

STABLE CATEGORIES AND RECONSTRUCTION

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1. INTRODUCTION

This work is an attempt towards a Morita theory for stable equivalences between self-injective algebras. More precisely, given two self-injective algebras A and B and an equivalence between their stable categories, consider the set \mathcal{S} of images of simple B -modules inside the stable category of A . That set satisfies some obvious properties of Hom-spaces and it generates the stable category of A . Keep now only \mathcal{S} and A . Can B be reconstructed? We show how to reconstruct the graded algebra associated to the radical filtration of (an algebra Morita equivalent to) B . It would be interesting to develop further an obstruction theory for the existence of an algebra B with that given filtration, starting only with \mathcal{S} (this might be studied in terms of localization of A_∞ -algebras). Note that a result of Linckelmann [Li] shows that, if we consider only stable equivalence of Morita type, then B is characterized by \mathcal{S} — but this result does not provide a reconstruction of B from \mathcal{S} .

We also study a similar problem in the more general setting of a triangulated category \mathcal{T} . Given a finite set \mathcal{S} of objects satisfying Hom-properties analogous to those satisfied by the set of simple modules in the derived category of a ring and assuming that the set generates \mathcal{T} , we construct a t -structure on \mathcal{T} . In the case $\mathcal{T} = D^b(A)$ and A is a symmetric algebra, the first author has shown [Ri] that there is a symmetric algebra B with an equivalence $D^b(B) \xrightarrow{\sim} D^b(A)$ sending the set of simple B -modules to \mathcal{S} . The case of a self-injective algebra leads to a slightly more general situation: there is a finite dimensional differential graded algebra B with $H^i(B) = 0$ for $i > 0$ and for $i \ll 0$ with the same property as above.

2. NOTATIONS

Let \mathcal{C} be an additive category. Given S a set of objects of \mathcal{C} , we denote by $\text{add } S$ the full subcategory of \mathcal{C} of objects isomorphic to finite direct sums of objects of S .

Let k be a field and A a finite dimensional k -algebra. We say that A is split if the endomorphism ring of every simple A -module is k . We denote by $A\text{-mod}$ the category of finitely generated left A -modules and by $D^b(A)$ its derived category. For A self-injective, we denote by $A\text{-stab}$ the stable category, the quotient of $A\text{-mod}$ by projective modules. Given M an A -module, we denote by ΩM the kernel of a projective cover of M and by $\Omega^{-1}M$ the cokernel of an injective hull of M .

3. SIMPLE GENERATORS FOR TRIANGULATED CATEGORIES

3.1. Category of filtered objects. Let \mathcal{T} be a triangulated category and \mathcal{S} a full subcategory of \mathcal{T} .

We define a category \mathcal{F} as follows.

- Its objects are diagrams

$$M = (\cdots \rightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \xrightarrow{\varepsilon_0} N_0)$$

where M_i is an object of \mathcal{T} , $M_i = 0$ for $i \gg 0$, such that

- (i) $M_1 \xrightarrow{f_1} M_0 \xrightarrow{\varepsilon_0} N_0$ is the beginning of a distinguished triangle
- (ii) for all $i \geq 1$, the cone N_{i-1} of f_i is in $\text{add } \mathcal{S}$
- (iii) the canonical map $\text{Hom}(N_0, S) \rightarrow \text{Hom}(M_0, S)$ is surjective for all $S \in \mathcal{S}$
- (iv) the canonical map $\text{Hom}(N_i, S) \rightarrow \text{Hom}(M_i, S)$ is bijective for all $S \in \mathcal{S}$ and $i \geq 1$.

Note that $\varepsilon_i : M_i \rightarrow N_i = \text{cone}(f_{i+1})$ is well defined up to unique isomorphism for $i \geq 1$ thanks to property (iv). For $i \geq 0$, we define a new object $M_{\geq i}$ of \mathcal{F} as $\cdots \rightarrow M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{\varepsilon_i} N_i$.

- Given another diagram M' , we define $\text{Hom}_{\mathcal{F}}(M, M')_0$ as the subspace of $\text{Hom}(N_0, N'_0)$ consisting of those maps g such that there is $h : M_0 \rightarrow M'_0$ with $\varepsilon'_0 h = g \varepsilon_0$.

We put $\text{Hom}_{\mathcal{F}}(M, M')_i = \text{Hom}_{\mathcal{F}}(M, M'_{\geq i})_0$ and $\text{Hom}_{\mathcal{F}}(M, M') = \bigoplus_{i \geq 0} \text{Hom}_{\mathcal{F}}(M, M')_i$.

- Let now $g_0 \in \text{Hom}(N_0, N'_0)$. By (iv), there are maps h_0, h_1, \dots and g_1, g_2, \dots making the following diagrams commutative

$$\begin{array}{ccccccc} M_{i+1} & \xrightarrow{f_{i+1}} & M_i & \xrightarrow{\varepsilon_i} & N_i & \longrightarrow & M_{i+1}[1] \\ h_{i+1} \downarrow & & h_i \downarrow & & g_i \downarrow & & h_{i+1}[1] \downarrow \\ M'_{i+1} & \xrightarrow{f'_{i+1}} & M'_i & \xrightarrow{\varepsilon'_i} & N'_i & \longrightarrow & M'_{i+1}[1] \end{array}$$

Lemma 3.1. *The maps $g_i : N_i \rightarrow N'_i$ (for $i \geq 1$) depend only on g_0 .*

Proof. Let us show that g_1 depends only on g_0 . The general case is similar, by induction.

Let $p : M_0 \rightarrow M'_1$. We have to show that the only map $q : N_1 \rightarrow N'_1$ such that $\varepsilon'_1 p f_1 = q \varepsilon_1$ is $q = 0$.

By (iii), there is $r : N_0 \rightarrow N'_1$ such that $\varepsilon'_1 p = r \varepsilon_0$. So, $\varepsilon'_1 p f_1 = r \varepsilon_0 f_1 = 0$. By (iv), we deduce from $q \varepsilon_1 = 0$ that $q = 0$.

$$\begin{array}{ccc} M_1 & \xrightarrow{\varepsilon_1} & N_1 \\ & \searrow f_1 & \vdots \\ & & M_0 \xrightarrow{\varepsilon_0} N_0 \\ & \swarrow p & \searrow r \\ M'_1 & \xrightarrow{\varepsilon'_1} & N'_1 \end{array}$$

□

Let $g_0 \in \text{Hom}_{\mathcal{F}}(M, M')_i$ and $g'_0 \in \text{Hom}_{\mathcal{F}}(M', M'')_j$. We define the product $g'_0 g_0$ as the composition $N_0 \xrightarrow{g_0} N'_i \xrightarrow{g'_i} N''_{i+j}$.

Lemma 3.2. *Assume $\text{Hom}(S, T[n]) = 0$ for all $S, T \in \mathcal{S}$ and $n < 0$. Let M be an object of \mathcal{F} . Then, the canonical map $\text{Hom}(N_0, S) \rightarrow \text{Hom}(M_0, S)$ is an isomorphism.*

Proof. By induction on $-i$, we see that $\text{Hom}(M_i, S[n]) = 0$ for $n < 0$ and $S \in \mathcal{S}$. It follows that $\text{Hom}(M_1[1], S) = 0$, hence the canonical map $\text{Hom}(N_0, S) \rightarrow \text{Hom}(M_0, S)$ is injective, as well as being surjective by assumption. \square

3.2. t -structures. Let k be a field and assume \mathcal{T} is a k -linear triangulated category.

We assume from now on the following

- Hypothesis 1.** (1) $\text{Hom}(S, T) = k^{\delta_{S,T}}$ for $S, T \in \mathcal{S}$
 (2) \mathcal{S} generates \mathcal{T} as a triangulated category
 (3) $\text{Hom}(S, T[n]) = 0$ for $S, T \in \mathcal{S}$ and $n < 0$.

3.2.1.

