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# Smooth Solution to the 1-Dimensional Spin Equations of Antiferromagnets

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**Abstract** In this paper, the global existence and uniqueness of a smooth solution to the periodic initial-value problem of the spin equations of antiferromagnets in 1 dimension are proved.

**Keywords** Spin equations of antiferromagnets, Smooth solutions, Existence, Uniqueness
**MR(2000) Subject Classification** 35D10, 35Q80

## 1 Introduction

Most crystals have magnetically ordered structures. This means that in the absence of an external magnetic field, the mean magnetic moment of at least one of the atoms in each unit cell of the crystal is non-zero.

In the simplest type of magnetically ordered crystals, i.e., ferromagnets such as Fe, Ni, Co and Dy, the mean magnetic moments of all atoms have the same orientation provided that the temperature of the ferromagnet does not exceed a critical value, i.e., the Curie temperature.

The equation of motion for the magnetic moment $\vec{m}$ is

$$\vec{m}_t = \vec{m} \times \Delta\vec{m}.$$

The corresponding spin equation is

$$\vec{m}_t = \Delta\vec{m} + \vec{m} \times \Delta\vec{m}.$$

The equation with Gilbert damping reads as

$$\vec{m}_t = \vec{m} \times (\vec{m} \times \Delta\vec{m}) + \vec{m} \times \Delta\vec{m}.$$

The above equations have been widely discussed by physicists and mathematicians (see [1–4]). Recently the discussions have been extended to the inhomogeneous Heisenberg chain

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equations [5] and the compressible Heisenberg chain equations [6]. These models were introduced in [1, 7–9].

In antiferromagnets, the mean atomic magnetic moments compensate each other within each unit cell (in zero external magnetic field). In other words, an antiferromagnet consists of a set of sublattices (called magnetic sublattices), each of which has a non-zero mean magnetic moment provided that the temperature of the antiferromagnet is less than a critical temperature, known as the Néel temperature.

Without taking dissipation into account, the equations of motion for the magnetizations $\vec{m}_1$ and $\vec{m}_2$ of the two magnetic sublattics read as follows (see [1] or [3]):

$$\begin{cases} \vec{m}_t = 2k_1\vec{m}\times\Delta\vec{m} + k_{11}\vec{m}\times\Delta\vec{n}, \\ \vec{n}_t = 2k_2\vec{n}\times\Delta\vec{n} + k_{22}\vec{n}\times\Delta\vec{m}. \end{cases}$$

The equations of motion for the spin wave of antiferromagnets take the following form (see also [1] or [3]):

$$\begin{cases} \vec{m}_t = \Delta\vec{m} + 2k_1\vec{m}\times\Delta\vec{m} + k_{11}\vec{m}\times\Delta\vec{n}, \\ \vec{n}_t = \Delta\vec{n} + 2k_2\vec{n}\times\Delta\vec{n} + k_{22}\vec{n}\times\Delta\vec{m}, \end{cases}$$

where $k_1, k_2, k_{11}, k_{22}$ are constants. Note that in the above equations, $\vec{m}, \vec{n}$ are both three-dimensional vectors. Using the Galerkin method, we proved in [10] that the above equations with periodic initial values in $d$-dimensions admit at least one global weak solution. Moreover, we also gave some regularity results for the weak solutions in [10].

In this paper, we intend to establish the existence and uniqueness of the smooth solution to the following periodic initial-value problem of spin equations of antiferromagnets:

$$\begin{cases} \vec{m}_t = \vec{m}_{xx} + 2k_1\vec{m}\times\vec{m}_{xx} + k_{11}\vec{m}\times\vec{n}_{xx}, & (x,t)\in\Omega\times\mathbb{R}_+, \\ \vec{n}_t = \vec{n}_{xx} + 2k_2\vec{n}\times\vec{n}_{xx} + k_{22}\vec{n}\times\vec{m}_{xx}, & (x,t)\in\Omega\times\mathbb{R}_+, \end{cases} \tag{1.1}$$

with conditions

$$\begin{cases} \vec{m}(x,0) = \vec{m}_0(x), & \vec{n}(x,0) = \vec{n}_0(x), \quad x\in\Omega, \\ \vec{m}(x+2D,t) = \vec{m}(x,t), & (x,t)\in\mathbb{R}^1\times\mathbb{R}_+, \\ \vec{n}(x+2D,t) = \vec{n}(x,t), & (x,t)\in\mathbb{R}^1\times\mathbb{R}_+, \\ \|\vec{m}_0(x)\|_{C^\infty(\Omega)} \le K, & \|\vec{n}_0(x)\|_{C^\infty(\Omega)} \le K, \end{cases} \tag{1.2}$$

where $D > 0$, $\Omega\subset\mathbb{R}^1$ is an internal with width $2D$. $\vec{m}_0(x+2D) = \vec{m}_0(x)$, $\vec{n}_0(x+2D) = \vec{n}_0(x)$. We assume, in this paper, that $k_1 \ne 0, k_2 \ne 0$, $\frac{k_{11}}{2k_1} = \frac{k_{22}}{2k_2}$ and $0 < \alpha < 1$, where $\alpha = \frac{k_{11}}{2k_1} = \frac{k_{22}}{2k_2}$.

## 2 Existence of Local Smooth Solutions

To get the existence of local smooth solution of (1.1)–(1.2), we apply the difference method. We need the following well-known lemmas:

**Lemma 2.1** [4] *Let $q, r$ be real numbers and $j, m$ be integers such that $1 \le q, r \le \infty$, $0 \le j < m$. If $u \in W^{m,r}(\Omega)\cap L^q(\Omega)$, then*

$$\|D^j u\|_p \le C\|u\|_q^{1-\alpha}\|D^m u\|_r^{\alpha},$$

*where* $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $p \ge 1$, $\frac{j}{m} \le \alpha \le 1$ *and*

$$\frac{1}{p} - j = \frac{1-\alpha}{q} + \alpha\Big(\frac{1}{r} - m\Big), \quad \Omega \subset R^1.$$

**Lemma 2.2** [11] *Let $p$ be a real number and $j, m$ be integers such that $2 \le p \le \infty$, $0 \le j < m$. Then*

$$\|\delta^j u_h\|_p \le C\|u_h\|_2^{1-\alpha}\left(\|\delta^m u_h\|_2 + \frac{\|u_h\|_2}{(2D)^m}\right)^{\alpha},$$

*where* $u_h = \{u_j = u(x_j) \mid j = 0,1,2,\ldots,J\}$, $x_j = jh$, $h = 2D/J$, $\alpha = \frac{1}{m}(j + \frac{1}{2} - \frac{1}{p})$,

$$\|\delta^k u_h\|_p = \left(\sum_{i=0}^{J-k}\left|\frac{\Delta_+^k u_i}{h^k}\right|^p h\right)^{\frac{1}{p}}.$$

