SERIES EVALUATION OF A QUARTIC INTEGRAL

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Abstract. We present a new single sum series evaluation of Moll's quartic integral and present two new generalizations.

In a beautiful personal story [6] Victor Moll describes his encounter with certain quartic integral and derives its evaluation and goes on to study analytic and number theoretic properties (*log-concavity*, *p-adic valuations*, *location of the zeros*, etc.) of a polynomial associated with the evaluation of the integral [1,2,4,5,6,7]. In this article we use the Almkvist-Zeilberger algorithm ([3,8,9,10]) to derive a new series evaluation of this integral. In addition, we give two new generalizations of the identity. In [1], T. Amdeberhan and V. Moll presented a survey of old and new proofs of the evaluation and the formula:

Theorem 1 [T. Amdeberhan and V. Moll, [1]]:

$$\int_0^\infty \frac{dx}{(x^4+2x^2a+1)^{m+1}} = \frac{\pi}{2}\frac{\binom{2m}{m}}{4^m(2(a+1))^{m+1/2}}\,{}_2F_1\left(\begin{matrix}-m, m+1\\ -m+1/2\end{matrix}\,;\,(a+1)/2\right) .$$

where

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\,;\,x\right) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k(1)_k}x^k$$

and $(z)_k = z(z+1)(z+2)\ldots(z+k-1)$.

The polynomial associated with the evaluation of the integral that is the subject of study in [1,2,3,4] is

$$P_m(a) = \frac{\binom{2m}{m}}{4^m}\,{}_2F_1\left(\begin{matrix}-m, m+1\\ -m+1/2\end{matrix}\,;\,(a+1)/2\right) .$$

Next we state the main results of this article:

Theorem 2:

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \quad = \quad \frac{1}{4}\sum_{l=0}^{\infty}(-1)^l\frac{2^l(\frac{l}{2}-\frac{3}{4})!(m+\frac{l}{2}-\frac{1}{4})!}{l!m!}a^l .$$

Proof:

We use the Almkvist-Zeilberger algorithm ([3,8,9,10]), and the reader is assumed to be familiar with this method. In particular, we used Zeilberger's Maple package EKHAD8 (procedure AZc) that computes differential operators and certificates for single variable hyper-exponential functions accompanying [3], available from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/EKHAD` .

We cleverly construct the (certificate) function

$$R(x,a) = -\frac{x(4m+3+4ax^2m+2ax^2-x^4)}{(x^4+2ax^2+1)}$$

with the motives

$$-4m-3-4a(2m+3)D_a(F(x,a)) - 4(a^2-1)D_a^2F(x,a) = D_x(R(x,a)F(x,a)) \ ,$$

where $F(x,a)$ is the integrand and D_a is differentiation operator with respect to the variable a. If we integrate both sides with respect to x on the limits of integration and observe that the right-hand side vanishes, we get a differential operator

$$-4m-3-4a(2m+3)D_a - 4(a^2-1)D_a^2 \ ,$$

that annihilates the left side of the theorem. Using the standard technique (or use Paul Zimmermann and Bruno Salvy's $gfun$ from Maple library if you wish) of translating a differential equation satisfied by a power series into a recurrence relation for its coefficients $a_l(m)$, we get

$$(-4l^2+(-8m-8)l-4m-3)a_l(m) + (4l^2+12l+8)a_l(m)(l+2) = 0 \ ,$$

a homogeneous recurrence relation satisfied by the discrete coefficient function $a_l(m)$. Finally, the theorem follows by solving the recurrence relation with the initial conditions calculated directly from the integral: $a_0(m) = I(0,m)$ and $a_1(m) = I'(0,m)$, where $I(a,m)$ is the the integral on the left. Q.E.D.