Lemma 3.3. *Given $N \in \mathcal{T}$, there is a sequence $0 = M_r \xrightarrow{f_r} \cdots \rightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$ and $d : \mathbf{Z}_{>0} \rightarrow \mathbf{Z}$ non increasing such that $\text{cone}(f_i)[d(i)] \in \mathcal{S}$.*

For such a sequence, the maps $M_{r-1} \rightarrow N$ and $N \rightarrow \text{cone}(f_1)$ are non zero.

Proof. Since \mathcal{T} is generated by \mathcal{S} , there is a sequence $0 = M_r \rightarrow \cdots \rightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$ and $d : \mathbf{Z}_{>0} \rightarrow \mathbf{Z}$ such that $\text{cone}(f_i)[d(i)] \in \mathcal{S}$.

We put $N_i = \text{cone}(f_i) = S_i[-d(i)]$ with $S_i \in \mathcal{S}$. Take i such that $d(i) > d(i-1)$. Let T be the cone of $f_{i-1}f_i : M_i \rightarrow M_{i-2}$. The octahedral axiom gives a distinguished triangle $S_i[-d(i)] \rightarrow T \rightarrow S_{i-1}[-d(i-1)] \rightsquigarrow$.

Assume the morphism $S_{i-1}[-d(i-1)] \rightarrow S_i[-d(i)+1]$ is non zero. Then it is an isomorphism and $d(i) = d(i-1) + 1$. It follows that $T = 0$ and $f_{i-1}f_i$ is an isomorphism. Consequently,

$$0 = M_r \rightarrow \cdots \rightarrow M_{i+1} \xrightarrow{f_{i-1}f_i f_{i+1}} M_{i-2} \rightarrow \cdots \rightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$$

is a new sequence with successive cones being shifts of objects of \mathcal{S} .

By induction, we can assume that the morphism $S_{i-1}[-d(i-1)] \rightarrow S_i[-d(i)+1]$ is zero. Then, $T \simeq N_i \oplus N_{i-1}$. There is an object M'_{i-1} and distinguished triangles $M_i \rightarrow M'_{i-1} \rightarrow N_{i-1} \rightsquigarrow$ and $M'_{i-1} \rightarrow M_{i-2} \rightarrow N_i \rightsquigarrow$. Put $M'_j = M_j$ for $j \neq i-1$. So,

$$0 = M'_r \rightarrow \cdots \rightarrow M'_2 \rightarrow M'_1 \rightarrow M'_0 = N$$

is a new sequence with the same cones as in the original sequence except the i and $i-1$ ones which have been swapped. By induction, we can reorder the cones in the sequence so that d is non increasing.

Assume the map $M_{r-1} \rightarrow N$ is zero. Let T be its cone. Then $T \simeq N \oplus M_{r-1}[1]$. Note that T is filtered by the $S_i[-d(i)]$ with $-d(i) < -d(r) + 1$, hence $\text{Hom}(M_{r-1}[1], T) = 0$. So we have a contradiction. The case of the map $N \rightarrow N_1$ is similar. \square

Let $\mathcal{T}^{\leq 0}$ (resp. $\mathcal{T}^{> 0}$) be the full subcategory of objects N in \mathcal{T} such that there is a sequence $0 = M_r \rightarrow \cdots \rightarrow M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 = N$ with $\text{cone}(f_i)$ a direct sum of objects $S[r]$ with $S \in \mathcal{S}$ and $r \geq 0$ (resp. $r < 0$).

Proposition 3.4. *$(\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0})$ is a bounded t -structure on \mathcal{T} .*

Proof. By induction, we see there is no non-zero map from an object of $\mathcal{T}^{\leq 0}$ to an object of $\mathcal{T}^{> 0}$. Furthermore, we have $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{> 0} \subseteq \mathcal{T}^{> 0}[1]$.

Let $N \in \mathcal{T}$. Pick a sequence as in Lemma 3.3. Take s such that $d(s) > 0$ and $d(s+1) \leq 0$. Let L be the cone of $f_1 \cdots f_s : M_s \rightarrow N$. We have a distinguished triangle

$$M_s \rightarrow N \rightarrow L \rightsquigarrow$$

with $M_s \in \mathcal{T}^{\leq 0}$ and $L \in \mathcal{T}^{> 0}$. □

We have a characterization of $\mathcal{T}^{\geq 0}$ and $\mathcal{T}^{\leq 0}$:

Proposition 3.5. *Let $N \in \mathcal{T}$. Then, $N \in \mathcal{T}^{\leq 0}$ if and only if $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$.*

Similarly, $N \in \mathcal{T}^{\geq 0}$ if and only if $\text{Hom}(S[i], N) = 0$ for $S \in \mathcal{S}$ and $i > 0$.

Proof. We have $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$, if $N \in \mathcal{S}[r]$ with $r \geq 0$. By induction, it follows that if $N \in \mathcal{T}^{\leq 0}$, then $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$.

Assume now $\text{Hom}(N, S[i]) = 0$ for $S \in \mathcal{S}$ and $i < 0$. Pick a filtration of N as in Lemma 3.3. Then, $d(1) \leq 0$, hence $d(i) \leq 0$ for all i and $N \in \mathcal{T}^{\leq 0}$.

The other case is similar. □

Note that the heart \mathcal{A} of the t -structure is artinian and noetherian. Its set of simple objects is \mathcal{S} .

Remark 3.6. Assume \mathcal{T} can be generated by a finite set of objects. Then, there is a finite subcategory \mathcal{S}' of \mathcal{S} generating \mathcal{T} . It follows immediately from condition (i) that $\mathcal{S} = \mathcal{S}'$. So, \mathcal{S} has only finitely many objects.

3.2.2. In §3.2.2, we assume $\mathcal{T} = D^b(A)$ where A is a finite dimensional k -algebra. By Remark 3.6, \mathcal{S} is finite (note that \mathcal{T} is generated by the simple A -modules, up to isomorphism).

Proposition 3.7. *Let $S \in \mathcal{S}$. There is a bounded complex of finitely generated injective A -modules $I_S(S) \in \mathcal{T}^{\geq 0}$ such that, given $T \in \mathcal{S}$ and $i \in \mathbf{Z}$, we have*

$$\text{Hom}_{D^b(A)}(T, I_S(S)[i]) = \begin{cases} k & \text{for } i = 0 \text{ and } S = T \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, there is a bounded complex of finitely generated projective A -modules $P_S(S) \in \mathcal{T}^{\leq 0}$ such that, given $T \in \mathcal{S}$ and $i \in \mathbf{Z}$, we have

$$\text{Hom}_{D^b(A)}(P_S(S)[i], T) = \begin{cases} k & \text{for } i = 0 \text{ and } S = T \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The construction of a complex $I_S(S)$ of A -modules with the Hom property is [Ri, §5] (note that the proof of [Ri, Lemma 5.4] is valid for non-symmetric algebras). It is in $\mathcal{T}^{\geq 0}$ by Proposition 3.5. Since $\bigoplus_{i \in \mathbf{Z}} \dim \text{Hom}_{D^b(A)}(V, I_S(S)[i]) = 0$ for all simple A -modules V , we deduce that $I_S(S)$ is isomorphic to a bounded complex of finitely generated injective A -modules.

The second case follows from the first one by passing to A^{opp} and taking the k -duals of elements of \mathcal{S} . □

We denote by $\tau^{> 0}$, etc... the truncation functors and ${}^t H^0$ the H^0 -functor associated to the t -structure constructed in §3.2.1.