**Lemma 2.3** [11] *Let* $u_h = \{u_j \mid j = 0, \pm1, \pm2, \ldots, \pm J, \ldots\}$, $v_h = \{v_j \mid j = 0, \pm1, \pm2, \ldots, \pm J, \ldots\}$ *such that* $u_{j+J} = u_j$, $v_{j+J} = v_j$. *We have:*

$$\text{(i)} \quad \sum_{j=0}^{J-1} u_j\Delta_+ v_j = -\sum_{j=1}^{J} v_j \Delta_- u_j,$$

$$\text{(ii)} \quad \sum_{j=1}^{J} u_j \Delta_+\Delta_- v_j = -\sum_{j=0}^{J-1}\Delta_+ u_j \Delta_+ v_j,$$

$$\text{(iii)} \quad \Delta_+(u_j v_j) = u_{j+1}\Delta_+ v_j + v_j \Delta_+ u_j,$$

*where $\Delta_+$, $\Delta_-$ denote the forward and backward difference, respectively.*

We use the difference method to prove the local existence of a smooth solution of (1.1)–(1.2). We establish the following difference-differential equation:

$$\begin{cases} \dfrac{d\vec{m}_j}{dt} = \dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2} + 2k_1\vec{m}_j \times \dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2} + k_{11}\vec{m}_j \times \dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}, \\ \dfrac{d\vec{n}_j}{dt} = \dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2} + 2k_2\vec{n}_j \times \dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2} + k_{22}\vec{n}_j \times \dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}, \end{cases} \tag{2.1}$$

with

$$\vec{m}_j|_{t=0} = \vec{m}_{0j} = \vec{m}_0(jh), \quad \vec{n}_j|_{t=0} = \vec{n}_{0j} = \vec{n}_0(jh), \tag{2.2}$$

$$\vec{m}_{j+J} = \vec{m}_j, \quad \vec{n}_{j+J} = \vec{n}_j, \tag{2.3}$$

where $j = 0, \pm1, \ldots, \pm J, \ldots$, $h = 2D/J$, $J > 0$.

It is clear that the initial-value problem for the ordinary differential equation (2.1)–(2.3) admits a local smooth solution. For such a solution, we shall give some estimates uniformly in $h$ and then get a local smooth solution to problem (1.1)–(1.2). In this section we always denote a local smooth solution of (2.1)–(2.3) by $(\vec{m}_j, \vec{n}_j)$, which exists in $[0, T_0]$.

**Lemma 2.4** *If $\vec{m}_0(x), \vec{n}_0(x) \in L^2(\Omega)$, then there are constants $C > 0$, $0 < T \le T_0$ independent of $h$ such that*

$$\sup_{0\le t\le T}[\|\vec{m}_h(t)\|_2^2 + \|\vec{n}_h(t)\|_2^2] \le C, \tag{2.4}$$

$$\int_0^T [\|\delta\vec{m}_h(t)\|_2^2 + \|\delta\vec{n}_h(t)\|_2^2] \le C. \tag{2.5}$$

*Proof* Multiplying the first equation of (2.1) by $\vec{m}_j h$, the second by $\vec{n}_j h$ and summing from $j=1$ to $J$, we have

$$\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{J}[|\vec{m}_j|^2 h + |\vec{n}_j|^2 h] + \sum_{j=0}^{J-1}\left|\frac{\Delta_+\vec{m}_j}{h}\right|^2 h + \sum_{j=0}^{J-1}\left|\frac{\Delta_+\vec{n}_j}{h}\right|^2 h = 0.$$

This gives

$$\frac{1}{2}\frac{d}{dt}[\|\vec{m}_h\|_2^2 + \|\vec{n}_h\|_2^2] + [\|\delta\vec{m}_h\|_2^2 + \|\delta\vec{n}_h\|_2^2] = 0, \tag{2.6}$$

and then yields (2.4)–(2.5).

**Lemma 2.5** *If $\vec{m}_0(x), \vec{n}_0(x) \in H^1(\Omega)$, then there are constants $C > 0$, $0 < T \le T_0$ independent of $h$ such that*

$$\sup_{0\le t\le T}[\|\delta\vec{m}_h(t)\|_2^2 + \|\delta\vec{n}_h(t)\|_2^2] \le C, \tag{2.7}$$

$$\int_0^T[\|\delta^2\vec{m}_h(t)\|_2^2 + \|\delta^2\vec{n}_h(t)\|_2^2] \le C. \tag{2.8}$$

*Proof* Multiplying the first equation of (2.1) by $\frac{\Delta_+\Delta_-\vec{m}_j}{h}$, the second by $\frac{\Delta_+\Delta_-\vec{n}_j}{h}$ and summing from $j=1$ to $J$, we have

$$\begin{cases} \sum\limits_{j=1}^{J}\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\cdot\dfrac{d\vec{m}_j}{dt}h \\ \quad = \sum\limits_{j=1}^{J}\left|\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\right|^2 h + k_{11}\sum\limits_{j=1}^{J}\left(\vec{m}_j \times \dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\right)\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}h, \\ \sum\limits_{j=1}^{J}\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\cdot\dfrac{d\vec{n}_j}{dt}h \\ \quad = \sum\limits_{j=1}^{J}\left|\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\right|^2 h + k_{22}\sum\limits_{j=1}^{J}\left(\vec{n}_j \times \dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\right)\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}h. \end{cases} \tag{2.9}$$

Multiplying the first equation of (2.1) by $\frac{\Delta_+\Delta_-\vec{n}_j}{h}$, the second by $\frac{\Delta_+\Delta_-\vec{m}_j}{h}$ and summing from $j=1$ to $J$, we have

$$\begin{cases} \sum\limits_{j=1}^{J}\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\cdot\dfrac{d\vec{m}_j}{dt}h \\ \quad = \sum\limits_{j=1}^{J}\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}h - 2k_1\sum\limits_{j=1}^{J}\left(\vec{m}_j \times \dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\right)\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}h, \\ \sum\limits_{j=1}^{J}\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\cdot\dfrac{d\vec{n}_j}{dt}h \\ \quad = \sum\limits_{j=1}^{J}\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}h - 2k_2\sum\limits_{j=1}^{J}\left(\vec{n}_j \times \dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\right)\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}h. \end{cases} \tag{2.10}$$

(2.10) gives

$$\begin{cases} \sum_{j=1}^{J}\Big(\vec m_j\times\frac{\Delta_+\Delta_-\vec n_j}{h^2}\Big)\frac{\Delta_+\Delta_-\vec m_j}{h^2}h \\ \qquad =\frac{1}{2k_1}\Big[\sum_{j=1}^{J}\frac{\Delta_+\Delta_-\vec m_j}{h^2}\frac{\Delta_+\Delta_-\vec n_j}{h^2}h-\sum_{j=1}^{J}\frac{\Delta_+\Delta_-\vec n_j}{h^2}\cdot\frac{d\vec m_j}{dt}h\Big], \\ \sum_{j=1}^{J}\Big(\vec n_j\times\frac{\Delta_+\Delta_-\vec m_j}{h^2}\Big)\frac{\Delta_+\Delta_-\vec n_j}{h^2}h \\ \qquad =\frac{1}{2k_2}\Big[\sum_{j=1}^{J}\frac{\Delta_+\Delta_-\vec m_j}{h^2}\frac{\Delta_+\Delta_-\vec n_j}{h^2}h-\sum_{j=1}^{J}\frac{\Delta_+\Delta_-\vec m_j}{h^2}\cdot\frac{d\vec n_j}{dt}h\Big]. \end{cases} \tag{2.11}$$

