Palindromic Permutations and Generalized Smarandache Palindromic Permutations [*][†]

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Abstract

The idea of left(right) palindromic permutations(LPPs,RPPs) and left(right) generalized Smarandache palindromic permutations(LGSPPs,RGSPPs) are introduced in symmetric groups S_n of degree n. It is shown that in S_n, there exist a LPP and a RPP and they are unique(this fact is demonstrated using S_2 and S_3). The dihedral group D_n is shown to be generated by a RGSPP and a LGSPP(this is observed to be true in S_3) but the geometric interpretations of a RGSPP and a LGSPP are found not to be rotation and reflection respectively. In S_3, each permutation is at least a RGSPP or a LGSPP. There are 4 RGSPPs and 4 LGSPPs in S_3, while 2 permutations are both RGSPPs and LGSPPs. A permutation in S_n is shown to be a LPP or RPP(LGSPP or RGSPP) if and only if its inverse is a LPP or RPP(LGSPP or RGSPP) respectively. Problems for future studies are raised.

1 Introduction

According to Ashbacher and Neirynck [1], an integer is said to be a palindrome if it reads the same forwards and backwards. For example, 12321 is a palindromic number. They also stated that it is easy to prove that the density of the palindromes is zero in the set of positive integers and they went ahead to answer the question on the density of generalized Smarandache palindromes (GSPs) by showing that the density of GSPs in the positive integers is approximately 0.11. Gregory [2], Smarandache [8] and Ramsharan [7] defined a generalized Smarandache palindrome (GSP) as any integer or number of the form

$$a_1a_2a_3\cdots a_na_n\cdots a_3a_2a_1 \qquad \text{or} \qquad a_1a_2a_3\cdots a_{n-1}a_na_{n-1}\cdots a_3a_2a_1$$

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where all $a_1, a_2, a_3, \cdots a_n \in \mathbb{N}$ having one or more digits. On the other hand, Hu [3] calls any integer or number of this form a Smarandache generalized palindrome(SGP). His naming will not be used here the first naming will be adopted.

Numbers of this form have also been considered by Khoshnevisan [4], [5] and [6]. For the sake of clarification, it must be mentioned that the possibility of the trivial case of enclosing the entire number is excluded. For example, 12345 can be written as (12345). In this case, the number is simply said to be a palindrome or a palindromic number as it was mentioned earlier on. So, every number is a GSP. But this possibility is eliminated by requiring that each number be split into at least two segments if it is not a regular palindrome. Trivially, since each regular palindrome is also a GSP and there are GSPs that are not regular palindromes, there are more GSPs than there are regular palindromes. As mentioned by Gregory [2], very interesting GSPs are formed from smarandacheian sequences. For an illustration he cited the smarandacheian sequence

$$11, 1221, 123321, \cdots, 12345678987654321, 1234567891010987654321,$$

$$123456789101111110987654321, \cdots$$

and observed that all terms are all GSPs. He also mentioned that it has been proved that the GSP 1234567891010987654321 is a prime and concluded his work by possing the question of 'How many primes are in the GSP sequence above?'.

Special mappings such as morphisms(homomorphisms, endomorphisms, automorphisms, isomorphisms e.t.c) have been useful in the study of the properties of most algebraic structures(e.g groupoids, quasigroups, loops, semigroups, groups e.tc.). In this work, the notion of palindromic permutations and generalized Smarandache palindromic permutations are introduced and studied using the symmetric group on the set $\mathbb{N}$ and this can now be viewed as the study of some palindromes and generalized Smarandache palindromes of numbers.

The idea of left(right) palindromic permutations(LPPs,RPPs) and left(right) generalized Smarandache palindromic permutations(LGSPPs,RGSPPs) are introduced in symmetric groups S_n of degree n. It is shown that in S_n, there exist a LPP and a RPP and they are unique. The dihedral group D_n is shown to be generated by a RGSPP and a LGSPP but the geometric interpretations of a RGSPP and a LGSPP are found not to be rotation and reflection respectively. In S_3, each permutation is at least a RGSPP or a LGSPP. There are 4 RGSPPs and 4 LGSPPs in S_3, while 2 permutations are both RGSPPs and LGSPPs. A permutation in S_n is shown to be a LPP or RPP(LGSPP or RGSPP) if and only if its inverse is a LPP or RPP(LGSPP or RGSPP) respectively. Some of these results are demonstrated with S_2 and S_3. Problems for future studies are raised.

