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# Factorizable Rpp Monoids

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**Abstract**

It is well known that the rpp semigroups are generalized regular semigroups. In order to further investigate the structure of rpp semigroups, we introduce the type-$(\mathcal{I}, \mathcal{L}^*)$ factorizable monoids. In this paper, we study such factorizable rpp monoids admitting a left univocal type-$(\mathcal{I}, \mathcal{L}^*)$ factorization. In particular, we prove that a semigroup $S$ is a factorizable rpp monoid admitting a left univocal type-$(\mathcal{I}, \mathcal{L}^*)$ factorization if and only if $S$ is isomorphic to a semidirect product of a band and a left cancellative monoid. Some results of Catino and Tolo on factorizable semigroups are extended and amplified.

**Mathematics Subject Classification:** 20M10

**Keywords:** Rpp semigroups, Left cancellative monoids, Factorizable semigroups, Semidirect product.

# 1 Introduction

A semigroup $S$ is called *right principal projective*, in short, *rpp* if for all $a \in S$, $aS^1$ regarded as an $S^1$-system is projective. We can also dually define the *left principal projective* semigroups (in short, *lpp*) *semigroups*. It was shown by Fountain [3] that a semigroup $S$ is rpp if and only if every $\mathcal{L}^*$-class of $S$ contains at least one idempotent. According to Fountain [3] and [4], a semigroup $S$ is called *abundant* if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of $S$ contains an idempotent. Equivalently, $S$ is an abundant semigroup if and only if $S$ is both an rpp and lpp semigroup. An abundant semigroup $S$ is called *quasi-adequate* if $E(S)$ ,the set of idempotents of $S$, forms a subsemigroup of $S$. Regular semigroups are abundant semigroups and orthodox semigroups quasi-adequate semigroups.

For an abundant semigroup $S$, Lawson [9] defined a partial order " $\leq$ " on $S$ by $a \leq b$ if and only if for some $e, f \in E(S)$, $a = eb = bf$. Later on,Guo [5] called an abundant semigroup $S$ *F-abundant* if for every $a \in S$, the congruence class of the minimal cancellative congruence $\sigma$ containing $a$ has a greatest element $m_a$ with respect to the partial order " $\leq$ ". It is clear that F-abundant semigroups are generalized F-regular semigroups (for F-regular semigroups, see [2], [10] etc.) Recently, Zhang-Chen [13] considered a special subclass of F-abundant semigroups, namely, the u-IC abundant semigroups. On the other hand,Ni-Chen-Guo [11] have provided a method of construction for the general F-abundant semigroups. Along this direction, Guo-Li-Shum [6] have further generalized F-abundant semigroups to F-rpp semigroups.

According to Catino[1],a semigroup $S$ is called a *factorizable semigroup* if there exist some proper subsemigroups $U$ and $V$ of $S$ such that $S = UV$, where $UV$ is the set $\{uv \mid u \in U, v \in V\}$. In this case, we call the pair $(U, V)$ a *factorization pair* of the factorizable semigroup $S$. A factorization pair $(U, V)$ of a factorizable semigroup $S$ is called *[left; right] univocal* if

$$(\forall u_1, u_2 \in U, v_1, v_2 \in V) \quad u_1v_1 = u_2v_2 \\ \Rightarrow [u_1 = u_2;\ v_1 = v_2]u_1 = u_2 \text{ and } v_1 = v_2.$$

We shall call a semigroup $S$ a *[left; right] univocal factorizable semigroup* if no confusion arises. For researches in this topic,the readers are referred to Catino[1] and Tolo [12].

In this paper, we study a class of left univocal factorizable rpp semigroups, namely, the left univocal factorizable rpp monoids of type-$(\mathcal{I}, \mathcal{L}^*)$. This class of left univocal factorizable rpp semigroups is an F-rpp semigroup. Moreover, the u-IC quasi-adequate semigroups are special cases of factorable rpp monoids of type $(\mathcal{I}, \mathcal{L}^*)$. We shall prove that a monoid is a left univocal factorizable rpp monoid of type-$(\mathcal{I}, \mathcal{L}^*)$ if and only if it is a semidirect product of a band and a left cancellative monoid.