Substituting (2.11) into (2.9), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[\sum_{j=0}^{J-1}\Big(\Big|\frac{\Delta_+\vec m_j}{h}\Big|^2h\Big)+\sum_{j=0}^{J-1}\Big(\Big|\frac{\Delta_+\vec n_j}{h}\Big|^2h\Big)+\alpha\sum_{j=0}^{J-1}\frac{\Delta_+\vec m_j}{h}\frac{\Delta_+\vec n_j}{h}h\Big] \\ &\quad+\sum_{j=1}^{J}\Big|\frac{\Delta_+\Delta_-\vec m_j}{h^2}\Big|^2h+\sum_{j=1}^{J}\Big|\frac{\Delta_+\Delta_-\vec n_j}{h^2}\Big|^2h \\ &=-2\alpha\sum_{j=1}^{J}\frac{\Delta_+\Delta_-\vec m_j}{h^2}\cdot\frac{\Delta_+\Delta_-\vec n_j}{h^2}h \\ &\le\alpha\Big[\sum_{j=1}^{J}\Big|\frac{\Delta_+\Delta_-\vec m_j}{h^2}\Big|^2h+\sum_{j=1}^{J}\Big|\frac{\Delta_+\Delta_-\vec n_j}{h^2}\Big|^2h\Big]. \end{aligned} \tag{2.12}$$

Since $0<\alpha<1$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[(\|\delta\vec m_h\|_2^2+\|\delta\vec n_h\|_2^2)+\alpha\sum_{j=0}^{J-1}\frac{\Delta_+\vec m_j}{h}\cdot\frac{\Delta_+\vec n_j}{h}h\Big] \\ &\quad+(1-\alpha)[\|\delta^2\vec m_h\|_2^2+\|\delta^2\vec n_h\|_2^2]\le 0. \end{aligned} \tag{2.13}$$

Integrating (2.13) over $[0,t]$ $(0\le t\le T)$, we obtain

$$\begin{aligned} &\frac{1}{2}(\|\delta\vec m_h\|_2^2+\|\delta\vec n_h\|_2^2)+(1-\alpha)\int_0^t(\|\delta^2\vec m_h\|_2^2+\|\delta^2\vec n_h\|_2^2)dt \\ &\le\alpha\sum_{j=0}^{J-1}\frac{\Delta_+\vec m_j}{h}\cdot\frac{\Delta_+\vec n_j}{h}h\Big|_t^0+\frac{1}{2}(\|\delta\vec m_{0h}\|_2^2+\|\delta\vec n_{0h}\|_2^2) \\ &\le\frac{1+\alpha}{2}(\|\delta\vec m_{0h}\|_2^2+\|\delta\vec n_{0h}\|_2^2)-\frac{\alpha}{2}(\|\delta\vec m_h\|_2^2+\|\delta\vec n_h\|_2^2). \end{aligned} \tag{2.14}$$

Therefore we have

$$\frac{1+\alpha}{2}(\|\delta\vec m_h\|_2^2+\|\delta\vec n_h\|_2^2)+(1-\alpha)\int_0^t(\|\delta^2\vec m_h\|_2^2+\|\delta^2\vec n_h\|_2^2)dt\le C. \tag{2.15}$$

**Corollary 2.1** *We have, for some constant $C>0$, independent of $h$,*

$$\sup_{0\le t\le T;1\le j\le J}[|\vec m_j|+|\vec n_j|]\le C. \tag{2.16}$$

**Lemma 2.6** *If $\vec m_0(x),\vec n_0(x)\in H^2(\Omega)$, then there are constants $C>0$, $0<T\le T_0$ indepen-*

*dent of h such that*

$$\sup_{0\le t\le T}[\|\delta^2\vec{m}_h(t)\|_2^2+\|\delta^2\vec{n}_h(t)\|_2^2]\le C, \tag{2.17}$$

$$\int_0^T[\|\delta^3\vec{m}_h(t)\|_2^2+\|\delta^3\vec{n}_h(t)\|_2^2]\le C. \tag{2.18}$$

*Proof* It follows from (2.1) that

$$\begin{cases} \dfrac{d\Delta_+\vec{m}_j}{dt}=\dfrac{\Delta_+^2\Delta_-\vec{m}_j}{h^2} \\ \qquad +2k_1\Delta_+\Big(\vec{m}_j\times\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)+k_{11}\Delta_+\Big(\vec{m}_j\times\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big), \\ \dfrac{d\Delta_+\vec{n}_j}{dt}=\dfrac{\Delta_+^2\Delta_-\vec{n}_j}{h^2} \\ \qquad +2k_2\Delta_+\Big(\vec{n}_j\times\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)+k_{22}\Delta_+\Big(\vec{n}_j\times\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big). \end{cases} \tag{2.19}$$

Multiplying the first equation of (2.19) by $\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}$, the second by $\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}$ and summing from $j=1$ to $J$, we have

$$\begin{cases} \sum\limits_{j=1}^J\dfrac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}\cdot\dfrac{d\Delta_+\vec{m}_j}{dt} \\ \quad =\sum\limits_{j=1}^J\Big|\dfrac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}\Big|^2h+2k_1\sum\limits_{j=1}^J\Delta_+\Big(\vec{m}_j\times\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\dfrac{\Delta_+^2\Delta_-\vec{m}_j}{h^3} \\ \qquad +k_{11}\sum\limits_{j=1}^J\Delta_+\Big(\vec{m}_j\times\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\dfrac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}, \\ \sum\limits_{j=1}^J\dfrac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}\cdot\dfrac{d\Delta_+\vec{n}_j}{dt} \\ \quad =\sum\limits_{j=1}^J\Big|\dfrac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}\Big|^2h+2k_2\sum\limits_{j=1}^J\Delta_+\Big(\vec{n}_j\times\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\dfrac{\Delta_+^2\Delta_-\vec{n}_j}{h^3} \\ \qquad +k_{22}\sum\limits_{j=1}^J\Delta_+\Big(\vec{n}_j\times\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\dfrac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}. \end{cases} \tag{2.20}$$

