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Comparing SOS and SDP relaxations of sensor network localization

João Gouveia · Ting Kei Pong

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Abstract We investigate the relationships between various sum of squares (SOS) and semidefinite programming (SDP) relaxations for the sensor network localization problem. In particular, we show that Biswas and Ye's SDP relaxation is equivalent to the degree one SOS relaxation of Kim et al. We also show that Nie's sparse-SOS relaxation is stronger than the edge-based semidefinite programming (ESDP) relaxation, and that the trace test for accuracy, which is very useful for SDP and ESDP relaxations, can be extended to the sparse-SOS relaxation.

Key words. Sensor network localization, semidefinite programming relaxation, sum of squares relaxation, individual trace.

1 Introduction

In its basic form, the sensor network localization problem is that of finding the coordinates of some sensors $x_i = (x_i^1, x_i^2)^T \in \mathbb{R}^2, i = 1, \dots, m$, while given the Cartesian coordinates of $n-m$ points in $\mathbb{R}^2$, $x_{m+1}, \dots, x_n$, which we call *anchors*, and the Euclidean distances $\|x_i - x_j\|$, for all $(i,j) \in \mathcal{A}$, where $\mathcal{A} \subset \{(i,j) \in \mathbb{N}^2 : 1 \leq i < j \leq n\}$ is the set of *edges*. We say that the two points x_i and x_j are *neighbors* if $(i,j) \in \mathcal{A}$. In practice, measured distances

João Gouveia
Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195, USA, and CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail: jgouveia@math.washington.edu

Ting Kei Pong
Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195, USA
E-mail: tkpong@math.washington.edu

may be inexact, and we only know some estimated d_{ij}, where

$$d_{ij}^2 = \|x_i^{\text{true}} - x_j^{\text{true}}\|^2 + \delta_{ij} \quad \forall (i,j) \in \mathcal{A},$$

$\delta = (\delta_{ij})_{(i,j)\in\mathcal{A}} \in \mathbb{R}^{|\mathcal{A}|}$ denotes the measurement noise, and x_i^{true} denotes the true position of the ith point. When $\delta = 0$, we call this problem the *noiseless* sensor network localization problem.

The sensor network localization problem is NP-hard in general, thus efforts have been directed at solving this problem approximately. One approach involves solving a convex relaxation, and then refining the resulting solution through local improvement. Examples of this approach are second-order cone programming (SOCP) relaxations [4,13], semidefinite programming (SDP) relaxations [1–3,5–7,11], edge-based semidefinite programming (ESDP) relaxations [10,14] and sum of squares (SOS) relaxations [8]. We will compare the SOS relaxations with the SDP type relaxations, and in particular show that the SOS relaxations are tighter.

2 Notation

Throughout this paper, sensor positions x_i are 2×1 vectors, $\mathcal{S}^n$ denotes the space of $n \times n$ real symmetric matrices, and ^{T} denotes transpose. For a vector $v \in \mathbb{R}^p$, $\|v\|$ denotes the Euclidean norm of v. For $A \in \mathbb{R}^{p\times q}$, a_{ij} denotes the (i,j)th entry of A. For $A, B \in \mathcal{S}^p$, $A \succeq B$ means $A - B$ is positive semidefinite. For $A \in \mathcal{S}^p$ and an index set $\mathcal{I}$, $A_{\mathcal{I}} = \left(a_{ij}\right)_{i,j\in\mathcal{I}}$ denotes the principal submatrix of A comprising the rows and columns of A indexed by $\mathcal{I}$.

Any instance of the sensor network localization problem has an associated graph structure, namely the graph $\mathcal{G} = (\{1,\ldots,n\},\mathcal{A})$. We will work under the standard assumptions that every connected component of $\mathcal{G}$ has at least one index corresponding to an anchor and that each sensor connects to at least one other sensor. The first assumption is justified since if a connected component has no anchors, all associated sensors are clearly not localizable, i.e. their positions are not uniquely determined from the known distances; while the second assumption is reasonable since if a sensor is only connected to anchors, determining its location can be treated as a separate problem. We partition the set $\mathcal{A}$ of edges into the sets $\mathcal{A}^s = \{(i,j) \in \mathcal{A} : i < j \leq m\}$ (edges from a sensor to a sensor) and $\mathcal{A}^a = \{(i,j) \in \mathcal{A} : i \leq m < j\}$ (edges from a sensor to an anchor). The set β^k will be the set of all monomials in variables $\{x_i^1, x_i^2 : i = 1,...,m\}$ with degree up to k, while for $(i,j) \in \mathcal{A}$, the set β_{ij}^k will denote the set of all monomials of degree up to k in variables $\{x_i^1, x_i^2, x_j^1, x_j^2\}$ if $(i,j) \in \mathcal{A}^s$, or in variables $\{x_i^1, x_i^2\}$ if $(i,j) \in \mathcal{A}^a$. Let β be any set of monomials, we define ξ_β to be the column vector indexed by β with polynomial entries such that for each $s \in \beta$, $[\xi_\beta]_s = s(x)$. Let Γ be the set of monomials obtained by taking all possible pairwise products of the elements of β. Then