Lemma 3.8. *The object ${}^t H^0(I_S(S))$ of \mathcal{A} is an injective hull of S and ${}^t H^0(P_S(S))$ is a projective cover of S .*

Proof. We have a distinguished triangle

$${}^tH^0(I_{\mathcal{S}}(S)) \rightarrow I_{\mathcal{S}}(S) \rightarrow \tau^{>0}I_{\mathcal{S}}(S) \rightsquigarrow .$$

Let $N \in \mathcal{A}$. We have $\text{Hom}(N, \tau^{>0}I_{\mathcal{S}}(S)) = 0$ and $\text{Hom}(N, I_{\mathcal{S}}(S)[1]) = 0$, so we deduce that $\text{Hom}(N, {}^tH^0(I_{\mathcal{S}}(S))[1]) = 0$. It follows that $\text{Ext}_{\mathcal{A}}^1(N, {}^tH^0(I_{\mathcal{S}}(S))) = 0$, hence ${}^tH^0(I_{\mathcal{S}}(S))$ is injective. Since $\text{Hom}(T, (\tau^{>0}I_{\mathcal{S}}(S))[-1]) = 0$, we have $\text{Hom}(T, {}^tH^0(I_{\mathcal{S}}(S))) \xrightarrow{\sim} \text{Hom}(T, I_{\mathcal{S}}(S)) = k^{\delta_{ST}}$ for $T \in \mathcal{S}$. So ${}^tH^0(I_{\mathcal{S}}(S))$ is an injective hull of S . The projective case is similar. \square

Let us consider the finite dimensional differential graded algebra

$$B = \text{End}_{\mathcal{A}}^{\bullet}\left(\bigoplus_S P_{\mathcal{S}}(S)\right) = \bigoplus_i \text{Hom}_{\mathcal{A}}\left(\bigoplus_S P_{\mathcal{S}}(S), \bigoplus_S P_{\mathcal{S}}(S)[i]\right).$$

Denote by $D^b(B)$ the derived category of finite dimensional differential graded B -modules.

Theorem 3.9. *We have $H^i(B) = 0$ for $i > 0$ and for $i \ll 0$. We have $H^0(B)\text{-mod} \simeq \mathcal{A}$ and $D^b(B) \simeq D^b(A)$.*

Proof. Let $N \in \mathcal{T}$ and consider a filtration of N as in Lemma 3.3. Take $S \in \mathcal{S}$ such that $S[i]$ is isomorphic to the cone of $M_d \rightarrow M_{d-1}$. Then, $\text{Hom}(P_{\mathcal{S}}(S)[i], N) \neq 0$. It follows that the right orthogonal category of $\{P_{\mathcal{S}}(S)[i]\}_{S \in \mathcal{S}, i \in \mathbf{Z}}$ is zero. Since the $P_{\mathcal{S}}(S)$ are perfect, it follows that $\bigoplus_S P_{\mathcal{S}}(S)$ generates the category of perfect complexes of A -modules as a triangulated category closed under taking direct summands [Nee, Lemma 2.2]. The functor $\text{Hom}_{\mathcal{A}}^{\bullet}\left(\bigoplus_S P_{\mathcal{S}}(S), -\right)$ gives an equivalence $D^b(A) \xrightarrow{\sim} D^b(B)$ [Ke, Theorem 4.3].

Let $C = \bigoplus_{S \in \mathcal{S}} P_{\mathcal{S}}(S)$ and $N = {}^tH^0(C)$. We have a distinguished triangle $\tau^{<0}C \rightarrow C \rightarrow N \rightsquigarrow$. We have $\text{Hom}(\tau^{<0}C, N[i]) = 0$ for $i \leq 0$. We deduce that the canonical morphism $\text{Hom}(N, N) \rightarrow \text{Hom}(C, N)$ is an isomorphism. We have $\text{Hom}(C, (\tau^{<0}C)[i]) = 0$ for $i \geq 0$ since $\tau^{<0}C$ is filtered by objects in $\mathcal{S}[d]$, $d > 0$ (cf Proposition 3.7). It follows that the canonical morphism $\text{Hom}(C, C) \rightarrow \text{Hom}(C, N)$ is an isomorphism.

This shows that the canonical morphism $\text{End}(C) \rightarrow \text{End}({}^tH^0(C))$ is an isomorphism. By Lemma 3.8, ${}^tH^0(C)$ is a progenerator for \mathcal{A} . So $H^0(B)\text{-mod} \simeq \mathcal{A}$.

Note that $H^i(B) = 0$ for $i \ll 0$ because $\bigoplus_S P_{\mathcal{S}}(S)$ is bounded. Since $P_{\mathcal{S}}(S)$ is filtered by objects in $\mathcal{S}[d]$ with $d \geq 0$, it follows from Proposition 3.7 that $\text{Hom}(P_{\mathcal{S}}(T), P_{\mathcal{S}}(S)[i]) = 0$ for $i > 0$. So, $H^i(B) = 0$ for $i > 0$. \square

The following proposition is clear.

Proposition 3.10. *Let B be a dg-algebra with $H^i(B) = 0$ for $i > 0$ and for $i \ll 0$. Let C be the sub-dg-algebra of B given by $C^i = B^i$ for $i < 0$, $C^0 = \ker d^0$ and $C^i = 0$ for $i > 0$. Then the restriction $D(B) \rightarrow D(C)$ is an equivalence.*

Let \mathcal{S} be a complete set of representatives of isomorphism classes of simple $H^0(B)$ -modules (viewed as dg- C -modules). Then \mathcal{S} satisfies Hypothesis 1. Furthermore, $\mathcal{A} \simeq H^0(B)\text{-mod}$.

So we have a bijection between

- the sets \mathcal{S} (up to isomorphism) satisfying Hypothesis 1
- the equivalences $D^b(B) \xrightarrow{\sim} D^b(A)$ where B is a dg-algebra with $H^i(B) = 0$ for $i > 0$ and for $i \ll 0$ and where B is well-defined up to quasi-isomorphism and the equivalence is taken modulo self-equivalences of $D^b(B)$ that fix the isomorphism classes of simple $H^0(B)$ -modules.

We recover a result of Al-Nofayee [Al, Theorem 4] :

Proposition 3.11. *Assume A is self-injective with Nakayama functor ν . The following are equivalent*

- $H^i(B) = 0$ for $i \neq 0$
- $\nu(\mathcal{S}) = \mathcal{S}$ (up to isomorphism).

Proof. Note that \mathcal{S} is stable under ν if and only if $\{P_S(S)\}_{S \in \mathcal{S}}$ is stable under ν (up to isomorphism). Given $S, T \in \mathcal{S}$ and $i \in \mathbf{Z}$, we have

$$\mathrm{Hom}_{D^b(A)}(P_S(S), P_S(T)[i])^* \simeq \mathrm{Hom}_{D^b(A)}(P_S(T), \nu(P_S(S))[-i]).$$

If \mathcal{S} is stable under ν , then $\mathrm{Hom}_{D^b(A)}(P_S(T), \nu(P_S(S))[-i]) = 0$ for $i > 0$, hence $H^{<0}(B) = 0$.

Assume now $H^{<0}(B) = 0$. Then, viewed as an object of $D^b(B)$, $\nu(P_S(S))$ is concentrated in degree 0. Since it is perfect, it is isomorphic to a projective indecomposable module, hence to $P_S(S')$ for some $S' \in \mathcal{S}$. So, \mathcal{S} is stable under ν . \square

We recover now the main result of [AlRi]:

Corollary 3.12. *Let A be a self-injective algebra and B an algebra derived equivalent to A . Then B is self-injective.*

From Proposition 3.11, we recover [Ri, Theorem 5.1] :

Theorem 3.13. *If A is symmetric then $H^i(B) = 0$ for $i \neq 0$, i.e., there is an equivalence $D^b(\mathcal{A}) \xrightarrow{\sim} D^b(A)$ where \mathcal{S} is the set of images of the simple objects of \mathcal{A} .*

Remark 3.14. Theorem 3.13 does not hold in general for a self-injective algebra. Take $A = k[\varepsilon]/(\varepsilon^2) \rtimes \mu_2$, where $\mu_2 = \{\pm 1\}$ acts on $k[\varepsilon]/(\varepsilon^2)$ by multiplication on ε . Assume k does not have characteristic 2. This is a self-injective algebra which is not symmetric. The Nakayama functor swaps the two simple A -modules U and V .

Let P_U (resp. P_V) be a projective cover of U (resp. V). Take $S = U$ and $T = P_U[1]$. Then, the set $\mathcal{S} = \{S, T\}$ satisfies Hypothesis 1. We have $I_S(T) \simeq T$ and $I_S(S) \simeq 0 \rightarrow P_U \rightarrow P_V \rightarrow 0$, a complex with homology V in degree 0 and -1 .