Hence, we have

$$\begin{cases} \dfrac12\dfrac{d}{dt}\sum\limits_{j=0}^{J-1}\Big|\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big|^2h+\sum\limits_{j=1}^J\Big|\dfrac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}\Big|^2h \\ \quad =-2k_1\sum\limits_{j=1}^J\Big(\dfrac{\Delta_+\vec{m}_j}{h}\times\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\dfrac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}h \\ \qquad -k_{11}\sum\limits_{j=1}^J\Big(\dfrac{\Delta_+\vec{m}_j}{h}\times\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\dfrac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}h \\ \qquad -k_{11}\sum\limits_{j=1}^J\Big(\vec{m}_j\times\dfrac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}\Big)\dfrac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}h, \\ \dfrac12\dfrac{d}{dt}\sum\limits_{j=0}^{J-1}\Big|\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big|^2h+\sum\limits_{j=1}^J\Big|\dfrac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}\Big|^2h \\ \quad =-2k_2\sum\limits_{j=1}^J\Big(\dfrac{\Delta_+\vec{n}_j}{h}\times\dfrac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\dfrac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h \\ \qquad -k_{22}\sum\limits_{j=1}^J\Big(\dfrac{\Delta_+\vec{n}_j}{h}\times\dfrac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\dfrac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h \\ \qquad -k_{22}\sum\limits_{j=1}^J\Big(\vec{n}_j\times\dfrac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}\Big)\dfrac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h. \end{cases} \tag{2.21}$$

Multiplying the first equation of (2.19) by $\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}$, the second by $\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}$ and summing from $j=1$ to $J$, we have

$$\begin{cases} \sum_{j=1}^{J}\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}\cdot\frac{d\Delta_+\vec{m}_j}{dt} \\ \quad =\sum_{j=1}^{J}\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^2}\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}+2k_1\sum_{j=1}^{J}\Delta_+\Big(\vec{m}_j\times\frac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3} \\ \qquad +k_{11}\sum_{j=1}^{J}\Delta_+\Big(\vec{m}_j\times\frac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}, \\ \sum_{j=1}^{J}\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}\cdot\frac{d\Delta_+\vec{n}_j}{dt} \\ \quad =\sum_{j=1}^{J}\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^2}\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}+2k_2\sum_{j=1}^{J}\Delta_+\Big(\vec{n}_j\times\frac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3} \\ \qquad +k_{22}\sum_{j=1}^{J}\Delta_+\Big(\vec{n}_j\times\frac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}. \end{cases} \tag{2.22}$$

That is,

$$\begin{cases} \sum_{j=1}^{J}\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}\cdot\frac{d}{dt}\frac{\Delta_+\vec{m}_j}{h}h \\ \quad =\sum_{j=1}^{J}\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h+2k_1\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{m}_j}{h}\times\frac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h \\ \qquad -2k_1\sum_{j=1}^{J}\Big(\vec{m}_j\times\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}\Big)\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}h \\ \qquad +k_{11}\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{m}_j}{h}\times\frac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h, \\ \sum_{j=1}^{J}\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}\cdot\frac{d}{dt}\frac{\Delta_+\vec{n}_j}{h}h \\ \quad =\sum_{j=1}^{J}\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h+2k_2\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{n}_j}{h}\times\frac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}h \\ \qquad -2k_2\sum_{j=1}^{J}\Big(\vec{n}_j\times\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}\Big)\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h \\ \qquad +k_{22}\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{n}_j}{h}\times\frac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}h. \end{cases} \tag{2.23}$$

Combining (2.21) with (2.23) we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\delta^2\vec{m}_h\|_2^2+\|\delta^2\vec{n}_h\|_2^2)+(\|\delta^3\vec{m}_h\|_2^2+\|\delta^3\vec{n}_h\|_2^2) \\ &=-\alpha\frac{d}{dt}\sum_{j=0}^{J-1}\frac{\Delta_+\Delta_-\vec{m}_j}{h^2}\cdot\frac{\Delta_+\Delta_-\vec{n}_j}{h^2}h-2\alpha\sum_{j=1}^{J}\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}\cdot\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h \\ &\quad-2k_1\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{m}_j}{h}\times\frac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}h-2k_2\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{n}_j}{h}\times\frac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h \\ &\quad-k_{11}\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{m}_j}{h}\times\frac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}h-k_{22}\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{n}_j}{h}\times\frac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h \end{aligned}$$

$$
\begin{aligned}
&-k_{11}\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{m}_j}{h}\times\frac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h-k_{22}\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{n}_j}{h}\times\frac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}h\\
&-\alpha k_{11}\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{m}_j}{h}\times\frac{\Delta_+\Delta_-\vec{n}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{n}_j}{h^3}h\\
&-\alpha k_{22}\sum_{j=1}^{J}\Big(\frac{\Delta_+\vec{n}_j}{h}\times\frac{\Delta_+\Delta_-\vec{m}_j}{h^2}\Big)\frac{\Delta_+^2\Delta_-\vec{m}_j}{h^3}h. \qquad (2.24)
\end{aligned}
$$

It follows from (2.24) that

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}(\|\delta^2\vec{m}_h\|_2^2+\|\delta^2\vec{n}_h\|_2^2)+\alpha\sum_{j=0}^{J-1}\frac{\Delta_+\Delta_-\vec{m}_j}{h^2}\cdot\frac{\Delta_+\Delta_-\vec{n}_j}{h^2}h\right]\\
&\quad+\frac{1-\alpha}{2}(\|\delta^3\vec{m}_h\|_2^2+\|\delta^3\vec{n}_h\|_2^2)\\
&\le C+C\max_{1\le j\le J}\left\{\Big|\frac{\Delta_+\vec{m}_j}{h}\Big|^2,\Big|\frac{\Delta_+\vec{n}_j}{h}\Big|^2\right\}\left[\|\delta^2\vec{m}_h\|_2^2+\|\delta^2\vec{n}_h\|_2^2\right]. \qquad (2.25)
\end{aligned}
$$

Using Lemma 2.2 and Lemma 2.5 we have

$$
\|\delta\vec{m}_h\|_\infty^2\le C\|\delta\vec{m}_h\|_2[\|\delta^2\vec{m}_h\|_2^2+\|\delta^2\vec{n}_h\|_2^2]^{\frac{1}{2}}\le C[\|\delta^2\vec{m}_h\|_2^2+\|\delta^2\vec{n}_h\|_2^2]^{\frac{1}{2}}. \qquad (2.26)
$$

Inserting (2.26) into (2.25), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}(\|\delta^2\vec{m}_h\|_2^2+\|\delta^2\vec{n}_h\|_2^2)+\alpha\sum_{j=0}^{J-1}\frac{\Delta_+\Delta_-\vec{m}_j}{h^2}\cdot\frac{\Delta_+\Delta_-\vec{n}_j}{h^2}h\right]\\
&\quad+\frac{1-\alpha}{2}(\|\delta^3\vec{m}_h\|_2^2+\|\delta^3\vec{n}_h\|_2^2)\\
&\le C+C\left[\|\delta^2\vec{m}_h\|_2^2+\|\delta^2\vec{n}_h\|_2^2\right]^{\frac{3}{2}}. \qquad (2.27)
\end{aligned}
$$

(2.27) and the Gronwall inequality yield the conclusion of the lemma.

