# HYPERBOLIC SETS WITH NONEMPTY INTERIOR

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ABSTRACT. In this paper we study hyperbolic sets with nonempty interior. We prove the folklore theorem that every transitive hyperbolic set with interior is Anosov. We also show that on a compact surface every locally maximal hyperbolic set with nonempty interior is Anosov. Finally, we give examples of hyperbolic sets with nonempty interior for a non-Anosov diffeomorphism.

## 1. INTRODUCTION

For a diffeomorphism $f$ of a closed connected manifold a hyperbolic set $\Lambda$ is a compact $f$-invariant set whose tangent space splits into invariant uniformly contracting and uniformly expanding directions. On a compact manifold these sets often possess a very rich structure. The pioneering article by Smale [8] states many of the standard results for hyperbolic sets. Hyperbolic sets with nonempty interior are quite special. Indeed, we have:

**Theorem 1.** *Let $f : M \to M$ be a diffeomorphism of a compact manifold $M$. If $f$ has a transitive hyperbolic set $\Lambda$ with nonempty interior, then $\Lambda = M$ and $f$ is Anosov.*

Theorem 1 appears to be a well known folklore theorem. We could find no proof of it in the literature, so one is provided.

Our second result shows that the hypothesis of transitivity in Theorem 1 can be replaced with local maximality and low dimensionality.

We recall the definition of locally maximal hyperbolic sets. Let $f : M \to M$ be a diffeomorphism of a compact smooth manifold $M$. A hyperbolic set $\Lambda$ is called *locally maximal* (or *isolated*) if there exists a neighborhood $V$ of $\Lambda$ in $M$ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V)$.

**Theorem 2.** *Let $f : M \to M$ be a diffeomorphism of a compact surface $M$. If $f$ has a locally maximal hyperbolic set $\Lambda$ with nonempty interior, then $\Lambda = M$, $M$ is the 2-torus, and $f$ is Anosov.*

The assumption of local maximality in Theorem 2 is a nontrivial one: In [3] it is shown that not every hyperbolic set of a surface is contained in a locally maximal hyperbolic set.

Finally, we construct hyperbolic sets, similar to the ones constructed in [3], containing nonempty interior that are not Anosov, so some hypothesis in addition to nonempty interior is necessary in Theorems 1 and 2. It follows from Theorem 2 that these hyperbolic sets are not included in locally maximal ones.

**Theorem 3.** *There exists a diffeomorphism of a compact smooth surface and hyperbolic set $\Lambda$ such that $\Lambda$ contains nonempty interior and is not contained in any locally maximal hyperbolic set.*

## 2. Background

In this section we provide background definitions and concepts. First, we define different types of recurrence which will be useful throughout.

Let $M$ be a manifold and $f$ a homeomorphism of $M$. A point $x \in M$ is *nonwandering* if for any open set $U$ containing $x$ there is an $N > 0$ such that $f^N(U) \cap U \neq \emptyset$. Denote the set of all nonwandering points as $\text{NW}(f)$. An $\epsilon$-*chain* from a point $x$ to a point $y$ for a map $f$ is a sequence $\{x = x_0, ..., x_n = y\}$ such that the $d(f(x_{j-1}), x_j) < \epsilon$ for all $1 \leq j \leq n$. The *chain recurrent set of* $f$ is denoted $\mathcal{R}(f)$ and defined by:

$$\mathcal{R}(f) = \{x \in M \mid \text{ there is an } \epsilon\text{-chain from } x \text{ to } x \text{ for all } \epsilon > 0\}.$$

The proof of Theorem 2 will rely heavily on the structure of $\mathcal{R}(f|_\Lambda)$. For any set $\Lambda$ the following inclusions hold:

$$\text{cl}(\text{Per}(f|_\Lambda)) \subset \text{NW}(f|_\Lambda) \subset \mathcal{R}(f|_\Lambda).$$