## 2 Preliminaries

Throughout this paper, we adopt the terminologies and notions given in [1], [4] and [7]. Now we cite some known results in the literature which will be used in the sequel.

To begin with, we first state some facts concerning the Green *-relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on the semigroup $S$.

**Lemma 2.1** [4] *Let $S$ be an semigroup and $a, b \in S$. Then the following statements are equivalent:*

*(1) $a\mathcal{L}^*$ $[\mathcal{R}^*]$ $b$.*

*(2) For all $x, y \in S^1$, $ax = ay \Leftrightarrow bx = by$ $[xa = ya \Leftrightarrow xb = yb]$.*

**Lemma 2.2** [4] *Let $S$ be an semigroup and $a, e^2 = e \in S$. Then the following statements are equivalent:*

*(1) $a\mathcal{L}^*$ $[\mathcal{R}^*]$ $e$.*

*(2) $a = ae$ $[a = ea]$ and $ax = ay \Rightarrow ex = ey$ $[xa = ya \Rightarrow xe = ye]$ for all $x, y \in S^1$.*

Evidently, the Green $*$-relation $\mathcal{L}^*$ is a right congruence on $S$ while the Green $*$-relation $\mathcal{R}^*$ is a left congruence on $S$. In general, $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. And, for the regular elements $a$ and $b$ of $S$, $a\mathcal{L}^*b$ $[a\mathcal{R}^*b]$ if and only if $a\mathcal{L}b$ $[a\mathcal{R}b]$ (for more details, see [4] ). For the sake of brevity, we denote by $a^\dagger$ $[a^*]$ the typical idempotents related to $a$ by $\mathcal{R}^*$ $[\mathcal{L}^*]$ and use $E(S)$ to denote the set of all idempotents of the semigroup $S$. If $\mathcal{K}$ is one of the Green $*$-relations $\mathcal{L}^*, \mathcal{R}^*$ and $\mathcal{H}^*$ on $S$, then we denote the $\mathcal{K}$-class of $S$ containing $a$ by $K_a$.

Now suppose that $S$ is an rpp semigroup. For $x, y \in S$, we define

> $x \leq_\ell y$ if and only if $L^*_x \leq L^*_y$, that is, $L^*(x) \subseteq L^*(y)$, and there exists $f \in E(S)$ such that $x = yf$.

It was pointed out by Lawson [9] that $x \leq_\ell y$ if and only if there exist $y^*$ and an uniquely determined $f \in \omega(y^*)$ such that $x = yf$, where $\omega(y^*) = \{e \in E(S) \mid e\omega y^*\}$ (here $\omega$ is defined on $E(S)$ by $e\omega f \Leftrightarrow e = ef = fe$). In this case, for a fixed $y^*$, the idempotent $f$ is unique. In fact, if $g \in \omega(y^*)$ and $x = yg$, then $yf = yg$ and $y^*f = y^*g$. Thus $f = g$. In addition, it is noteworthy that any idempotent related to $y$ by $\mathcal{L}^*$ acts as the same as $y^*$.

A congruence $\rho$ on a semigroup $S$ is called a *left cancellative monoid congruence* on $S$ if $S/\rho$ is a left cancellative monoid. It is easy to check that the meet of left cancellative monoid congruences on the semigroup $S$ is also a left cancellative monoid congruence on $S$. Note that the universal relation is of course a left cancellative monoid congruence. Thus there exists a smallest left cancellative monoid congruence on an rpp semigroup $S$. We shall denote by $\sigma_S$ the smallest left cancellative monoid congruence on $S$ if exists. Also, we write $\sigma_S$ as $\sigma$ if there is no ambiguity.

An rpp semigroup $S$ is called *F-rpp* if each $\sigma$-class of $S$ has a greatest element with respect to the partial $\leq_\ell$ (see [6], for the detail).

Now we assume that $S$ is an F-rpp semigroup. We denote by $m_s$ the greatest element in the $\sigma$-class of $S$ containing $s$ with respect to $\leq_\ell$, and $M_S = \{m_s \mid s \in S\}$. In general, with respect to the multiplication of $S$, $M_S$ need not be a semigroup,however, it is easy to see that under the following multiplication "$*$",

$$m_s * m_t = m_{st},$$

$M_S$ forms a left cancellative monoid which is isomorphic to $S/\sigma$.