But before then, some definitions and basic results on symmetric groups in classical group theory which shall be employed and used are highlighted first.

2 Preliminaries

Definition 2.1 *Let X be a non-empty set. The group of all permutations of X under composition of mappings is called the **symmetric group** on X and is denoted by S_X. A* *subgroup of S_X is called a permutation group on X.*

It is easily seen that a bijection $X \simeq Y$ induces in a natural way an isomorphism $S_X \cong S_Y$. If $|X| = n$, S_X is denoted by S_n and called the *symmetric group of degree n.*

A permutation $\sigma \in S_n$ can be exhibited in the form

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix},$$

consisting of two rows of integers; the top row has integers $1, 2, \cdots, n$ usually(but not necessarily) in their natural order, and the bottom row has $\sigma(i)$ below i for each $i = 1, 2, \cdots, n$. This is called a two-row notation for a permutation. There is a simpler, one-row notation for a special kind of permutation called *cycle.*

Definition 2.2 *Let $\sigma \in S_n$. If there exists a list of distinct integers $x_1, \cdots, x_r \in \mathbb{N}$ such that*

$$\begin{aligned} &\sigma(x_i) = x_{i+1}, \qquad i = 1, \cdots, r-1, \\ &\sigma(x_r) = x_1, \\ &\sigma(x) = x \text{ if } x \notin \{x_1, \cdots, x_r\}, \end{aligned}$$

then σ is called a cycle of length r and denoted by $(x_1 \cdots x_r)$.

Remark 2.1 *A cycle of length 2 is called a transposition. In other words , a cycle $(x_1 \cdots x_r)$ moves the integers $x_1, \cdots, x_r$ one step around a circle and leaves every other integer in $\mathbb{N}$. If $\sigma(x) = x$, we say σ does not move x. Trivially, any cycle of length 1 is the identity mapping I or e. Note that the one-row notation for a cycle does not indicate the degree n, which has to be understood from the context.*

Definition 2.3 *Let X be a set of points in space, so that the distance $d(x, y)$ between points x and y is given for all $x, y \in X$. A permutation σ of X is called a* ***symmetry*** *of X if*

$$d(\sigma(x), \sigma(y)) = d(x, y) \ \forall \ x, y \in X.$$

Let X be the set of points on the vertices of a regular polygon which are labelled $\{1, 2, \cdots, n\}$ i.e $X = \{1, 2, \cdots, n\}$.

The group of symmetries of a regular polygon P_n of n sides is called the ***dihedral group of degree*** *n and denoted D_n.*

Remark 2.2 *It must be noted that D_n is a subgroup of S_n i.e $D_n \leq S_n$.*

Definition 2.4 *Let S_n be a symmetric group of degree n. If $\sigma \in S_n$ such that*

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix},$$

then

1. *the number* $N_\lambda(\sigma) = 12\cdots n\sigma(n)\cdots\sigma(1)$ *is called the left palindromic value(LPV) of* σ.
2. *the number* $N_\rho(\sigma) = 12\cdots n\sigma(1)\cdots\sigma(n)$ *is called the right palindromic value(RPV) of* σ.

Definition 2.5 *Let* $\sigma \in S_X$ *such that*

$$\sigma = \left(\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{array} \right).$$

If $X = \mathbb{N}$, *then*

1. σ *is called a left palindromic permutation(LPP) if and only if the number* $N_\lambda(\sigma)$ *is a palindrome.*
$$PP_\lambda(S_X) = \{\sigma \in S_X \ : \ \sigma \textit{ is a LPP}\}$$
2. σ *is called a right palindromic permutation(RPP) if and only if the number* $N_\rho(\sigma)$ *is a palindrome.*
$$PP_\rho(S_X) = \{\sigma \in S_X \ : \ \sigma \textit{ is a RPP}\}$$
3. σ *is called a palindromic permutation(PP) if and only if it is both a LPP and a RPP.*

$$PP(S_X) = \{\sigma \in S_X \ : \ \sigma \textit{ is a LPP and a RPP }\} = PP_\lambda(S_X)\bigcap PP_\rho(S_X)$$

Definition 2.6 *Let* $\sigma \in S_X$ *such that*

$$\sigma = \left(\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{array} \right).$$

