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INFINITELY MANY EQUIVALENT VERSIONS OF THE GRACEFUL TREE CONJECTURE

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A *graceful labeling of a graph* with q edges is a labeling of its vertices using the integers in $[0, q]$, such that no two vertices are assigned the same label and each edge is uniquely identified by the absolute difference between the labels of its endpoints. The well known Graceful Tree Conjecture (GTC) states that all trees are graceful, and it remains open. It was proved in 1999 by Broersma and Hoede that there is an equivalent conjecture for GTC stating that all trees containing a perfect matching are strongly graceful (graceful with an extra condition). In this paper we extend the above result by showing that there exist infinitely many equivalent versions of the GTC. Moreover we verify these infinitely many equivalent conjectures of GTC for trees of diameter at most 7. Among others we are also able to identify new graceful trees and in particular generalize the Δ-construction of Stanton-Zarnke (and later Koh-Rogers-Tan) for building graceful trees through two smaller given graceful trees.

1. INTRODUCTION AND BACKGROUND

Throughout this paper, by a graph we mean an undirected finite graph without multiple edges and loops. Moreover, we use standard graph theory terminologies and notations. Let G be a graph with q edges. A one-to-one function $f : V(G) \to \{0, 1, \dots, q\}$ from the vertex set $V(G)$ (if any) is said to be a *graceful labeling* of G, if the absolute value $|f(u)-f(v)|$ is assigned to the edge uv as its label and the resulting induced edge labels are pairwise distinct. This is equivalent to requiring the set of induced edge labels to be exactly $\{1, 2, \dots, q\}$. A graph admitting such a graceful labeling is called a *graceful graph*. Since the focus of this paper

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is on the graceful labeling of trees instead of that of general graphs, we rephrase the definition of graceful labeling for trees as follows: A tree T with p vertices and $p-1$ edges is called *graceful* if there exists a bijection f from the vertex set $V(T)$ onto $\{1, 2, \ldots, p\}$ such that the induced edge labels are exactly $1, 2, \ldots, p-1$, where the induced edge label for an edge uv is the absolute value of the difference of two end vertex labels, that is, $|f(u) - f(v)|$.

This graceful labeling problem was posed by Rosa [**10**] under the term *β-valuation*. The term graceful labeling was coined by Golomb [**5**]. Rosa introduced β-valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs, in particular, to address a conjecture of Ringel [**9**] which states that the complete graph K_{2n+1} can be decomposed into $2n+1$ subgraphs that are all isomorphic to a given tree with n edges. For this reason, Kotzig, Ringel, and Rosa raised the **Graceful Tree Conjecture (GTC)** (which implies the above conjecture of Ringel) that every tree is graceful [**10**]. This conjecture is one of most challenging problems in graph theory and it remains open. For a history of this problem as well as recent advancements, we refer the readers to [**3, 4**].

Among the trees known to be graceful are: caterpillars (a *caterpillar* is a tree with the property that the removal of its endpoints leaves a path); lobsters with a perfect matching (a *lobster* is a tree with the property that the removal of its endpoints leaves a caterpillar); trees with at most four end-vertices; trees with diameter at most 5; symmetrical trees (that is, a rooted tree in which every level contains vertices of the same degree); rooted trees where the roots having odd degree and the lengths of the paths from the root to the leaves differ by at most one and all the internal vertices have the same parity; rooted trees with diameter D where every vertex has even degree except for one root and the leaves in level $\left\lfloor \frac{D}{2} \right\rfloor$; trees having an even or quasi even degree sequence, etc. For more references about graceful graphs, readers may see the dynamic survey by J. Gallian [**4**] and more papers for graceful labeling in literature given in the references. We purposely keep the references short and we refer the readers to [**4**] for an extensive reference list.

One's initial approach to the Graceful Tree Conjecture might be to apply induction; that is, to decompose a tree into several smaller trees. However there is no known method to piece the graceful labeling functions of the small trees into a graceful labeling function of the original tree. Nevertheless, under special conditions, this may be possible. For example, Stanton and Zarnke [**11**] (and later Koh, Rogers and Tan [**7**]) gave a construction to form a bigger graceful tree from two given graceful trees if the two trees satisfy certain conditions. In this paper, we generalize this construction.