We now have the following result:

**Lemma 2.3** [6] *Let $S$ be an F-rpp semigroup. Then the following statements hold:*

*(1) $E(S)$ is a band with identity.*

*(2) For all $m \in M_S$ and $m^*$, $m^*E(S) \subseteq E(S)m^*$ and $E(S)m \subseteq mE(S)$.*

*(3) For all $s \in S$ and for each $m_s^*$, there exists a unique $f_s \in \omega(m_s^*)$ such that $s = m_s f_s$.*

Let $S$ be a monoid with identity 1. Denote by $H_1^*$ the $\mathcal{H}^*$-class of $S$ containing 1. Then $H_1^*$ is a cancellative monoid with identity 1. And for $a \in S$, we write $A^*(a) = \{b \in S \mid a = ebf, e\mathcal{R}^*a\mathcal{L}^*f, e, f \in E(S)\}$ and $U^*(a) = A^*(a) \cap H_1^*$. Now, we call an IC quasi-adequate monoid $S$ a *u-IC quasi-adequate semigroup* if $|U^*(a)| = 1$, for all $a \in S$.

The following result was obtained by Zhang and Chen in [13].

**Lemma 2.4** [13] *Let $S$ be an F-abundant semigroup. Then $S$ is a u-IC quasi-adequate semigroup if and only if $M_S = H_1^*$.*

# 3 Definitions and characterizations

**Definition 3.1** *We call a factorization $(A, B)$ of a factorizable monoid $S$* ***type-***$(\mathcal{I}, \mathcal{L}^*)$ *if the following conditions are satisfied:*

*(1) A is contained in the* $\mathcal{L}^*$*-class of* $S$ *containing* $1$*, where* $1$ *is the identity of* $S$*.*

*(2)* $B = E(S)$*.*

*(3) For each* $a \in A$*,* $Ba \subseteq aB$*.*

*We also call* $S$ *a* ***factorizable monoid of type-***$(\mathcal{I}, \mathcal{L}^*)$***.***

By Definition 3.1, the following corollary is immediate.

**Corollary 3.2** *Let* $(A, B)$ *be a factorization of a factorizable monoid* $S$*. If* $(A, B)$ *is type-*$(\mathcal{I}, \mathcal{L}^*)$*, then* $E(S)$ *is a band.*

**Proposition 3.3** *Let* $S$ *be a factorizable rpp monoid admitting a left univocal type-*$(\mathcal{I}, \mathcal{L}^*)$ *factorization* $(A, B)$*. Then the factorization* $(A, B)$ *is univocal.*

*Proof.* Let $a_1, a_2 \in A, b_1, b_2 \in B$ and $a_1 b_1 = a_2 b_2$. Then by our hypothesis, $a_1 b_1 = a_1 b_2$ and $1b_1 = 1b_2$, since $a_1 \mathcal{L}^* 1$. That is, $b_1 = b_2$. Thus $(A, B)$ is univocal. □

In the following proposition, we describe the relationship between the factorizable rpp monoid admitting a left univocal type-$(\mathcal{I}, \mathcal{L}^*)$ factorization and the F-rpp semigroups.

**Proposition 3.4** *Let* $S$ *be a factorizable monoid with factorization* $(A, B)$ *and with identity* $1$*.Then* $S$ *is a factorizable rpp monoid admitting a left univocal type-*$(\mathcal{I}, \mathcal{L}^*)$ *factorization* $(A, B)$ *if and only if* $S$ *is an F-rpp semigroup with* $A = M_S$ *and* $B = E(S)$*.*

*Proof.* Suppose that $S$ is an F-rpp semigroup with $A = M_S$ and $B = E(S)$. Then by Lemma 2.3 (3), $(A, B)$ is a left univocal factorization of $S$ since $m^* = 1$ and $\omega(m^*) = E(S)$, for all $m \in A$. Again by Lemma 2.3 (2), $(A, B)$ is type-$(\mathcal{I}, \mathcal{L}^*)$. This proves the sufficient part.