If $X = \mathbb{N}$, *then*

1. σ *is called a left generalized Smarandache palindromic permutation(LGSPP) if and only if the number* $N_\lambda(\sigma)$ *is a GSP.*
$$GSPP_\lambda(S_X) = \{\sigma \in S_X \ : \ \sigma \textit{ is a LGSPP}\}$$
2. σ *is called a right generalized Smarandache palindromic permutation(RGSPP) if and only if the number* $N_\rho(\sigma)$ *is a GSP.*
$$GSPP_\rho(S_X) = \{\sigma \in S_X \ : \ \sigma \textit{ is a RGSPP}\}$$
3. σ *is called a generalized Smarandache palindromic permutation(GSPP) if and only if it is both a LGSPP and a GSPP.*
$$GSPP(S_X) = \{\sigma \in S_X \ : \ \sigma \textit{ is a LGSPP and a RGSPP }\} = GSPP_\lambda(S_X)\bigcap GSPP_\rho(S_X)$$

Theorem 2.1 *(Cayley Theorem)*

Every group is isomorphic to a permutation group.

Theorem 2.2 *The dihedral group D_n is a group of order $2n$ generated by two elements σ, τ satisfying $\sigma^n = e = \tau^2$ and $\tau\sigma = \sigma^{n-1}\tau$, where*

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix} \qquad \textit{and} \qquad \tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & n & \cdots & 2 \end{pmatrix}.$$

3 Main Results

Theorem 3.1 *In any symmetric group S_n of degree n, there exists*

1. *a LPP and it is unique.*
2. *a RPP and it is unique.*

But there does not exist a PP.

Proof

Let $\sigma \in S_n$, then

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{pmatrix}.$$

1. When

   $$\sigma(n) = n, \sigma(n-1) = n-1, \cdots, \sigma(2) = 2, \sigma(1) = 1$$

   then the number

   $$N_\lambda(\sigma) = 12\cdots n\sigma(n)\cdots\sigma(2)\sigma(1) = 12\cdots nn\cdots 21$$

   is a palindrome which implies $\sigma \in PP_\lambda(S_n)$. So, there exists a LPP. The uniqueness is as follows. Observe that

   $$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} = I.$$

   Since S_n is a group for all $n \in \mathbb{N}$ and I is the identity element(mapping), then it must be unique.

2. When

   $$\sigma(1) = n, \sigma(2) = n-1, \cdots, \sigma(n-1) = 2, \sigma(n) = 1$$

   then the number

   $$N_\rho(\sigma) = 12\cdots n\sigma(1)\cdots\sigma(n-1)\sigma(n) = 12\cdots nn\cdots 21$$

   is a palindrome which implies $\sigma \in PP_\rho(S_n)$. So, there exists a RPP. The uniqueness is as follows. If there exist two of such, say σ_1 and σ_2 in S_n, then

   $$\sigma_1 = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1(1) & \sigma_1(2) & \cdots & \sigma_1(n) \end{pmatrix} \qquad \text{and} \qquad \sigma_2 = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_2(1) & \sigma_2(2) & \cdots & \sigma_2(n) \end{pmatrix}$$

such that

$$N_\rho(\sigma_1) = 12\cdots n\sigma_1(1)\cdots\sigma_1(n-1)\sigma_1(n)$$

and

$$N_\rho(\sigma_2) = 12\cdots n\sigma_2(1)\cdots\sigma_2(n-1)\sigma_2(n)$$

are palindromes which implies

$$\sigma_1(1) = n, \sigma_1(2) = n-1, \cdots, \sigma_1(n-1) = 2, \sigma_1(n) = 1$$

and

$$\sigma_2(1) = n, \sigma_2(2) = n-1, \cdots, \sigma_2(n-1) = 2, \sigma_2(n) = 1.$$

So, $\sigma_1 = \sigma_2$, thus σ is unique.

The proof of the last part is as follows. Let us assume by contradiction that there exists a PP $\sigma \in S_n$. Then if

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix},$$

$$N_\lambda(\sigma) = 12\cdots n\sigma(n)\cdots\sigma(2)\sigma(1)$$

and

$$N_\rho(\sigma) = 12\cdots n\sigma(1)\cdots\sigma(n-1)\sigma(n)$$

are palindromes. So that $\sigma \in S_n$ is a PP. Consequently,

$$n = \sigma(n) = 1, n-1 = \sigma(n-1) = 2, \cdots, 1 = \sigma(1) = n,$$

so that σ is not a bijection which means $\sigma \notin S_n$. This is a contradiction. Hence, no PP exist.