A tree T, containing a perfect matching M, is *strongly graceful* if T has a graceful labeling f such that $f(u) + f(v) = n + 1$ for every edge uv in M. Broersma and Hoede [**1**] conjectured that every tree containing a perfect matching is strongly graceful; moreover, they showed that this new conjecture is equivalent to the Graceful Tree Conjecture. We generalize the above result to obtain infinitely many equivalent versions of GTC in this article. In particular we raise a strongly graceful tree conjecture associated with a "gracefully similar system." Such a gracefully similar system is precisely the tool we need to extend the result of Stanton and Zarnke [**11**] that we mentioned earlier. It is worth noting, as pointed out to us by the anonymous referee, that the latest and most recent version of Δ-construction can be found in the paper of Burzio and Ferrarese [**2**]. Moreover, the referee also gave an interesting remark about the Δ-construction that we think is worthwhile to reproduce here as we think it will be beneficial to other researchers who are starting research in this area: "Although this paper ([**1**]) was published after that of Burzio and Ferrarese, Broersma and Hoede were probably not aware of this result because of the long wait times between review and publication at Ars Combinatoria, the journal to which they submitted their paper. It is likely that they wrote their paper before Burzio and Ferrarese. The reason for this inference is that the work of Broersma and Hoede is really another result about the Δ-construction. However, since this observation has not been noted in the literature until very recently, papers have been published on strongly graceful graphs (e.g. ...) in which the authors make no reference to and do not view how their results relate to the Δ-construction. If they did, perhaps they would see many simplifications of their proofs."

2. A CONJECTURE

Let (T, λ) be a fixed tree of order k with a given graceful labeling λ. Without loss of generality we may name the vertices via $\lambda(v_j) = j$ for $v_j \in V(T)$, $1 \leq j \leq k$, such that the differences $|\lambda(v_i) - \lambda(v_j)| = |i - j|$ are all distinct for $v_i v_j \in E(T)$. Later in this paper, we interchangeably use v_j and $\lambda(v_j) = j$, $1 \leq j \leq k$, to represent the vertices, that is, we refer to the vertices by their labels.

We define two graceful trees (T, λ) and (T', λ'), both of order k, to be *gracefully similar* if for bipartitions (A, B) and (A', B') for T and T' respectively, we have $\lambda(A) = \lambda'(A')$, $\lambda(B) = \lambda'(B')$ and moreover $\{\lambda(a) - \lambda(b) : a \in A, b \in B, ab \in E(T)\} = \{\lambda'(a') - \lambda'(b') : a' \in A', b' \in B', a'b' \in E(T')\}$. We denote $T \sim T'$ if T and T' are gracefully similar. We note that each tree has only one bipartition of the vertex set. Although $|\lambda(a) - \lambda(b)|$'s and $|\lambda'(a') - \lambda'(b')|$'s are the same since both λ and λ' are graceful labelings, we are requiring the two sets to be the same without the absolute value sign. Figure 1 gives an example of four gracefully similar trees of order 6 $\{(T_i, \lambda_i) \mid 1 \leq i \leq 4\}$ that are mutually similar with each other. Note that the sets $\lambda_i(A) = \{1, 2, 3\}$ and $\lambda_i(B) = \{4, 5, 6\}$ are the same, and the difference sets $\{\lambda_i(a) - \lambda_i(b) : a \in A, b \in B, ab \in E(T_i)\}$ are all the same one: $\{-1, -2, -3, -4, -5\}$, for each $1 \leq i \leq 4$.

We call $\{(T_k^i, \lambda_i) \mid 1 \leq i \leq n\}$ a *gracefully similar system*, if T_k^i is a tree of order k with a given graceful labeling λ_i for each $1 \leq i \leq n$, and $T_k^i \sim T_k^j$ for all $1 \leq i \neq j \leq n$. So the four trees in Figure 1 form a gracefully similar system.