**Corollary 2.2** *Under the conditions in Lemma 2.6, we have, for some $C>0$, $0<T\le T_0$ independent of $h$,*

$$
\sup_{0\le t\le T;1\le j\le J}\left(\Big|\frac{\Delta_+\vec{m}_j(t)}{h}\Big|+\Big|\frac{\Delta_+\vec{n}_j(t)}{h}\Big|\right)\le C, \qquad (2.28)
$$

$$
\int_0^T(\|\delta\vec{m}_{ht}(t)\|_2^2+\|\delta\vec{n}_{ht}(t)\|_2^2)dt\le C. \qquad (2.29)
$$

By the induction method we have:

**Lemma 2.7** *If $\vec{m}_0(x),\vec{n}_0(x)\in H^k(\Omega)$, then there are constants $0<T\le T_0$, $C>0$ independent of $h$ such that*

$$
\sup_{0\le t\le T}(\|\delta^k\vec{m}_h\|_2+\|\delta^k\vec{n}_h\|_2)\le C, \qquad (2.30)
$$

$$\sup_{0\le t\le T}(\|\delta^{k-2}\vec{m}_{ht}\|_2+\|\delta^{k-2}\vec{n}_{ht}\|_2)\le C,\quad (k\ge 2),\tag{2.31}$$

$$\sup_{0\le t\le T}(\|\delta^{k-4}\vec{m}_{htt}\|_2+\|\delta^{k-4}\vec{n}_{htt}\|_2)\le C,\quad (k\ge 4).\tag{2.31}$$

From Lemma 2.7, the *a priori* estimates for solutions to the difference differential equation (2.1)–(2.3), using the method in [4], we conclude that there exists a constant $T>0$ such that problem (1.1)–(1.2) admits a smooth solution in $R^1\times[0,T]$ $(0<T\le T_0)$. That is:

**Theorem 2.1** *Let $\vec{m}_0(x),\vec{n}_0(x)\in H^k(\Omega)$. Then (1.1)–(1.2) admits a local smooth solution $(\vec{m}(x,t),\vec{n}(x,t))$ in $[0,T]$ with $T$ depending on $k$*

$$\vec{m}(x,t),\vec{n}(x,t)\in\Big(\bigcap_{s=0}^{[\frac{k}{2}]}W^s_\infty(0,T;H^{k-2s}(\Omega))\Big)\bigcap\Big(\bigcap_{s=0}^{[\frac{k+1}{2}]}H^s(0,T;H^{k+1-2s}(\Omega))\Big).$$

## 3 Global Existence of Smooth Solutions

In the previous sections we have got the local existence of smooth solutions to (1.1)–(1.2). Now in this section we shall discuss the global existence of a smooth solution by extending the local smooth solution via the *a priori* estimates and get the uniqueness of a smooth solution.

**Lemma 3.1** *Let $(\vec{m},\vec{n})$ be a smooth solution of (1.1)–(1.2). Then for any $T>0$ there is a constant $K_1>0$ depending only on $\|\vec{m}_0\|_{H^1(\Omega)}$ and $\|\vec{n}_0\|_{H^1(\Omega)}$, but independent of $T$ and $D$ such that*

$$\sup_{0\le t\le T}\|\vec{m}(\cdot,t),\vec{n}(\cdot,t)\|_{L^2(\Omega)}\le K_1;\tag{3.1}$$

$$\sup_{0\le t\le T}\|\vec{m}_x(\cdot,t),\vec{n}_x(\cdot,t)\|_{L^2(\Omega)}\le K_1;\tag{3.2}$$

$$\int_0^T\int_\Omega[|\vec{m}_{xx}(\cdot,t)|^2+|\vec{n}_{xx}(\cdot,t)|^2]dxdt\le K_1;\tag{3.3}$$

$$\|\vec{m}_t,\vec{n}_t\|_{L^2(\Omega_T)}\le K_1.\tag{3.4}$$

*Proof* Multiplying the first equation of (1.1) by $\vec{m}$ and the second by $\vec{n}$, one easily gets (3.1).

Multiplying the first equation of (1.1) by $\vec{m}_{xx}$ and the second by $\vec{n}_{xx}$, we have

$$\begin{cases}\vec{m}_{xx}\vec{m}_t=|\vec{m}_{xx}|^2+k_{11}(\vec{m}\times\vec{n}_{xx})\vec{m}_{xx},\\ \vec{n}_{xx}\vec{n}_t=|\vec{n}_{xx}|^2+k_{22}(\vec{n}\times\vec{m}_{xx})\vec{n}_{xx},\end{cases}$$

that is,

$$\begin{cases}\vec{m}_{xx}\vec{m}_t=|\vec{m}_{xx}|^2-k_{11}(\vec{m}\times\vec{m}_{xx})\vec{n}_{xx},\\ \vec{n}_{xx}\vec{n}_t=|\vec{n}_{xx}|^2-k_{22}(\vec{n}\times\vec{n}_{xx})\vec{m}_{xx}.\end{cases}\tag{3.5}$$

Multiplying the first equation of (1.1) by $\vec{n}_{xx}$ and the second by $\vec{m}_{xx}$, we have

$$\begin{cases}\vec{n}_{xx}\vec{m}_t=\vec{m}_{xx}\vec{n}_{xx}+2k_1(\vec{m}\times\vec{m}_{xx})\vec{n}_{xx},\\ \\ \vec{m}_{xx}\vec{n}_t=\vec{n}_{xx}\vec{m}_{xx}+2k_2(\vec{n}\times\vec{n}_{xx})\vec{m}_{xx},\end{cases}$$

and this implies

$$\begin{cases} (\vec{m}\times\vec{m}_{xx})\vec{n}_{xx} = \dfrac{1}{2k_1}\vec{n}_{xx}\vec{m}_t - \dfrac{1}{2k_1}\vec{m}_{xx}\vec{n}_{xx}, \\ (\vec{n}\times\vec{n}_{xx})\vec{m}_{xx} = \dfrac{1}{2k_2}\vec{m}_{xx}\vec{n}_t - \dfrac{1}{2k_2}\vec{n}_{xx}\vec{m}_{xx}. \end{cases} \tag{3.6}$$

Combining (3.5) with (3.6), we have

$$\begin{cases} \vec{m}_{xx}\vec{m}_t = |\vec{m}_{xx}|^2 - \dfrac{k_{11}}{2k_1}\vec{n}_{xx}\vec{m}_t + \dfrac{k_{11}}{2k_1}\vec{m}_{xx}\vec{n}_{xx}, \\ \vec{n}_{xx}\vec{n}_t = |\vec{n}_{xx}|^2 - \dfrac{k_{22}}{2k_2}\vec{m}_{xx}\vec{n}_t + \dfrac{k_{22}}{2k_2}\vec{n}_{xx}\vec{m}_{xx}. \end{cases} \tag{3.7}$$

Integrating (3.7) over $\Omega$ and using the periodic conditions (1.2), one gets

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\displaystyle\int_\Omega |\vec{m}_x|^2 + \int_\Omega |\vec{m}_{xx}|^2 = -\dfrac{k_{11}}{2k_1}\int_\Omega \vec{n}_{xx}\vec{m}_t - \dfrac{k_{11}}{2k_1}\int_\Omega \vec{m}_{xx}\vec{n}_{xx}, \\ \dfrac{1}{2}\dfrac{d}{dt}\displaystyle\int_\Omega |\vec{n}_x|^2 + \int_\Omega |\vec{n}_{xx}|^2 = -\dfrac{k_{22}}{2k_2}\int_\Omega \vec{m}_{xx}\vec{n}_t - \dfrac{k_{22}}{2k_2}\int_\Omega \vec{m}_{xx}\vec{n}_{xx}. \end{cases} \tag{3.8}$$

