

Low-Energy Theorems from Holography

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Abstract

In the context of gauge/gravity duality, we verify two types of gauge theory low-energy theorems, the dilation Ward identities and the decoupling of heavy flavor. First, we provide an analytic proof of non-trivial dilation Ward identities for a theory holographically dual to a background with gluon condensate (the self-dual Liu–Tseytlin background). In this way an important class of low-energy theorems for correlators of different operators with the trace of the energy-momentum tensor is established, which so far has been studied in field theory only. Another low-energy relationship, the so-called decoupling theorem, is numerically shown to hold universally in three holographic models involving both the quark and the gluon condensate. We show this by comparing the ratio of the quark and gluon condensates in three different examples of gravity backgrounds with non-trivial dilaton flow. As a by-product of our study, we also obtain gauge field condensate contributions to meson transport coefficients.

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1 Introduction

On the long road towards a holographic description of QCD, there are some milestones corresponding to exact relations which have to be satisfied also in any holographic model. These are so called low-energy theorems [1] (see e.g. [2] for review). In field theory these are statements which impose restrictions on the various correlators. The purpose of this work is to compare holography to field theory by considering the low-energy theorems concerning one- and two-point functions of a strongly coupled gauge theory on both sides of the correspondence. We report nice non-trivial agreement in two important cases: the dilation Ward identities and the decoupling theorem for the heavy flavor. Recently the validity of a related class of theorems (QCD sum rules) was shown holographically in [3] at finite temperature. Apart from demonstrating

the validity of low-energy theorems, a particular result of our analysis is a statement on the IR universality of theories dual to three scale-dependent backgrounds with non-trivial dilaton flow.

First, we aim at realizing the QCD low-energy theorems explicitly, for instance

$$\int d^4x \langle T(x)O(0) \rangle = -\dim(O)\langle O \rangle, \quad (1)$$

where $T = T_\mu^\mu$ is energy-momentum trace on the boundary. This is trivially satisfied in the conformal case: The right-hand side is expected to be zero in a conformal field theory where all condensates vanish. For an explicit expression for the correlators of energy-momentum components see e.g. [4]. Thus for a nontrivial test we need a background which is different from AdS in the IR, dual to a non-conformal field theory, for instance with a gluon condensate. There are a number of models which generalize the original AdS/CFT correspondence to the backgrounds corresponding to non-vacuum states of $\mathcal{N} = 4$ SYM or to non-conformal and non-supersymmetric theories. We use the self-dual background by Liu and Tseytlin [5] with non-zero expectation value of the gluon operator $\langle \text{tr}G^2 \rangle$ in this part of our work. To perform the test of dilation Ward identities, we calculate the two-point correlators $\langle \text{tr}G^2(x)\text{tr}G^2(0) \rangle$, $\langle \text{tr}G^2(x)\text{tr}G\tilde{G}(0) \rangle$, $\langle T(x)\text{tr}G^2(0) \rangle$, $\langle T(x)\text{tr}G\tilde{G}(0) \rangle$, $\langle T_{\mu\nu}(x)T_{\alpha\beta}(0) \rangle$ in this background.

The analysis of correlators is easily performed for non-zero frequency. In this way we reproduce the results for transport coefficients, extending the analysis to the case of the non-conformal backgrounds considered. First of all, we calculate the η/s ratio of shear viscosity over entropy via $\langle T_{xy}T_{xy} \rangle$, which was performed for the conformal case in [6, 7, 8, 9]. Here we find using suitable holographic renormalization that condensate corrections to $\frac{\eta}{s}|_{T \rightarrow 0} = \frac{1}{4\pi}$ are absent in the Liu-Tseytlin background, i.e. the nonzero VEV of the gluon field strength $\langle \text{tr}G^2 \rangle$ does not affect the value of η/s .

Secondly, we check the relationship between two-point and one point functions in gauge theory with fundamental fermions, known as decoupling relation

$$\langle \frac{\alpha_s}{\pi} \text{tr}G^2 \rangle = -12m \langle \bar{q}q \rangle. \quad (2)$$

Fundamental fermions are introduced in our system via probe $D7$ branes, see e.g. [10]. The $D7$ branes represent the fundamental degrees of freedom, being convenient locations for the fundamental strings to end and to be thus endowed with a global $SU(N_f)$ flavor symmetry in the Maldacena limit. The length of the strings corresponds to the quark mass, and the subleading term in the asymptotics of the embedding coordinates

to the condensate. A non-trivial test of the theorem considered is possible only for an IR-non-trivial metric. For that purpose we use three different dilaton flow backgrounds with gluon condensate: the self-dual Liu–Tseytlin background mentioned above [5], the Gubser–Kehagias–Sfetsos background [11], [12] and the Constable–Myers background [13]. All of these are examples for non-trivial dilaton flows. A remarkable universality among the three models and agreement with standard field theory is observed.

Let us now present the two low-energy theorems discussed in this paper.

Dilation Ward Identity. It was argued in [1] that the following dilation Ward identity holds within field theory,

$$\lim_{q \rightarrow 0} i \int e^{iqx} d^4x \left\langle T \left\{ \mathcal{O}(x), \frac{\beta(\alpha_s)}{4\alpha_s} \text{tr}G^2(0) \right\} \right\rangle = (-d) \langle \mathcal{O} \rangle [1 + \text{mass-dependent terms}], \quad (3)$$

where d is the canonical dimension of the operator \mathcal{O} , $T\{\cdot, \cdot\}$ stands for the time ordered product and the one-loop beta-function is normalized as $\beta(\alpha_s) = -\frac{b\alpha_s^2}{2\pi}$, $b = \frac{11}{3}N_c - \frac{2}{3}N_f$. Identities for higher correlators are also available:

$$i^2 \int d^4x d^4y \left\langle T \left\{ \mathcal{O}(x), \frac{\beta(\alpha_s)}{4\alpha_s} \text{tr}G^2(y), \frac{\beta(\alpha_s)}{4\alpha_s} \text{tr}G^2(0) \right\} \right\rangle = (-d)^2 \langle \mathcal{O} \rangle [1 + \text{mass-dep.}]. \quad (4)$$

For the gluon field strength operators we obtain:

$$i \int \left\langle T \left\{ \frac{3\alpha_s}{4\pi} \text{tr}G^2(x), \frac{3\alpha_s}{4\pi} \text{tr}G^2(0) \right\} \right\rangle = \frac{18}{b} \left\langle \frac{\alpha_s}{\pi} \text{tr}G^2 \right\rangle. \quad (5)$$

Decoupling Theorem. Novikov, Shifman, Vainshtein and Zakharov derived in [1] the following equation for light quarks by considering the regularity of the beta function

$$\frac{d}{dm_q} \left\langle \frac{\alpha_s}{\pi} \text{tr}G^2 \right\rangle = -\frac{24}{b} \langle \bar{q}q \rangle. \quad (6)$$

This low-energy theorem for heavy quarks is recovered also in an independent manner in [14]. Besides, for heavy quarks the following relation due to Shifman, Vainshtein and Zakharov holds

$$m \langle \bar{q}q \rangle = -\frac{1}{12} \left\langle \frac{\alpha_s}{\pi} \text{tr}G^2 \right\rangle. \quad (7)$$

The derivation of this equation is found in [15]. It expresses the continuity of the energy-momentum trace at the flavor number thresholds of the beta-function. The

factors 12 and 24 in the equations above are universal, they do not contain N_c or N_f . In this paper, we shown that relation (7) holds holographically in the three dilaton-flow backgrounds to great accuracy.

A related calculation, the holographic derivation of the Veneziano-Witten formula relating the mass of the η' meson and the topological susceptibility of pure Yang-Mills theory, was performed in [10]. The holographic conformal anomaly was previously considered under finite temperature in the 5-dimensional model with a dilaton potential adjusted in such way that both confinement and the correct UV behaviour of the coupling are reproduced [16].

This paper is organized as follows. In Section 2 we describe technicalities related to finding correlators. In Section 3 we describe the holographic description of the dilation Ward identities. Section 4 contains our main result – the derivation of the nonperturbative decoupling of the heavy flavor in the different dilaton flow models. In the last section we discuss the importance of having established the decoupling and scaling theorems holographically. Several necessary facts concerning the models are collected in Appendix A, while Appendix B concerns the derivation of the transport properties of the models under consideration.

2 Recipes of AdS/CFT

For later use, let us briefly review the AdS/CFT prescription for calculating two-point functions, emphasizing in particular the derivation of the gauge-fixing and the Gibbons-Hawking term. In the analysis of the boundary term we follow here very closely the analysis of [4]. A reader familiar with these technicalities can proceed directly to the next section.