We said earlier that one goal of this paper is to assemble a number of graceful labeled trees into a large graceful labeled tree. It turns out that the concept of gracefully similar system is important. We now define a concept that will be useful in such an assembling. We use the notation $d(H_1, H_2) = \min\{d(h_1, h_2) \mid h_1 \in V(H_1), h_2 \in V(H_2)\}$, we denote the distance for two subgraphs H_1 and H_2 of G. Let $\{(T_k^i, \lambda_i) \mid 1 \le i \le n\}$ be a fixed gracefully similar system. We say a tree T of order nk admits a *gracefully similar factor* associated with the gracefully similar system, if it contains $T_k^* = T_k^1 \cup T_k^2 \cup \cdots \cup T_k^n$ as a spanning forest, and also for each edge e in $E(T) - E(T_k^*)$ there exist T_k^s and T_k^t with $d(T_k^s, T_k^t) = 1$, where $1 \le s \ne t \le n$, and a unique r, $1 \le r \le k$, such that $e = v_{sr}v_{tr}$, where $v_{ij} \in V(T_k^i)$ is the j-th vertex in T_k^i defined via $\lambda_i(v_{ij}) = j$, for $1 \le i \le n$ and $1 \le j \le k$.

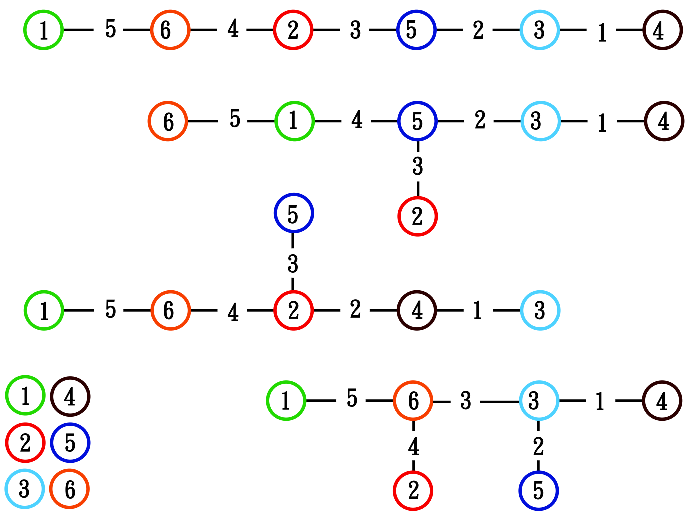

Figure 1. Four mutually gracefully similar trees of order 6

Although the definition of a gracefully similar factor seems complicated, it is actually quite natural. Perhaps it is best to describe the definition intuitively as T can be factored into $\{(T_k^i, \lambda_i) \mid 1 \le i \le n\}$ if T can be obtained by adding $n-1$ edges to "hook" up the n trees (T_k^i's) into one large tree such that each new edge uv is between two trees, say u in T_k^i and v in T_k^j with $\lambda_i(u) = \lambda_j(v)$. An example for a tree T of order nk which admits a *gracefully similar factor* associated with the gracefully similar system $\{(T_k^i, \lambda_i) \mid 1 \le i \le n\}$ is given in Figure 2, where $k = 6$ and $n = 4$. Note that it is built from four mutually gracefully similar trees of order 6 in Figure 1 and one has many different options for edges connecting these T_k^i's.

Let $\{(T_k^i, \lambda_i) \mid 1 \le i \le n\}$ be a gracefully similar system and $T_k^* = T_k^1 \cup T_k^2 \cup \cdots \cup T_k^n$ be a gracefully similar factor of T. We say a bijection f (vertex labeling) from the vertex set $V(T)$ onto $\{1, 2, \ldots, |V(T)|\}$ is a *strongly graceful labeling* of T if (1) f is a graceful labeling, and (2) f satisfies the following conditions:

$$f(v_{ij}) = j + (i-1)k \text{ if } v_{ij} \in A_i \text{ and } f(v_{ij}) = j + (n-i)k \text{ if } v_{ij} \in B_i$$

for $v_{ij} \in V(T_k^i) = A_i \cup B_i$, where $1 \le i \le n$ and $1 \le j \le k$. We note that (2) is a condition on assigning a label based on the labelings of the smaller tree. This can be seen as a possible way to assemble smaller graceful trees into a large tree. We remark that the ordering of the trees in a gracefully similar system is important. We further note this definition of strong gracefulness generalizes the one for a tree with a perfect matching that we mentioned previously. We are now ready for our conjecture.