A point $y$ is an $\omega$-*limit point* of $x$ provided there is a sequence $\{f^{n_j}(x)\}_{j=0}^{\infty}$ such that $n_j$ goes to infinity as $j$ goes to infinity and $\lim_{j\to\infty} d(f^{n_j}(x), y) = 0$. The $\omega$-*limit set* of $x$ is denoted by $\omega(x, f)$ and consists of all $\omega$-limit points of $x$ for $f$. The $\alpha$-*limit set* is defined similarly, with $n_j$ going to negative infinity, and is denoted by $\alpha(x, f)$. For a set $X$ we define the set of $\omega$-limit points to points in $X$ as

$$\omega(X, f) = \{y \in M \mid y \in \omega(x, f) \text{ for some } x \in X\}.$$

Similarly, we define the $\alpha$-limit points as

$$\alpha(X, f) = \{y \in M \mid y \in \alpha(x, f) \text{ for some } x \in X\}.$$

We now review some basic definitions and facts from hyperbolic dynamics. Let $M$ be a smooth manifold, $U \subset M$ an open set, and $f : U \to M$ a $C^1$ diffeomorphism onto its image.

**Definition:** A compact $f$-invariant set $\Lambda \subset M$ is called a *hyperbolic set* for $f$ if there is a $Df$-invariant splitting $T_\Lambda M = E^u \oplus E^s$ and positive constants $C$ and $\lambda < 1$ such that, for any point $x \in \Lambda$ and any $n \in \mathbb{N}$,

$$\|Df_x^n v\| \leq C\lambda^n \|v\|, \text{ for } v \in E_x^s, \text{ and}$$
$$\|Df_x^{-n} v\| \leq C\lambda^n \|v\|, \text{ for } v \in E_x^u.$$

A diffeomorphism $f$ is *Anosov* if $M$ is a hyperbolic set for $f$. Note, it is always possible to make a smooth change of the metric near the hyperbolic set so that $C = 1$. Such a metric is called an *adapted metric.*

For $\epsilon > 0$ sufficiently small and $x \in \Lambda$ the *local stable and unstable manifolds* are respectively:

$$W_\epsilon^s(x) = \{y \in M \mid \text{for all } n \in \mathbb{N}, d(f^n(x), f^n(y)) \leq \epsilon\}, \text{ and}$$
$$W_\epsilon^u(x) = \{y \in M \mid \text{for all } n \in \mathbb{N}, d(f^{-n}(x), f^{-n}(y)) \leq \epsilon\}.$$

The *stable and unstable manifolds* are respectively:

$$W^s(x) = \bigcup_{n \geq 0} f^{-n}\left(W_\epsilon^s(f^n(x), f)\right), \text{ and}$$
$$W^u(x) = \bigcup_{n \geq 0} f^{n}\left(W_\epsilon^u(f^{-n}(x), f)\right).$$

For $\Lambda$ a hyperbolic set of a $C^r$ diffeomorphism, the stable and unstable manifolds are injectively immersed $C^r$ submanifolds characterized by uniform contraction and uniform expansion under forward iterates of $f$, respectively.

If $\Lambda$ is an invariant set of a manifold $M$, the *stable manifold* of $\Lambda$ denoted $W^s(\Lambda)$, is defined to be all points $x \in M$ such that $\omega(x) \subset \Lambda$. Similarly, the *unstable manifold* of $\Lambda$, is defined to be all points $x \in M$ such that $\alpha(x) \subset \Lambda$. A useful well known result concerning locally maximal hyperbolic sets is the following:

**Lemma 1.** *Let $\Lambda$ be a locally maximal hyperbolic invariant set. Then,*

$$W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x), \text{ and}$$
$$W^u(\Lambda) = \bigcup_{x \in \Lambda} W^u(x).$$

Locally maximal hyperbolic sets have some special properties which will be used in proving Theorem 2. First, we have the Shadowing Theorem, see [6, p. 415]. Let $f$ be a homeomorphism of a compact manifold. Let $\{x_j\}_{j=j_1}^{j_2}$ be an $\epsilon$-chain for $f$. A point $y$ $\delta$-*shadows* $\{x_j\}_{j=j_1}^{j_2}$ provided $d(f^j(y), x_j) < \delta$ for $j_1 \leq j \leq j_2$.