Example 3.1 *Let us consider the symmetric group S_2 of degree 2. There are two permutations of the set $\{1,2\}$ given by*

$$I = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \textit{and} \qquad \delta = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

$$N_\rho(I) = 1212 = (12)(12), N_\lambda(I) = 1221 \textit{ or } N_\lambda(I) = 1(22)1,$$

$$N_\rho(\delta) = 1221 \textit{ or } N_\rho(\delta) = (12)(21) \textit{ and } N_\lambda(\delta) = 1212 = (12)(12).$$

So, I and δ are both RGSPPs and LGSPPs which implies I and δ are GSPPs i.e $I, \delta \in GSPP_\rho(S_2)$ and $I, \delta \in GSPP_\lambda(S_2) \Rightarrow I, \delta \in GSPP(S_2)$. Therefore, $GSPP(S_2) = S_2$. Furthermore, it can be seen that the result in Theorem 3.1 is true for S_2 because only I is a LPP and only δ is a RPP. There is definitely no PP as the theorem says.

Example 3.2 *Let us consider the symmetric group S_3 of degree 3. There are six permutations of the set $\{1, 2, 3\}$ given by*

$$e = I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \tau_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

As claimed in Theorem 3.1, the unique LPP in S_3 is I while the unique RPP in S is τ_2. There is no PP as the theorem says.

Lemma 3.1 *In S_3, the following are true.*

1. *At least $\sigma \in GSPP_\rho(S_3)$ or $\sigma \in GSPP_\lambda(S_3)$ $\forall$ $\sigma \in S_3$.*
2. *$|GSPP_\rho(S_3)| = 4$, $|GSPP_\lambda(S_3)| = 4$ and $|GSPP(S_3)| = 2$.*

Proof
Observe the following :

$$N_\lambda(I) = 123321,\ N_\rho(I) = 123123 = (123)(123).$$

$$N_\lambda(\sigma_1) = 123132,\ N_\rho(\sigma_1) = 123231 = 1(23)(23)1.$$

$$N_\lambda(\sigma_2) = 123213,\ N_\rho(\sigma_2) = 123312 = (12)(33)(12).$$

$$N_\lambda(\tau_1) = 123231 = 1(23)(23)1,\ N_\rho(\tau_1) = 123132.$$

$$N_\lambda(\tau_2) = 123123 = (123)(123),\ N_\rho(\tau_2) = 123321 = 123321.$$

$$N_\lambda(\tau_3) = 123312 = (12)(33)(12),\ N_\rho(\tau_3) = 123213.$$

So, $GSPP_\lambda(S_3) = \{I, \tau_1, \tau_2, \tau_3\}$ and $GSPP_\rho(S_3) = \{I, \sigma_1, \sigma_2, \tau_2\}$. Thus, 1. is true. Therefore, $|GSPP_\rho(S_3)| = 4$, $|GSPP_\lambda(S_3)| = 4$ and $|GSPP(S_3)| = |GSPP_\rho(S_3) \bigcap GSPP_\lambda(S_3)| = 2$. So, 2. is true.

Lemma 3.2 *S_3 is generated by a RGSPP and a LGSPP.*

Proof
Recall from Example 3.2 that

$$S_3 = \{I = e, \sigma_1, \sigma_2, \tau_1, \tau_2, \tau_3\}.$$

If $\sigma = \sigma_1$ and $\tau = \tau_1$, then it is easy to verify that

$$\sigma^2 = \sigma_2,\ \sigma^3 = e,\ \tau^2 = e,\ \sigma\tau = \tau_3,\ \sigma^2\tau = \tau_2 = \tau\sigma \text{ hence,}$$

$$S_3 = \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau_3\} \Rightarrow S_3 = \langle\sigma, \tau\rangle.$$

From the proof Lemma 3.1, σ is a RGSPP and τ is a LGSPP. This justifies the claim.