The dg-algebra B has homology $H^0(B)$ isomorphic to the path algebra of the quiver $\bullet \longrightarrow \bullet$, $H^{-1}(B) = k$ and $H^i(B) = 0$ for $i \neq 0, -1$.

The derived category of the hereditary algebra $H^0(B)$ is not equivalent to $D^b(A)$.

3.3. Graded of an abelian category. Let \mathcal{A} be an abelian k -linear artinian and noetherian category with finitely many simple objects up to isomorphism and \mathcal{S} a complete set of representatives of isomorphism classes of simple objects. We assume \mathcal{A} is split, i.e., endomorphism rings of simple objects are isomorphic to k . Let $\mathcal{T} = D^b(\mathcal{A})$.

Let $\mathrm{gr}\mathcal{A}$ be the category with objects the objects of \mathcal{A} and where $\mathrm{Hom}_{\mathrm{gr}\mathcal{A}}(M, N)$ is the graded vector space associated to the filtration of $\mathrm{Hom}_{\mathcal{A}}(M, N)$ given by $\mathrm{Hom}_{\mathcal{A}}(M, N)^i = \{f \mid \mathrm{im} f \subseteq \mathrm{rad}^i N\}$.

Given M in \mathcal{A} , let $M_i = \mathrm{rad}^i M$, $f_i : M_i \rightarrow M_{i-1}$ the inclusion, $N_0 = M/M_1$ and $\varepsilon_0 : M \rightarrow M/M_1$ the projection. This defines an object of \mathcal{F} .

We obtain a functor $\mathrm{gr}\mathcal{A} \rightarrow \mathcal{F}$.

Proposition 3.15. *The canonical functor $\mathrm{gr}\mathcal{A} \rightarrow \mathcal{F}$ is an equivalence.*

Proof. The image of $\text{Hom}_{\mathcal{A}}(N, N')$ in $\text{Hom}_{\mathcal{A}}(N, N'_0)$ is isomorphic to the quotient of $\text{Hom}_{\mathcal{A}}(N, N')$ by $\text{Hom}_{\mathcal{A}}(N, \text{rad } N')$ and it follows that the functor is fully faithful.

Let us show that it is essentially surjective. Let $M \in \mathcal{F}$. Let $r \geq 0$ such that $M_{r+1} = 0$. Then, $M_r \xrightarrow{\sim} N_r$ has homology concentrated in degree 0 and is semi-simple. By induction on $-i$, it follows from the distinguished triangle $M_{i+1} \rightarrow M_i \rightarrow N_i \rightsquigarrow$ that M_i has homology concentrated in degree 0.

Note that we have an exact sequence $0 \rightarrow H^0 M_{i+1} \rightarrow H^0 M_i \rightarrow H^0 N_i \rightarrow 0$. Since the canonical map $\text{Hom}(H^0 N_i, S) \rightarrow \text{Hom}(H^0 M_i, S)$ is bijective for any simple S , it follows that $H^0 N_i$ is the largest semi-simple quotient of $H^0 M_i$. So, $M_i \xrightarrow{\sim} \text{rad}^i M_0$ and M comes from an object of \mathcal{A} . \square

4. SIMPLE GENERATORS FOR STABLE CATEGORIES

4.1. From equivalences. Let k be a field and A a split self-injective k -algebra with no projective simple module.

Let B be another split self-injective k -algebra with no projective simple module, and let $F : B\text{-stab} \xrightarrow{\sim} A\text{-stab}$ be an equivalence of triangulated categories. Let \mathcal{S}' be a complete set of representatives of isomorphism classes of simple B -modules. For $L \in \mathcal{S}'$, let L' be an indecomposable A -module isomorphic to $F(L)$ in $A\text{-stab}$. Let $\mathcal{S} = \{L'\}_{L \in \mathcal{S}'}$. Then,

- (i) $\text{Hom}_{A\text{-stab}}(S, T) = k^{\delta_{S,T}}$ for $S, T \in \mathcal{S}$
- (ii) Every object M of $A\text{-stab}$ has a filtration $0 = M_r \rightarrow M_{r-1} \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 = M$ such that the cone of $M_i \rightarrow M_{i-1}$ is isomorphic to an object of \mathcal{S} .

Note that (ii) is equivalent to

- (ii') Given M in $A\text{-mod}$, there is a projective module P such that $M \oplus P$ has a filtration $0 = N_r \subset N_{r-1} \subset \cdots \subset N_1 \subset N_0 = M \oplus P$ with the property that N_i/N_{i-1} is isomorphic (in $A\text{-mod}$) to an object of \mathcal{S} .

Linckelmann has shown the following [Li, Theorem 2.1 (iii)] :

Proposition 4.1. *Assume that F is induced by an exact functor $B\text{-mod} \rightarrow A\text{-mod}$. If \mathcal{S} consists of simple modules, then there is a direct summand of F that is an equivalence $B\text{-mod} \xrightarrow{\sim} A\text{-mod}$.*

We deduce :

Corollary 4.2. *Let B_1, B_2 be split self-injective algebras with no projective simple modules and $G_i : B_i\text{-mod} \rightarrow A\text{-mod}$ exact functors inducing stable equivalences. Assume $\mathcal{S}_1 = \mathcal{S}_2$ (up to isomorphism). Then, B_1 and B_2 are Morita equivalent.*

So, if we assume in addition that F comes from an exact functor G between module categories, then B is determined by \mathcal{S} , up to Morita equivalence.

The functor G is isomorphic to $X \otimes_B -$ where X is an (A, B) -bimodule. We can (and will) choose G so that X has no non-zero projective direct summand. Then, $G(L)$ is indecomposable for L simple [Li, Theorem 2.1 (ii)], so $\mathcal{S} = \{G(L)\}_{L \in \mathcal{S}'}$, up to isomorphism.

Proposition 4.3. *An A -module M is in the image of G if and only if there is a filtration $0 = M_r \subset M_{r-1} \subset \cdots \subset M_1 \subset M_0 = M$ such that M_i/M_{i-1} is isomorphic to an object of \mathcal{S} .*

Proof. Take L a B -module. Then the image by G of a filtration of L whose successive quotients are simple provides a filtration as required.

Conversely, we proceed by induction on r . We have an exact sequence $0 \rightarrow G(N) \rightarrow M \rightarrow G(L) \rightarrow 0$ and a corresponding element $\zeta \in \text{Ext}_A^1(G(L), G(N))$. We have an isomorphism $\text{Ext}_B^1(L, N) \xrightarrow{\sim} \text{Ext}_A^1(G(L), G(N))$ and we take ζ' to be the inverse image of ζ under this isomorphism. This gives an exact sequence $0 \rightarrow N \rightarrow M' \rightarrow L \rightarrow 0$, and hence an exact sequence $0 \rightarrow G(N) \rightarrow G(M') \rightarrow G(L) \rightarrow 0$ with class ζ . It follows that $M \simeq G(M')$ and we are done. \square

4.2. Filtrable objects.

4.2.1. Given two A -modules M and N , we write $M \sim N$ to denote the existence of an isomorphism between M and N in A -stab. Given $f, g \in \text{Hom}_A(M, N)$, we write $f \sim g$ if $f - g$ is a projective map.

Lemma 4.4. *Let $f, f' : M \rightarrow N$ be two surjective maps with $f \sim g$. Then there is $\sigma \in \text{Aut}_A(M)$ with $f' = f\sigma$ and $\sigma \sim \text{id}_M$.*

Similarly, let $f, f' : N \rightarrow M$ be two injective maps with $f \sim g$. Then there is $\sigma \in \text{Aut}_A(M)$ with $f' = \sigma f$ and $\sigma \sim \text{id}_M$.