$$\xi_\beta \xi_\beta^T = \sum_{s\in\Gamma} s(x) A_s, \tag{1}$$

for some $|\beta| \times |\beta|$ real symmetric matrices A_s. Given a real vector y indexed by a set of monomials containing Γ, we define the moment matrix of y with respect to β as

$$M_\beta(y) = \sum_{s \in \Gamma} y_s A_s,$$

a linearization of (1).

3 SOS relaxations

The sensor network localization can be formulated as the following polynomial optimization problem:

$$v_{\text{opt}} := \min_{x_1,\dots,x_m} p(x) := \sum_{(i,j)\in\mathcal{A}} (\|x_i - x_j\|^2 - d_{ij}^2)^2. \tag{2}$$

Let $\mathcal{P} \subseteq \mathbb{R}^{2\times m}$ be the set of minimizers to problem (2). This is an unconstrained polynomial optimization problem and can be relaxed using sums of squares, as proposed in [8] by Nie:

$$\begin{aligned} v_{\text{sos}} := &\max_{q_i,\gamma} \gamma \\ &\text{s.t. } \; p(x) - \gamma = \textstyle\sum_{i=1}^r q_i(x)^2, \end{aligned} \tag{3}$$

where $q_i(x)$ are arbitrary polynomials. It is well known that this problem can be reformulated as the following SDP

$$\begin{aligned} v_{\text{sos}} := &\max_{W,\gamma} \gamma \\ &\text{s.t. } \; p(x) - \gamma = \xi_{\beta^2}^T W \xi_{\beta^2}, \quad W \succeq 0. \end{aligned} \tag{4}$$

Write $p(x) = \sum_{s\in\beta^4} p_s s(x)$, the dual of (4) can then be written as

$$\begin{aligned} v_{\text{mom}}^d := &\min_y \sum_{s\in\beta^4} p_s y_s \\ &\text{s.t. } \; M_{\beta^2}(y) \succeq 0 \\ &\qquad y_1 = 1, \end{aligned} \tag{5}$$

where y is a real vector indexed by β^4. The solution set of (5) is denoted by $\mathcal{S}_{\text{mom}}$ and the sensor x_i is recovered from a solution $y \in \mathcal{S}_{\text{mom}}$ by setting $x_i = (y_{x_i^1}, y_{x_i^2})^T$. The set of all sensor positions (each sensor position is denoted by a 2×1 vector) obtained this way is denoted by $\mathcal{P}_{\text{mom}} \subseteq \mathbb{R}^{2\times m}$.

In view of the special structure of (2), Nie proposed considering the following sparse-SOS relaxation:

$$\begin{aligned} v_{\text{spsos}} := &\max_{W_{ij},\gamma} \gamma \\ &\text{s.t. } \; p(x) - \gamma = \sum_{(i,j)\in\mathcal{A}^s} \xi_{\beta_{ij}^2}^T W_{ij} \xi_{\beta_{ij}^2} \\ &\qquad W_{ij} \succeq 0 \quad \forall (i,j) \in \mathcal{A}^s. \end{aligned} \tag{6}$$

This corresponds to demanding not only that $p(x)-\gamma$ is a sum of squares, but also that each of its summands is the square of a polynomial depending only on x_i and x_j, for some $(i,j)\in\mathcal{A}^s$. The dual of (6) is

$$\begin{aligned} v_{\text{spmom}} := \min_{y} \; & \sum_{(i,j)\in\mathcal{A}} \sum_{s\in\beta^4_{ij}} p^{ij}_s y_s \\ \text{s.t.} \;\; & M_{\beta^2_{ij}}(y) \succeq 0 \quad \forall (i,j)\in\mathcal{A}^s \\ & y_1 = 1, \end{aligned} \tag{7}$$

where

$$(\|x_i-x_j\|^2-d^2_{ij})^2 =: \sum_{s\in\beta^4_{ij}} p^{ij}_s s(x) \quad \forall (i,j)\in\mathcal{A}.$$