Remark 3.1 *In Lemma 3.2, S_3 is generated by a RGSPP and a LGSPP. Could this statement be true for all S_n of degree n? Or could it be true for some subgroups of S_n? Also, it is interesting to know the geometric meaning of a RGSPP and a LGSPP. So two questions are possed and the two are answered.*

Question 3.1

1. *Is the symmetric group S_n of degree n generated by a RGSPP and a LGSPP? If not, what permutation group(s) is generated by a RGSPP and a LGSPP?*
2. *Are the geometric interpretations of a RGSPP and a LGSPP rotation and reflection respectively?*

Theorem 3.2 *The dihedral group D_n is generated by a RGSPP and a LGSPP i.e $D_n = \langle \sigma, \tau \rangle$ where $\sigma \in GSPP_\rho(S_n)$ and $\tau \in GSPP_\lambda(S_n)$.*

Proof
Recall from Theorem2.2 that the dihedral group $D_n = \langle \sigma, \tau \rangle$ where

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & 1 \end{pmatrix} \qquad \text{and} \qquad \tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & n & \cdots & 2 \end{pmatrix}.$$

Observe that

$$N_\rho(\sigma) = 123\cdots n23\cdots n1 = 1(23\cdots n)(23\cdots n)1,\ N_\lambda(\sigma) = 123\cdots n1n\cdots 32.$$

$$N_\rho(\tau) = 12\cdots n1n\cdots 2,\ N_\lambda(\tau) = 12\cdots n2\cdots n1 = 1(2\cdots n)(2\cdots n)1.$$

So, $\sigma \in GSPP_\rho(S_n)$ and $\tau \in GSPP_\lambda(S_n)$. Therefore, the dihedral group D_n is generated by a RGSPP and a LGSPP.

Remark 3.2 *In Lemma 3.2, it was shown that S_3 is generated by a RGSPP and a LGSPP. Considering Theorem 3.2 when $n = 3$, it can be deduced that D_3 will be generated by a RGSPP and a LGSPP. Recall that $|D_3| = 2 \times 3 = 6$, so $S_3 = D_3$. Thus Theorem 3.2 generalizes Lemma 3.2.*

Rotations and Reflections Geometrically, in Theorem 3.2, σ is a rotation of the regular polygon P_n through an angle $\frac{2\pi}{n}$ in its own plane, and τ is a reflection (or a turning over) in the diameter through the vertex 1. It looks like a RGSPP and a LGSPP are formed by rotation and reflection respectively. But there is a contradiction in S_4 which can be traced from a subgroup of S_4 particularly the Klein four-group. The Klein four-group is the group of symmetries of a four sided non-regular polygon(rectangle). The elements are:

$$e = I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\text{and} \qquad \delta_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Observe the following:

$$N_\rho(\delta_1) = 12343412 = (12)(34)(34)(12),\ N_\lambda(\delta_1) = 12342143.$$

$$N_\rho(\delta_2) = 12342143 = 12342143,\ N_\lambda(\delta_2) = 12343412 = (12)(34)(34)(12).$$

$$N_\rho(\delta_3) = 12344321 = 123(44)321,\ N_\lambda(\delta_3) = 12341234 = (1234)(1234).$$

So, δ_1 is a RGSPP while δ_2 is a LGSPP and δ_3 is a GSPP. Geometrically, δ_1 is a rotation through an angle of π while δ_2 and δ_3 are reflections in the axes of symmetry parallel to the sides. Thus δ_3 which is a GSPP is both a reflection and a rotation, which is impossible. Therefore, the geometric meaning of a RGSPP and a LGSPP are not rotation and reflection respectively. It is difficult to really ascertain the geometric meaning of a RGSPP and a LGSPP if at all it exist.

How beautiful will it be if $GSPP_\rho(S_n)$, $PP_\rho(S_n)$, $GSPP_\lambda(S_n)$, $PP_\lambda(S_n)$, $GSPP(S_n)$ and $PP(S_n)$ form algebraic structures under the operation of map composition.

Theorem 3.3 *Let S_n be a symmetric group of degree n. If $\sigma \in S_n$, then*

1. $\sigma \in PP_\lambda(S_n) \Leftrightarrow \sigma^{-1} \in PP_\lambda(S_n)$.
2. $\sigma \in PP_\rho(S_n) \Leftrightarrow \sigma^{-1} \in PP_\rho(S_n)$.
3. $I \in PP_\lambda(S_n)$.