Add the two terms of (3.8) to give

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega [|\vec{m}_x|^2 + |\vec{n}_x|^2] + \int_\Omega [|\vec{m}_{xx}|^2 + |\vec{n}_{xx}|^2] \\ &= -\alpha\frac{d}{dt}\int_\Omega (\vec{m}_x\cdot\vec{n}_x) - 2\alpha\int_\Omega (\vec{m}_{xx}\cdot\vec{n}_{xx}) \\ &\le -\alpha\frac{d}{dt}\int_\Omega (\vec{m}_x\cdot\vec{n}_x) + \alpha\int_\Omega [|\vec{m}_{xx}|^2 + |\vec{n}_{xx}|^2];\ 0<\alpha<1. \end{aligned} \tag{3.9}$$

Integrating (3.9) from 0 to $T$, we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega [|\vec{m}_x|^2 + |\vec{n}_x|^2] + (1-\alpha)\int_0^T\int_\Omega [|\vec{m}_{xx}|^2 + |\vec{n}_{xx}|^2] \\ &\le -\alpha\int_\Omega (\vec{m}_{0x}\cdot\vec{n}_{0x}) - \alpha\int_\Omega (\vec{m}_x\cdot\vec{n}_x) + \frac{1}{2}\int_\Omega [|\vec{m}_{0x}|^2 + |\vec{n}_{0x}|^2]; \end{aligned}$$

and hence

$$\frac{1-\alpha}{2}\int_\Omega [|\vec{m}_x|^2 + |\vec{n}_x|^2] + (1-\alpha)\int_0^T\int_\Omega [|\vec{m}_{xx}|^2 + |\vec{n}_{xx}|^2] \le K_1,$$

with $K_1$ depending only on $k_1, k_2,\ k_{11},\ k_{22},\ \|\vec{m}_{0x}\|_{L^2(\Omega)}$ and $\|\vec{n}_{0x}\|_{L^2(\Omega)}$ but independent of $T$ and $D$. This lemma is proved.

From the above lemma and the embedding theorem we have:

**Lemma 3.2** *Let $(\vec{m},\vec{n})$ be a smooth solution of* (1.1)–(1.2)*. Then there is a constant $K_1>0$ as in Lemma* 3.1 *such that*

$$\|\vec{m},\vec{n}\|_{L^\infty(\Omega_T)} \le K_1. \tag{3.10}$$

**Lemma 3.3** *Let $(\vec{m},\vec{n})$ be a smooth solution of* (1.1)–(1.2)*. Then, for any $T>0$, there is a constant $K_2>0$ depending only on $\|\vec{m}_0\|_{H^2(\Omega)}$ and $\|\vec{n}_0\|_{H^2(\Omega)}$, but independent of $D$ such that*

$$\sup_{0\le t\le T}\|\vec{m}_{xx}(\cdot,t),\vec{n}_{xx}(\cdot,t)\|_{L^2(\Omega)}\le K_2, \tag{3.11}$$

$$\int_0^T\int_\Omega[|\vec{m}_{xxx}(\cdot,t)|^2+|\vec{n}_{xxx}(\cdot,t)|^2]dxdt\le K_2, \tag{3.12}$$

$$\|\vec{m}_{xt},\vec{n}_{xt}\|_{L^2(\Omega_T)}\le K_2; \tag{3.13}$$

*and then, from the embedding inequality,*

$$\|\vec{m}_x,\vec{n}_x\|_{L^\infty(\Omega_T)}\le K_2. \tag{3.14}$$

*Proof* Differentiating (1.1) with respect to $x$ two times and then multiplying the first one by $\vec{m}_{xx}$ and the second one by $\vec{n}_{xx}$, we get

$$\begin{cases}\vec{m}_{xx}\vec{m}_{xxt}=\vec{m}_{xxxx}\vec{m}_{xx}+2k_1(\vec{m}\times\vec{m}_{xx})_{xx}\vec{m}_{xx}\\ \qquad\qquad +k_{11}(\vec{m}\times\vec{n}_{xx})_{xx}\vec{m}_{xx},\\ \vec{n}_{xx}\vec{n}_{xxt}=\vec{n}_{xxxx}\vec{n}_{xx}+2k_2(\vec{n}\times\vec{n}_{xx})_{xx}\vec{n}_{xx}\\ \qquad\qquad +k_{22}(\vec{n}\times\vec{m}_{xx})_{xx}\vec{n}_{xx},\end{cases}$$

that is,

$$\begin{cases}\frac{1}{2}\frac{d}{dt}\int_\Omega|\vec{m}_{xx}|^2+\int_\Omega|\vec{m}_{xxx}|^2=-2k_1\int_\Omega(\vec{m}_x\times\vec{m}_{xx})\vec{m}_{xxx}\\ \qquad\qquad -k_{11}\int_\Omega(\vec{m}\times\vec{n}_{xxx}+\vec{m}_x\times\vec{n}_{xx})\vec{m}_{xxx},\\ \frac{1}{2}\frac{d}{dt}\int_\Omega|\vec{n}_{xx}|^2+\int_\Omega|\vec{n}_{xxx}|^2=-2k_2\int_\Omega(\vec{n}_x\times\vec{n}_{xx})\vec{n}_{xxx}\\ \qquad\qquad -k_{22}\int_\Omega(\vec{n}\times\vec{m}_{xxx}+\vec{n}_x\times\vec{m}_{xx})\vec{n}_{xxx}.\end{cases} \tag{3.15}$$

Differentiating (1.1) with respect to $x$ two times and then multiplying the first one by $\vec{n}_{xx}$ and the second one by $\vec{m}_{xx}$, we get

$$\begin{cases}\vec{n}_{xx}\vec{m}_{xxt}=\vec{m}_{xxxx}\vec{n}_{xx}+2k_1(\vec{m}\times\vec{m}_{xx})_{xx}\vec{n}_{xx}\\ \qquad\qquad +k_{11}(\vec{m}\times\vec{n}_{xx})_{xx}\vec{n}_{xx},\\ \vec{m}_{xx}\vec{n}_{xxt}=\vec{n}_{xxxx}\vec{m}_{xx}+2k_2(\vec{n}\times\vec{n}_{xx})_{xx}\vec{m}_{xx}\\ \qquad\qquad +k_{22}(\vec{n}\times\vec{m}_{xx})_{xx}\vec{m}_{xx},\end{cases}$$