**Theorem 4.** *(Shadowing Theorem) If $\Lambda$ is a locally maximal hyperbolic set, then given any $\delta > 0$ there exists an $\epsilon > 0$ and $\eta > 0$ such that if $\{x_j\}_{j=j_1}^{j_2}$ is an $\epsilon$-chain for $f$ with $d(x_j, \Lambda) < \eta$, then there is a $y$ which $\delta$-shadows $\{x_j\}_{j=j_1}^{j^2}$. If the $\epsilon$-chain is periodic, then $y$ is periodic. If $j_2 = -j_1 = \infty$, then $y$ is unique and $y \in \Lambda$.*

The Shadowing Theorem implies the following:

**Corollary 1.** *If $\Lambda$ is a locally maximal hyperbolic set of a diffeomorphism $f$, then* $\mathrm{cl}(\mathrm{Per}(f|_\Lambda)) = \mathrm{NW}(f|_\Lambda) = \mathcal{R}(f|_\Lambda)$.

A standard result is the following Spectral Decomposition Theorem [4, p. 575]. (Note in [4] the result is stated for the nonwandering set, but from the above corollary this is equal to the chain recurrent set.)

**Theorem 5.** *(Spectral Decomposition) Let $M$ be a Riemannian manifold, $U \subset M$ open, $f : U \to M$ a diffeomorphic embedding, and $\Lambda \subset U$ a compact locally maximal hyperbolic set for $f$. Then there exist disjoint closed sets $\Lambda_1, ..., \Lambda_m$ and a permutation $\sigma$ of $\{1, ..., m\}$ such that $\mathcal{R}(f|_\Lambda) = \bigcup_{i=1}^m \Lambda_i$, $f(\Lambda_i) = \Lambda_{\sigma(i)}$, and when $\sigma^k(i) = i$ then $f^k|_{\Lambda_i}$ is topologically mixing.*

A set $X$ is *topologically mixing* for $f$ provided that, for any open sets $U$ and $V$ in $X$, there is a positive integer $n_0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$. Note that if $X$ is topologically mixing for $f$, then $X$ is topologically mixing for $f^k$ for any $k \in \mathbb{N}$. Also, if a set $X$ is topologically mixing for a diffeomorphism $f$, then $X$ is topologically transitive for $f$.

For two points $p$ and $q$ in a hyperbolic set $\Lambda$ denote the set of points in the transverse intersection of $W^s(p)$ and $W^u(q)$ as $W^s(p) \pitchfork W^u(q)$. Two hyperbolic periodic points $p$ and $q$ contain a *transverse heteroclinic point* if $W^s(p) \pitchfork W^u(q) \neq \emptyset$.

For $\Lambda$ a locally maximal hyperbolic set define a relation on $\mathrm{Per}(f|_\Lambda)$ by $x \sim y$ if $W^u(x) \pitchfork W^s(y) \neq \emptyset$ and $W^s(x) \pitchfork W^u(y) \neq \emptyset$. This is an equivalence relation on $\Lambda$ and each set $\Lambda_i$ from the Spectral Decomposition Theorem is the closure of an equivalence class. Two points $x, y \in \mathcal{R}(f|_\Lambda)$ are *heteroclinically related* if $x$ and $y$ are both in the same $\Lambda_i$.

Throughout we will use the fact that locally maximal hyperbolic sets possess a local product structure. A hyperbolic set possesses a *local product structure* provided there exist constants $\delta, \epsilon > 0$ such that if $x, x' \in \Lambda$ and $d(x, x') < \delta$, then $W^s_\epsilon(x, f)$ and $W^u_\epsilon(x, f)$ intersect in exactly one point which is contained in $\Lambda$.

**Proposition 1.** [4, p. 581] *For a hyperbolic set locally maximal and possessing local product structure are equivalent conditions.*

Let $\Lambda$ be a locally maximal hyperbolic set and let the collection $\{\Lambda_i\}_{i=1}^m$ be given by the Spectral Decomposition Theorem. We define a binary relation $\ll$ by

$$\Lambda_i \ll \Lambda_j \text{ if } W^u(\Lambda_i) \pitchfork W^s(\Lambda_j) \setminus \bigcup_{l=1}^m \Lambda_l \neq \emptyset.$$