We consider the general rules for two-point functions and calculate the matrix of correlators

$$M_{ij} = \langle O_i O_j \rangle|_{(p)} = \frac{\delta^2 S_{full}}{\delta \bar{\Phi}_i(p) \delta \bar{\Phi}_j(-p)}. \quad (8)$$

The standard wisdom on finding Green function of the fields present is to set the action of the type

$$S_{bulk} = \int d^4x dz \phi'^2 g^{zz} \sqrt{g} \quad (9)$$

out onto the boundary as

$$S_{boundary} = \int d^4x \phi \phi' g^{zz} \sqrt{g}|_{z \rightarrow 0} . \quad (10)$$

The correlator in terms of bulk-to-boundary Green functions $G(x, z)$ of the field ϕ is given by

$$\langle O(x)O(0) \rangle = G(x, z) \partial_z G(0, z)|_{z=0} . \quad (11)$$

In our case two additional difficulties arise. First, the correct boundary term should be supplemented by the Gibbons–Hawking term [4], which makes a theory defined on manifold with boundary globally diffeomorphism-invariant. Second, the bilinear action of fields' fluctuations is non-diagonal, this means that we shall be dealing with a matrix of Green functions rather than with separately-treatable ones.

Let us define Green function matrix. Namely, if field Φ_i has a bulk solution $\Phi_i(z)$, satisfying $z^{\delta_i} \Phi_i(z)|_{z \rightarrow 0} = \bar{\Phi}_i$, then by definition

$$K_{ij}(z) = \frac{\delta \Phi_j(z)}{\delta \bar{\Phi}_i} . \quad (12)$$

Let us establish the correct boundary term. The full action of our bulk theory is actually [4]

$$S_{full} = S_{10d} + S_{div} + S_{4d} \quad (13)$$

where the Gibbons–Hawking term

$$S_{4d} = -2\partial_z \int d^4x \sqrt{-g_4} - c \int d^4x \sqrt{-g_4} , \quad (14)$$

is here given by

$$g_4 = \det(g_{ij}), \quad i = 0, 1, 2, 3. \quad (15)$$

The constant c can be fixed arbitrarily to our convenience, e.g. as in eq. (4.15) in [4]. The other piece which one has to take into account is the full divergence term S_{div} , which does not affect equations of motion, but does change the appearance of the action and makes it diagonal in terms of physical degrees of freedom of the graviton. It is the well-known fluctuation term

$$S_{div} = \frac{3}{2} \partial_\mu W^\mu , \quad (16)$$

the vector W^μ is (see [17], Vol.II, §96)

$$W^\mu = \sqrt{-g} \left(g^{\alpha\beta} \delta \Gamma_{\alpha\beta}^\mu - g^{\alpha\mu} \delta \Gamma_{\alpha\beta}^\beta \right) , \quad (17)$$

where $\delta\Gamma_{\alpha\beta}^\mu = \Gamma_{\alpha\beta}^\mu(g+h) - \Gamma_{\alpha\beta}^\mu(g)$. This constitutes the gauge-fixing prescription for our problem.

Consider now the second variation of these actions in fluctuation fields; denote these second-order expressions as $S_{10d}^{(2)}$, $S_{div}^{(2)}$, $S_{4d}^{(2)}$ respectively; they contain both fields and their derivatives. The two-point correlator is then

$$\langle O_i O_j \rangle = K_{ik} \frac{\partial^2 \mathcal{L}}{\partial \Phi'_k \partial \Phi'_m} \partial_z K_{jm} + K_{ik} \frac{\partial^2 S_{4d}^{(2)}}{\partial \Phi_k \partial \Phi'_m} \partial_z K_{jm} + K_{ik} \frac{\partial^2 S_{4d}^{(2)}}{\partial \Phi_k \partial \Phi_m} K_{jm}, \quad (18)$$

here \mathcal{L} is Lagrangian density of the bulk action:

$$S_{bulk} = S_{10d}^{(2)} + S_{div}^{(2)} = \int dz \mathcal{L}. \quad (19)$$

The above structure is obvious from the following reasons. Consider the bulk action

$$\delta^2 S_{bulk} = \frac{\delta \Phi_m(z)}{\delta \bar{\Phi}_j} \frac{\delta^2 S_{bulk}}{\delta \Phi_m \delta \Phi_k} \frac{\delta \Phi_k(z)}{\delta \bar{\Phi}_i}, \quad (20)$$

where

$$\frac{\delta^2 S_{bulk}}{\delta \Phi_m \delta \Phi_k} = \int dz \left[\frac{\partial^2 L}{\partial \Phi'_m \partial \Phi'_k} \partial_z \delta \Phi_m \partial_z \delta \Phi_k + \frac{\partial^2 L}{\partial \Phi_m \partial \Phi'_k} \delta \Phi_m \partial_z \delta \Phi_k + \frac{\partial^2 L}{\partial \Phi_m \partial \Phi_k} \delta \Phi_m \delta \Phi_k \right]. \quad (21)$$

Taking into account that Green functions of field fluctuations by definition satisfy equations:

$$\left[-\partial_z \frac{\partial^2 L}{\partial \Phi'_m \partial \Phi'_k} \partial_z + \frac{\partial^2 L}{\partial \Phi_m \partial \Phi'_k} \partial_z + \frac{\partial^2 L}{\partial \Phi_m \partial \Phi_k} \right] \delta \Phi_k(z) = 0, \quad (22)$$

one sees that the only contribution of S_{bulk} into the correlator will be, after taking off the derivative and integration, the term:

$$\delta^2 S_{bulk} = \delta \Phi_m(z) \frac{\partial^2 L}{\partial \Phi'_m \partial \Phi'_k} \partial_z \delta \Phi_k(z). \quad (23)$$

Now remembering the definition of Green function matrix

$$K_{mj} = \frac{\delta \Phi_m(z)}{\delta \bar{\Phi}_j}, \quad (24)$$

we arrive exactly at (18). Then there is the purely boundary term (Hawking-Gibbons term). It does not require the above procedure, since it already sits on 4d. Then it contributes the following:

$$\delta^2 S_{4d} = \frac{\partial^2 S_{4d}}{\partial \Phi'_m \partial \Phi_k} \partial_z \delta \Phi_m \delta \Phi_k + \frac{\partial^2 S_{4d}}{\partial \Phi_m \partial \Phi_k} \delta \Phi_m \delta \Phi_k. \quad (25)$$

The action S_{4d} contains no more than one derivative term, which is due to normal differentiating of extrinsic curvature, thus $\frac{\partial^2 L}{\partial \Phi'^2} = 0$. This contributes the other two terms into the correlator (18).

3 Low-Energy Theorems

In this Section we calculate the matrix of the two-point correlators for the gluonic operators and components of the energy-momentum tensor. Then we compare these to one-point correlators and find that the correct scaling relations from field theory are satisfied on the gravity side. We begin by introducing the Liu–Tseytlin model in which we will perform our calculations in this section.

Liu–Tseytlin model. In the Einstein frame the bulk action of *IIB* superstring theory is [5]

$$S_{10} = \frac{1}{g_s^2 (2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{g_{10}} \left(R - \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} e^{2\phi} (\partial_\mu C)^2 - \frac{1}{2} |F_5|^2 \right), \quad (26)$$

where R is the curvature, ϕ is dilaton, F_5 is 5-form and C is axion.

The Liu–Tseytlin model is a generalized background for holography those which possesses self-duality. It describes a field-theory flow from a strongly-coupled conformal theory in the UV to a theory with condensate $\text{tr } G^2$ in the IR. By virtue of self-duality it is still supersymmetric. However, it possesses a scale parameter, which makes it closer to real-world physics. The self-duality is provided by the presence of a non-trivial axion field. Despite the presence of the scale, it is conformal in the UV; in the IR the dilaton singularity is determined by the gluon condensate $\text{tr } G^2$. Within supergravity this background is understood as “smeared” $D(-1)$ brane with a usual stack of $D3$ -branes. Since $D(-1)$ brane is an instanton in 10D, the resulting 4d theory can be considered as having an instanton-gas type of vacuum, which is advantageous for QCD purposes. Moreover, this background is confining (in the sense

of Wilson loop linear behavior at large temporal separation), and the string tension is proportional to the condensate. Of course, we do not claim to produce any real QCD results in this framework, but we believe it to be a very useful toy model.

For the Liu–Tseytlin background [5] metric in Einstein frame looks like the standard conformal solution

$$ds^2 = g_{0\mu\nu} dx^\mu dx^\nu = R^2 \left(\frac{dx^{\mu 2}}{\sqrt{h_3}} + \sqrt{h_3} \frac{dz^2 + z^2 d\Omega_5^2}{z^4} \right), \quad (27)$$

but the dilaton is modified by the smeared instanton (nonzero density of $D(-1)$)

$$e^\phi = h_{-1}, \quad (28)$$

and an axion is present

$$C_0 = \frac{1}{h_{-1}} - 1; \quad (29)$$

the $D3$ and $D(-1)$ form-factors are:

$$h_3 = z^4, \quad (30)$$

and

$$h_{-1} = 1 + qz^4. \quad (31)$$

The parameter q is the crucial quantity for us, since it measures the degree of IR-non-conformality of the theory (remember that in the UV, the theory is conformal and its β -function is zero).