Conversely, let $S$ be a factorizable rpp monoid admitting a left univocal type-$(\mathcal{I}, \mathcal{L}^*)$ factorization $(A, B)$. We now prove that

$$\rho = \{(a, b) \in S \times S \mid a = me, b = mf \text{ for some } e, f \in E(S) \text{ and } m \in A\}$$

is a left cancellative monoid congruence on $S$. Obviously, $\rho$ is both reflexive and symmetric. If $a\rho b$ and $b\rho c$, then there exist elements $m, n \in A$ and $e, f, g, h \in E(S)$ such that $a = me, b = mf = ng$ and $c = nh$. The second equality implies that $m = n$ and $g = f$ since $(A, B)$ is a univocal factorization of $S$.

Thus $\rho$ is transitive so that $\rho$ is an equivalence relation on $S$. Note that for $e, f, g \in E(S)$ and $m, n \in A$, we have

$$mnhf = (me)(nf) = (me)(ng) = mnhg$$

with $h \in E(S)$ such that $en = nh$ (by Definition 3.1 (3)). Thus $\rho$ is a left congruence on $S$. Similarly, we can prove that $\rho$ is a right congruence on $S$. Thus, $\rho$ is indeed a congruence on $S$.

Now let $a, b, c \in S$ and $(ab)\rho = (ac)\rho$. Then, by our assumptions, there exist elements $m, n, p \in A$ and $e, f, g \in E(S)$ such that $a = me, b = nf$ and $c = pg$. Hence $ab = menf = mnhf$ with $h \in E(S)$, and similarly $ac = mpkg$ where $k \in E(S)$. On the other hand, since $(ab)\rho = (ac)\rho$, we have $q \in A$ and $u, v \in E(S)$ such that $ab = qu$ and $ac = qv$. Observe that $(A, B)$ is a univocal factorization of $S$. Since $m\mathcal{L}^*1$, $mn = q = mp$ and $1n = 1p$ and consequently, $n = p$. This shows that $b\rho c$, and thereby $\rho$ is a left cancellative monoid congruence on $S$.

Consider that $\sigma_S$ is the smallest left cancellative monoid congruence on $S$. We have $\sigma_S \subseteq \rho$. On the other hand, it is clear that $\rho \subseteq \sigma_S$ since $e\sigma_S 1$ for all $e \in E(S)$. Hence, $\sigma_S = \rho$.

Note that each element in $A$ is related to 1 by $\mathcal{L}^*$ and that for all $e \in E(S)$, $e\omega 1$. Thus $me \leq_\ell m$. On the other hand, by the above proof, every $\sigma$-class of $S$ consists of the elements of the form $me$ with $m \in A$ and $e \in E(S)$, and hence every element of $A$ is the greatest element of some $\sigma$-class of $S$ with respect to " $\leq_\ell$ ".

Now, it is not difficult to see that $S$ is an F-rpp semigroup with $M_S = A$. □

It was observed by Guo [6] that an F-abundant semigroup is an IC abundant F-rpp semigroup and vice versa. Thus,from the above observation and Lemma 2.4, the following proposition is immediate.

**Proposition 3.5** *A semigroup is a u-IC quasi-adequate semigroup if and only if it is a factorizable rpp semigroup one of whose factorizations is left univocal of type-*$(\mathcal{I}, \mathcal{L}^*)$*, and which is an IC abundant semigroup.*

*Proof.* ($\Rightarrow$) Assume that $S$ is a u-IC quasi-adequate semigroup. Note that in an IC abundant semigroup " $\leq = \leq_\ell$ " (for detail, see [9]). Thus by Lemma 2.4, $S$ is an F-rpp semigroup and $M_S = H_1^*$ ($\subseteq L_1^*$). This implies that $S$ is a factorizable monoid admitting a left univocal type-$(\mathcal{I}, \mathcal{L}^*)$, by Proposition 3.4.