A *$k$-cycle* is a sequence of distinct sets $\Lambda_{i_1}, ..., \Lambda_{i_{k-1}}$ in $\{\Lambda_i\}_{i=1}^m$ such that

$$\Lambda_{i_1} \ll \Lambda_{i_2} \ll \cdots \ll \Lambda_{i_{k-1}} \ll \Lambda_{i_1}.$$

**Theorem 6.** *Let $\Lambda$ be a locally maximal hyperbolic set and let $\Lambda_1, ..., \Lambda_m$ be given by the Spectral Decomposition Theorem. If the sets $f^k(\Lambda_i) = \Lambda_i$ for some $k \in \mathbb{N}$, and for all $i \in \{1, ..., m\}$, then each $\Lambda_i$ is a locally maximal hyperbolic set for $f^k$ and the relation $\ll$ as defined above has at most 1-cycles restricted to $\Lambda$.*

**Proof.** Fix $\delta$ and $\epsilon$ as given by the local product structure of $\Lambda$. Under the action of $f^k$ each $\Lambda_i$ is a hyperbolic set. Given two periodic points $p, q \in \Lambda_i$ such that $d(p, q) < \delta$ it is easy to see that the points in $W^s_\epsilon(p) \cap W^u_\epsilon(q)$ and $W^s_\epsilon(q) \cap W^u_\epsilon(p)$ are both contained in $\mathcal{R}(f|_\Lambda)$. Since periodic points of $f^k|_\Lambda$ are dense in $\Lambda_i$ each $\Lambda_i$ is a locally maximal hyperbolic set under $f^k$.

We next show that the relation $\ll$ has no $l$-cycles for $l > 1$. We will show that if there were an $l$-cycle for $l > 1$, then each of the sets in the cycle would be heteroclinically related. Since each $\Lambda_i$ is the closure of a heteroclinic class it follows that the cycle is a 1-cycle. The following lemma will help establish the heteroclinic relation.

**Lemma 2.** *For $p$ a periodic point in $\Lambda_i$ the set $W^s(p)$ is dense in $W^s(\Lambda_i)$ and the set $W^u(p)$ is dense in $W^u(\Lambda_i)$.*

**Proof of Lemma.** Let $p \in \Lambda_i$ be periodic. Any other periodic point $q \in \Lambda_i$ is heteroclinically related to $p$ and the Inclination Lemma [6, p. 203] then shows that $W^s(p)$ accumulates on $W^s(q)$. From this it follows that $W^s(p)$ is dense in $W^s(\Lambda_i)$. Similarly, $W^u(p)$ is dense in $W^u(\Lambda_i)$. □

We now return to the proof of the theorem. Let $p \in \Lambda_{i_1}$ be a periodic point. Next assume that $f$ has a cycle

$$\Lambda_{i_1} \ll \Lambda_{i_2} \ll \cdots \ll \Lambda_{i_{l-1}} \ll \Lambda_{i_1}.$$

The density of periodic points in $\Lambda_i$ for each $i \in \{1, ..., m\}$ and the above lemma imply that for any $j \in \{1, ..., l-1\}$ and any periodic point $q_j \in \Lambda_{i_j}$, the point $q_j$ is heteroclinically related to $p$. We then have that $\Lambda_{i_j} = \Lambda_{i_1}$ for all $j$. Hence, the relation $\ll$ as defined above has at most 1-cycles restricted to $\Lambda$. □

The above theorem implies that we can talk of the lowest and highest elements in the relation $\ll$. This will be useful in Section 3 in proving the existence of attractors and repellers.

In the proof of Theorem 2 it will be useful to know more about the structure of hyperbolic attractors contained in compact smooth Riemannian surfaces. Most of this material is a review of [2].

A set $\Lambda_a$ is a *hyperbolic attractor* provided $\Lambda_a$ is a hyperbolic set, $f|_{\Lambda_a}$ is transitive, a neighborhood $V$ of $\Lambda_a$ exists such that $f(\mathrm{cl}(V)) \subset V$, and $\Lambda_a = \bigcap_{n \in \mathbb{N}} f^n(V)$. The neighborhood $V$ is an *attracting neighborhood* for $\Lambda_a$. A hyperbolic attractor is *nontrivial* if it is not a periodic orbit.