Strongly Graceful Tree Conjecture (SGTC). *Let $k \geq 2$. For a fixed gracefully similar system $\{(T_k^i, \lambda_i) \mid 1 \leq i \leq n\}$, every nontrivial tree admitting a gracefully similar factor is strongly graceful, for each $k \geq 2$.*

We note that this conjecture implies that there are infinitely many equivalent versions of GTC, since T_k^i could be assigned as any given known graceful tree, say a path P_k, a caterpillar, etc.; in particular the case $T_k^i = P_2$ for all i coincides with previous result of Broersma and Hoede [**1**]. In the following sections we prove the equivalence of GTC and SGTC for T_k^i with $k \geq 2$, and also verify the SGTC for trees of diameter no more than a bound which is determined by the diameters of T_k^i's.

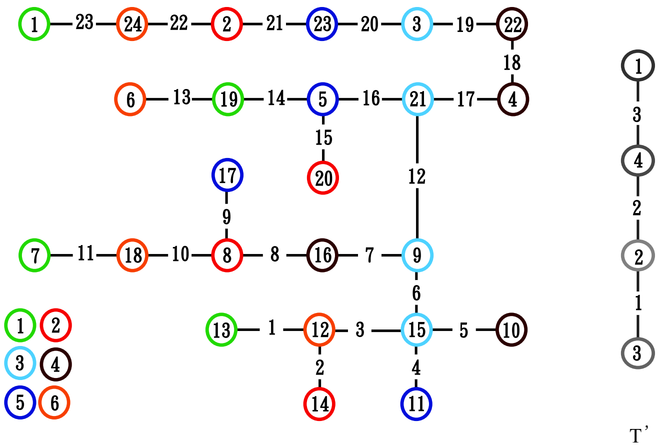

Figure 2. A strongly graceful tree built by four Gracefully Similar Trees of order 6

3. MAIN RESULTS

Let $\{(T_k^i, \lambda_i) \mid 1 \leq i \leq n\}$ be a gracefully similar system and $T_k^* = T_k^1 \cup T_k^2 \cup \cdots \cup T_k^n$ be a gracefully similar factor of T. Then T has nk vertices. Let T' be obtained from T by contracting each T_k^i into a vertex. Then T' is a tree of order n and it is called the *contraction tree* of T with respected to $\{(T_k^i, \lambda_i) \mid 1 \leq i \leq n\}$. In Figure 2, T' is a path as indicated. We remark that the edges of the contraction tree T' are in one-to-one correspondence with the edges in $E(T) - E(T_k^*)$. The next result will show that under certain conditions, we can assemble smaller graceful trees into a large graceful tree.

Theorem 3.1. *Let T be a tree of order nk admitting a gracefully similar factor based upon a gracefully similar system $\{(T_k^i, \lambda_i) \mid 1 \le i \le n\}$. If the contraction tree T' of T is graceful, then T is strongly graceful.*

Proof. With notations defined above, let f be a vertex labeling of T such that $f(v_{ij}) = j + (i-1)k$, if $v_{ij} \in A_i$, and $f(v_{ij}) = j + (n-i)k$, if $v_{ij} \in B_i$ for $1 \le i \le n$ and $1 \le j \le k$. To show f is strongly graceful, it suffices to show that f is graceful. Let the graceful labeling of T' be λ', and identify $u_i \in V(T')$ with T_k^i via $\lambda'(u_i) = i$ for each $1 \le i \le n$. For an edge $u_iu_j \in E(T')$ is in one-to-one correspondence with some unique edge $v_{im}v_{jm} \in E(T) - E(T_k^*)$ where $v_{im} \in V(T_k^i)$ and $v_{jm} \in V(T_k^j)$. Note that v_{im} and v_{jm} are either simultaneously in A_i and A_j respectively, or simultaneously in B_i and B_j respectively. Thus for edges in $E(T) - E(T_k^*)$, the induced edge labels $|f(v_{im}) - f(v_{jm})| = k|i-j|$ are distinct, since $|\lambda'(u_i) - \lambda'(u_j)| = |i-j|$ are distinct for edges $u_iu_j \in E(T')$. Also note that particularly the induced edge labels $|f(v_{im}) - f(v_{jm})|$ in $E(T) - E(T_k^*)$ are multiples of k.

