ v_k \;\middle|\; \ln\left(1 + \frac{s_k^n}{1 + \sum_{j \neq k} \alpha_{kj} s_j^n}\right) \geq \lambda,\ 1 \leq k \leq K \right\},\ n = 1, 2, 3. \tag{14}$$

Notice that for any feasible solution of (13), it holds that

$$\sum_{n=1}^{3} \ln\left(1 + \frac{s_k^n}{1 + \sum_{j \neq k} \alpha_{kj} s_j^n}\right) \leq \sum_{n=1}^{3} \ln\left(1 + s_k^n\right) \leq 3 \ln 2,$$

for all $k = 1, ..., K$. Suppose that $\{s_k^n \mid n = 1, 2, 3; k = 1, ..., K\}$ is an optimal solution for (P_{pf}) with optimal value f^*. Then, for each given k, $1 \leq k \leq K$, it follows that

$$\sum_{n=1}^{3} \ln\left(1 + \frac{s_k^n}{1 + \sum_{j \neq k} \alpha_{kj} s_j^n}\right) \geq \frac{\exp(f^*)}{(3 \ln 2)^{K-1}} \geq \frac{(\ln 2)^K}{(3 \ln 2)^{K-1}} = 3\lambda.$$

By the above inequality and (14), we have $\cup_{n=1}^{3} V^n = V$. What remains to be seen is that each V^n forms an independent set. For this purpose, take any two vertices $v_k, v_l \in V^n$, and we wish to show that in this case it $(v_k, v_l) \notin E$. First, we note that

$$s_k^n \geq \frac{s_k^n}{1 + \sum_{j \neq k} \alpha_{kj} s_j^n} \geq \exp(\lambda) - 1 \geq \lambda,$$

and similarly, $s_l^n \geq \lambda$. Suppose by contradiction that $(v_k, v_l) \notin E$ and so $\alpha_{kj} = 1/\lambda^2$. Then,

$$\frac{\lambda}{1 + \lambda} = \frac{1}{1 + \frac{1}{\lambda^2}\lambda} \geq \frac{1}{1 + \frac{1}{\lambda^2} s_l^n} \geq \frac{s_k^n}{1 + \sum_{j \neq k} \alpha_{kj} s_j^n} \geq \lambda,$$

which is clearly a contradiction. Thus, the vertices in V^n $(n = 1, 2, 3)$ are independent, establishing the NP-hardness as desired. [Notice that we cannot claim strong NP-hardness since the transformation outlined above involves exponentially large numbers (in terms of K), although their binary lengths remain polynomial.].

Appendix E Proof of Theorem 6

Let $K = 2$. The case of $H(u) = H_1(u) = \sum_{n=1}^{N} u_n$ (sum-rate) has been considered in [8]. Below, we treat the other three cases using the same polynomial transformation from the equipartition problem: given a set of N (even) positive integers a_1, a_2,..., a_N, determine if there exists a subset $S \subset \{1, 2, ..., N\}$ of $N/2$ numbers such that

$$\sum_{n \in S} a_n = \sum_{n \notin S} a_n = \frac{1}{2} \sum_{n=1}^{N} a_n.$$

Recall that it has been shown [8] that if

$$\alpha_{12}^n \alpha_{21}^n \geq 1/4, \quad \text{for all } n$$

then the optimal solution of the two-user sum-rate maximization problem must be FDMA, satisfying $s_1^n s_2^n = 0$, for all n. Furthermore, given an even integer N and a set of N positive integers $a_1, a_2, ..., a_N$, we can construct a two-user communication system as follows: let there be a total of N frequency tones, and let the channel noise powers for the two users be $\sigma_1^n = \sigma_2^n = a_n$, for $n = 1, 2, ..., N$. We also set the crosstalk coefficients $\alpha_{12}^n = \alpha_{21}^n = 1.01$ for all n, and let $P_1 = P_2 = P := (N+1)^3 \sigma_M$, with $\sigma_M := \max_n a_n$. In this case, problem (P_d^N) is reduced to the following:

$$\begin{aligned} H_i^{fdma} := \text{maximize} \quad & H_i(u_1, u_2) \\ \text{subject to} \quad & u_1 = \sum_{n=1}^N \ln\left(1 + \frac{s_1^n}{a_n}\right), \quad u_2 = \sum_{n=1}^N \ln\left(1 + \frac{s_2^n}{a_n}\right) \\ & \mathbf{s} \in \mathcal{S}, \quad \sum_{n=1}^N s_1^n \le P, \quad \sum_{n=1}^N s_2^n \le P, \end{aligned} \tag{15}$$

where $i = 1, 2, 3, 4$. Let u_1^*, u_2^* denote the optimal rates of user 1 and user 2 respectively. Next we consider a convex relaxation of (15), with $i = 1$, by dropping the nonconvex FDMA constraint $\mathbf{s} \in \mathcal{S}$, and by combining the two separate power constraints as a single one:

$$\begin{aligned} R_{relax} := \text{maximize} \quad & H_1(u_1, u_2) \\ \text{subject to} \quad & u_1 = u_2 = \sum_{n=1}^N \ln\left(1 + \frac{s^n}{a_n}\right) \\ & \sum_{n=1}^N s^n \le 2P, \quad s^n \ge 0, \quad \forall\, n. \end{aligned} \tag{16}$$

Notice that the relaxed problem (16) is a standard single user rate maximization problem, so R_{relax} can be evaluated easily using convex optimization (or the classical Karush-Kuhn-Tucker optimality condition). For the case $H(u) = H_1(u)$ (sum-rate maximization), it was shown [8] that

$$H_1^{fdma} \le R_{relax} = N \ln\left(\frac{2P + \sum_{n=1}^N a_n}{N}\right) - \sum_{n=1}^N \ln a_n$$

and the equality holds if and only if $u_1^* = u_2^*$. Moreover, the latter holds if and only if the equipartition problem has a 'yes' answer. For other three cases of system utility functions ($i = 2, 3, 4$), we have

$$H_4^{fdma}(u_1, u_2) \le H_3^{fdma}(u_1, u_2) \le H_2^{fdma}(u_1, u_2) \le H_1^{fdma}(u_1, u_2)$$

for all $u_1, u_2 \ge 0$, where the equalities hold if and only if $u_1 = u_2$. Thus, for $i = 2, 3, 4$, we can conclude that $H_i^{fdma} \le R_{relax}$, with equality holding exactly when $u_1^* = u_2^*$, or equivalently when the equipartition problem has a 'yes' answer. This implies the NP-hardness of the spectrum management problem (P_d^N) in the two-user case for all three system utility functions $H_2(u)$, $H_3(u)$ and $H_4(u)$.

Appendix F   Proof of Theorem 8

In what follows, we only prove that the optimal value of (P_d^N) converges to that of (P_c), as the dual case is similar. Let $v_d^N(P)$ and $v(P)$ denote the optimal values of (P_d^N) and (P_c) respectively. Suppose $v(P)$ is attained at Lebesgue integrable functions $\{s_1(f), s_2(f), ..., s_K(f)\}$. Then, we have $s_i(f) \geq 0$ and

$$\begin{aligned} v(P) &= H(u_1, u_2, ..., u_K), \\ u_i &= \int_0^1 R_i(s_1(f), s_2(f), \ldots, s_K(f))df, \quad i = 1, 2, ..., K, \\ P_i &\geq \int_0^1 s_i(f)df, \quad i = 1, 2, ..., K. \end{aligned}$$

By the definition of Lebesgue integral, for each $\epsilon > 0$, there exists some $N_1 > 0$ and a partition of the nonnegative real line

$$0 = R_i^0 < R_i^1 < R_i^2 < \cdots < R_i^{N_1} < R_i^{N_1+1} = \infty$$

such that

$$\left| u_i - \sum_{n=1}^{N_1} R_i^n \mu(A_i^n) \right| \leq \epsilon,$$

where $\mu(\cdot)$ denotes the Lebesgue measure and A_i^n is the inverse image of interval $[R_i^n, R_i^{n+1})$ under mapping R_i:

$$A_i^n = R_i^{-1}([R_i^n, R_i^{n+1})), \quad n = 1, 2, ..., N_1.$$

Notice that the sets $\{A_i^1, A_i^2, ..., A_i^{N_1}\}$ are Lebesgue measurable and together they form a partition of the unit interval $[0, 1] = \bigcup_{n=1}^{N_1} A_i^n$. Similarly, there exists some $N_2 > 0$ and a partition of the nonnegative real line

$$0 = s_i^0 < s_i^1 < s_i^2 < \cdots < s_i^{N_1} < s_i^{N_1+1} = \infty$$

such that

$$\left| \int_0^1 s_i(f)df - \sum_{n=1}^{N_2} s_i^n \mu(B_i^n) \right| \leq \epsilon,$$

where B_i^n is the inverse image of interval $[R_i^n, R_i^{n+1})$ under mapping $s_i(f)$:

$$B_i^n = s_i^{-1}([s_i^n, s_i^{n+1})), \quad n = 1, 2, ..., N_2.$$

Thus, each user $i = 1, 2, ..., K$ has two partitions $\{A_i^1, A_i^2, ..., A_i^{N_1}\}$, $\{B_i^1, B_i^2, ..., B_i^{N_2}\}$ of the unit interval $[0, 1]$. By a further refinement of these partitions for all i if necessary, we assume that the partitions are identical for all users. For simplicity, let $\{A^1, A^2, ..., A^{N_3}\}$ denote the partition common for all users. Then we have

$$\left| u_i - \sum_{n=1}^{N_3} \bar{R}_i^n \mu(A^n) \right| \leq \epsilon, \quad \forall\ i = 1, 2, ..., K, \tag{17}$$

and

$$\left| \int_0^1 s_i(f) df - \sum_{n=1}^{N_3} \bar{s}_i^n \mu(A^n) \right| \leq \epsilon, \quad \forall\ i = 1, 2, ..., K, \tag{18}$$

where $\bar{R}_i^n \in \{R_i^1, R_i^2, \cdots, R_i^{N_1}\}$ and $\bar{s}_i^n \in \{s_i^1, s_i^2, \cdots, s_i^{N_2}\}$ for $n = 1, 2, ..., N_3$. Since $\bar{R}_i^n$ is constant over A^n and is defined by the values of $s_i(f)$ over A^n which are $\{\bar{s}_i^n\}_{i=1}^K$, we have

$$\bar{R}_i^n = R_i(\bar{s}_1^n, \bar{s}_2^n, \cdots, \bar{s}_K^n), \quad \forall\ 1 \leq i \leq K,\ 1 \leq n \leq N_3. \tag{19}$$

Let

$$M = \max_{1 \leq i \leq K} \{R_i^{N_1}, s_i^{N_2}\} = \max_{1 \leq i \leq K, 1 \leq n \leq N_3} \{\bar{R}_i^n, \bar{s}_i^n\}.$$

Since each A^n is Lebesgue measurable, it follows that there exists a finite union of disjoint intervals $A^n(1), ..., A^n(j_n)$ which form an almost exact approximation of A^n in the sense that

$$\mu\left(A^n \triangle \left(\bigcup_{j=1}^{j_n} A^n(j) \right) \right) \leq \frac{\epsilon}{MN_3},$$

where $\triangle$ denotes set difference operator. This implies that

$$\left| \mu(A^n) - \sum_{j=1}^{j_n} \mu(A^n(j)) \right| \leq \frac{\epsilon}{MN_3}. \tag{20}$$

Consider a uniform partition of the unit $[0, 1] = \left\{0, \frac{1}{N}, \frac{2}{N}, ..., \frac{N-1}{N}, 1\right\}$. We need to approximate the end points of the intervals $A^n(j)$ simultaneously by rational numbers of the form i/N. By Dirichlet theorem for simultaneous Diophantine approximation, there exists a sufficiently large integer N_4 such that for all $N = kN_4$ (multiples of N_4) each interval $A^n(j)$ can be well-approximated by a finite interval of the form $I_N^n(j) = [\frac{i_j}{N}, \frac{k_j}{N}]$, with i_j, k_j being integers in the interval $[0, N]$, such that

$$\left| \mu(A^n(j)) - \frac{|k_j - i_j|}{N} \right| \leq \frac{\epsilon}{MNj_n}. \tag{21}$$