Comparing the right-hand side of our theorem with that of (*theorem 1*), we get

$$P_m(a) = \frac{2^{m+3/2}(a+1)^{m+1/2}}{4\pi}\sum_{l=0}^{\infty}(-1)^l\frac{2^l(\frac{l}{2}-\frac{3}{4})!(m+\frac{l}{2}-\frac{1}{4})!}{l!m!}a^l \qquad (Polypart)$$

Using Newton's Binomial theorem,

$$(1+a)^{m+1/2} = \sum_{k=0}^{\infty}\binom{m+1/2}{k}a^k \ ,$$

and multiplication of series, the coefficient of a^n, $d_n(m)$, in the polynomial $P_m(a)$ is

$$d_n(m) = \frac{2^{m+3/2}}{4\pi} \sum_{k+l=n} \binom{m+\frac{1}{2}}{k} (-1)^l \frac{2^l(\frac{l}{2}-\frac{3}{4})!(m+\frac{l}{2}-\frac{1}{4})!}{l!m!}$$
$$= \frac{2^{m+3/2}}{4\pi} \sum_{l=0}^{n} \binom{m+\frac{1}{2}}{n-l} (-1)^l \frac{2^l(\frac{l}{2}-\frac{3}{4})!(m+\frac{l}{2}-\frac{1}{4})!}{l!m!} .$$

Next, we give the first of two generalizations in which 2 in the integral of *theorem 2* is replaced by any integer n for which the integral exists.

Theorem 3:

$$\int_0^\infty \frac{dx}{(x^{2n}+nax^n+1)^{m+1}} = \frac{1}{2n} \sum_{l=0}^{\infty} (-1)^l \frac{n^l(\frac{l}{2}-\frac{2n-1}{2n})!(\frac{l}{2}+m-\frac{1}{2n})!}{l!m!} a^l .$$

any integer n for which the integral exists.

Proof:

First, we make the change of variables $z = x^n$ and the question reduces to evaluating

$$\int_0^\infty \frac{dz}{n(z^2+2az+1)^{m+1} z^{1-1/n}} .$$

Then, EKHAD gives a differential operator

$$-2n - 2nm + 1 - (2m+3)n^2 a D_a - n^2(a^2-1)D_a^2 .$$

with certificate function

$$R(z,a) = -\frac{nz(2n+2nm-1+nzma+naz-az-z^2)}{z^2+az+1} .$$

That is,

$$(-2n-2nm+1-(2m+3)n^2aD_a - n^2(a^2-1)D_a^2)F(x,a) = D_x(R(x,a)F(x,a)) .$$

where $F(x,a)$ is the integrand and D_a is differentiation operator with respect to the variable a. Now integrate both sides and convert the resulting differential operator for the series into a recurrence relation for the coefficients and solve. Q.E.D.

The second generalization where n is replaced by any parameter α for which the integral exists whose proof follows from *theorem 3* by writing αa as $n\left(\frac{a\alpha}{n}\right)$.

Theorem 4:

$$\int_0^\infty \frac{dx}{(x^{2n}+\alpha ax^n+1)^{m+1}} = \frac{1}{2} \sum_{l=0}^{\infty} (-1)^l n^{l-1} (\frac{\alpha}{n})^l \frac{(\frac{l}{2}-\frac{2n-1}{2n})!(\frac{l}{2}+m-\frac{1}{2n})!}{l!m!} a^l .$$

for any integer $n > 0$ and indeterminate α for which the integral exists.

Problem: Find analogous polynomial *Polypart* as in theorem 1 associated with the evaluation of the generalization in *theorem 3* if it exists, i.e. find $closedForm(n,m,a)$ such that

$$P_m^n(a) \quad := \quad closedForm(n,m,a) \times \frac{1}{2n}\sum_{l=0}^{\infty}(-1)^l \frac{n^l(\frac{l}{2}-\frac{2n-1}{2n})!(\frac{l}{2}+m-\frac{1}{2n})!}{l!m!}a^l \ ,$$

is a polynomial in a.

For the special case $n=2$ (*Polypart*), $closedForm(2,m,a) = \frac{\pi}{2}\frac{1}{(2(a+1))^{m+1/2}}$.

References

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