Proof. Let $L = \ker f$ and $L' = \ker f'$. Let $L = L_0 \oplus P$ and $L' = L'_0 \oplus P'$ with P, P' projective and L_0, L'_0 without non-zero projective direct summands. We have an isomorphism $\bar{\alpha}_0 \in \text{Hom}_{A\text{-stab}}(L_0, L'_0)$ in A -stab giving rise to an isomorphism of distinguished triangles in A -stab

$$\begin{array}{ccccccc} L_0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \Omega^{-1}L_0 \\ \bar{\alpha}_0 \downarrow \sim & & \parallel & & \parallel & & \Omega^{-1}(\bar{\alpha}_0) \downarrow \sim \\ L'_0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \Omega^{-1}L'_0 \end{array}$$

Let $\alpha_0 \in \text{Hom}_A(L_0, L'_0)$ lifting $\bar{\alpha}_0$. This is an isomorphism. There is now a commutative diagram of A -modules, where the exact rows come from the elements of $\text{Ext}_A^1(N, L_0)$ and $\text{Ext}_A^1(N, L'_0)$ defined above :

$$\begin{array}{ccccccc} 0 & \longrightarrow & L_0 & \longrightarrow & M_0 & \longrightarrow & N \longrightarrow 0 \\ & & \alpha_0 \downarrow \sim & & \sigma_0 \downarrow \sim & & \parallel \\ 0 & \longrightarrow & L'_0 & \longrightarrow & M'_0 & \longrightarrow & N \longrightarrow 0 \end{array}$$

We have $M \simeq M_0 \oplus P \simeq M'_0 \oplus P'$, hence $P \simeq P'$. Let $\alpha : L \xrightarrow{\sim} L'$ extending α_0 . Then there is $\sigma : M \xrightarrow{\sim} M$ making the following diagram commute

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\ & & \alpha \downarrow \sim & & \sigma \downarrow \sim & & \parallel \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \end{array}$$

and we are done.

The second part of the lemma has a similar proof — it can also be deduced from the first part by duality. \square

4.2.2.

Hypothesis 2. Let \mathcal{S} be a finite set of indecomposable finitely generated A -modules such that $\text{Hom}_{A\text{-stab}}(S, T) = k^{\delta_{S,T}}$ for $S, T \in \mathcal{S}$.

An \mathcal{S} -filtration for an A -module M is a filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ such that $\bar{M}_i = M_i/M_{i+1}$ is in $\text{add}(\mathcal{S})$ for $0 \leq i \leq r-1$.

We say that M is *filtrable* if it admits an \mathcal{S} -filtration.

Lemma 4.5. *Let M be a non-projective filtrable A -module. Then there is $S \in \mathcal{S}$ such that $\text{Hom}_{A\text{-stab}}(M, S) \neq 0$ (resp. such that $\text{Hom}_{A\text{-stab}}(S, M) \neq 0$).*

Proof. Assume $\text{Hom}_{A\text{-stab}}(M, S) = 0$ for all $S \in \mathcal{S}$. Since M is filtrable, it follows that $\text{End}_{A\text{-stab}}(M) = 0$, and hence M is projective, which is not true. The second case is similar. \square

Lemma 4.6. *Let M be a filtrable module and $S \in \mathcal{S}$. Given $f : M \rightarrow S$ non-projective, there is $g : M \rightarrow S$ surjective with filtrable kernel such that $f \sim g$. Similarly, given $f : S \rightarrow M$ non-projective, there is $g : S \rightarrow M$ injective with filtrable cokernel such that $f \sim g$.*

Proof. We proceed by induction on the number of terms in a filtration of M . The result is clear if $M \in \mathcal{S}$.

Let $0 \rightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} T \rightarrow 0$ be an exact sequence with $T \in \mathcal{S}$ and N filtrable.

Assume first $f\alpha : N \rightarrow S$ is projective. Then there is $p : M \rightarrow S$ projective and $g : T \rightarrow S$ with $f - p = g\beta$. Since g is not projective, it is an isomorphism. Consequently, $f - p$ is surjective and its kernel is isomorphic to N by Lemma 4.4, so we are done.

Assume now $f\alpha : N \rightarrow S$ is not projective. By induction, there is $q : N \rightarrow S$ projective such that $f\alpha + q$ is surjective with filtrable kernel N' . Since $\alpha : N \rightarrow M$ is injective, there is a projective map $p : M \rightarrow S$ with $q = p\alpha$. Now, we have an exact sequence $0 \rightarrow N/N' \xrightarrow{\bar{\alpha}} M/\alpha(N') \rightarrow T \rightarrow 0$ and a non-projective surjection $f + p : M/\alpha(N') \rightarrow S$. Since $(f + p)\bar{\alpha} : N/N' \rightarrow S$ is an isomorphism, it follows that the kernel of the map $M/\alpha(N') \rightarrow S$ is isomorphic to T . Since N' is filtrable, it follows that $\ker(f + p)$ is filtrable and we are done. The second assertion follows by duality. \square

From Lemmas 4.4 and 4.6, we deduce :

Lemma 4.7. *Let $S \in \mathcal{S}$ and let M be a filtrable module.*

If $f : M \rightarrow S$ be a surjective and non-projective map, then $\ker f$ is filtrable.

Similarly, if $g : S \rightarrow M$ is injective and non-projective, then $\text{coker } g$ is filtrable.

From Lemmas 4.5 and 4.6, we deduce :

Lemma 4.8. *Let M be filtrable non-projective. Then there is a submodule S of M , with $S \in \mathcal{S}$, such that M/S is filtrable and the inclusion $S \rightarrow M$ is not projective. Similarly, there is a filtrable submodule N of M such that $M/N \in \mathcal{S}$ and $M \rightarrow M/N$ is not projective.*

Proposition 4.9. *Let M be an A -module with a decomposition $M \sim M'_1 \oplus M'_2$ in the stable category. If M is filtrable then there is a decomposition $M = M_1 \oplus M_2$ such that M_i is filtrable and $M_i \sim M'_i$*

Proof. We can assume M is not projective, for otherwise the proposition is trivial. We prove the proposition by induction on the dimension of M .

Let $M = T_1 \oplus T_2 \oplus P$ with P projective, T_i without non-zero projective direct summand and $T_i \sim M'_i$. Denote by $\pi : M \rightarrow T_1$ the projection.

By Lemma 4.5, there is $S \in \mathcal{S}$ such that $\mathrm{Hom}_{A\text{-stab}}(M, S) \neq 0$. Hence, $\mathrm{Hom}_{A\text{-stab}}(T_i, S) \neq 0$ for $i = 1$ or $i = 2$. Assume for instance $i = 1$. Pick a non-projective map $\alpha : T_1 \rightarrow S$. So, $\alpha\pi : M \rightarrow S$ is not projective. By Lemma 4.6, there is a surjective map $\beta : M \rightarrow S$ with $\beta \sim \alpha\pi$ and $N = \ker \beta$ filtrable. Then $N \sim L \oplus T_2$, where L is the kernel of $\alpha + p : T_1 \oplus P_S \rightarrow S$ and $p : P_S \rightarrow S$ is a projective cover of S . By induction, we have $N = N_1 \oplus N_2$ with N_i filtrable and $N_1 \sim L$, $N_2 \sim T_2$. Now, the map $S \rightarrow L[1]$ gives a map $S \rightarrow N_1[1]$ (in $A\text{-stab}$). Let M_1 be the extension of S by N_1 corresponding to that map. Then $M \simeq M_1 \oplus N_2$, the modules M_1 and N_2 are filtrable, $M_1 \sim M'_1$, and $N_2 \sim M'_2$. \square

Let M be a filtrable module. We say that M has no projective remainder if there is no direct sum decomposition $M = N \oplus P$ with $P \neq 0$ projective and N filtrable.

Lemma 4.10. *Let M be a filtrable module with no projective remainder and let $S \in \mathcal{S}$.*

For $f : M \rightarrow S$ surjective, $\ker f$ is filtrable if and only if f is non-projective.

For $f : S \rightarrow M$ injective, $\mathrm{coker} f$ is filtrable if and only if f is non-projective.