Note that $M_{\beta^2_{ij}}(y)\succeq 0$ for $(i,j)\in\mathcal{A}^s$ implies $M_{\beta^2_{ij}}(y)\succeq 0$ for $(i,j)\in\mathcal{A}^a$, since each sensor is connected to at least one other sensor. The solution set of (7) is denoted by $\mathcal{S}_{\text{spmom}}$. The sensor x_i is recovered from a solution y of (7) by setting $x_i = (y_{x^1_i}, y_{x^2_i})^T$. The set of all sensor positions obtained this way is denoted by $\mathcal{P}_{\text{spmom}} \subseteq \mathbb{R}^{2\times m}$. It was shown in [8, Theorem 3.4] that $v_{\text{spsos}} = v_{\text{spmom}}$ and it is easy to see that, in the noiseless case, $\mathcal{P}\subseteq\mathcal{P}_{\text{mom}}\subseteq\mathcal{P}_{\text{spmom}}$ and that $\mathcal{S}_{\text{mom}}\subseteq\mathcal{S}_{\text{spmom}}$. A general study of these sparse SOS relaxations can be found in [9].

A different SOS relaxation is proposed in [5]. There, the original problem is formulated as

$$v_{\text{opt}} := \min_{x_1,\ldots,x_m} p(x) := \sum_{(i,j)\in\mathcal{A}} \left| \|x_i-x_j\|^2 - d^2_{ij} \right|, \tag{8}$$

and a degree one SOS relaxation is used. More specifically, let

$$\|x_i-x_j\|^2-d^2_{ij} =: \sum_{s\in\beta^2} p^{ij}_s s(x) \quad \forall (i,j)\in\mathcal{A}.$$

The relaxation is given by

$$\begin{aligned} v^1_{\text{mom}} := \min_{y} \; & \sum_{(i,j)\in\mathcal{A}} \left| \sum_{s\in\beta^2} p^{ij}_s y_s \right| \\ \text{s.t.} \;\; & M_{\beta^1}(y) \succeq 0 \\ & y_1 = 1, \end{aligned} \tag{9}$$

where $M_{\beta^1}(y)$ is the moment matrix generated by moment vector $y=(y_s)_{s\in\beta^2}$.

4 Relationship between SOS and SDP relaxations

In the SDP approach of Biswas and Ye [1,2], instead of (2), the sensor network localization problem is formulated as in (8). Letting $X := \begin{pmatrix} x_1 & \cdots & x_m \end{pmatrix}$ and I_2 denote the 2×2 identity matrix, Biswas and Ye considered the following SDP relaxation of (8):

$$\begin{aligned} \min_Z \; & \sum_{(i,j)\in\mathcal{A}} \left|\ell_{ij}(Z) - d_{ij}^2\right| \\ \text{s.t.} \;\; & Z = \begin{pmatrix} U & X^T \\ X & I_2 \end{pmatrix} \succeq 0, \end{aligned} \tag{10}$$

where $U = \begin{pmatrix} u_{ij} \end{pmatrix}_{1\le i,j\le m}$ and

$$\ell_{ij}(Z) := \begin{cases} u_{ii} - 2u_{ij} + u_{jj} & \text{if } i < j \le m; \\ u_{ii} - 2x_i^T x_j + \|x_j\|^2 & \text{if } i \le m < j. \end{cases}$$

The solution set of (10) is denoted by $\mathcal{S}_{\text{sdp}}$, while the set of the corresponding recovered sensor positions, given by X, is denoted by $\mathcal{P}_{\text{sdp}}$. Our first result shows that the SDP relaxation (10) is equivalent to the degree one SOS relaxation (9). The proof presented is a simplification of the original argument and is due to Paul Tseng.