On the other hand, we consider the remaining edge labels for edges in $E(T_k^*)$. Assume $v_{ia}v_{ib} \in E(T_k^i)$ and $v_{jc}v_{jd} \in E(T_k^j)$, where $v_{ia} \in A_i$, $v_{ib} \in B_i$, $v_{jc} \in A_j$, and $v_{jd} \in B_j$, for $1 \le i, j \le n$ and $1 \le a, b, c, d \le k$. We prove in the following that the induced edge labels are all distinct over the edges in T_k^*. Assume $(i, a, b) \ne (j, c, d)$. Suppose on the contrary that $|f(v_{ia}) - f(v_{ib})| = |f(v_{jc}) - f(v_{jd})|$. Then there are two possibilities as follows:

Case 1. $(n-2i+1)k + (b-a) = (n-2j+1)k + (d-c)$. Then after some algebraic simplifications, we have $2(j-i)k = -(b-a) + (d-c)$. However $-2(k-1) \le -(b-a) + (d-c) \le 2(k-1)$ and $-(b-a) + (d-c)$ is a multiple of $2k$. Thus $-(b-a) + (d-c) = 0$. So $i = j$, and hence we have $a = c$ and $b = d$ since $T_k^i = T_k^j$ are graceful. This gives a contradiction.

Case 2. $(n-2i+1)k + (b-a) = -[(n-2j+1)k + (d-c)]$. Again this reduces to $2(n-j-i+1)k = -[(b-a)+(d-c)] = 0$. However we claim that $(b-a)+(d-c) = 0$ is impossible. Otherwise, we have$(b-a) = -(d-c)$. But T_k^i and T_k^j are gracefully similar, which means that these two sets $I = \{-(a'-b') : v_{ia'} \in A_i, v_{ib'} \in B_i, v_{ia'}v_{ib'} \in E(T_k^i)\}$ and $J = \{-(c'-d') : v_{jc'} \in A_j, v_{jd'} \in B_j, v_{jc'}v_{jd'} \in E(T_k^j)\}$ are the same, one would then have $(d-c) \in J$ and $-(d-c) = (b-a) \in I = J$. Therefore both $(d-c) \in J$ and $-(d-c) \in J$, a contradiction. (If both $d-c$ and $-(d-c)$ are in J, then both T_k^i and T_k^j have an induced edge label $|d-c|$ twice.) Hence the claim is true. (The reader may observe that here it is enough to utilize the fact that T_k^i and T_k^j are graceful if $i + j = n+1$.)

In summary, we have $|f(v_{ia}) - f(v_{ib})| \ne |f(v_{jc}) - f(v_{jd})|$ whenever $(i, a, b) \ne (j, c, d)$, that is, the induced edge labels are all distinct for edges in $E(T_k^*)$. Furthermore, it is not hard to see that these induced edge labels are not multiples of k, since $|f(v_{i,a}) - f(v_{i,b})| = |(a-b) + (2i-n-1)k|$ and $-(k-1) \le (a-b) \le (k-1)$. Combining all cases above, we see that all induced edge labels are distinct. That is f is a graceful labeling, hence a strongly graceful labeling. □

We remark that by using Theorem 3.1, one may easily construct new (strongly) graceful graphs through smaller graceful trees which form a gracefully similar system $\{(T_k^i, \lambda_i) \mid 1 \leq i \leq n\}$ in the theorem; see Figure 2. Note that in case $T_k^i = T_k$ and $\lambda_i = \lambda$ for all $1 \leq i \leq n$, it reduces to the case of the Δ-construction created by Stanton and Zarnke [**11**] in 1973. On the other hand more general situations can be considered. Note that in the above proof, graceful similarity is not necessarily needed for all T_k^i's, but only needed for those T_k^i and T_k^j with $i+j = n+1$, as we have remarked earlier in Case 2 of the proof of Theorem 3.1. This means one may use more general graceful trees, which are not gracefully similar, to construct larger strongly graceful trees. We decide not to use the more general definition as it will make the definition more complicated. Instead, we simply make a remark regarding this. This also implies other results in this paper can be generalized accordingly with a similar fashion.

Remark. Theorem 3.1 can be strengthened by requiring two trees in the system to be gracefully similar if $i + j = n + 1$.

Conversely we have the following.