Proof. Assume f is projective. Then there is a decomposition $M = N \oplus P$ and $f = (0, g)$ with P projective. Now, $\ker f = N \oplus \ker g$. If $\ker f$ is filtrable, then it follows from Lemma 4.9 that M has a non-zero projective submodule whose quotient is filtrable.

The converse is given by Lemma 4.7. The second part of the Lemma has a similar proof. \square

Lemma 4.11. *Let $M = M_0 \oplus M_1$ with M and M_0 filtrable and such that M_0 has no projective remainder. Then M_1 is filtrable.*

Proof. We proceed by induction on $\dim M_0$ — the result is clear for $M_0 = 0$. Assume $M_0 \neq 0$. Let $f : M_0 \rightarrow S$ be a surjection with $S \in \mathcal{S}$ and $\ker f$ filtrable. By Lemma 4.10, f is not projective. Then $f' : M \xrightarrow{\mathrm{can}} M_0 \xrightarrow{f} S$ is a non-projective surjection. By Lemma 4.7, $\ker f'$ is filtrable. We have $\ker f' = \ker f \oplus M_1$ and we are done. \square

4.2.3. We now turn to filtrations by objects in $\mathrm{add}(\mathcal{S})$.

Lemma 4.12. *Let M be a filtrable module and N a filtrable submodule of M such that $M/N \in \mathrm{add} \mathcal{S}$. Then, N is minimal with these properties if and only if N has no projective remainder and the canonical map $\mathrm{Hom}_{A\text{-stab}}(M/N, S) \rightarrow \mathrm{Hom}_{A\text{-stab}}(M, S)$ is surjective for every $S \in \mathcal{S}$.*

Proof. Let N be a minimal filtrable submodule of M such that $M/N \in \mathrm{add} \mathcal{S}$. Denote by $i : N \rightarrow M$ the injection and $p : M \rightarrow M/N$ the quotient map.

Let $S \in \mathcal{S}$. Fix $f_1, \dots, f_r : M/N \rightarrow S$ such that $\sum_i f_i : M/N \rightarrow S^r$ is surjective and $\ker \sum_i f_i$ has no direct summand isomorphic to S . Let T be the subspace of $\mathrm{Hom}_{A\text{-stab}}(M, S)$ generated by $f_1 p, \dots, f_r p$. Assume this is a proper subspace, so there is $f' : M \rightarrow S$ whose image in $\mathrm{Hom}_{A\text{-stab}}(M, S)$ is not in T . Then $f'i : N \rightarrow S$ is not projective, hence there is a projective map $q : N \rightarrow S$ such that $f'i + q$ is surjective and has filtrable kernel N' (Lemma 4.6). There is $q' : M \rightarrow S$ projective such that $q = q'i$. Now, $M/N' \simeq M/N \oplus S$ and this contradicts the minimality of N . It follows that the canonical map $\mathrm{Hom}_{A\text{-stab}}(M/N, S) \rightarrow \mathrm{Hom}_{A\text{-stab}}(M, S)$ is surjective. Assume $N = N' \oplus P$ with N' filtrable with no projective remainder and P projective.

Proof. The first part of the proposition follows from Lemmas 4.12 and 4.13.

Let $\tau \in \text{Aut}(N)$ such that $\tau = \text{id}_N + p$ with $p : N \rightarrow N$ projective. Then there is a projective map $q : M \rightarrow N$ with $p = qi$. Let $\sigma = \text{id}_M + q$. Then $\sigma|_N = \tau$. Now, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N \longrightarrow 0 \\ & & \tau \downarrow \sim & & \sigma \downarrow & & \text{id} \downarrow \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N \longrightarrow 0 \end{array}$$

and hence σ is an automorphism of M .

Let N' be a minimal filtrable submodule of M such that $M/N' \in \text{add } \mathcal{S}$. Then we have shown that $M/N \xrightarrow{\sim} M/N'$ and that via such an isomorphism, the maps $M \rightarrow M/N$ and $M \rightarrow M/N'$ are stably equal. Now, Lemma 4.4 shows there is $\sigma \in \text{Aut}(M)$ with $N' = \sigma(N)$ and $\sigma \sim \text{id}_M$. \square

Let M be filtrable. An \mathcal{S} -radical filtration of M is a filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ such that M_i is a minimal filtrable submodule of M_{i-1} with $M_{i-1}/M_i \in \text{add } \mathcal{S}$.

Proposition 4.15. *Let M be a filtrable A -module with no projective remainder. Let $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ and $0 = M'_{r'} \subseteq M'_{r'-1} \subseteq \cdots \subseteq M'_0 = M$ be two \mathcal{S} -radical filtrations of M . Then, $r = r'$ and there is an automorphism of M that swaps the two filtrations and that is stably the identity.*

Proof. We prove this lemma by induction on the dimension of M . By Proposition 4.14, there is $\sigma \in \text{Aut}(M)$ such that $\sigma(M'_1) = M_1$ and $\sigma \sim \text{id}_M$. Now, by induction, we have $r = r'$ and there is $\tau \in \text{Aut}(M_1)$ such that $\tau\sigma(M'_i) = M_i$ for $i > 0$ and $\tau \sim \text{id}_{M_1}$. By Proposition 4.14, there is $\tau' \in \text{Aut}(M)$ such that $\tau'_{|M_1} = \tau$ and $\tau' \sim \text{id}_M$. Now, $\tau'\sigma$ sends M'_i onto M_i . \square

Remark 4.16. A filtrable projective module can have two \mathcal{S} -radical filtrations with non-isomorphic layers.

Consider $A = k\mathfrak{A}_4$, the group algebra of the alternating group of degree 4 and assume k has characteristic 2 and contains a cubic root of 1. Let B be the principal block of $k\mathfrak{A}_5$. Then, the restriction functor is a stable equivalence between B and A . Let \mathcal{S} be the set of images of the simple B -modules. Denote by k the trivial A -module and by k_+ , k_- the non-trivial simple A -modules. Then $\mathcal{S} = \{k, S_+, S_-\}$ where S_ε is a non-trivial extension of ε by $-\varepsilon$. Let P and P' be the two projective indecomposable B -modules that don't have k as a quotient. Then $\text{Res}_{\mathfrak{A}_4} P \simeq \text{Res}_{\mathfrak{A}_4} P'$. This projective module has two \mathcal{S} -radical filtrations with non-isomorphic layers : one coming from the radical filtration of P and one coming from the radical filtration of P' .

While \mathcal{S} -radical filtrations are not unique in general for filtrable modules with a projective remainder, there are some cases where uniqueness still holds :

Proposition 4.17. *Assume A is a symmetric algebra. Let $0 \rightarrow S \rightarrow M \rightarrow T \rightarrow 0$ and $0 \rightarrow S' \rightarrow M \rightarrow T' \rightarrow 0$ be two exact sequences with $S, S', T, T' \in \mathcal{S}$. Assume that the sequences don't both split. Then there is an automorphism of M swapping the two exact sequences.*

Proof. If M is non-projective, then this is a consequence of Proposition 4.14.

Assume M is projective. Since A is symmetric, we have a non-projective map $T \simeq \Omega^{-1}S \rightarrow S$. It follows that $S = T$. Similarly, $T' = S'$. We have exact sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}(S', S) \rightarrow \text{Hom}(S', M) \rightarrow \text{Hom}(S', S) \rightarrow \text{Ext}^1(S', S) \rightarrow 0 \\ 0 \rightarrow \text{Hom}(S', S') \rightarrow \text{Hom}(S', M) \rightarrow \text{Hom}(S', S') \rightarrow \text{Ext}^1(S', S') \rightarrow 0 \end{aligned}$$

We have $\Omega^{-1}S' \simeq S'$, and hence $\dim \text{Ext}^1(S', S') = 1$. Consequently, $\dim \text{Hom}(S', M)$ is an odd integer. It follows that $\text{Ext}^1(S', S) \neq 0$, hence $\text{Hom}_{A\text{-stab}}(S', S) \neq 0$, so $S' = S$ and we are done by Lemma 4.4. \square

Lemma 4.18. *Let $0 = M_r \subset M_{r-1} \subset \cdots \subset M_0 = M$ be a filtration of M with $M_{i-1}/M_i \in \text{add } \mathcal{S}$.*

- (i) *If M has no projective remainder, then M_i has no projective remainder, for all i .*
- (ii) *If the filtration is an \mathcal{S} -radical filtration, then M_i has no projective remainder for $i \geq 1$.*

Proof. Consider an exact sequence $0 \rightarrow N \oplus P \rightarrow M \rightarrow L \rightarrow 0$ of filtrable modules with P projective and N filtrable. Then there is an extension M' of L by N such that $M = M' \oplus P$ and M' is filtrable. The first part of the lemma follows.