Theorem 1 **(a)** *Let Z be a feasible solution of* (10)*, then there is a vector y indexed by β^2 that is feasible for* (8) *and has the same objective value.*
(b) *If y is a feasible solution of* (8)*, then there exists Z that is feasible for* (10) *and has the same objective value.*

Proof (a) Let $Z = \begin{pmatrix} U & X^T \\ X & I_2 \end{pmatrix}$, where $X = \begin{pmatrix} x_1 & \cdots & x_m \end{pmatrix}$. Define y by setting $y_1 = 1$, $y_{x_i^k} = x_i^k$, $y_{x_i^1 x_j^k} = x_i^1 x_j^k$ and $y_{x_i^2 x_j^2} = u_{ij} - x_i^1 x_j^1$ for all $i, j = 1, ..., m$ and $k = 1, 2$. Let v_k be the vector $(x_1^k \ldots x_m^k)$, for $k = 1, 2$, then we have

$$M_{\beta^1}(y) = \begin{pmatrix} 1 \\ v_1^T \\ v_2^T \end{pmatrix} \begin{pmatrix} 1 & v_1 & v_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & U - X^T X \end{pmatrix}.$$

The first matrix is positive semidefinite of rank 1 while the second matrix is positive semidefinite since, by (10) and a basic property of Schur complement, $U - X^T X \succeq 0$. Thus y is a feasible solution of (8) and it is easy to check that y gives the same objective value as Z.

(b) Consider the submatrices U_1 and U_2 of $M_{\beta^1}(y)$ indexed by $\{1, x_1^1, ..., x_m^1\}$ and $\{1, x_1^2, ..., x_m^2\}$ respectively. Let $w_k = \begin{pmatrix} y_{x_1^k} & \cdots & y_{x_m^k} \end{pmatrix}$, $k = 1, 2$. By the same property of Schur complement as above, we have $U_k \succeq w_k^T w_k$ for $k = 1, 2$. Let $U = U_1 + U_2$ and $X = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, then

$$U = U_1 + U_2 \succeq w_1^T w_1 + w_2^T w_2 = X^T X,$$

hence $Z = \begin{pmatrix} U & X^T \\ X & I_2 \end{pmatrix}$ is a feasible solution of (10). Again, it is easy to check that Z gives the same objective value as y. ∎

The SDP relaxation (10) can be further relaxed to the ESDP relaxation by requiring only the principal submatrices of Z associated with $\mathcal{A}^s$ to be positive semidefinite, as proposed in [14]. Specifically, the ESDP relaxation is

$$\begin{array}{ll} \min\limits_{Z} & \sum\limits_{(i,j)\in\mathcal{A}} \left|\ell_{ij}(Z) - d_{ij}^2\right| \\ \text{s.t.} & \begin{pmatrix} u_{ii} & u_{ij} & x_i^T \\ u_{ij} & u_{jj} & x_j^T \\ x_i & x_j & I_2 \end{pmatrix} \succeq 0 \ \forall (i,j) \in \mathcal{A}^s, \end{array} \tag{11}$$

where Z stands for the matrix $\begin{pmatrix} U & X^T \\ X & I_2 \end{pmatrix}$, and whose solution set we will denote by $\mathcal{S}_{\text{esdp}}$. As usual we will denote by $\mathcal{P}_{\text{esdp}}$ the set of corresponding recovered sensor positions X. Then we have the following result.

Theorem 2 *In the noiseless case, $\mathcal{P}_{\text{spmom}} \subseteq \mathcal{P}_{\text{esdp}}$.*

Proof Take any $X = \left(x_1 \cdots x_m\right) \in \mathcal{P}_{\text{spmom}}$ and the corresponding $y \in \mathcal{S}_{\text{spmom}}$. Then for each $(i,j) \in \mathcal{A}^s$, it holds that $M_{\beta_{ij}^2}(y) \succeq 0$. Hence, both $M_{\{1,x_i^2,x_j^2\}}(y)$ and $M_{\{1,x_i^1,x_j^1\}}(y)$, being principal submatrices of $M_{\beta_{ij}^2}(y)$, are positive semidefinite. For $(i,j) \in \mathcal{A}^s$, define $u_{kl} := y_{x_k^1 x_l^1} + y_{x_k^2 x_l^2}$ for $k,l \in \{i,j\}$. We claim that

$$\begin{pmatrix} u_{ii} & u_{ij} & x_i^T \\ u_{ij} & u_{jj} & x_j^T \\ x_i & x_j & I_2 \end{pmatrix} \succeq 0.$$