The Tseytlin-Liu background has been successfully used for a number of applications, e.g. calculating meson spectra [18, 19, 20, 21]. In all these applications, its relevance to QCD has been demonstrated. In [22] a finite-temperature extension of the [5] solution has been found, which has been a further motivation to apply it to realistic high-energy quark-gluon plasmas. We shall employ Liu-Tseytlin background to test dilation Ward identities in Section 1 and decoupling relation in Section 4.

Holographic normalization of the operators

Here we consider normalization of the gluon field strength operator; the normalization of the quark operators will be considered in the next Section. According to the AdS/CFT dictionary we state that the fluctuation $\delta\phi(z, Q)$ of dilaton field

$$\phi(z, Q) = \phi_0(z) + \delta\phi(z, Q) \quad (32)$$

is dual to the operator O_ϕ , proportional to the QCD scalar gluonic operator

$$\text{tr}(G^2) \equiv \frac{1}{c_\phi} O_\phi. \quad (33)$$

We can fix the normalization constant c_ϕ by comparing the two-point functions

$$\langle O_\phi O_\phi \rangle = c_\phi^2 \langle \text{tr}(G^2) \text{tr}(G^2) \rangle. \quad (34)$$

At large momenta the leading behavior of gluonic correlator in QCD is [23]:

$$\langle \text{tr}(G^2)(Q) \text{tr}(G^2)(Q) \rangle = \frac{N_c^2 - 1}{4\pi^2} Q^4 \ln(Q^2 \epsilon^2). \quad (35)$$

To obtain a two-point function from holography we take the second variation of the action computed on a classical solution. In the vicinity of the boundary of AdS_5 the action (26) for the fluctuation is:

$$S_5 = \frac{\pi^3 R^8}{g_s^2 (2\pi)^7 \alpha'^4} \int d^4 x dz \frac{1}{z^3} \frac{1}{2} [-(\partial_z \delta\phi)^2 - \partial_\mu \delta\phi \partial^\mu \delta\phi + 2e^{2\phi_0} \delta\phi (\partial_z C)^2]. \quad (36)$$

Here we have taken the near boundary limit $r \gg L$ (so that $r^2 \simeq \rho^2$) and changed coordinates $z = \frac{R^2}{r^2}$. π^3 is the volume of the S_5 sphere, R^8 came from the determinant of the metric ($\sqrt{g} = \frac{R^{10}}{z^5}$). The last term containing the profile of axion field is negligible at the boundary (small z) because $\partial_z C(z) \sim z^3$. We can find the bulk-to-boundary propagator of $\phi(z, Q)$ at small z and large Q^2 . It is

$$\varphi(z, Q) = \frac{Q^2 z^2}{2} K_2(Qz), \quad \varphi(0, Q) = 1, \quad (37)$$

where K_i is McDonald function of the second kind. Now we can compute the second variation of the action. It is

$$\langle O_\phi O_\phi \rangle = \frac{\delta^2 S_{cl}}{\delta\phi_0 \delta\phi_0} = \frac{\pi^3 R^8}{g_s^2 (2\pi)^7 \alpha'^4} \frac{1}{2} \varphi(z, Q) \frac{\partial_z \varphi(z, Q)}{z^3} \Big|_{z=\epsilon} = \frac{N_c^2}{4(2\pi)^2} \frac{1}{8} Q^4 \ln(Q^2 \epsilon^2), \quad (38)$$

where we used the definition $R^4 = 4\pi g_s \alpha'^2 N_c$ and the asymptotic of McDonald function. Comparing this result with the expression of QCD we find

$$O_\phi = \frac{1}{4\sqrt{2}} \text{tr}(G^2). \quad (39)$$

To establish a relation between gluon condensate and the expansion coefficient of the dilaton field we compute the vacuum expectation value of O_ϕ at zero momentum taking the first variation of the action with respect to the boundary value of the field ϕ_0 . At zero momentum near the boundary the dilaton field behaves as

$$\phi(z) = \phi_0 + \phi_4 z^4. \quad (40)$$

For the dual operator given by (39) we find

$$\langle O_\phi \rangle = \frac{\delta S_{cl}}{\delta \phi_0} = \frac{\pi^3 R^8}{g_s^2 (2\pi)^7 \alpha'^4} \frac{1}{2} \varphi(z, Q) \frac{\partial_z \phi(z, Q)}{z^3} \Big|_{z=\epsilon} = \frac{N_c^2}{4(2\pi)^2} 4\phi_4. \quad (41)$$

From (39) and (40) we get the expression for the gluon condensate

$$\langle \text{tr}(G^2) \rangle \equiv 4\sqrt{2} O_\phi = N_c^2 \frac{4\sqrt{2}}{(2\pi)^2} \phi_4. \quad (42)$$

In the Liu-Tseytlin model the infinitesimal fluctuations of the fields on the bulk couple to the operators $\text{tr}G^2$, $\text{tr}G\tilde{G}$, $T_{\mu\nu}$ in the boundary $\mathcal{N} = 4$ SYM theory. Moreover, in the Liu-Tseytlin model the dilaton field behaves as $e^\phi = 1 + qz^4$, so the parameter of solution ϕ_4 in (42) equals q and the scalar and pseudoscalar gluon condensates are nontrivial and equal to the value given in (42), i.e.

$$\langle \text{tr}G^2 \rangle = \langle \text{tr}G\tilde{G} \rangle = N_c^2 \frac{4\sqrt{2}}{(2\pi)^2} q. \quad (43)$$

Correlators at Zero Frequency Fluctuation terms are defined as

$$\begin{aligned} \phi &= \phi_c + \varphi, \\ C &= C_0 + \xi, \\ g &= g_{0\mu\nu} + h_{\mu\nu}. \end{aligned} \quad (44)$$

We consider the following interaction term to provide a correspondence with the boundary theory:

$$S_{int} = \int d^4x \left[\frac{1}{2} T_{\mu\nu} \bar{h}^{\mu\nu} - e^{-\phi_c} \left(\bar{\varphi} \frac{\text{tr}G^2}{4\sqrt{2}} + \bar{\xi} \frac{\text{tr}G\tilde{G}}{4\sqrt{2}} \right) \right], \quad (45)$$

which, after introduction of useful self-dual and anti-self-dual components

$$G^\pm = \frac{G \pm \tilde{G}}{2} \quad (46)$$

and splitting axion and dilaton fluctuations into a new couple of variables

$$\eta^\pm = \varphi \pm \xi, \quad (47)$$

becomes

$$S_{int} = \int d^4x \left[\frac{1}{2} T_{\mu\nu} \bar{h}^{\mu\nu} - \frac{e^{-\phi_c}}{4\sqrt{2}} (\bar{\eta}^+ \text{tr} G^{+2} + \bar{\eta}^- \text{tr} G^{-2}) \right]. \quad (48)$$

Here bars denote four-dimensional sources, which are proportional to boundary values of five-dimensional fields:

$$\bar{h}_{\mu\nu} = z^2 h_{\mu\nu}|_{z=0}, \quad \bar{\eta}^\pm = \eta^\pm|_{z=0}, \quad \bar{\varphi} = \varphi|_{z=0}. \quad (49)$$

Fluctuations of F_5 are fully determined by h_μ^μ , thus there is no independent source for them.

Let us choose the gauge $h_{5\mu} = 0$, $k^\mu h_{\mu\nu} = 0$, $u^\mu h_{\mu\nu} = 0$, where wave-vector $k = (\omega, 0, 0, k)$, constant vector u is $u = (1, 0, 0, 0)$. We work with five fields:

$$\bar{\Phi}_i = (\eta^+, \bar{h}_{11} + \bar{h}_{22}, \bar{h}_{11} - \bar{h}_{22}, \bar{h}_{12}, \eta^-), \quad (50)$$

$i = 1, \dots, 5$, each coupled to the corresponding \mathcal{O}_i operator²

$$\mathcal{O}_i = \left(\frac{\text{tr} G^{+2}}{4\sqrt{2}}, \frac{1}{8} T_\mu^\mu, \frac{3}{8} T_{11} - \frac{1}{8} T_{22} - \frac{1}{8} T_{33} - \frac{1}{8} T_{00}, T_{xy}, \frac{\text{tr} G^{-2}}{4\sqrt{2}} \right), \quad (51)$$

with G^+ and G^- the self-dual and anti-self-dual parts of G , respectively, via

$$S_{int} = \int d^4x dz \sum_{i=1}^5 \mathcal{O}_i \Phi_i. \quad (52)$$

The relevant part of the fluctuation action in the bulk is

$$S_{10d+div}^{(2), \text{double deriv.}} = \int d^4x dz \left(\frac{1}{z^3} \Phi_1' \Phi_5' + \frac{z}{8} \Phi_2'^2 + \frac{z}{8} \Phi_3'^2 + \frac{z}{2} \Phi_4'^2 \right). \quad (53)$$

One should not be misled by its diagonal structure; besides the diagonal terms with double derivatives, the full bilinear action contains terms which make it non-diagonal.