that is,

$$\begin{cases}\int_\Omega\vec{n}_{xx}\vec{m}_{xxt}+\int_\Omega\vec{m}_{xxx}\vec{n}_{xxx}=-k_{11}\int_\Omega(\vec{m}_x\times\vec{n}_{xx})\vec{n}_{xxx}\\ \qquad -2k_1\int_\Omega(\vec{m}\times\vec{m}_{xxx}+\vec{m}_x\times\vec{m}_{xx})\vec{n}_{xxx},\\ \int_\Omega\vec{m}_{xx}\vec{n}_{xxt}+\int_\Omega\vec{m}_{xxx}\vec{n}_{xxx}=-k_{22}\int_\Omega(\vec{n}_x\times\vec{m}_{xx})\vec{m}_{xxx}\\ \qquad -2k_2\int_\Omega(\vec{n}\times\vec{n}_{xxx}+\vec{n}_x\times\vec{n}_{xx})\vec{m}_{xxx},\end{cases}$$

or

$$\left\{\begin{array}{l}\displaystyle\int_\Omega \vec{n}_{xx}\vec{m}_{xxt}+\int_\Omega \vec{m}_{xxx}\vec{n}_{xxx}=-k_{11}\int_\Omega(\vec{m}_x\times\vec{n}_{xx})\vec{n}_{xxx}\\ \qquad\displaystyle +2k_1\int_\Omega(\vec{m}\times\vec{n}_{xxx})\vec{m}_{xxx}-2k_1\int_\Omega(\vec{m}_x\times\vec{m}_{xx})\vec{n}_{xxx},\\ \displaystyle\int_\Omega \vec{m}_{xx}\vec{n}_{xxt}+\int_\Omega \vec{m}_{xxx}\vec{n}_{xxx}=-k_{22}\int_\Omega(\vec{n}_x\times\vec{m}_{xx})\vec{m}_{xxx}\\ \qquad\displaystyle +2k_2\int_\Omega(\vec{n}\times\vec{m}_{xxx})\vec{n}_{xxx}-2k_2\int_\Omega(\vec{n}_x\times\vec{n}_{xx})\vec{m}_{xxx}.\end{array}\right. \tag{3.16}$$

Substituting (3.16) into (3.15) and then putting the two equalities together, we have

$$\begin{aligned}&\frac{1}{2\alpha}\frac{d}{dt}\int_\Omega[|\vec{m}_{xx}|^2+|\vec{n}_{xx}|^2]+\frac{1}{\alpha}\int_\Omega[|\vec{m}_{xxx}|^2+|\vec{n}_{xxx}|^2]+\frac{d}{dt}\int_\Omega(\vec{m}_{xx}\vec{n}_{xx})\\ &=-2\int_\Omega\vec{m}_{xxx}\vec{n}_{xxx}-\frac{2k_1}{\alpha}\int_\Omega(\vec{m}_x\times\vec{m}_{xx})\vec{m}_{xxx}-2k_1\int_\Omega(\vec{m}_x\times\vec{n}_{xx})\vec{m}_{xxx}\\ &\quad-\frac{2k_2}{\alpha}\int_\Omega(\vec{n}_x\times\vec{n}_{xx})\vec{n}_{xxx}-2k_2\int_\Omega(\vec{n}_x\times\vec{m}_{xx})\vec{n}_{xxx}\\ &\quad-2k_1\int_\Omega(\vec{m}_x\times\vec{m}_{xx})\vec{n}_{xxx}-2k_2\int_\Omega(\vec{n}_x\times\vec{n}_{xx})\vec{m}_{xxx}\\ &\quad-k_{11}\int_\Omega(\vec{m}_x\times\vec{n}_{xx})\vec{n}_{xxx}-k_{22}\int_\Omega(\vec{n}_x\times\vec{m}_{xx})\vec{m}_{xxx}.\end{aligned}\tag{3.17}$$

We get

$$\begin{aligned}&\frac{1}{2\alpha}\frac{d}{dt}\int_\Omega[|\vec{m}_{xx}|^2+|\vec{n}_{xx}|^2+(\vec{m}_{xx}\vec{n}_{xx})]+\Big(\frac{1}{\alpha}-1\Big)\int_\Omega[|\vec{m}_{xxx}|^2+|\vec{n}_{xxx}|^2]\\ &\quad\le C\Big(\sup_\Omega|\vec{m}_x|+\sup_\Omega|\vec{n}_x|\Big)(\|\vec{m}_{xx}\|_2+\|\vec{n}_{xx}\|_2)(\|\vec{m}_{xxx}\|_2+\|\vec{n}_{xxx}\|_2).\end{aligned}\tag{3.18}$$

Using the following interpolation inequalities (note that the spatial dimension is 1) in (3.18)

$$\begin{aligned}&\|\vec{m}_x\|_{L^\infty(\Omega)}\le C\|\vec{m}_x\|_2^{3/4}\|\vec{m}_{xx}\|_2^{1/4},\quad \|\vec{m}_{xx}\|_{L^2(\Omega)}\le C\|\vec{m}_x\|_2^{1/2}\|\vec{m}_{xxx}\|_2^{1/2},\\ &\|\vec{n}_x\|_{L^\infty(\Omega)}\le C\|\vec{m}_x\|_2^{3/4}\|\vec{m}_{xx}\|_2^{1/4},\quad \|\vec{n}_{xx}\|_{L^2(\Omega)}\le C\|\vec{m}_x\|_2^{1/2}\|\vec{m}_{xxx}\|_2^{1/2},\end{aligned}$$

we obtain

$$\frac{d}{dt}\int_\Omega[|\vec{m}_{xx}|^2+|\vec{n}_{xx}|^2+(\vec{m}_{xx}\vec{n}_{xx})]+\int_\Omega[|\vec{m}_{xxx}|^2+|\vec{n}_{xxx}|^2]\le C.\tag{3.19}$$

The lemma is proved.

From the above lemma and the embedding theorem we have:

**Lemma 3.4** *Suppose the conditions of Lemma* 3.3 *hold. Let* $(\vec{m},\vec{n})$ *be a smooth solution of* (1.1)–(1.2). *Then there is a constant* $K_2>0$ *as in Lemma* 3.3 *such that*

$$\|\vec{m}_x,\vec{n}_x\|_{L^\infty(\Omega_T)}\le K_2.\tag{3.20}$$

By induction we have:

**Lemma 3.5** *Let* $\vec{m}_0,\vec{n}_0\in H^i(\Omega)$ $(i\ge 4)$ *such that* $\vec{m}_0(x-D)=\vec{m}_0(x+D)$, $\vec{n}_0(x-D)=\vec{n}_0(x+D)$. *Let* $(\vec{m},\vec{n})$ *be a smooth solution of* (1.1)–(1.2). *Then, for any* $T>0$, *there is a* *constant $K_i > 0$ depending only on $\|\vec{m}_0\|_{H^i(\Omega)}$ and $\|\vec{n}_0\|_{H^i(\Omega)}$, but independent of $D$, such that*

$$\sup_{0\le t\le T} \|\vec{m}_{x^{i-2s}t^s}(\cdot,t), \vec{n}_{x^{i-2s}t^s}(\cdot,t)\|_{L^2(\Omega)} \le K_i,\ (i \ge 2s), \tag{3.21}$$

$$\int_0^T \int_\Omega [|\vec{m}_{x^{i+1-2s}t^s}(\cdot,t)|^2 + |\vec{n}_{x^{i+1-2s}t^s}(\cdot,t)|^2] dxdt \le K_i,\ i+1 \ge 2s, \tag{3.22}$$