To see this, it suffices to show that the Schur complement of I_2,

$$\begin{pmatrix} u_{ii} & u_{ij} \\ u_{ij} & u_{jj} \end{pmatrix} - \begin{pmatrix} \|x_i\|^2 & x_i^T x_j \\ x_i^T x_j & \|x_j\|^2 \end{pmatrix},$$

is positive semidefinite. But this matrix is the sum of the Schur complement of 1 in $M_{\{1,x_i^1,x_j^1\}}(y)$, which is

$$\begin{pmatrix} y_{(x_i^1)^2} - (x_i^1)^2 & y_{x_i^1 x_j^1} - x_i^1 x_j^1 \\ y_{(x_j^1)^2} - x_i^1 x_j^1 & y_{x_i^1 x_j^1} - (x_j^1)^2 \end{pmatrix},$$

and the Schur complement of 1 in $M_{\{1,x_i^2,x_j^2\}}(y)$, which is

$$\begin{pmatrix} y_{(x_i^2)^2} - (x_i^2)^2 & y_{x_i^2 x_j^2} - x_i^2 x_j^2 \\ y_{(x_j^2)^2} - x_i^2 x_j^2 & y_{x_i^2 x_j^2} - (x_j^2)^2 \end{pmatrix}.$$

Both matrices are positive semidefinite since $M_{\{1,x_i^2,x_j^2\}}(y)$ and $M_{\{1,x_i^1,x_j^1\}}(y)$ are. Hence, the claim follows.

Define $Z := \begin{pmatrix} U & X^T \\ X & I_2 \end{pmatrix}$. Then Z is feasible for (11). We shall show that $Z \in \mathcal{S}_{\text{esdp}}$. To this end, let q^{ij} be a column vector such that

$$\|x_i - x_j\|^2 - d_{ij}^2 =: \sum_{s \in \beta_{ij}^2} q_s^{ij} s(x) \quad \forall (i,j) \in \mathcal{A}.$$

Since $M_{\beta_{ij}^2}(y) \succeq 0$ for all $(i,j) \in \mathcal{A}$, it follows that

$$\begin{pmatrix} 1 & \sum_{s \in \beta_{ij}^2} q_s^{ij} y_s \\ \sum_{s \in \beta_{ij}^2} q_s^{ij} y_s & \sum_{s \in \beta_{ij}^4} p_s^{ij} y_s \end{pmatrix} = \begin{pmatrix} e_1^T \\ q^T \end{pmatrix} M_{\beta_{ij}^2}(y) \begin{pmatrix} e_1 & q \end{pmatrix} \succeq 0$$

for all $(i,j) \in \mathcal{A}$, where e_1 is the vector that is one in the first entry and zero otherwise. This last relation implies

$$\sum_{s \in \beta_{ij}^4} p_s^{ij} y_s \geq \left(\sum_{s \in \beta_{ij}^2} q_s^{ij} y_s \right)^2 = \left(\ell_{ij}(Z) - d_{ij}^2 \right)^2 \quad \forall (i,j) \in \mathcal{A}.$$

Since $y \in \mathcal{S}_{\text{spmom}}$, the noiseless assumption implies that $\sum_{(i,j)\in\mathcal{A}} \left(\ell_{ij}(Z) - d_{ij}^2 \right)^2 = 0$, and hence Z solves (11). This proves that $\mathcal{P}_{\text{spmom}} \subseteq \mathcal{P}_{\text{esdp}}$. ■

From the above proof, we obtain the following corollary.

Corollary 1 *Consider the noiseless case. Let* $y \in \mathcal{S}_{\text{spmom}}$. *For* $(i,j) \in \mathcal{A}^s$, *define* $x_k^s := y_{x_k^s}$, $u_{kl} := y_{x_k^1 x_l^1} + y_{x_k^2 x_l^2}$, *for* $s = 1, 2$ *and* $k, l \in \{i, j\}$. *Then* $Z := \begin{pmatrix} U & X^T \\ X & I_2 \end{pmatrix} \in \mathcal{S}_{\text{esdp}}$.

Remark 1 Similarly one can show that, in the noiseless case, $\mathcal{P}_{\text{mom}} \subseteq \mathcal{P}_{\text{sdp}}$.

5 Testing accuracy of individual sensors

As in [10], [13] and [14], one is interested in identifying sensors whose recovered locations remain the same for all solutions since, in the noiseless case, these sensors will turn out to be in their true position. Hence, we are interested in the following set

$$\mathcal{I}_{\text{spmom}} := \left\{ i \in \{1, ..., m\} \mid (y_{x_i^1}, y_{x_i^2}) \text{ is invariant over } \mathcal{S}_{\text{spmom}}. \right\}$$

In order to identify elements in $\mathcal{I}_{\text{spmom}}$, we consider a version of individual trace for SOS relaxations.