($\Leftarrow$) Suppose that $S$ is a factorizable rpp semigroup. Assume that one of its factorizations is left univocal and is also of type-$(\mathcal{I}, \mathcal{L}^*)$, say $(A, B)$ which is IC abundant. Then, by Proposition 3.4, $S$ is an F-rpp semigroup with $A = M_S$

and $B = E(S)$. On the other hand, by the using the same arguments as in Proposition 3.5, we see that $S$ is an F-abundant semigroup. Note that $S$ is IC abundant so that on $S$, we have " $\leq = \leq_\ell$ ". Thus, $M_S = H_1^*$, that is, $A = H_1^*$. Now, by Lemma 2.4, $S$ is a u-IC quasi-adequate semigroup. □

# 4 A structure theorem

In this section we shall establish a semidirect product theorem for the factorizable rpp monoids admitting a left univocal type-$(\mathcal{I}, \mathcal{L}^*)$ factorizations.

Let $S$ be a semigroup and $T$ a monoid. Denote by $End(S)$ the semigroup of endomorphisms of $S$ into itself. Let

$$\varphi : T \to End(S) \text{ defined by } x \mapsto \varphi_x$$

be a monoid homomorphism from $T$ into $End(S)$. Then, we define a multiplication " $\circ$ " on the set $S \times T$ such that for $(a, x), (b, y) \in S \times T$,

$$(a, x) \circ (b, y) = ((a\varphi_y)b, xy).$$

It is easy to check that $(S \times T, \circ)$ is a semigroup. We call this semigroup *the semidirect product of $S$ by $T$ with the structure mapping $\varphi$*, and we denote this semigroup by $S \times_\varphi T$. If, in addition, the component $S$ is a monoid, then the semidirect product $S \times_\varphi T$ is also a monoid with identity $(\iota, 1)$, where $\iota$ and 1 are the identities of $S$ and $T$, respectively. (for more details, see [8, Definitions 1.8]).

**Proposition 4.1** *Let $B \times_\varphi M$ be the semidirect product of a band $B$ with identity $\iota$ by a left cancellative monoid $M$ with identity 1 for the structure mapping $\varphi$. Then*

*(1) $E(B \times_\varphi M) = \{(e, 1) \in B \times_\varphi M\}$. Moreover, $E(B \times_\varphi M)$ is isomorphic to $B$.*

*(2) For all $(e, m), (f, n) \in B \times_\varphi M$, $(e, m)\mathcal{L}^*(f, n)$ if and only if $e\mathcal{L}f$. Moreover, $L^*_{(\iota,1)} = \{(\iota, m) \mid m \in M\}$.*

*(3) For all $(e, m), (f, n) \in B \times_\varphi M$, $(e, m) \leq_\ell (f, n)$ if and only if $e\omega f$ and $m = n$.*

*(4) $B \times_\varphi M$ is a factorizable rpp monoid with left univocal type $(\mathcal{I}, \mathcal{L}^*)$ factorization $(\{\iota\} \times M, E(B \times_\varphi M))$.*

*Proof.* (1) If $(e,x) \in E(B \times_\varphi M)$, then $(e,x) = (e,x)^2 = ((e\varphi_x)e, x^2)$ and $x^2 = x$. Thus $x = 1$ because $M$ is a left cancellative monoid. This leads to $E(B \times_\varphi M) \subseteq \{(e,1) \in B \times_\varphi M$ and $e \in B\}$. By the multiplication of semidirect product, the reverse inclusion is obvious. Thus,

$$E(B \times_\Phi M) = \{(e,1) \in B \times_\varphi M \text{ and } e \in B\}.$$

The remaining proof is clear.

(2) Let $(e,x) \in B \times_\varphi M$. Suppose that $(f,y),(g,z) \in (B \times_\varphi M)^1$ and $(e,x)(f,y) = (e,x)(g,z)$, that is, $((e\varphi_y)f, xy) = ((e\varphi_z)g, xz)$. Then $(e\varphi_y)f = (e\varphi_z)g$ and $xy = xz$. The latter equality implies that $y = z$ since $M$ is a left cancellative monoid. Thus

$$(e,1)(f,y) = ((e\varphi_y)f, y) = ((e\varphi_z)g, z) = (e,1)(g,z).$$

On the other hand, we have $(e,x)(e,1) = (e,x)$. Consequently, by Lemma 2.2, $(e,x)\mathcal{L}^*(e,1)$, and $B \times_\varphi M$ is an rpp semigroup. Note that any idempotent related to the identity by $\mathcal{L}^*$ must be the identity. Now, it is clear that $L^*_{(\iota,1)} = \{(\iota,m) \mid m \in M\}$.