Theorem 3.2. *Let T be a tree of order nk admitting a gracefully similar factor based upon a gracefully similar system $\{(T_k^i, \lambda_i) \mid 1 \leq i \leq n\}$. If T is strongly graceful, then the contraction tree T' of the tree T is graceful.*

Proof. Let f be the strongly graceful labeling of T defined as above. Identify T_k^i with a vertex u_i in T' for each $1 \leq i \leq n$ and assign the label i to the vertex u_i in T', via $\lambda(u_i) = i$. As above for an edge $u_i u_j \in E(T')$ is in one-to-one correspondence with some unique edge $v_{im} v_{jm} \in E(T) - E(T_k^*)$ where $v_{im} \in V(T_k^i)$ and $v_{jm} \in V(T_k^j)$. Thus the induced edge label $|f(v_{im}) - f(v_{jm})| = k|i - j|$ are all distinct since f is strongly graceful. Therefore $|i-j| = |\lambda(u_i) - \lambda(u_j)|$ are all distinct, hence λ is graceful. ☐

We are in a position to state our second main result:

Theorem 3.3. *The Graceful Tree Conjecture* **GTC** *is equivalent to the Strongly Graceful Tree Conjecture* **SGTC** *for $k \geq 2$.*

Proof. (**GTC** ⇒ **SGTC**) Assume T is an arbitrary tree with a gracefully similar factor based upon a gracefully similar system $\{(T_k^i, \lambda_i) \mid 1 \leq i \leq n\}$. Then its contraction tree T' is a tree, which is graceful by **GTC**. Therefore by Theorem 3.1, we see T is strongly graceful.

(**SGTC** ⇒ **GTC**) Assume T is an arbitrary tree. For a gracefully similar system $\{(T_k^i, \lambda_i) \mid 1 \leq i \leq n\}$, consider the extension $\widetilde{T}$ of the tree T by attaching each T_k^i at each vertex of T over the corresponding positions, say attaching the first (with the vertex order determined by the graceful-ness) vertex of T_k^i if one will, which makes $\widetilde{T}$ to be a tree with a gracefully similar factor, and its contraction tree is T. By **SGTC** the extension $\widetilde{T}$ admits a strongly graceful labeling f, thus the contraction tree T is also graceful by Theorem 3.2.

REMARK. When $T_k^i = P_2$ for each i, the above Theorem reduces to previous result of BROERSMA and HOEDE [**1**], which is the equivalence of the GTC and the conjecture that every tree with a perfect matching is strongly graceful.

4. MORE EXAMPLES OF NEW GRACEFUL TREES

In this section we make use of the results established in the previous section to identify new classes of strongly graceful trees, hence graceful graphs. Note that for simplicity, we assume the gracefully similar system $\{(T_k^i, \lambda_i) \mid 1 \le i \le n\}$ is made of the same graceful tree (T_k, λ), that is, $T_k^i = T_k$ and $\lambda_i = \lambda$ for all $1 \le i \le n$. This is exactly the Δ-construction created by STANTON and ZARNKE [**11**] when $n = 2$. All results below can be generalized with different gracefully similar T_k^i's. Note that in such special case, gracefully similar factors are called *graceful T_k-factors* and strongly graceful is referred as *strongly T_k-graceful*.

First we start with the following definition, which generalizes the notion of caterpillars and lobsters: Let m be a non-negative integer. A tree T is called an *m-distant tree* if it becomes a path after at least m recursive steps of leaf removal, where one step of leaf removal for a tree T means removing all leaves from T. Therefore a 0-distant tree is a path, a 1-distant tree is a caterpillar (not a path), and a 2-distant tree is a lobster (neither a caterpillar nor a path). We note that every tree is an m-distant tree of some m.

Lemma 4.4. *Let T_k be a graceful tree with diameter $D(T_k)$, and T be a m-distant tree with a graceful T_k-factor. Let T' be the contraction m'-distant tree of T. Then $m \ge \left\lceil \frac{D(T_k)}{2} \right\rceil + m'$.*

Proof. Note that it takes at least $\left\lceil \frac{D(T_k)}{2} \right\rceil + 1$ steps of removing leaves from T, in order to remove one leaf of T'. Then we see that for T with a graceful T_k-factor, one needs at least $\left\lceil \frac{D(T_k)}{2} \right\rceil + 1 + (m' - 1) = \left\lceil \frac{D(T_k)}{2} \right\rceil + m'$ steps of removing leaves to make T become a path. Then $m \ge \left\lceil \frac{D(T_k)}{2} \right\rceil + m'$.