Definition 1 For any $y \in \mathcal{S}_{\text{spmom}}$, the i-th individual trace of y is defined as

$$\text{Tr}_i(y) := y_{(x_i^1)^2} + y_{(x_i^2)^2} - (y_{x_i^1})^2 - (y_{x_i^2})^2.$$

Note that the trace is always nonnegative since $y_{(x_i^k)^2} - (y_{x_i^k})^2$ is the determinant of a principal submatrix of $M_{\beta_{ij}^2}(y)$, for $k = 1, 2$. We have the following simple result, generalizing the zero trace test to the setting of SOS relaxations. The proof parallels that of [13, Proposition 4.1].

Theorem 3 *If* $\mathrm{Tr}_i(y) = 0$ *for some* y *in the relative interior of* $\mathcal{S}_{\text{spmom}}$*, then* $i \in \mathcal{I}_{\text{spmom}}$*.*

Proof We shall show that $y_{x_i^1}$ is invariant over $\mathcal{S}_{\text{spmom}}$. The proof for $y_{x_i^2}$ is similar. Note that $\mathrm{Tr}_i(y) = 0$ implies $y_{(x_i^1)^2} = (y_{x_i^1})^2$. Take any $w \in \mathcal{S}_{\text{spmom}}$. Since y is in the relative interior of $\mathcal{S}_{\text{spmom}}$, there exists $\epsilon > 0$ so that both

$$\eta := y + \epsilon(w - y) \text{ and } \zeta := y - \epsilon(w - y)$$

belong to $\mathcal{S}_{\text{spmom}}$. Thus, $y = \frac{\eta+\zeta}{2}$ and hence

$$\begin{aligned} 0 = y_{(x_i^1)^2} - (y_{x_i^1})^2 &= \frac{1}{2}[\eta_{(x_i^1)^2} - (\eta_{x_i^1})^2] + \frac{1}{2}[\zeta_{(x_i^1)^2} - (\zeta_{x_i^1})^2] + \frac{1}{4}(\eta_{x_i^1} - \zeta_{x_i^1})^2 \\ &\geq \frac{1}{4}(\eta_{x_i^1} - \zeta_{x_i^1})^2 = \epsilon^2(w_{x_i^1} - y_{x_i^1})^2. \end{aligned}$$

This shows that $w_{x_i^1} = y_{x_i^1}$ and the proof is complete. ■

It is not known whether the converse of Theorem 3 is true. Nonetheless, we are able to establish a partial converse to the theorem in the noiseless case. This follows from the fact that if $y \in \mathcal{S}_{\text{spmom}}$ and $Z \in \mathcal{S}_{\text{esdp}}$ is obtained from y according to Corollary 1 then the trace $\mathrm{Tr}_i(y)$ equals the ESDP trace $\mathrm{tr}_i(Z)$ as defined in [14]. Then the proofs of [10, Lemmas 2,3] follow through and we get the following result.

Lemma 1 *In the noiseless case, let* $i \leq m$ *and* $y \in \mathcal{S}_{\text{spmom}}$ *be such that the corresponding recovered sensor positions verify* $\|x_i - x_j\| = d_{ij}$*. Then if* $j > m$*, we have* $\mathrm{Tr}_i(y) = 0$*, and if* $j \leq m$*, we have* $\mathrm{Tr}_i(y) = \mathrm{Tr}_j(y)$*.*

Now, the next theorem follows from Lemma 1 by a simple induction argument.

Theorem 4 *In the noiseless case, let* $i \in \mathcal{I}_{\text{spmom}}$ *be such that there exists a path with nodes in* $\mathcal{I}_{\text{spmom}}$ *connecting* x_i *to an anchor. Then* $\mathrm{Tr}_i(y) = 0$ *for all* $y \in \mathcal{S}_{\text{spmom}}$*.*

6 Numerical examples

Theorem 2 says that in the noiseless case, the sparse-SOS relaxation is at least as strong as the ESDP relaxation. We illustrate this fact in Example 1, which is taken from [14, Example 1]. In [14], the same example was used to illustrate that the SDP relaxation is stronger than the ESDP relaxation. All computations presented were done with SeDuMi1.05 [12] interfaced in Matlab7.7.