The boundary Gibbons-Hawking action term is

$$S_{4d}^{(2), \text{derivatives}} = \int d^4x \frac{1}{8} (4c h_{xy}(z)^2 + 16z h'_{xy}(z) h_{xy}(z) + \Phi_2(z) (c \Phi_2(z) + 4z \Phi_2'(z))). \quad (54)$$

²Some of these operators, e.g. the \mathcal{O}_3 are not of immediate interest; however, it costs no additional effort to incorporate them into the calculation, so we work the correlators out for them as well.

The full system of equations upon Green functions (22) in the given background (27)–(31) is cumbersome and therefore is given in the Appendix B, eq.(119). Note that for the F_5 form we always have $\delta F = -2/r^3\Phi_2$, which solves automatically the equations of motion for this field and at the same time retains the constancy of the Ramond-Ramond flow $\int_{S^5} F_5 = N_c$.

It is instructive to start with zero-frequency correlators (setting $\omega = 0$ in (119) in Appendix B). Subsequently, we introduce finite frequencies ω . In this case we find oscillatory solutions (Bessel functions) (121) instead of the rational ones (120). The limit $\omega \rightarrow 0$ of the finite frequency result coincides with our previously found result at $\omega = 0$ and thus provides an additional check of the validity for our procedure.

The solutions (121) contain ten modes labelled by coefficients C_i , $i = 1 \dots 10$. One would expect that out of the ten modes five must be IR finite, yet quite unexpectedly six there are six IR finite modes ($C_1, C_4, C_5, C_6, C_7, C_9$), and the remaining four are infinite. An extra constraint is therefore necessary to make the Green function matrix (12) a well-defined 5×5 matrix. We require that the resulting correlator matrix be symmetric, which is equivalent to the condition $C_5 = C_6/2$, which removes exactly one redundant degree of freedom.

The Green function matrix is then (recall that $c_\phi = \frac{1}{4\sqrt{2}}$):

$$K_{ij} = \begin{array}{c|ccccc} & c_\phi \text{tr}G^{+2} & \frac{1}{8}T_\mu^\mu & \mathcal{O}_3 & \mathcal{O}_4 & c_\phi \text{tr}G^{-2} \\ \hline c_\phi \text{tr}G^{+2} & qz^4 - q\epsilon^4 + 1 & 0 & 0 & 0 & 0 \\ \hline \frac{1}{8}T_\mu^\mu & 0 & \frac{1}{z^2} & 0 & 0 & 0 \\ \hline \mathcal{O}_3 & 0 & 0 & \frac{1}{z^2} & 0 & 0 \\ \hline \mathcal{O}_4 & 0 & 0 & 0 & \frac{1}{z^2} & 0 \\ \hline c_\phi \text{tr}G^{-2} & -2q(\epsilon^4 - z^4) & 0 & 0 & 0 & qz^4 - q\epsilon^4 + 1 \end{array} \quad (55)$$

with the \mathcal{O}_i as given by (51). q is the non-conformality parameter defined in (31).

As a result, combining our knowledge of Green function matrix (55), the boundary action (54) and the derivative piece of the bulk action (53) we obtain the matrix:

$$M = \begin{array}{c|ccccc} & c_\phi \text{tr}G^{+2} & \frac{1}{8}T_\mu^\mu & \mathcal{O}_3 & \mathcal{O}_4 & c_\phi \text{tr}G^{-2} \\ \hline c_\phi \text{tr}G^{+2} & -4q & -2q & 0 & 0 & -2q \\ \hline \frac{1}{8}T_\mu^\mu & -2q & -\frac{1}{4\epsilon^4} & 0 & 0 & 0 \\ \hline \mathcal{O}_3 & 0 & 0 & -\frac{1}{4\epsilon^4} & 0 & 0 \\ \hline \mathcal{O}_4 & 0 & 0 & 0 & -\frac{1}{\epsilon^4} & 0 \\ \hline c_\phi \text{tr}G^{-2} & -2q & 0 & 0 & 0 & 0 \end{array}, \quad (56)$$

which contains information on the correlators of \mathcal{O}_i , \mathcal{O}_j via the following relation

$$\langle \mathcal{O}_i \mathcal{O}_j \rangle = \frac{N_c^2}{16\pi^2} M_{ij}. \quad (57)$$

Some comments are due here. The singular terms $\frac{1}{\epsilon^4}$ are expected due to the divergencies on the field theory side; they are subtracted by a holographic renormalization procedure, analogously to field-theoretical subtraction. The asymmetry in $\mathcal{O}_1 \leftrightarrow \mathcal{O}_5$ is also expected: what we consider is a self-dual configuration, therefore, the self-dual and the anti-self-dual operators have different properties.

Using the matrix elements obtained above, we can now establish the low-energy theorems. After normalization according to (39) we have

$$\left\{ \begin{array}{l} \int d^4x \langle \text{tr} G^{+2}(x) T(0) \rangle = 4 \langle \text{tr} G^{+2}(0) \rangle, \\ \int d^4x \langle \text{tr} G^{-2}(x) T(0) \rangle = 0, \\ \int d^4x \langle \text{tr} G^2(x) \text{tr} G^2(0) \rangle = \frac{1}{2} \langle \text{tr} G^2 \rangle, \\ \int d^4x \langle \text{tr} G \tilde{G}(x) \text{tr} G \tilde{G}(0) \rangle = 0. \end{array} \right. \quad (58)$$

where $T = T_{\nu\nu}$. Here we see that the first and the second lines of the equations above (58) constitute exactly the statement of the low-energy theorems

$$\langle \hat{O} T \rangle = \dim(O) \langle \hat{O} \rangle. \quad (59)$$

Note that $\langle \text{tr} (G^-)^2 \rangle = 0$.

The third line of (58) must be compared to the field-theoretical result

$$\int \langle \text{tr} G^2 \text{tr} G^2 \rangle \sim 1/\beta_0 \langle \text{tr} G^2 \rangle, \quad (60)$$

where in standard perturbation theory, β_0 is the one-loop coefficient of the beta-function. This equation reflects a breaking of the conformal symmetry. For the Liu–Tseytlin model the standard beta function vanishes. Nevertheless, the massive parameter q generates additional terms in the effective action. This gives rise to the contribution $T_{\mu\mu} \sim \text{tr} G^{-2}$ to the trace of the energy-momentum tensor at the operator

level. This is consistent with the low-energy theorem given by the third line of (58). On the other hand, $\langle \text{tr}G^{-2} \rangle = 0$, thus the expectation value of the energy-momentum tensor and the vacuum energy vanish, ensuring consistency with supersymmetry.

The fourth relation in (58) implies that the topological susceptibility of the vacuum, which is proportional to this correlator [24], vanishes in the Liu–Tseytlin model, which is in the agreement with the fact that the model is supersymmetric³.

Correlators at Finite Frequency Now let us analyze the finite-frequency solutions. The solutions are given in Appendix, eq. (121); only relevant modes shown. Unlike the $\omega = 0$ solutions, which were exact solutions, here $\Phi_2(z)$ and $\Phi_5(z)$ are powerlog expansions in ω and r . Since we are interested in the near-UV behaviour of Green functions, and eventually expand correlator matrix in powers of ω , this approximation is reasonable. The matrix of correlators becomes:

$$M = \begin{array}{c} \begin{array}{|c|c|c|c|c|c|} \hline & c_\phi \text{tr}G^{+2} & \frac{1}{8}T_\mu^\mu & \mathcal{O}_3 & \mathcal{O}_4 & c_\phi \text{tr}G^{-2} \\ \hline c_\phi \text{tr}G^{+2} & -4q & -2q & 0 & 0 & \frac{\log(\omega e)\omega^4}{8} - 2q \\ \hline \frac{1}{8}T_\mu^\mu & -2q & -\frac{\log(\omega e)\omega^4}{32} & 0 & 0 & 0 \\ \hline \mathcal{O}_3 & 0 & 0 & -\frac{\log(\omega e)\omega^4}{32} & 0 & 0 \\ \hline \mathcal{O}_4 & 0 & 0 & 0 & -\frac{\log(\omega e)\omega^4}{32} & 0 \\ \hline c_\phi \text{tr}G^{-2} & \frac{\log(\omega e)\omega^4}{8} - 2q & 0 & 0 & 0 & 0 \\ \hline \end{array} \\ (61) \end{array}$$

The most interesting physical implication of this correlator matrix comes from the $\langle T_{xy}T_{xy} \rangle$ element. It is proportional to $\frac{\eta}{s}|_{T=0}$, and here we observe its independence of q . This fact is not trivial from dimensional considerations, since we possess another dimensionful parameter, the frequency ω . Thus we have established

$$\frac{\eta}{s}(q, \omega) \Big|_{T=0} = \frac{1}{4\pi}. \quad (62)$$

As a bonus of this calculation, in the Appendix A we easily elaborate the matrix of quarkonium transport coefficient based on the above correlator matrix.