Theorem 4.5. *Let T_k be a graceful tree with diameter $D(T_k)$, and T be an m-distant tree with a graceful T_k-factor. If T is an m-distant tree for $m \le \left\lceil \frac{D(T_k)}{2} \right\rceil + 1$, then T is strongly T_k-graceful.*

Proof. It suffices to show that the contraction tree T' of the m-distant tree T is either 0 or 1-distant tree, since paths or caterpillars are known to be graceful, then by Theorem 3.1 we are done. Therefore we assume that the contraction tree T' is a m'-distant tree for $m' \ge 2$. By Lemma 4.4, one needs at least $\left\lceil \frac{D(T_k)}{2} \right\rceil + m' \ge \left\lceil \frac{D(T_k)}{2} \right\rceil + 2$ steps of removing leaves to make T become a path. This is a contradiction since T is an m-distant tree for $m \le \left\lceil \frac{D(T_k)}{2} \right\rceil + 1$. This completes the proof.

Remark. When $T_k = P_2$, and note also that $D(P_2) = 1$, the above theorem implies that all lobsters with a perfect matching is graceful, a result previously shown in [**8**].

In 2009 Yao et al. [**12**] showed that all trees admitting a perfect matching (that is admitting a graceful P_2-factor) of diameter $D \leq 5$ are strongly graceful. We improve the result with the following:

Theorem 4.6. *All trees admitting a perfect matching of diameter $D \leq 7$ is strongly graceful.*

In fact we prove a more general situation as follows, which verifies the Strongly Graceful Tree Conjecture (SGTC) for such trees with diameter no more than a bound determined by the diameter of T_k.

Lemma 4.7. *Let T be a tree with a graceful T_k-factor, and with diameter $D(T)$. Let T' be the contraction tree of T with diameter $D(T')$. Then $D(T) \leq 2\left\lceil \frac{D(T_k)}{2} \right\rceil + D(T')$.*

Proof. There exists a path P of length $D(T')$ in T'. Consider the pull back of the path P in T, its length is at least $2\left\lceil \frac{D(T_k)}{2} \right\rceil + D(T')$, where the part $2\left\lceil \frac{D(T_k)}{2} \right\rceil$ is contributed by looking at the pull back of the two end vertices (two copies of T_k's) of the path P. Then $D(T) \leq 2\left\lceil \frac{D(T_k)}{2} \right\rceil + D(T')$.

Theorem 4.8. *Let T be a tree with a graceful T_k-factor, and with diameter $D(T)$. Then T is strongly T_k-graceful if $D(T) \leq 2\left\lceil \frac{D(T_k)}{2} \right\rceil + 5$.*

Proof. It suffices to show by Theorem 3.1 that the diameter of the contraction tree T' satisfies $D(T') \leq 5$ since all trees of diameter no more than 5 is graceful as proved in [**6**]. Suppose that $D(T') \geq 6$, then by Lemma 4.7 $D(T) \geq 2\left\lceil \frac{D(T_k)}{2} \right\rceil + 6$, a contradiction.

Remark. Note that in case of $T_k = P_2$, the diameter $D(P_2) = 1$. Thus Theorem 4.8 reduces to Theorem 4.6, which generalizes previous result in [**12**].

We now consider a refinement of graceful labeling. Let T be a of a tree T of order k with A and B be its bipartition sets. An *α-labeling* T is a graceful labeling f such that $f(u) < f(v)$ for all $u \in A$ and $v \in B$; that is, $\max\{f(u) : u \in A\} < \min\{f(v) : v \in B\}$. In other words, there exists an integer m, $1 \leq m \leq k$, such that $f(A) = \{1, 2, \ldots, m\}$ and $f(B) = \{m+1, \ldots, k\}$. A tree T is *equitable* if for the bipartition of $V(T) = A \cup B$ one has that $||A| - |B|| \leq 1$. Finally a graceful T_k-factor is an **α-factor** if T_k is admitting an α-labeling. We now give a result that gives a relationship among these concepts.