In this way, we have

$$\begin{aligned}\mu\left(A^n\triangle\left(\bigcup_{j=1}^{j_n} I_N^n(j)\right)\right) &\leq \mu\left(A^n\triangle\left(\bigcup_{j=1}^{j_n} A^n(j)\right)\right)+\mu\left(\left(\bigcup_{j=1}^{j_n} I_N^n(j)\right)\triangle\left(\bigcup_{j=1}^{j_n} A^n(j)\right)\right)\\ &\leq \frac{\epsilon}{MN_3}+\sum_{j=1}^{j_n}\mu\left(I_N^n(j)\triangle A^n(j)\right)\\ &\leq \frac{\epsilon}{MN_3}+\sum_{j=1}^{j_n}\frac{\epsilon}{MN_3 j_n}=\frac{2\epsilon}{MN_3},\end{aligned}$$

where the last step follows from (20)–(21). Therefore, we obtain

$$\left|\mu(A^n)-\mu\left(\bigcup_{j=1}^{j_n} I_N^n(j)\right)\right|\leq\frac{2\epsilon}{MN_3}.$$

In other words, each set A^n can be approximated by a finite union of intervals of the form I_N^i taken from a uniform partition of unit interval $[0,1]$, provided N is sufficiently large (and multiple of N_4). Substituting these approximations into integral estimates (17)–(18), we obtain

$$\begin{aligned}\left|u_i-\frac{1}{N}\sum_{n=1}^{N}\hat{R}_i^n\right| &\leq \left|u_i-\sum_{n=1}^{N_3}\bar{R}_i^n\mu(A^n)\right|+\max_n\bar{R}_i^n\sum_{n=1}^{N_3}\left|\mu(A^n)-\mu\left(\bigcup_{j=1}^{j_n} I_N^n(j)\right)\right|\\ &\leq \epsilon+M\times N_3\times\frac{2\epsilon}{MN_3}=3\epsilon,\quad\forall\ i=1,2,...,K,\end{aligned}$$

where $\hat{R}_i^n\in\{R_i^1,R_i^2,...,R_i^{N_1}\}$ for all $n=1,2,...,N$. Similarly, we have

$$\left|\int_0^1 s_i(f)df-\frac{1}{N}\sum_{n=1}^{N}\hat{s}_i^n\right|\leq 3\epsilon,\quad\forall\ i=1,2,...,K,$$

where $\hat{s}_i^n\in\{s_i^1,s_i^2,...,s_i^{N_2}\}$ for all $n=1,2,...,N$, which further implies

$$\frac{1}{N}\sum_{n=1}^{N}\hat{s}_i^n\leq\int_0^1 s_i(f)df+3\epsilon\leq P_i+3\epsilon. \tag{22}$$

Moreover, it follows from (19) that

$$\hat{R}_i^n=R_i(\hat{s}_1^n,\hat{s}_2^n,\cdots,\hat{s}_K^n),\quad\forall\ 1\leq i\leq K,\ 1\leq n\leq N,$$

which implies

$$\left|u_i-\frac{1}{N}\sum_{n=1}^{N}R_i(\hat{s}_1^n,\hat{s}_2^n,\cdots,\hat{s}_K^n)\right|\leq 3\epsilon.$$

Since the objective function $H(u_1,u_2,...,u_K)$ is continuous, the above estimate and (22) show that

$$\lim_{\substack{N=kN_4\\ k\to\infty}}|P_N^*-P_\infty^*|\leq\delta(\epsilon),$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. In a similar fashion, there exists $N_5 > 0$ such that

$$\lim_{\substack{N=kN_5 \\ k\to\infty}} |D_N^* - D_\infty^*| \le \delta(\epsilon).$$

Since $P_\infty^* = D_\infty^*$ (Corollary 1), it follows that

$$\lim_{\substack{N=kN_4N_5 \\ k\to\infty}} |D_N^* - P_N^*| \le 2\delta(\epsilon).$$

Letting $\epsilon \to 0$, we obtain $\liminf_{N\to\infty} |D_N^* - P_N^*| = 0$ as desired.