*where $i, s$ are non-negative integers.*

Finally we obtain from the extension method the following global existence theorem:

**Theorem 3.1** *Let $\vec{m}_0, \vec{n}_0 \in H^i(\Omega)$ $(i \ge 4)$ such that $\vec{m}_0(x-D) = \vec{m}_0(x+D)$, $\vec{n}_0(x-D) = \vec{n}_0(x+D)$. Then problem (1.1)–(1.2) admits at least one smooth solution $(\vec{m}(x,t), \vec{n}(x,t))$ with $\vec{m}, \vec{n} \in G(T)$, where*

$$G(T) = \sum_{s=0}^{[i/2]} W^s_\infty(0,T; H^{k-2s}(\Omega)) \cap \sum_{s=0}^{[(i+1)/2]} H^s(0,T; H^{i+1-2s}(\Omega)). \tag{3.23}$$

## 4 Uniqueness of the Smooth Solution

In this section we prove that the smooth solution obtained in Section 3 is unique.

**Theorem 4.1** *Let $\vec{m}_0(x), \vec{n}_0(x) \in H^i(\Omega)$ $(i \ge 4)$ and problem (1.1)–(1.2) admit two smooth solutions $(\vec{m}_1(x,t), \vec{n}_1(x,t))$ and $(\vec{m}_2(x,t), \vec{n}_2(x,t))$ as in Theorem 3.1. Then $\vec{m}_1(x,t) = \vec{m}_2(x,t)$, $\vec{n}_1(x,t) = \vec{n}_2(x,t)$.*

*Proof* Set $w_1(x,t) = \vec{m}_1(x,t) - \vec{m}_2(x,t)$, $w_2(x,t) = \vec{n}_1(x,t) - \vec{n}_2(x,t)$. Then $(w_1(x,t), w_2(x,t))$ satisfies the following problem:

$$\begin{cases} w_{1t} = w_{1xx} + 2k_1(\vec{m}_1 \times \vec{m}_{1xx} - \vec{m}_2 \times \vec{m}_{2xx}) \\ \qquad + k_{11}(\vec{m}_1 \times \vec{n}_{1xx} - \vec{m}_2 \times \vec{n}_{2xx}), \quad (x,t) \in \Omega \times \mathbb{R}_+, \\ w_{2t} = w_{2xx} + 2k_2(\vec{n}_1 \times \vec{n}_{1xx} - \vec{n}_2 \times \vec{n}_{2xx}) \\ \qquad + k_{22}(\vec{n}_1 \times \vec{m}_{1xx} - \vec{n}_2 \times \vec{m}_{2xx}), \qquad (x,t) \in \Omega \times \mathbb{R}_+, \\ w_i(x-D,t) = w_i(x+D,t),\ i = 1,2, \qquad (x,t) \in \mathbb{R}^1 \times \mathbb{R}_+, \\ w_1(x,0) = w_2(x,0) = 0, \quad x \in \Omega. \end{cases} \tag{4.1}$$

Multiplying the first equation of (4.1) and the second one of (4.1) by $w_1(x,t)$ and $w_2(x,t)$, we obtain

$$\begin{cases} w_1 w_{1t} = w_1 w_{1xx} + 2k_1[(\vec{m}_1 \times w_{1x})_x w_1 - (\vec{m}_{1x} \times w_{1x}) w_1] \\ \qquad + k_{11}[(\vec{m}_1 \times w_{2x})_x w_1 - (\vec{m}_{1x} \times w_{2x}) w_1], \quad (x,t) \in \Omega \times \mathbb{R}_+, \\ w_2 w_{2t} = w_2 w_{2xx} + 2k_2[(\vec{n}_1 \times w_{2x})_x w_2 - (\vec{n}_{1x} \times w_{2x}) w_2] \\ \qquad + k_{22}[(\vec{n}_1 \times w_{1x})_x w_2 - (\vec{n}_{1x} \times w_{1x}) w_2], \quad (x,t) \in \Omega \times \mathbb{R}_+. \end{cases} \tag{4.2}$$

Integrating (4.2) over $\Omega$, we get

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\displaystyle\int_\Omega |w_1|^2 + \int_\Omega |w_{1x}|^2 = -2k_1 \int_\Omega (\vec{m}_{1x} \times w_{1x}) w_1 \\ \qquad -k_{11} \displaystyle\int_\Omega (\vec{m}_1 \times w_{2x}) w_{1x} - k_{11} \int_\Omega (\vec{m}_{1x} \times w_{2x}) w_1, \\ \dfrac{1}{2}\dfrac{d}{dt}\displaystyle\int_\Omega |w_2|^2 + \int_\Omega |w_{2x}|^2 = -2k_2 \int_\Omega (\vec{n}_{1x} \times w_{2x}) w_2 \\ \qquad -k_{22} \displaystyle\int_\Omega (\vec{n}_1 \times w_{1x}) w_{2x} - k_{22} \int_\Omega (\vec{n}_{1x} \times w_{1x}) w_2. \end{cases} \tag{4.3}$$

Noting that in (4.3), $\|\vec{m}\|_\infty$, $\|\vec{n}\|_\infty$, $\|\vec{m}_x\|_\infty$, $\|\vec{n}_x\|_\infty$, $\|w_1\|_\infty$, $\|w_2\|_\infty$, $\|w_{1x}\|_\infty$, $\|w_{2x}\|_\infty$ are all bounded, then we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w_1|^2 + \int_\Omega |w_{1x}|^2 \le \frac{1}{2}\int_\Omega |w_{1x}|^2 + C\int_\Omega |w_1|^2, \tag{4.4}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w_2|^2 + \int_\Omega |w_{2x}|^2 \le \frac{1}{2}\int_\Omega |w_{2x}|^2 + C\int_\Omega |w_2|^2. \tag{4.5}$$

(4.4), (4.5) and the Gronwall inequality yield the desired conclusion.

**Remark** Since the previous estimates are all independent of $D$, we know that, by letting $D \to +\infty$, the following theorem on the unique solvability of Cauchy problem holds:

**Theorem 4.2** *Let $\vec{m}_0, \vec{n}_0 \in H^i(\mathbb{R}^1)$ $(i \ge 4)$. Then Equation* (1.1) *with the following initial condition:*

$$\vec{m}(x,t) = \vec{m}_0(x),\ \vec{n}(x,t) = \vec{n}_0(x),\ x \in \mathbb{R}^1,$$

*admits at least one smooth solution $(\vec{m}(x,t), \vec{n}(x,t))$ with $\vec{m}, \vec{n} \in G(T)$, where*

$$G(T) = \sum_{s=0}^{[i/2]} W_\infty^s(0,T;H^{k-2s}(\mathbb{R}^1)) \cap \sum_{s=0}^{[(i+1)/2]} H^s(0,T;H^{i+1-2s}(\mathbb{R}^1)). \tag{4.6}$$

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