³Note that in the D4/D6 model [10] the topological susceptibility does not vanish. However there is no contradiction between these facts, since the model of [10] breaks supersymmetry (similarly to Sakai-Sugimoto model), whereas Liu-Tseytlin model retains supersymmetry.

4 Holographic Decoupling of the Heavy Flavor

4.1 Physics of Decoupling

In this Section we holographically derive the central result of this paper, which is known as “decoupling relation”. It can be found in [15]:

$$\frac{\alpha_s}{\pi} \langle G_{\mu\nu}^a G_{\mu\nu}^a \rangle = -12m_q \langle \bar{q}q \rangle. \quad (63)$$

The derivation of this relation is somewhat intuitive, but let us still restate the arguments by Shifman, Vainshtein and Zakharov. For vacuum expectation values of the different operators pertinent to light quarks the parameter of expansion is quark mass. For heavy quarks we expand in the inverse quark mass and set external momentum to $Q^2 \sim 0$. Let us suppose there exists a quark for which both expansions, small and large m are true. As it is in particular a “heavy” quark, the quark condensate can be done perturbatively from the triangle diagram with gluons as “vacuum sources”, shown in Fig. (1).

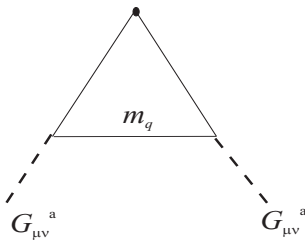


Figure 1: Vacuum diagram with heavy quarks depicting $\langle \bar{q}q \rangle$ as gluon-driven quantity.

One can understand the argument from which the relation (7) emerges as follows. Consider the trace of energy-momentum tensor of a gauge theory. For low quark mass there is beta-function contribution from the quark, for heavy quark there is only the gluonic contribution to the beta-function, yet there is quark chiral condensate is present:

$$\theta_\mu^\mu = \begin{cases} \left(\frac{11}{3}N_c - \frac{2}{3} \right) \frac{\alpha_s}{8\pi} \text{tr}G^2, & \text{above threshold,} \\ \left(\frac{11}{3}N_c \right) \frac{\alpha_s}{8\pi} \text{tr}G^2 + m\bar{q}q, & \text{below threshold.} \end{cases} \quad (64)$$

When the two are equated at some intermediate scale, the necessary relation (7) appears. Equating small and large m domains happens on the ground that we select the scale at which the heavy quarks “decouple” from the one-loop polarization operator. Hence this theorem is also known as decoupling relation. A picture of condensate as function of quark mass is given in [1].

4.2 Decoupling in Specific Backgrounds

We now establish relation (63) holographically by considering different backgrounds, those of Constable and Myers [13], of Gubser [11] and of Liu and Tseytlin. The Liu and Tseytlin background (27) was already discussed above in the Section 1. The Constable—Myers background in the Einstein frame has the metric

$$ds^2 = \frac{\left(\frac{b^4+r^4}{r^4-b^4}\right)^{\frac{1}{8b^4}}}{\sqrt{h_3}} dx_\mu^2 + \frac{(r^4 - b^4) \left(\frac{b^4+r^4}{r^4-b^4}\right)^{\frac{1}{4}\left(2-\frac{1}{2b^4}\right)} \sqrt{\left(\frac{b^4+r^4}{r^4-b^4}\right)^{\frac{1}{2b^4}} - 1}}{r^4} (dr^2 + r^2 d\Omega_5^2), \quad (65)$$

where

$$h_3 = \left(\frac{b^4 + r^4}{r^4 - b^4}\right)^{\frac{1}{2b^4}} - 1, \quad (66)$$

and the dilaton is

$$e^\phi = \left(\frac{b^4 + r^4}{r^4 - b^4}\right)^{\frac{1}{2}\sqrt{10-\frac{1}{4b^8}}}, \quad (67)$$

axion is zero, and $F_5 = \epsilon_5 \frac{1}{h_3}$, where ϵ_5 is the unitary antisymmetric tensor in the S_5 directions.

The chiral condensate and meson spectrum involving a Goldstone boson were obtained in [25] by embedding a D7 brane probe into a Constable–Myers background. Masses of heavy-light mesons in this background in D7 model were obtained in [26]. The quark condensate, pion decay constant and the higher order Gasser- Leutwyler coefficients were calculated for D7 model in this background in [27]. D7 embeddings were argued to be stable in this background [28, 29].

One of the first non-conformal backgrounds introduced into AdS/CFT was considered by Gubser [11]:

$$ds^2 = \sqrt[4]{1 - \frac{b^8}{r^8}} r^2 dx_\mu^2 + \frac{1}{r^2} (dr^2 + r^2 d\Omega_5^2), \quad (68)$$

dilaton in this background is

$$e^\phi = \left(\frac{\frac{r^4}{b^4} + 1}{\frac{r^4}{b^4} - 1} \right)^{\sqrt{\frac{3}{2}}}, \quad (69)$$

and the axion is zero. Originally it was intended to model confinement, yet it became also useful for introducing the gluon condensate. Shortly before Gubser, this background was also obtained by Kehagias and Sfetsos [12] in a less convenient parametrization.

Introduction of fundamental fields. We are modelling the fundamental fermionic degrees of freedom by embedding the D7 brane into one of the three backgrounds described above. The Dirac–Born–Infeld action for the D7 brane embedding in Einstein frame is given by

$$S_{D7} = \frac{1}{g_s(2\pi)^7 \alpha'^4} \int d^8\xi e^\phi \sqrt{\det_{\alpha\beta} (\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu})}. \quad (70)$$

The embedding of D7 is made as shown in the following table:

$AdS_5 \times S^5$	0	1	2	3	4	5	6	7	8	9
D7	+	+	+	+	+	+	+	+	-	-

(71)

One can get an image of the corresponding physics in Fig. (4.2), where string modes generating specific sectors of the spectrum are shown.

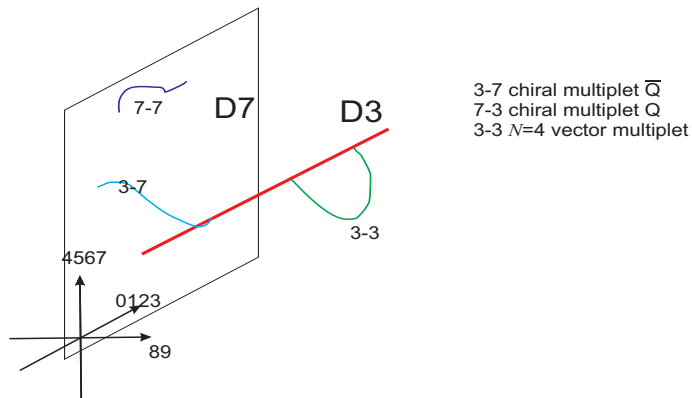


Figure 2: Scheme of the D3-D7 geometry and the corresponding string/field modes.

We look for embeddings of the form

$$X^9 = w(\rho), \quad X^8 = 0, \quad (72)$$

where embedding function w , worldsheet coordinates ξ_i and target space coordinates r, ρ are related as follows

$$\begin{aligned} w^2(\rho) &= r^2 - \rho^2, \\ \rho &= \sqrt{\xi_5^2 + \xi_6^2 + \xi_7^2 + \xi_8^2}. \end{aligned} \tag{73}$$

N_f quark flavours can be considered introducing N_f corresponding $D7$ branes with embedding coordinates $w_i, i = 1 \dots N_f$. If the quark masses are equal, $D7$ branes form a stack and the action (70) is multiplied by the factor N_f . In the following we restrict ourselves to the case of just one flavour for simplicity, considering only one embedding coordinate $w(\rho)$. Using these definitions we easily construct the equations of motion for $w(\rho)$,

$$\begin{aligned} 2\rho g_{00}(r)w'(\rho) (w'(\rho)^2 + 1) g'_{55}(r) - 2w(\rho) (w'(\rho)^2 + 1) (g_{55}(r)g'_{00}(r) + g_{00}(r)g'_{55}(r)) + \\ + g_{55}(r) (2\rho g'_{00}(r)w'(\rho)^3 + 2\rho g'_{00}(r)w'(\rho) + r g_{00}(r)w''(\rho)) = 0, \end{aligned} \tag{74}$$

where the corresponding g_{ii} should be taken for each respective metric. We solve them numerically at different values of the vacuum parameters and fields, corresponding to the boundary conditions at $\rho \rightarrow \infty$; a typical embedding is shown in Fig. (3).

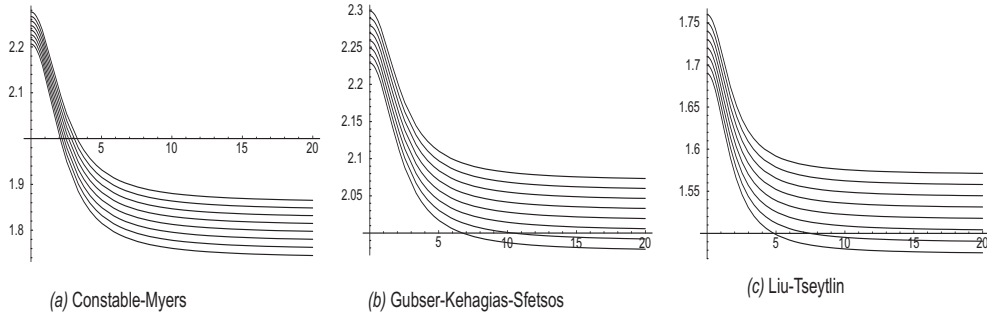


Figure 3: Typical embeddings of $D7$ branes.