Proof

1. $\sigma \in PP_\lambda(S_n)$ implies
$$N_\lambda(\sigma) = 12\cdots n\sigma(n)\cdots\sigma(2)\sigma(1)$$
is a palindrome. Consequently,
$$\sigma(n) = n, \sigma(n-1) = n-1, \cdots, \sigma(2) = 2, \sigma(1) = 1.$$
So,
$$N_\lambda(\sigma^{-1}) = \sigma(1)\sigma(2)\cdots\sigma(n)n\cdots 21 = 12\cdots nn\cdots 21 \Rightarrow \sigma^{-1} \in PP_\lambda(S_n).$$
The converse is similarly proved by carrying out the reverse of the procedure above.

2. $\sigma \in PP_\rho(S_n)$ implies
$$N_\rho(\sigma) = 12\cdots n\sigma(1)\cdots\sigma(n-1)\sigma(n)$$
is a palindrome. Consequently,
$$\sigma(1) = n, \sigma(2) = n-1, \cdots, \sigma(n-1) = 2, \sigma(n) = 1.$$
So,
$$N_\rho(\sigma^{-1}) = \sigma(1)\cdots\sigma(n-1)\sigma(n)12\cdots n = n\cdots 2112\cdots n \Rightarrow \sigma^{-1} \in PP_\rho(S_n).$$
The converse is similarly proved by carrying out the reverse of the procedure above.

3.

$$I = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

$$N_\lambda(I) = 12\cdots nn\cdots 21 \Rightarrow I \in PP_\lambda(S_n).$$

Theorem 3.4 *Let S_n be a symmetric group of degree n. If $\sigma \in S_n$, then*

1. $\sigma \in GSPP_\lambda(S_n) \Leftrightarrow \sigma^{-1} \in GSPP_\lambda(S_n)$.
2. $\sigma \in GSPP_\rho(S_n) \Leftrightarrow \sigma^{-1} \in GSPP_\rho(S_n)$.
3. $I \in GSPP(S_n)$.

Proof
If $\sigma \in S_n$, then

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

So,

$$N_\lambda(\sigma) = 12\cdots n\sigma(n)\cdots\sigma(2)\sigma(1)$$

and

$$N_\rho(\sigma) = 12\cdots n\sigma(1)\cdots\sigma(n-1)\sigma(n)$$

are numbers with even number of digits whether n is an even or odd number. Thus, $N_\rho(\sigma)$ and $N_\lambda(\sigma)$ are GSPs defined by

$$a_1a_2a_3\cdots a_na_n\cdots a_3a_2a_1$$

and not

$$a_1a_2a_3\cdots a_{n-1}a_na_{n-1}\cdots a_3a_2a_1$$

where all $a_1, a_2, a_3, \cdots a_n \in \mathbb{N}$ having one or more digits because the first has even number of digits(or grouped digits) while the second has odd number of digits(or grouped digits). The following grouping notations will be used:

$$(a_i)_{i=1}^n = a_1a_2a_3\cdots a_n \qquad \text{and} \qquad [a_i]_{i=1}^n = a_na_{n-1}a_{n-2}\cdots a_3a_2a_1.$$

Let $\sigma \in S_n$ such that

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_n) \end{pmatrix}$$

where $x_i \in \mathbb{N}\ \forall\ i \in \mathbb{N}$.

1. So, $\sigma \in GSPP_\lambda(S_n)$ implies

$$N_\lambda(\sigma) = (x_{i_1})_{i_1=1}^{n_1}(x_{i_2})_{i_2=(n_1+1)}^{n_2}(x_{i_3})_{i_3=(n_2+1)}^{n_3}\cdots(x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}(x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n}$$

$$[\sigma(x_{i_n})]_{i_n=(n_{n-1}+1)}^{n_n}[\sigma(x_{i_{n-1}})]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}\cdots[\sigma(x_{i_3})]_{i_3=(n_2+1)}^{n_3}[\sigma(x_{i_2})]_{i_2=(n_1+1)}^{n_2}[\sigma(x_{i_1})]_{i_1=1}^{n_1}$$

is a GSP where $x_{i_j} \in \mathbb{N}\ \forall\ i_j \in \mathbb{N}$, $j \in \mathbb{N}$ and $n_n = n$. The interval of integers [1,n] is partitioned into

$$[1, n] = [1, n_1] \cup [n_1 + 1, n_2] \cup \cdots \cup [n_{n-2} + 1, n_{n-1}] \cup [n_{n-1}, n_n].$$

The length of each grouping $(\cdot)_{i_j}^{n_j}$ or $[\cdot]_{i_j}^{n_j}$ is determined by the corresponding interval of integers $[n_i + 1, n_{i+1}]$ and it is a matter of choice in other to make the number $N_\lambda(\sigma)$ a GSP.