Assume now the filtration is an \mathcal{S} -radical filtration. Assume for some $i \geq 1$, we have $M_i = N \oplus P$ with N filtrable with no projective remainder and P projective and filtrable (Lemma 4.11). Then, $M = M' \oplus P$ with P filtrable by (i). There is an exact sequence $0 \rightarrow L \rightarrow P \rightarrow S \rightarrow 0$ with $S \in \mathcal{S}$ and L filtrable. Now, the canonical surjection $M' \oplus P \rightarrow M/M_1 \oplus S$ has filtrable kernel and this contradicts the minimality of M_1 . \square

Proposition 4.19. *Let M_1 and M_2 be two filtrable A -modules with no projective remainder. If $M_1 \sim M_2$, then $M_1 \simeq M_2$.*

Proof. We prove the proposition by induction on $\min(\dim M_1, \dim M_2)$. Fix an isomorphism ϕ from M_2 to M_1 in the stable category. Let $X = \bigoplus_{S \in \mathcal{S}} S \otimes \text{Hom}_{A\text{-stab}}(M_1, S)$ and $g_1 \in \text{Hom}_{A\text{-stab}}(M_1, X)$ be the canonical map. Let $g_2 = g_1\phi$. By Propositions 4.14 and 4.15, there are exact sequences

$$0 \rightarrow N_1 \rightarrow M_1 \xrightarrow{f_1} X \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N_2 \rightarrow M_2 \xrightarrow{f_2} X \rightarrow 0$$

with the image of f_i in the stable category equal to g_i . So, there is an isomorphism from N_2 to N_1 in the stable category compatible with ϕ . By Lemma 4.18, N_1 and N_2 have no projective remainder. By induction, we deduce that there is an isomorphism $N_2 \xrightarrow{\sim} N_1$ lifting the stable isomorphism. So, M_1 and M_2 are extensions of isomorphic modules, with the same class in Ext^1 , hence are isomorphic. \square

4.3. Generators and reconstruction.

4.3.1. We assume from now on that

Hypothesis 3. \mathcal{S} satisfies Hypothesis 2 and given $M \in A\text{-mod}$, there is a projective A -module P such that $M \oplus P$ is filtrable.

Proposition 4.20. *Let $S \in \mathcal{S}$. Let $P_S \rightarrow S$ be a projective cover of S and P minimal projective such that $\Omega S \oplus P$ is filtrable. Let $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = \Omega S \oplus P$ be an \mathcal{S} -radical filtration.*

Then $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 \subseteq P_S \oplus P$ is an \mathcal{S} -radical filtration.

If A is symmetric, then $M_{r-1} \simeq S$.

Proof. Let $f_1 : P_S \rightarrow S$ be a surjective map and $f = (f_1, 0) : P_S \oplus P \rightarrow S$. Let $T \in \mathcal{S}$ and $g : P_S \oplus P \rightarrow T$ such that we have an exact sequence $0 \rightarrow L \rightarrow P_S \oplus P \xrightarrow{f+g} S \oplus T \rightarrow 0$ with L filtrable.

We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & P_S \oplus P & \xrightarrow{f+g} & S \oplus T \longrightarrow 0 \\ & & \parallel & & \uparrow & & \downarrow (0, \text{id}) \\ 0 & \longrightarrow & L & \longrightarrow & \Omega S \oplus P & \longrightarrow & T \longrightarrow 0 \end{array}$$

The surjection $\Omega S \oplus P \rightarrow T$ is projective and has filtrable kernel. From Lemma 4.10, we get a contradiction to the minimality of P . It follows that $\Omega S \oplus P$ is a minimal submodule of $P_S \oplus P$ such that the quotient is in $\text{add } \mathcal{S}$.

We have $\text{Hom}_{A\text{-stab}}(T, \Omega S) \simeq \text{Hom}_{A\text{-stab}}(S, T)^*$, since A is symmetric. Now, $\text{Hom}_{A\text{-stab}}(M_{r-1}, \Omega S \oplus P) \neq 0$ by Lemma 4.10. The second part of the proposition follows. \square

Let M and N be two A -modules with filtrations $0 = M_r \subseteq M_{r-1} \subseteq \dots \subseteq M_0 = M$ and $0 = N_s \subseteq N_{s-1} \subseteq \dots \subseteq N_0 = N$. Let $\text{Hom}_A^f(M, N)$ be the subspace of $\text{Hom}_A(M, N)$ of filtered maps (*i.e.*, those g such that $g(M_i) \subseteq N_i$). We put $\bar{M}_i = M_i/M_{i+1}$. We denote by ϕ_i the composition of canonical maps $\phi_i : \text{Hom}_A^f(M, N) \rightarrow \text{Hom}_A(\bar{M}_i, \bar{N}_i) \rightarrow \text{Hom}_{A\text{-stab}}(\bar{M}_i, \bar{N}_i)$.

We view $N' = N_i$ as a filtered module with the induced filtration $0 = N'_{s-i} \subseteq N'_{s-i-1} = N_{s-1} \subseteq \dots \subseteq N'_1 = N_{i+1} \subseteq N'_0 = N'$.

Lemma 4.21. *Let M be a filtrable A -module with an \mathcal{S} -radical filtration and N be a filtrable A -module with an \mathcal{S} -filtration. Let $f \in \text{Hom}_A^f(M, N)$ with $\phi_0(f) = 0$. Then $\phi_i(f) = 0$ for all i .*

Proof. The map $\bar{f}_0 : \bar{M}_0 \rightarrow \bar{N}_0$ induced by f is projective. So there is a projective module P and a commutative diagram

$$\begin{array}{ccc} M & & N \\ \downarrow & \nearrow & \downarrow \\ \bar{M}_0 & \xrightarrow{\bar{f}_0} & \bar{N}_0 \end{array}$$

P

Let p be the composition $p : M \rightarrow \bar{M}_0 \rightarrow P \rightarrow N$. Then $f - p \sim f$, $f - p$ and f have the same restriction to M_1 , and $\overline{(f - p)}_0 = 0$. Consequently it is enough to prove the lemma in the case where $\bar{f}_0 = 0$.

From now on, we assume $\bar{f}_0 = 0$. Assume the map $\bar{f}_1 : \bar{M}_1 \rightarrow \bar{N}_1$ is not projective. So there is $S \in \mathcal{S}$ and a (split) surjection $g : \bar{N}_1 \rightarrow S$ such that $g\bar{f}_1 : \bar{M}_1 \rightarrow S$ is not projective. Let $s : S \rightarrow \bar{M}_1$ be a right inverse to g , and let L be the kernel of $g\bar{f}_1$.

We have an exact sequence $0 \rightarrow L \rightarrow M/M_2 \xrightarrow{(\text{can}, g\bar{f}_1)} \bar{M}_0 \oplus S \rightarrow 0$. So the inverse image of L in M_1 is a filtrable submodule of M with quotient isomorphic to $\bar{M}_0 \oplus S$. This contradicts the fact that M_1 is a minimal filtrable submodule of M such that $M/M_1 \in \text{add } \mathcal{S}$. So \bar{f}_1 is projective; *i.e.*, $\phi_1(f) = 0$.