4.3 Normalization of the “Quark” Operators

Following the same steps as in Section 3 we explore the scalar field w dual to the operator $\bar{q}q$, where q is the quark field. It is described by the action of the $D7$ brane (70), for which w_i is embedding coordinate. Here and after we are dealing only with flavour i and will omit this index where it is possible. The action for the fluctuations

of w is

$$S_5 = -\frac{2\pi^2 R^4}{g_s(2\pi)^7 \alpha'^4} \int d^4 x dz e^\phi \left[\frac{1}{2z} (\partial_z w)^2 + \frac{1}{2z} \partial_\mu w \partial^\mu w \right]. \quad (75)$$

Here we change coordinates the same way as in (36), $2\pi^2$ is a volume of 3-sphere R^4 comes again from the determinant of the metric $\sqrt{g^{(8)}} = \frac{R^6}{z^3}$. In the limit of large momenta near the boundary the bulk-to-boundary propagator is

$$\tilde{w}(z, Q) = Qz K_1(Qz), \quad \tilde{w}(0, Q) = 1. \quad (76)$$

The scalar field is dual to the operator O_w , which is proportional to $\bar{q}q = \frac{1}{c_w} O_w$. We compute two-point function of O_w to fix the normalization

$$\begin{aligned} \langle O_w O_w \rangle &= \frac{\delta^2 S_{8cl}}{\delta w_0 \delta w_0} = \frac{2\pi^2 R^4}{g_s(2\pi)^7 \alpha'^4} e^\phi \frac{1}{2} \tilde{w}(z, Q) \frac{\partial_z \tilde{w}(z, Q)}{z} \Big|_{z=\epsilon} \\ &= \frac{N_c}{2(2\pi)^4 \alpha'^2} \frac{1}{2} Q^2 \ln(Q^2 \epsilon^2) \Big|_{z=\epsilon}. \end{aligned} \quad (77)$$

Here the fact is used that $e^\phi|_{boundary} = 1$ [5] and again $R^4 = 4\pi g_s \alpha'^2 N_c$. We compare this result with the QCD calculation (see eq. 4.27 in [15]),

$$\langle \bar{q}q \bar{q}q \rangle = \frac{N_c}{16\pi^2} Q^2 \ln(Q^2 \epsilon^2), \quad (78)$$

and find

$$O_w = \frac{1}{2\pi\alpha'} \bar{q}q. \quad (79)$$

At this stage we can identify the boundary value of the field $w_0 = w|_{z=0}$. It is the source of $O_w = c_w(\bar{q}q)$, so it is proportional to the quark mass $w_0 = \frac{1}{c_w} M$. Thus we have

$$M = \frac{1}{2\pi\alpha'} w_0. \quad (80)$$

To identify quark condensate $\langle \bar{q}q \rangle$ we compute the expectation value of O_w at $Q = 0$. In this limit near the boundary the supergravity field takes the asymptotic form

$$w(z) = w_0 + w_2 z^2. \quad (81)$$

The result is

$$\langle O_w \rangle = \frac{\delta S_{8cl}}{\delta w_0} = \frac{2\pi^2 R^4}{g_s(2\pi)^7 \alpha'^4} e^\phi \frac{1}{2} \tilde{w}(z, Q) \frac{\partial_z w(z, Q)}{z} \Big|_{z=\epsilon} = \frac{N_c}{2(2\pi)^4 \alpha'^2} 2w_2. \quad (82)$$

The quark condensate is normalized as follows

$$\langle \bar{q}q \rangle = \frac{1}{c_w} \langle O_w \rangle = \frac{N_c}{(2\pi)^3 \alpha'} w_2. \quad (83)$$

To check the decoupling theorem, we have to study the relation $\frac{M \langle \bar{q}q \rangle}{\frac{g_{YM}^2}{4\pi^2} \langle \text{tr}(G^2) \rangle}$ for one specific quark flavour, i.e. $N_f = 1$. Expressed via the parameters of our model (42),(80),(83), it turns out to be

$$\frac{M \langle \bar{q}q \rangle}{\frac{g_{YM}^2}{4\pi^2} \langle \text{tr}(G^2) \rangle} = \frac{1}{N_c 4\sqrt{2} \alpha'^2 g_{YM}^2} \frac{w_0 w_2}{\phi_4}, \quad (84)$$

with coefficients w_0, w_2, ϕ_4 defined in (40), (81). It is convenient to express all coefficients via the expansion parameters in the coordinate $r = \frac{R^2}{z}$. We denote them by $\phi = \phi_0 + \frac{b_4}{r^4}, w = a + \frac{c}{r^2}$. Obviously, these are related to the former defined in (40), (81) by $\phi_4 = \frac{b_4}{R^8}, \omega_2 = \frac{c}{R^4}$. Recalling that $R^4 = 4\pi g_s \alpha'^2 N_c$ and $g_{YM}^2 = 4\pi g_s$, we obtain

$$\frac{M \langle \bar{q}q \rangle}{\frac{g_{YM}^2}{4\pi^2} \langle \text{tr}(G^2) \rangle} = \frac{1}{4\sqrt{2}} \frac{ac}{b_4}. \quad (85)$$

For the theorem

$$\frac{M \langle \bar{q}q \rangle}{\frac{g_{YM}^2}{4\pi^2} \langle \text{tr}(G^2) \rangle} = -\frac{1}{12} \quad (86)$$

to hold, the parameters a, b_4, c must satisfy

$$\frac{ac}{b_4} = -\frac{\sqrt{2}}{3}. \quad (87)$$

This relation, equivalent to the decoupling theorem, will be tested numerically below.

4.4 Numerics

We obtain numerically the dimensionless ratio of the solution coefficients $\frac{ac}{b_4}$, linearly related to the condensate ratio by (85). The ratio of the condensates obtained numerically is shown in Fig. (4) for the Gubser background. Similar pictures are obtained for the other two backgrounds. Each point in the parameter space represents an individual “measurement”, that is, a solution for a $D7$ -brane embedding at given gluon condensate and quark mass, from which the value for the quark condensate follows. By fitting the “experimental” points we estimate the value of the ratio and

a statistical error margin thereof. The mass is assumed to be the largest scale under consideration.

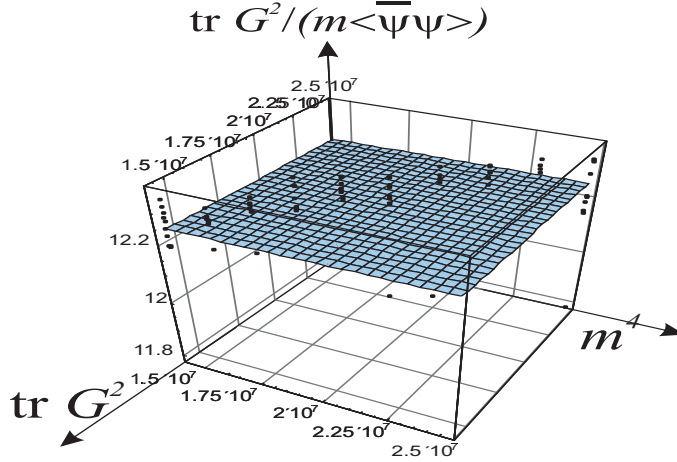


Figure 4: Dependence of the ratio $\frac{m\langle\bar{q}q\rangle}{\langle\text{tr}G^2\rangle}$ on quark mass

We obtain numerically the following results for the dimensionless ratio of the condensates we are looking for:

$$-\frac{\frac{g_{YM}^2 M}{4\pi^2} \langle\text{tr}(G^2)\rangle}{M\langle\bar{q}q\rangle} = \begin{cases} \text{Constable-Myers} & 12.0078 \pm 0.005 \\ \text{Gubser} & 12.25 \pm 0.01 \\ \text{Liu-Tseytlin} & 11.9192 \pm 0.0020 \end{cases} \quad (88)$$

Comparing the results to the correct analytic value $-\frac{g_{YM}^2 M}{4\pi^2} \frac{\langle\text{tr}(G^2)\rangle}{M\langle\bar{q}q\rangle} = 12$, we see agreement with good accuracy. The obvious universality of the three different metrics might signal that the decoupling theorem is insensitive to the details of the IR physics.

5 Discussion

Let us restate the main results of this study:

- We have established a universal constant value for the ratio $\frac{m\langle\bar{q}q\rangle}{\langle\text{tr}G^2\rangle}$ in the holographic duality with a good precision (0.5%), thus supporting the validity of the heavy flavor decoupling in holographic models of QCD.