Now that $N_\lambda(\sigma)$ is a GSP, the following are true:

$$\begin{array}{rcl}
(x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n} = [\sigma(x_{i_n})]_{i_n=(n_{n-1}+1)}^{n_n} & \Leftrightarrow & [x_{i_n}]_{i_n=(n_{n-1}+1)}^{n_n} = (\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n} \\
(x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} = [\sigma(x_{i_{n-1}})]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} & \Leftrightarrow & [x_{i_{n-1}}]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} = (\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} \\
\vdots & \vdots & \vdots \\
(x_{i_2})_{i_2=(n_1+1)}^{n_2} = [\sigma(x_{i_2})]_{i_2=(n_1+1)}^{n_2} & \Leftrightarrow & [x_{i_2}]_{i_2=(n_1+1)}^{n_2} = (\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2} \\
(x_{i_1})_{i_1=1}^{n_1} = [\sigma(x_{i_1})]_{i_1=1}^{n_1} & \Leftrightarrow & [x_{i_1}]_{i_1=1}^{n_1} = (\sigma(x_{i_1}))_{i_1=1}^{n_1}
\end{array}$$

Therefore, since

$$\sigma = \begin{pmatrix} x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \\ \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \end{pmatrix},$$

then

$$\sigma^{-1} = \begin{pmatrix} \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \\ x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \end{pmatrix},$$

so

$$N_\lambda(\sigma^{-1}) = (\sigma(x_{i_1}))_{i_1=1}^{n_1}(\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2}(\sigma(x_{i_3}))_{i_3=(n_2+1)}^{n_3}\cdots(\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}$$

$$(\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n}[x_{i_n}]_{i_n=(n_{n-1}+1)}^{n_n}[x_{i_{n-1}}]_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}\cdots[x_{i_3}]_{i_3=(n_2+1)}^{n_3}[x_{i_2}]_{i_2=(n_1+1)}^{n_2}[x_{i_1}]_{i_1=1}^{n_1}$$

is a GSP hence, $\sigma^{-1} \in GSPP_\lambda(S_n)$.

The converse can be proved in a similar way since $(\sigma^{-1})^{-1} = \sigma$.

2. Also, $\sigma \in GSPP_\rho(S_n)$ implies

$$N_\rho(\sigma) = (x_{i_1})_{i_1=1}^{n_1}(x_{i_2})_{i_2=(n_1+1)}^{n_2}(x_{i_3})_{i_3=(n_2+1)}^{n_3}\cdots(x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}(x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n}$$

$$(\sigma(x_{i_1}))_{i_1=1}^{n_1}(\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2}(\sigma(x_{i_3}))_{i_3=(n_2+1)}^{n_3}\cdots(\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}(\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n}$$

is a GSP where $x_{i_j} \in \mathbb{N}\ \forall\ i_j \in \mathbb{N}$, $j \in \mathbb{N}$ and $n_n = n$. The interval of integers [1,n] is partitioned into

$$[1, n] = [1, n_1] \cup [n_1 + 1, n_2] \cup \cdots \cup [n_{n-2} + 1, n_{n-1}] \cup [n_{n-1}, n_n].$$

The length of each grouping $(\cdot)_{i_j}^{n_j}$ is determined by the corresponding interval of integers $[n_i + 1, n_{i+1}]$ and it is a matter of choice in other to make the number $N_\rho(\sigma)$ a GSP.