We now prove by induction that $\phi_i(f) = 0$ for all i . Assume $\phi_d(f) = 0$. Then, we apply the result above to the filtered modules M_d and N_d to get $\phi_{d+1}(f) = 0$. \square

4.3.2. We define a category \mathcal{G} as follows.

- Its objects are A -modules together with a fixed \mathcal{S} -radical filtration.
- We define $\text{Hom}_{\mathcal{G}}(M, N)_i$ as the image of $\text{Hom}_A^f(M, N_i)$ in $\text{Hom}_{A\text{-stab}}(\bar{M}_0, \bar{N}_i)$. We put $\text{Hom}_{\mathcal{G}}(M, N) = \bigoplus_i \text{Hom}_{\mathcal{G}}(M, N)_i$.
- Let $f \in \text{Hom}_{\mathcal{G}}(M, N)_i$ and $g \in \text{Hom}_{\mathcal{G}}(L, M)_j$. Let $\tilde{f} : M \rightarrow N_i$ be a filtered map lifting f . It induces a map $\phi_j(\tilde{f}) \in \text{Hom}_{A\text{-stab}}(\bar{M}_j, \bar{N}_{i+j})$ independent of the choice of \tilde{f} (Lemma 4.21). We define the product fg to be $\phi_j(\tilde{f}) \circ \phi_0(g)$.

Given $S \in \mathcal{S}$, let $P_S \rightarrow S$ be a projective cover of S and Q_S projective minimal such that $\Omega S \oplus Q_S$ is filtrable. Fix a radical filtration of $P_S \oplus Q_S$ with first term $\Omega S \oplus Q_S$.

Let $M = \bigoplus_{S \in \mathcal{S}} (P_S \oplus Q_S)$. This comes with an \mathcal{S} -radical filtration. We have constructed a $\mathbf{Z}_{\geq 0}$ -graded k -algebra $\text{End}_{\mathcal{G}}(M)$.

The following Lemma is clear.

Lemma 4.22. *Let \mathcal{S} be a complete set of representatives of isomorphism classes of simple A -modules. Then we have an equivalence $\text{gr}(A\text{-mod}) \xrightarrow{\sim} \mathcal{G}$. If A is basic, then $\text{End}_{\mathcal{G}}(M)$ is isomorphic to the graded algebra associated with the radical filtration of A .*

We have now obtained our partial reconstruction result :

Theorem 4.23. *Let B be a selfinjective algebra with no simple projective module. Let M be an (A, B) -bimodule inducing a stable equivalence and having no projective direct summand. Let $\mathcal{S} = \{M \otimes_B L\}$ where L runs over a complete set of representatives of isomorphism classes of simple B -modules.*

Then, there is an equivalence $\text{gr}(B\text{-mod}) \xrightarrow{\sim} \mathcal{G}$. If B is basic, there is an isomorphism between the graded algebra associated with the radical filtration of B and $\text{End}_{\mathcal{G}}(M)$.

4.3.3. The category \mathcal{G} can be constructed directly as in §3.1, using only the stable category with its triangulated structure.

Proposition 4.24. *Let M be a module with an \mathcal{S} -filtration $0 = M_r \subseteq M_{r-1} \subseteq \dots \subseteq M_0 = M$. This is an \mathcal{S} -radical filtration if and only if*

- $\text{Hom}_{A\text{-stab}}(M_i/M_{i+1}, S) \rightarrow \text{Hom}_{A\text{-stab}}(M_i, S)$ is an isomorphism for all $S \in \mathcal{S}$ and $i > 0$,
- $\text{Hom}_{A\text{-stab}}(M_0/M_1, S) \rightarrow \text{Hom}_{A\text{-stab}}(M_0, S)$ is surjective for all $S \in \mathcal{S}$, and
- M_i has no projective remainder for $i > 0$.

Assume the filtration is an \mathcal{S} -radical filtration. Then M has no projective remainder if and only if $\text{Hom}_{A\text{-stab}}(M_0/M_1, S) \rightarrow \text{Hom}_{A\text{-stab}}(M_0, S)$ is an isomorphism.

Proof. Let M be a module with an \mathcal{S} -radical filtration $0 = M_r \subseteq M_{r-1} \subseteq \dots \subseteq M_0 = M$. The canonical map $\text{Hom}_{A\text{-stab}}(M_i/M_{i+1}, S) \rightarrow \text{Hom}_{A\text{-stab}}(M_i, S)$ is surjective for all $S \in \mathcal{S}$, by Lemma 4.12. Note that M_i has no projective remainder for $i > 0$, by Lemma 4.18. It follows that the canonical map $\text{Hom}_{A\text{-stab}}(M_i/M_{i+1}, S) \rightarrow \text{Hom}_{A\text{-stab}}(M_i, S)$ is an isomorphism for all $S \in \mathcal{S}$ (Lemma 4.13).

Let us now prove the other implication. Since M_i has no projective remainder for $i > 0$, it follows from Lemma 4.12 that $0 = M_r \subseteq M_{r-1} \subseteq \dots \subseteq M_1$ is an \mathcal{S} -radical filtration of M_1 .

Assume the filtration is an \mathcal{S} -radical filtration. If M has no projective remainder, then $\mathrm{Hom}_{A\text{-stab}}(M_0/M_1, S) \rightarrow \mathrm{Hom}_{A\text{-stab}}(M_0, S)$ is injective by Lemma 4.13.

Assume now that $\mathrm{Hom}_{A\text{-stab}}(M/M_1, S) \rightarrow \mathrm{Hom}_{A\text{-stab}}(M, S)$ is bijective. Assume $M = M' \oplus P$ with M' filtrable and P projective. We have $\mathrm{Hom}_{A\text{-stab}}(M/M_1, S) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-stab}}(M, S) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-stab}}(M', S)$. There is a surjective map $g : M' \rightarrow M/M_1$ with filtrable kernel such that the composition $M \xrightarrow{\mathrm{can}} M' \xrightarrow{g} M/M_1$ is equal to the canonical map $M \rightarrow M/M_1$ in the stable category, by Proposition 4.14. By Lemma 4.4, we have $M_1 \simeq \ker g \oplus P$. Since M_1 has no projective remainder by the first part of the proposition, we get $P = 0$, hence M has no projective remainder. \square

Let $\mathcal{T} = A\text{-stab}$. Note that \mathcal{S} is determined by its image in \mathcal{T} and it satisfies Hypothesis 3 if and only if $\mathrm{Hom}_{\mathcal{T}}(S, T) = k^{\delta_{ST}}$ for all $S, T \in \mathcal{S}$ and every object of \mathcal{T} is an iterated extension of objects of \mathcal{S} .

We have a functor $\mathcal{G} \rightarrow \mathcal{F}$: it sends a module M with an \mathcal{S} -radical filtration $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_0 = M$ to $\cdots \rightarrow 0 \rightarrow M_{r-1} \rightarrow \cdots \rightarrow M_1 \rightarrow M \rightarrow M/M_1$ (cf Proposition 4.24).

Proposition 4.25. *The canonical functor $\mathcal{G} \xrightarrow{\sim} \mathcal{F}$ is an equivalence.*

Proof. The functor is clearly fully faithful.

Start with $0 = N_r \xrightarrow{f_r} N_{r-1} \rightarrow \cdots \rightarrow N_1 \xrightarrow{f_1} N_0 \xrightarrow{\varepsilon_0} M_0$. Adding a projective direct summand to the N_i 's, we can lift the maps f_i to maps that are injective in the module category and such that the successive quotients have no projective direct summands. So we have a filtration $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_1 \subseteq M'_0$ such that M'_i/M'_{i+1} is stably isomorphic to a direct sum of objects of \mathcal{S} . Since it has no projective summand, it is actually isomorphic to a sum of objects of \mathcal{S} ; *i.e.*, we have an \mathcal{S} -filtration. Consider i maximal such that M'_i has a projective remainder. Then $0 = M'_r \subseteq M'_{r-1} \subseteq \cdots \subseteq M'_i$ is an \mathcal{S} -radical filtration by Proposition 4.24 (first part). The second part of Proposition 4.24 shows now that M'_i has no projective remainder, a contradiction. So the filtration is an \mathcal{S} -filtration. \square

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