- We have obtained a version of the low-energy theorem $\int \langle T\mathcal{O} \rangle = \dim(\mathcal{O})\langle \mathcal{O} \rangle$ satisfied in holography with condensates for the pure glue sector.

In addition we also find the following results

- A non-trivial relation between two-point and one-point functions $\int \langle G^2 G^2 \rangle = \text{const} \langle G^2 \rangle$ has been established.
- Shear and bulk viscosities have been shown to be independent of condensates.
- The quarkonium diffusion coefficient has been obtained both at non-zero temperature and for a non-vanishing condensate in the Appendix A.

The significance of establishing the decoupling ratio is its relevance to justifying phenomenological approaches to QCD via holography. We hope to find an analytic explanation of the amazing agreement which appears not to be a coincidence. Here we provide demonstration for a very important example of a statement which relates the quark and gluon sectors. This should encourage further development of realistic AdS/QCD constructions based on geometries with broken scale-invariance.

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A Quarkonium Transport in Self-Dual Background

A.1 Self-Dual Background at Zero Temperature

Here we review the method of [30] for calculating quarkonium transport properties. The basic result of this discussion is a decoupled structure, in which the contributions of the fermionic part of the action will be separated from those of the gluonic part according to the pattern

$$\text{meson kin. coeff.} = \left[\begin{array}{l} \text{meson mass shift} \\ \text{(D7 contribution)} \end{array} \right] \times \left[\begin{array}{l} \text{two-point correlator} \\ \text{(D3 contribution)} \end{array} \right]. \quad (89)$$

Consider a complex field φ of a slowly moving meson of velocity v , coupled to some operators of gluonic sector,

$$L = \varphi^+ v \partial_t \varphi + \sum_n c_n \varphi^+ \mathcal{O}_n \varphi, \quad (90)$$

where coefficients c_n are defined e.g. from $D7$ action of a dual model, which secures existence of mesons. The latter are understood as eigenmodes of fluctuations above the classical solution of $D7$ equations of motion. Interaction terms modify the spectra of eigenmodes in bulk; in terms of the boundary theory this amounts to meson mass shift. Coefficients C_n are then introduced as ‘‘susceptibility’’ of mass with regard to switching on the operator \mathcal{O}_n :

$$\delta M = -c_n \langle \mathcal{O}_n \rangle. \quad (91)$$

Considering one-particle dynamics we can obtain from (90)

$$\frac{dp_i}{dt} = \mathcal{F}_i, \quad (92)$$

where

$$\mathcal{F}_i = \int d^3x \varphi^+ \nabla_i c_n \mathcal{O}_n \varphi, \quad (93)$$

while correlator of two forces is directly related to transport coefficient

$$\kappa = \frac{1}{3} \int dt \langle \mathcal{F}(t) \mathcal{F}(0) \rangle. \quad (94)$$

One can integrate field φ out of these relations and obtain finally

$$\kappa = \frac{1}{3} \int k^2 d^3k c_n^2 \frac{2T}{\omega} \text{Im} \langle \mathcal{O}_n \mathcal{O}_n \rangle|_k, \quad (95)$$

where

$$\langle \mathcal{O}_n \mathcal{O}_n \rangle|_k = \int d^4x \theta(t) e^{i(\omega t - \vec{k} \cdot \vec{x})} \langle \mathcal{O}_n(x) \mathcal{O}_n(0) \rangle. \quad (96)$$

Here the contributions of flavor dynamics and pure gluodynamics are decoupled; below we proceed in calculating the gluodynamical part (the two-point correlator); the coefficients c_n being responsible for mass shifts are known in literature.

A.2 Self-Dual Background at Finite Temperature

It is possible to obtain quarkonium diffusion and relaxation coefficients at finite temperature and condensate, extending the work [30]⁴ to the background of [22]. This background has the metric

$$ds^2 = R^2 \left(\frac{1 - r^4 \pi^4 T^4}{r^2} dt^2 + \frac{dx_3^2}{r^2} + \frac{dr^2}{r^2(1 - r^4 \pi^4 T^4)} \right) + R^2 d\Omega_5^2, \quad (97)$$

dilaton is

$$e^\phi = 1 + \frac{q}{\pi^4 T^4} \log \left(\frac{1}{1 - r^4 \pi^4 T^4} \right), \quad (98)$$

axion is related to dilaton in the same way as in the zero-temperature Liu-Tseytlin background

$$C = e^{-\phi} - 1. \quad (99)$$

Quarkonium transport coefficients are quantities which feel both the fermionic piece of the action (some embedded brane) and the gluodynamics. From the former comes mass susceptibility to condensate, from the latter – correlators of interest. In principle, it would make a good sense to work in a back-reacted metric, however this we postpone till the method is fully technically developed for the well-controllable Ghoroku–Liu–Tseytlin metric.

For convenience we further use the variable

$$u = r^2 \pi^2 T^2, \quad (100)$$

which lives in the interval $(0, 1)$. We consider a reduced sector of the fluctuations, namely, those of fields η^+ , η^- , $h_{11} + h_{22}$. The equations of motion are given in Appendix A (122).

⁴We thank Derek Teaney for providing us with his unpublished Notes.

We see now that the problem of fields coupling to each other is additionally burdened by presence of finite temperature. Yet diagonalization of these equations is possible by means of the following functional transformation

$$\begin{cases} \bar{\eta}^+(u) &= \eta^+(u) \\ h(\bar{u}) &= h(u) + q(C_1 - \pi^2 \log(1 - u^2)) \eta^+(u) \\ \bar{\eta}^-(u) &= qh(u) \left(F_1 - \frac{\log(1-u^2)}{2\pi^2} \right) + \eta^-(u) + \\ &+ q \left(\frac{1}{4}q \log^2(1 - u^2) - \frac{qC_1 \log(1-u^2)}{2\pi^2} + C_2 \right) \eta^+(u) \end{cases} \quad (101)$$

Now for each of the variables we can write down an equation similar to that for the simple dilaton modes:

$$\varphi''(u) + \frac{u(u^3 + 6u + 4\omega^2 + 4k^2(u^2 - 1)) - 3}{4u^2(u^2 - 1)^2} \varphi(u) = 0, \quad (102)$$

for which transport coefficient is known; we calculated it independently, and found it to be in agreement with the previous results [30]

$$\frac{2\omega}{T} G_{\phi,\phi} = \pi^2 k^4 e^{-2C_\gamma k/T}, \quad (103)$$

where $C_\gamma = 4\sqrt{\frac{2}{\pi}}\Gamma\left(\frac{5}{4}\right)^2 \approx 2.62$. Knowledge of diagonalization matrix allows us to transform these results (at $q = 0$) into non-zero-condensate background:

$$\langle \Phi_i \Phi_j \rangle = (\hat{1} + qA) \langle \Phi'_i \Phi'_j \rangle_{q=0} (\hat{1} + qA)^+, \quad (104)$$

where zero-condensate solutions are rotated to non-zero-condensate by the following rotation matrix in mode space:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ \pi^2 & 0 & 0 \\ 0 & 1/2/\pi^2 & 0 \end{pmatrix}, \quad (105)$$

and the non-perturbed matrix of finite-temperature correlators is diagonal

$$\langle \Phi_i \Phi_j \rangle_{q=0} = \begin{pmatrix} \langle \text{tr} G^{+2} \text{tr} G^{+2} \rangle & 0 & 0 \\ 0 & \langle TT \rangle & 0 \\ 0 & 0 & \langle \text{tr} G^{-2} \text{tr} G^{-2} \rangle \end{pmatrix}, \quad (106)$$

whence one easily gets the mesonic transport coefficient by use of the following formula:

$$\kappa = \sum_{\mathcal{O}} c_{\mathcal{O}}^2 \frac{1}{3} \frac{\pi}{2} \int k^2 \frac{d^3 k}{(2\pi)^3} \frac{2\omega}{T} \langle \Phi_{\mathcal{O}} \Phi_{\mathcal{O}} \rangle, \quad (107)$$

where the respective mass susceptibility coefficients are obtained from considering the fermionic fluctuations coming from the embedded $D7$ brane piece of the action, and are defined via

$$\delta M = -c_{\mathcal{O}} \langle \mathcal{O} \rangle, \quad (108)$$

where M refers to the mass of quarkonium.