Now that $N_\rho(\sigma)$ is a GSP, the following are true:

$$\begin{aligned}
(x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n} &= (\sigma(x_{i_1}))_{i_1=1}^{n_1}\\
(x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}} &= (\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2}\\
\vdots \qquad & \vdots \qquad \vdots\\
(x_{i_2})_{i_2=(n_1+1)}^{n_2} &= (\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}\\
(x_{i_1})_{i_1=1}^{n_1} &= (\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n}
\end{aligned}$$

Therefore, since

$$\sigma = \begin{pmatrix} x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \\ \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \end{pmatrix},$$

then

$$\sigma^{-1} = \begin{pmatrix} \sigma(x_1) & \cdots & \sigma(x_{i_1}) & \cdots & \sigma(x_{n_1}) & \cdots & \sigma(x_{n_{n-1}+1}) & \cdots & \sigma(x_{j_k}) & \cdots & \sigma(x_{n_n}) \\ x_1 & \cdots & x_{i_1} & \cdots & x_{n_1} & \cdots & x_{n_{n-1}+1} & \cdots & x_{j_k} & \cdots & x_{n_n} \end{pmatrix},$$

so

$$N_\rho(\sigma^{-1}) = (\sigma(x_{i_1}))_{i_1=1}^{n_1}(\sigma(x_{i_2}))_{i_2=(n_1+1)}^{n_2}(\sigma(x_{i_3}))_{i_3=(n_2+1)}^{n_3}\cdots(\sigma(x_{i_{n-1}}))_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}$$

$$(\sigma(x_{i_n}))_{i_n=(n_{n-1}+1)}^{n_n}(x_{i_1})_{i_1=1}^{n_1}(x_{i_2})_{i_2=(n_1+1)}^{n_2}(x_{i_3})_{i_3=(n_2+1)}^{n_3}\cdots(x_{i_{n-1}})_{i_{n-1}=(n_{n-2}+1)}^{n_{n-1}}(x_{i_n})_{i_n=(n_{n-1}+1)}^{n_n}$$

is a GSP hence, $\sigma^{-1} \in GSPP_\rho(S_n)$.

The converse can be proved in a similar way since $(\sigma^{-1})^{-1} = \sigma$.

3.

$$I = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

$$N_\lambda(I) = 12\cdots nn\cdots 21 = 12\cdots(nn)\cdots 21 \Rightarrow I \in GSPP_\lambda(S_n) \text{ and}$$
$$N_\rho(I) = (12\cdots n)(12\cdots n) \Rightarrow I \in GSPP_\rho(S_n)$$

thus, $I \in GSPP(S_n)$.

4 Conclusion and Future studies

By Theorem 3.1, it is certainly true in every symmetric group S_n of degree n there exist at least a RGSPP and a LGSPP(although they are actually RPP and LPP). Following Example 3.1, there are 2 RGSPPs, 2 LGSPPs and 2 GSPPs in S_2 while from Lemma 3.1, there are 4 RGSPPs, 4 LGSPPs and 2 GSPPs in S_3. Also, it can be observed that

$$|GSPP_\rho(S_2)| + |GSPP_\lambda(S_2)| + |GSPP(S_2)| = 2! = |S_2| \text{ and}$$

$$|GSPP_\rho(S_3)| + |GSPP_\lambda(S_3)| + |GSPP(S_3)| = 3! = |S_3|.$$

The following problems are open for further studies.

Problem 4.1 *1. How many RGSPPs, LGSPPs and GSPPs are in S_n?*

2. Does there exist functions $f_1, f_2, f_3 : \mathbb{N} \to \mathbb{N}$ such that $|GSPP_\rho(S_n)| = f_1(n)$, $|GSPP_\lambda(S_n)| = f_2(n)$ and $|GSPP(S_n)| = f_3(n)$?

3. In general, does the formula

$$|GSPP_\rho(S_n)| + |GSPP_\lambda(S_n)| + |GSPP(S_n)| = n! = |S_n|?$$

hold. If not, for what other $n > 3$ is it true?

The GAP package or any other appropriate mathematical package could be helpful in investigating the solutions to them.

If the first question is answered, then the number of palindromes that can be formed from the set $\{1, 2, \cdots n\}$ can be known since in the elements of S_n, the bottom row gives all possible permutation of the integers $1, 2, \cdots n$.

The Cayley Theorem(Theorem2.1) can also be used to make a further study on generalized Smarandache palindromic permutations. In this work, $\mathbb{N}$ was the focus and it does not contain the integer zero. This weakness can be strengthened by considering the set $\mathbb{Z}_n = \{0, 1, 2, \cdots n-1\}\ \forall\ n \in \mathbb{N}$. Recall that $(\mathbb{Z}_n, +)$ is a group and so by Theorem2.1 $(\mathbb{Z}_n, +)$ is isomorphic to a permutation group particularly, one can consider a subgroup of the symmetric group $S_{\mathbb{Z}_n}$.

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