By using the same arguments as above, we have

$$(e,m)\mathcal{L}^*(f,n) \Longleftrightarrow (e,1)\mathcal{L}^*(f,1) \Longleftrightarrow (e,1)\mathcal{L}(f,1) \Longleftrightarrow e\mathcal{L}f.$$

(3) If $(e,x) \leq_\ell (f,y)$, then there exists $(e,1) \in \omega((f,1))$ such that $(e,x) = (f,y)(e,1)$. This shows that $(e,x) = ((f\varphi_1)e, 1y) = (fe, y)$, and hence $x = y$ and $e = fe$. On the other hand, by the multiplication of semidirect product, we have $(e,1) \in \omega((f,1))$ if and only if $e = ef = fe$, that is, $e \in \omega(f)$.

Conversely, if $x = y$ and $e\omega f$,then, by the above proof, $(e,1) \in \omega((f,1))$. By assumption, we can obtain that $(e,x) = (f,y)(e,1)$. Thus $(e,x) \leq_\ell (f,y)$.

(4) Define a relation $\rho$ on $B \times_\varphi M$ by: for all $(e,m),(f,n) \in B \times_\varphi M$

$$(e,m)\rho(f,n) \text{ if and only if } m = n.$$

Then, it is easy to check that $\rho$ is a congruence on $B \times_\varphi M$. On the other hand, it is easy to see that the mapping

$$\theta: \ B \times_\varphi M/\rho \to M \text{ defined by } (e,m)\rho \mapsto m$$

is a semigroup isomorphism. This shows that $\rho$ is a left cancellative monoid congruence on $B \times_\varphi M$. It is also clear that $\rho_{(e,m)} = B \times \{m\}$. Furthermore, $(\iota,m)$ is the greatest element of $\rho_{(e,m)}$ with respect to the Lawson order "$\leq_\ell$". Thus, $B \times_\varphi M$ is an F-rpp semigroup and $M_{B \times_\varphi M} = L^*_{(\iota,1)}$. By Proposition 3.4, $B \times_\varphi M$ is a factorizable rpp monoid with left univocal type-$(\mathcal{I}, \mathcal{L}^*)$ factorization $(\{\iota\} \times M, E(B \times_\varphi M))$. □

We now establish the following main theorem.

**Theorem 4.2** (**Main Theorem**) *Let $B$ be a band with an identity, and $M$ a left cancellative monoid. Let $\varphi$ be a monoid homomorphism of $M$ into $End(B)$. Then the semidirect product of $B$ by $M$ for the structure mapping $\varphi$ is a left univocal factorizable rpp monoid of type-$(\mathcal{I}, \mathcal{L}^*)$. Conversely, any left univocal factorizable rpp monoid of type-$(\mathcal{I}, \mathcal{L}^*)$ can be constructed by the above manner.*

*Proof.* By Proposition 4.1, we only need to prove the converse part. Now assume that $S$ is a factorizable rpp monoid with the identity 1 which admits a left univocal type-$(\mathcal{I}, \mathcal{L}^*)$ factorization $(M, E)$. Then, by Proposition 3.5, $S$ is an F-rpp semigroup with $M = M_S$ and $E = E(S)$. For $m \in M$ and $e \in E$, by Lemma 2.3(3), there exists a unique $(em)^\diamond \in E$ such that $em = m(em)^\diamond$, since $m^* = 1$ and $\omega(m^*) = E(S)$. Now, we consider the following mapping $\varphi_m$ :

$$\varphi_m : \ E \to E \ \text{ defined by } e \mapsto (em)^\diamond.$$

Let $f \in E$. Then $m(efm)^\diamond = efm = em(fm)^\diamond = m(em)^\diamond(fm)^\diamond$, and by the uniqueness of $(em)^\diamond$, $(efm)^\diamond = (em)^\diamond(fm)^\diamond$. Thus, $(ef)\varphi_m = (e\varphi_m)(f\varphi_m)$. This shows that $\varphi_m$ is a homomorphism of $E$ into itself.