The correlators themselves are obtained in the following way, which we illustrate on the example of dilaton. We consider three domains: UV, IR and the intermediate domain (we denote the latter QC for semiclassics, since semiclassical approximate solutions will be valid therein). The physical limitations are infalling boundary condition on the horizon and reflected wave in the UV, which reduces number of unknown coefficients from 6 to 4. Then, we have matching conditions separate for each of the modes in the matching regions between UV and QC, and between QC and IR. This provides additional 4 constraints, thus the system is fully defined. In the UV the general solution to EOM is

$$\phi = \frac{2uI_2(2\sqrt{u}\sqrt{k^2 - \omega^2}) C_1}{k^2 - \omega^2} + 2u(k^2 - \omega^2) K_2(2\sqrt{u}\sqrt{k^2 - \omega^2}) C_2. \quad (109)$$

Taking the UV asymptotic ($u \rightarrow 0$) of ϕ , we see that physical boundary conditions are $C_1 = B, C_2 = 1$, where B is related to correlator straightforwardly:

$$\frac{2\omega}{T} G_{\phi\phi} = \frac{ImB}{\omega}. \quad (110)$$

On the contrary, expanding it for large k , we get the form appropriate for matching with QC:

$$\phi = e^{-2k\sqrt{u}} \sqrt{\pi} k^{-29/2} u^{-5/4} - \frac{B e^{2k\sqrt{u}} k^{-9/2} u^{-5/4}}{\sqrt{\pi}}. \quad (111)$$

The semiclassical equation has the approximate potential

$$V_{QC} = \frac{k^2}{u(1-u^2)}, \quad (112)$$

which allows to obtain the wave-functions in the standard way

$$\psi_{1,2} = \frac{e^{\pm \int p dx}}{\sqrt{p}}, \quad (113)$$

where

$$p = \sqrt{V_{QC} - E}. \quad (114)$$

The semiclassical solution near $u = 0$ and $u = 1$ is

$$\begin{aligned}\phi_{QC,u \rightarrow 0} &= -\frac{ie^{-2k\sqrt{u}}(e^{4k\sqrt{u}}A_1 + A_2)}{\sqrt{k}\sqrt[4]{u}} \\ \phi_{QC,u \rightarrow 1} &= -\frac{ie^{-\sqrt{2}k(\sqrt{1-u}+1)}(e^{2\sqrt{2}k}A_1 + e^{2k\sqrt{2(1-u)}}A_2)}{\sqrt{k}\sqrt[4]{2-2u}}.\end{aligned}\tag{115}$$

The IR solution with infalling boundary condition has only one degree of freedom:

$$\phi_{IR} = \left(e^{\sqrt{2}k\sqrt{1-u}} \csc(\pi\omega + e^{-\sqrt{2}k\sqrt{1-u}})\right) \frac{\sqrt{\pi}C}{2^{3/4}\sqrt{k}\sqrt[4]{1-u}}.\tag{116}$$

Equating the QC solution branches with those of IR and UV solutions, we get

$$\text{Im}B = \pi^2 k^4 e^{-2C_\gamma k/T},\tag{117}$$

as already stated above. Taking the integral over phase space (107) and performing linear transformation of correlator matrix (104), we get for transport coefficient

$$\kappa = \frac{1}{3}T^9 \frac{60\Gamma\left(\frac{3}{4}\right)^6}{\pi^2\Gamma\left(\frac{1}{4}\right)^6} [c_{\text{tr}G^{+2}}(1 + 2q\pi^2) + c_T(1 + q\pi^2) + c_{\text{tr}G^{-2}}],\tag{118}$$

where c_i are found in [30], $c_{\text{tr}G^2} = \frac{8}{5\pi} \left(\frac{2\pi}{M_0}\right)^3$, $c_T = \frac{12}{5\pi} \left(\frac{2\pi}{M_0}\right)^3$, M_0 being the meson mass.

B Equations of Motion

Here we shown the equations of motion for Liu–Tseytlin background in the graviton, axion and dilaton sector, corresponding to the pure glue sector on the boundary. The definitions of the fields are contained in eqs. (51)- (53).

$$\left\{ \begin{array}{l}
z \left((q\omega z^4 + \omega)^2 - 32q^2 z^6 \right) \eta_+(z) + \\
+ (qz^4 + 1) \left((11qz^4 + 3) \eta'_+(z) - z(qz^4 + 1) \eta''_+(z) \right) = 0, \\
32q^2 \eta_+(z) z^6 + (qz^4 + 1) \left((qz^4 + 1) (z^2 \omega^2 + 4) \Phi_2(z) - \right. \\
\left. - z (8q\eta'_+(z) z^2 + (qz^4 + 1) (\Phi'_2(z) + z\Phi''_2(z))) \right) = 0, \\
(z^2 \omega^2 + 4) \Phi_2(z) - z (\Phi'_2(z) + z\Phi''_2(z)) = 0, \\
(z^2 \omega^2 + 4) h_{xy}(z) - z (h'_{xy}(z) + zh''_{xy}(z)) = 0, \\
-32q^2 \eta_+(z) z^7 + (q\omega z^4 + \omega)^2 \eta_-(z) z - \\
- (qz^4 + 1) (8q\Phi_2(z) z^5 + (4q\Phi'_2(z) z^5 + (qz^4 + 1) \eta''_-(z)) z + (5qz^4 - 3) \eta'_-(z)) = 0.
\end{array} \right. \quad (119)$$

Solutions for the EOM in the Liu–Tseytlin case at zero frequency $\omega = 0$ are:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \end{pmatrix} = \begin{pmatrix} C_2 (qz^4 + 1)^2 + C_1 (qz^4 + 1) \\ \frac{-q^2 C_2 z^8 + C_3 z^4 + C_4}{z^2} \\ \frac{C_8 z^4 + C_7}{z^2} \\ \frac{C_{10} z^4 + C_9}{z^2} \\ qC_5 - \frac{C_6 q^2 + (q(qz^4 + 2)z^4 + 2)(4q(C_1 + C_2) + 2C_3)}{4q(qz^4 + 1)} \end{pmatrix} \quad (120)$$

Solution modes for a non-zero frequency:

$$\Phi_1 = \frac{1}{2}q\omega^2 K_2(z\omega)C_1 z^6 + \frac{1}{2}\omega^2 K_2(z\omega)C_1 z^2,$$

$$\begin{aligned} \Phi_2 = C_1 & \left[\frac{\gamma q\omega^8 z^{10}}{6144} - \frac{161q\omega^8 z^{10}}{552960} + \right. \\ & + \frac{q\omega^8 \log(z)z^{10}}{6144} + \frac{q\omega^8 \log(\omega)z^{10}}{6144} - \frac{q\omega^8 \log(16)z^{10}}{92160} - \\ & - \frac{q\omega^8 \log(8)z^{10}}{27648} - \frac{q\omega^8 \log(4)z^{10}}{184320} + \frac{1}{192}\gamma q\omega^6 z^8 - \frac{169q\omega^6 z^8}{23040} + \\ & + \frac{1}{192}q\omega^6 \log(z)z^8 + \frac{1}{192}q\omega^6 \log(\omega)z^8 - \frac{1}{960}q\omega^6 \log(16)z^8 - \\ & - \frac{q\omega^6 \log(4)z^8}{1920} + \frac{1}{16}\gamma q\omega^4 z^6 - \frac{17}{384}q\omega^4 z^6 + \frac{1}{16}q\omega^4 \log(z)z^6 + \\ & \left. + \frac{1}{16}q\omega^4 \log(\omega)z^6 - \frac{1}{32}q\omega^4 \log(4)z^6 + \frac{1}{3}q\omega^2 z^4 \right] + \frac{1}{2}\omega^2 K_2(z\omega)C_2, \end{aligned}$$

$$\Phi_3 = \frac{1}{2}\omega^2 K_2(z\omega)C_7,$$

$$\Phi_4 = \frac{1}{2}\omega^2 K_2(z\omega)C_9,$$

$$\begin{aligned} \Phi_5 = & -\frac{1}{12}q\omega^2 C_1 z^6 + \frac{1}{6}q\omega^2 C_4 z^6 - qC_1 z^4 - \frac{8qI_2(z\omega)C_1 z^2}{(qz^4 + 1)\omega^2} - \frac{\omega^2 K_2(z\omega)C_1 z^2}{qz^4 + 1} + \\ & + \frac{4q^2 I_2(z\omega)C_6 z^2}{(qz^4 + 1)\omega^2} + \frac{q\omega^2 K_2(z\omega)C_6 z^2}{8(qz^4 + 1)}. \end{aligned} \tag{121}$$

The thermal version of the Liu–Tseytlin backgrounds leads to the following equations of motion:

$$\left\{ \begin{aligned} & \frac{(u(u^3 + 6u + 4\omega^2 + 4k^2(u^2 - 1)) - 3)\eta^+}{4u^2(u^2 - 1)^2} + \eta^{+\prime\prime} = 0, \\ & -4q(u^2 + 1)h(u)u^2 + 4(u^2 - 1)(2quh' + \pi^2(u^2 - 1)\eta^{-\prime\prime})u^2 + \\ & + \pi^2(u(u^3 + 6u + 4\omega^2 + 4k^2(u^2 - 1)) - 3)\eta^- = 0, \\ & 4\left(h''(u^2 - 1)^2 + 2\pi^2q(2u(u^2 - 1)\eta^{+\prime} - (u^2 + 1)\eta^+(u))\right)u^2 + \\ & + (u(u^3 + 6u + 4\omega^2 + 4k^2(u^2 - 1)) - 3)h = 0. \end{aligned} \right. \tag{122}$$

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