Theorem 4.9. *Let T_k be a fixed tree of order k admitting an α-labeling. Assume that T is a tree of order nk with a graceful T_k-factor (an α-factor), and its contraction tree T' of order n admits an α-labeling. Let f be the associated strongly T_k-graceful labeling. Then T' is equitable if and only if f is an α-labeling.*

Proof. With notations defined above, let f be a vertex labeling of T such that $f(v_{ij}) = \lambda(v_j) + (i-1)k = j + (i-1)k$, if $v_{ij} \in A_i$, and $f(v_{ij}) = \lambda(v_j) + (n-i)k = j+(n-i)k$, if $v_{ij} \in B_i$ for $1 \le i \le n$ and $1 \le j \le k$. Let A_i and B_i be the bipartition set of $V(T_k^i)$. Also let the graceful labeling of T' be λ', and identify $u_i \in V(T')$ with T_k^i via $\lambda'(u_i) = i$ for each $1 \le i \le n$. Let A and B be the bipartition sets of $V(T_k)$, and A' and B' be the bipartition sets of $V(T')$. Since T_k and T' both admit an α-labeling, we assume λ and λ' be their graceful labelings respectively, and also there are two constants k_1 and k_2 such that $\{\lambda(u)|\ u \in A\} = \{1, 2, \ldots, k_1\}$, $\{\lambda(u)|\ u \in B\} = \{k_1+1, k_1+2, \ldots, k_1+k\}$, and $\{\lambda'(u)|\ u \in A'\} = \{1, 2, \ldots, k_2\}$, $\{\lambda'(u)|\ u \in B'\} = \{k_2+1, k_2+2, \ldots, n\}$ respectively. Without loss of generality, one may assume $|A'| \ge |B'|$. With notations defined here and that mentioned before, we see that $\lambda'(u_i) = i \le k_2$ for $u_i \in A'$ corresponding to T_k^i, hence

$$f(v_{ij}) = j + (i-1)k \le k_1 + (k_2-1)k, \text{if } v_{ij} \in A_i, \tag{1}$$

$$f(v_{ij}) = j + (n-i)k \ge (k_1+1) + (n-k_2)k, \text{if } v_{ij} \in B_i. \tag{2}$$

Also we see that $\lambda'(u_i) = i \ge k_2 + 1$ for $u_i \in B'$ corresponding to T_k^i, hence

$$f(v_{ij}) = j + (i-1)k \ge k_2k + 1, \text{if } v_{ij} \in A_i, \tag{3}$$

$$f(v_{ij}) = j + (n-i)k \le (n-k_2)k, \text{if } v_{ij} \in B_i. \tag{4}$$

From the above inequalities (1), (2), (3), and (4), we have the following:

$$\begin{aligned} &T' \text{ is equitable} \\ &\Longleftrightarrow 0 \le |A'| - |B'| \le 1 \\ &\Longleftrightarrow 0 \le k_2 - (n-k_2) \le 1 \\ &\Longleftrightarrow n - k_2 \ge k_2 - 1 \text{ and } k_2 \ge n - k_2 \\ &\Longleftrightarrow \max\{k_1 + (k_2-1)k, (n-k_2)k\} < \min\{(k_1+1) + (n-k_2)k, k_2k+1\} \\ &\Longleftrightarrow \max\{f(v_{ij})|\ v_{ij} \in A^*\} < \min\{f(v_{ij})|\ v_{ij} \in B^*\}, \end{aligned}$$

where $A^* = \{v_{ij}|\ (v_{ij} \in A_i) \wedge (u_i \in A'), \text{ or } (v_{ij} \in B_i) \wedge (u_i \in B'), 1 \le i \le n, 1 \le j \le k\}$ and $B^* = \{v_{ij}|\ (v_{ij} \in B_i) \wedge (u_i \in A'), \text{ or } (v_{ij} \in A_i) \wedge (u_i \in B'), 1 \le i \le n, 1 \le j \le k\}$.

Thus T' is equitable is equivalent to T admits an α-labeling.

5. CONCLUDING REMARKS

In this article we give infinitely many equivalent versions of the Graceful Tree Conjecture as well as a scheme to assemble small graceful trees into a larger graceful tree under certain conditions. As a byproduct it is easy to identify new classes of graceful graphs by using the gracefully similar systems built upon known smaller graceful trees, which are gracefully similar. It would be interesting to explore and identify more related concepts and relationships among them. For example, it would be nice to try to figure out the specific conditions when a strongly graceful graph admits α-valuations, σ-valuations, ρ-valuations, which are hierarchically related to graceful labelings (β-valuations). Also like many variants of graceful labelings in literature, one may also study similar variants for strongly graceful labelings. A final note is that, it is still a puzzle to us that, what conditions would be needed to show that lobsters are graceful or more generally m-distant trees, $m \geq 3$ with a graceful T_k-factor are graceful.

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