Example 1 Let $n = 6$ and $m = 3$. The anchors are $x_4 = (-0.4, 0)^T$, $x_5 = (0.4, 0)^T$ and $x_6 = (0, 0.4)^T$, and the true positions of the sensors are $x_1 = (-0.05, 0.3)^T$, $x_2 = (-0.08, 0.2)^T$ and $x_3 = (0.2, 0.3)^T$. We have $\mathcal{A} = \{(1,2), (1,3), (1,4), (1,6), (2,3), (2,4), (2,6), (3,5), (3,6)\}$.

First we solve the ESDP relaxation (11); the result is inaccurate, as is shown in Figure 1, with RMSD being 6e-2; where RMSD stands for Root Mean Square Deviance, defined by

$$\text{RMSD} = \left(\frac{1}{m} \sum_{i=1}^{m} \|x_i - x_i^{\text{true}}\|^2 \right)^{\frac{1}{2}}. \tag{12}$$

However, the sparse-SOS relaxation seems to provide an accurate solution, as is shown in the figure, with RMSD 9e-5. This is also suggested by the small individual traces of the solution obtained by solving the sparse-SOS relaxation: 4e-7, 2e-6 and 1e-6. By Theorem 3, the sensors are likely accurately positioned, since SeDuMi likely returns a relative interior solution. On the other hand, the individual traces of the solution obtained by solving the ESDP relaxation turn out to be much larger: 1e-3, 7e-3 and 4e-3, so the solution is less likely to be accurate.

How does the SOS relaxation compare with the SDP relaxation? The next example shows a network that is not localizable by solving the SDP relaxation, yet is likely localizable by solving the SOS relaxation. This implies that the underlying graph has a unique realization in $\mathbb{R}^2$, but does not have a unique realization if we relax the dimension restriction. It is surprising that the SOS relaxation is strong enough to restrict the dimensionality of the realization.

Example 2 Let $n = 5$ and $m = 2$. The anchors are $x_3 = (0, 0)^T$, $x_4 = (0.5, 1)^T$ and $x_5 = (1, 0)^T$, and the true positions of the sensors are $x_1 = (0.4, 0.7)^T$ and $x_2 = (0.6, 0.7)^T$. We have $\mathcal{A} = \{(1,2), (1,4), (1,5), (2,3), (2,4)\}$.

First we solve the SDP relaxation (10); the result is inaccurate, as is shown in Figure 2, with RMSD being 1e-1. It can also be shown manually that the solution to this SDP relaxation is not unique. However, the SOS relaxation seems to provide an accurate solution, as is shown in the figure, with RMSD 2e-4. This is also suggested by the individual traces of the solution obtained by solving the SOS relaxation: both being 3e-6.

The only existing computational results on solving (7) for large scale problems are reported in [8]. The codes are written in Matlab, calling SeDuMi to solve the corresponding SDP. The solution time is large comparing with other existing methods for sensor network localization [3,5,6,10,14] that solve other convex relaxations. In view of Theorems 2 and 3, it is worth investigating efficient algorithms to solve for a relative interior solution of (7). Since (7) has a partial separable structure, one direction is to look for a distributed algorithm, like the LPCGD algorithm in [10], to solve (7). A distributed algorithm is important for applications like real time tracking. Since each edge is related to a 15 by 15 matrix in the sparse-SOS relaxation, it should take more time