Let $\varphi$ be a mapping which maps from M into End(E):

$$\varphi : \ M \to End(E) \text{ defined by } : m \mapsto \varphi_m.$$

Then,it is clear that $\varphi$ is a semigroup homomorphism. Now, let $m, n \in M$ and $e \in E$. Then we deduce that

$$mn(emn)^\diamond = emn = m(em)^\diamond n = mn((em)^\diamond n)^\diamond.$$

By the uniqueness of $(em)^\diamond$, we have $((em)^\diamond n)^\diamond = (emn)^\diamond$ and

$$e\varphi_m\varphi_n = (em)^\diamond\varphi_n = ((em)^\diamond n)^\diamond = (emn)^\diamond = e\varphi_{mn}.$$

Hence, $\varphi_{mn} = \varphi_m\varphi_n$, and thereby, $\varphi$ is a semigroup homomorphism. Denote by 1 the identity of $S$. Since $(e1)^\diamond = e$, we see that $\varphi_1$ is the identity of $E$. Thus $\varphi$ is a monoid homomorphism.

Let $s \in S$. Note that $s \leq_\ell m_s$. Then, by Lemma 2.3 (3), we have a unique $s^\diamond \in E$ such that $s = m_s s^\diamond$ because $m^* = 1$ and $\omega(m^*) = E(S)$. Since $\mathcal{L}^*$ is a right congruence and $m_s\mathcal{L}^*1$, $s = m_s s^\diamond \mathcal{L}^* 1 s^\diamond = s^\diamond$. Now, we define

$$\theta : \ S \to E \times_\varphi M \text{ by } s \mapsto (s^\diamond, m_s).$$

Obviously, $\theta$ is well defined. It remains to verify that $\theta$ is a semigroup isomorphism. If $(s^\diamond, m_s) = (t^\diamond, m_t)$, then $s^\diamond = t^\diamond$ and $m_s = m_t$. Hence $s = m_s s^\diamond = m_t t^\diamond = t$ and this shows that $\theta$ is injective. By Lemma 2.3, $(me)^\diamond = e$ and

$m_{me} = m$, for any $e \in E$ and $m \in M$. Thus, $(me)\theta = (e, m)$ and hence $\theta$ is surjective.

Finally, if $s, t \in S$, then $m_{st} = m_s m_t$. By routine computation, we have

$$m_{st}(st)^\diamond = st = m_s s^\diamond m_t t^\diamond = m_s m_t (s^\diamond m_t)^\diamond t^\diamond = m_{st}(s^\diamond \varphi_{m_t})t^\diamond.$$

Since $m_{st}\mathcal{L}^*1$, by Lemma 2.2, we deduce that

$$(s^\diamond \varphi_{m_t})t^\diamond = 1(s^\diamond \varphi_{m_t})t^\diamond = 1(st)^\diamond = (st)^\diamond$$

and

$$\begin{aligned}(s\theta)(t\theta) &= (s^\diamond, m_s)(t^\diamond, m_t) = ((s^\diamond \varphi_{m_t})t^\diamond, m_s m_t) \\ &= ((s^\diamond m_t)^\diamond t^\diamond, m_{st}) = ((st)^\diamond, m_{st}) = (st)\theta.\end{aligned}$$

This shows that $\theta$ is a semigroup homomorphism. Consequently, $\theta$ is an isomorphism. □

**Remark 4.3** By Lemma 2.4 and Proposition 3.5, we can easily see that a u-IC quasi-adequate semigroup $S$ is always a factorable rpp monoid admitting a left univocal type $(\mathcal{I}, \mathcal{L}^*)$ factorization $(H_1^*, E(S))$. Thus, by Theorem 4.2, we can see immediately that a u-IC quasi-adequate semigroup is a semidirect product of a band and a cancellative monoid, and vice versa. Thus, in this way, we can re-obtain the main result recently obtained by Zhang-Chen for IC quasi-adequate semigroups in [13].

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