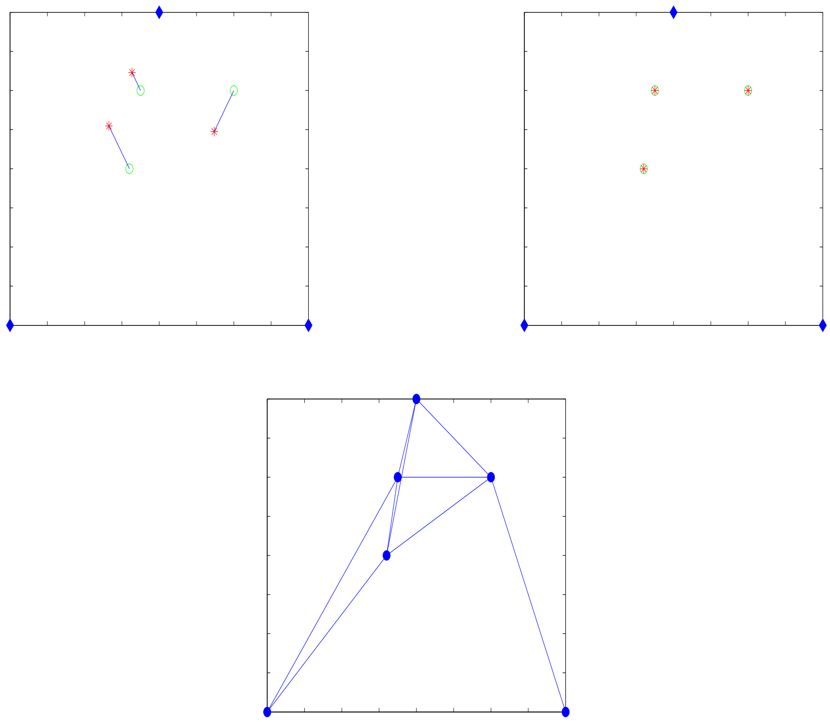

Fig. 1 The top left figure shows the anchor ("♦") and the solution found by solving ESDP relaxation (11). Each sensor position ("∗") found is joined to its true position ("○") by a line. The top right figure shows the same information for the solution found by solving sparse-SOS relaxation (6). The bottom figure shows the location of the points ("•") and the edges.

to solve (7) than to solve (11). However, sparse-SOS relaxation is stronger than the ESDP relaxation by Theorem 2: this is a tradeoff between solution accuracy and solution time.

A possible approach to save solution time and yet get higher accuracy would be to use this stronger convex relaxation to refine the solution obtained from solving the ESDP relaxation. Taking advantage of the existing trace test for ESDP, we take an ESDP solution, fix those sensors with small trace as new anchors, and run the sparse-SOS relaxation in the remaining reduced network. The advantage of this approach is that we would still have an accuracy certificate for the refined solution (the trace test), which is not common for existing refinement heuristics. Moreover, since the sparse-SOS relaxation is solved on the reduced network, the time taken to solve the problem should be smaller compared to solving the sparse-SOS relaxation on the whole network.

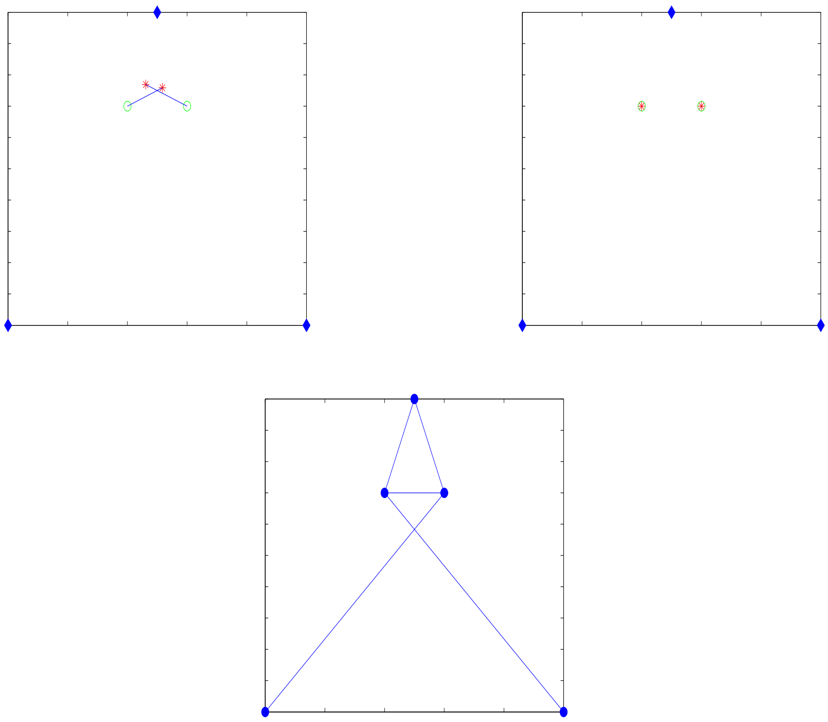

Fig. 2 The top left figure shows the anchor ("♦") and the solution found by solving SDP relaxation (10). Each sensor position ("∗") found is joined to its true position ("○") by a line. The top right figure shows the same information for the solution found by solving SOS relaxation (4). The bottom figure shows the location of the points ("•") and the edges.

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