Similarly, a set $\Lambda_r$ is a *hyperbolic repeller* provided $\Lambda_r$ is a hyperbolic set, $f|_{\Lambda_r}$ is transitive, a neighborhood $V$ of $\Lambda_r$ exists such that $f^{-1}(\mathrm{cl}(V)) \subset V$, and $\Lambda_r = \bigcap_{n\in\mathbb{N}} f^{-n}(V)$. A hyperbolic repeller is *nontrivial* if it is not a periodic orbit. The following standard result will be useful in the proof of Theorem 2.

**Proposition 2.** *Let $\Lambda$ be a hyperbolic attractor. Then, $W^u(p) \subset \Lambda$ for any point $p \in \Lambda$.*

Let $\Lambda_a$ be a hyperbolic attractor for a diffeomorphism $f$ of a compact surface $M$. If $x \in W^s(\Lambda_a)$, then $x \in W^s(y)$ for some $y \in \Lambda_a$. Denote by $W^s(x) = W^s(y)$ the stable manifold passing through $x$. A *stable separatrix* is a connected component of $W^s(x) - \{x\}$.

For $x \in W^u(\Lambda_a)$ there exists a homeomorphism $h$ of the open unit square $Q = (-1,1) \times (-1,1)$ to $M$ such that $h(0,0) = x$ and $h(Q) \cap W^u(\Lambda_a) = h((-1,1)\times F)$ where $F \subset (-1,1)$. Furthermore, for each $y \in F$ the set $h((-1,1) \times \{y\})$ is a neighborhood of $h(0,y)$ in $W^u(h(0,y))$. So for each point $x \in \Lambda_a$ and $N$ a sufficiently small neighborhood of $x$ the set $N \cap W^u(\Lambda_a)$ is a lamination. A point $x \in W^u(\Lambda_a)$ is a *u-border* if it belongs to an arc $h((-1,1) \times y)$, where $y$ is an extreme point of a component of the complement of $F$ in $(-1,1)$. Replacing stable with unstable we similarly define a point as an *s-border*.

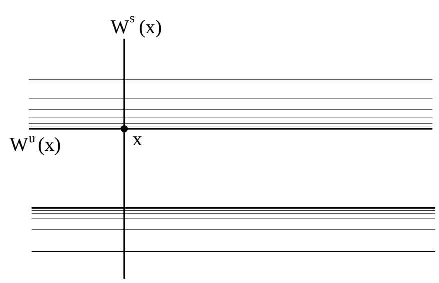


FIGURE 1. A $u$-border point

The following proposition follows from work of Palis and Newhouse [5] and stated explicitly in [2].

**Proposition 3.** *A hyperbolic attractor $\Lambda_a$ contains a u-border, but no s-border. If $\Lambda_a$ does not possess any border, then $f$ is Anosov.*

For $x \in \Lambda_a$ an *s-arch* is a subset of $W^s(x)$ homeomorphic to a closed interval, such that the end points of $\alpha$ intersect $\Lambda_a$ and no point in the interior of $\alpha$ intersects $\Lambda_a$. The following is shown in [2].

**Proposition 4.** *Let $S$ be a compact surface. If $\Lambda \subset S$ is a hyperbolic set containing a nontrivial hyperbolic attractor $\Lambda_a$ and $y \in \Lambda - \Lambda_a$ is contained in $W^s(\Lambda_a)$, then $y$ is contained in an $s$-arch.*

The following lemma restricts how the basins of transitive locally maximal hyperbolic sets and hyperbolic attractors can intersect.

**Lemma 3.** [2, p.52] *Let $f$ be a diffeomorphism of a compact surface without boundary containing a nontrivial hyperbolic attractor $\Lambda_a$ and transitive locally maximal hyperbolic basic set $K$ such that $W^u(K) \pitchfork W^s(\Lambda_a) = W^u(K) \cap W^s(\Lambda_a) \neq \emptyset$. If $x \in K$ and a separatrix $c$ of $W^u(x)$ intersects $W^s(\Lambda_a)$, then $x$ is periodic, $c$ is completely contained in the basin of attraction of $\Lambda_a$, and $\mathcal{O}(x) = K$.*

The following corollary will be used to prove Theorem 2.

**Corollary 2.** *Let $f$ be a diffeomorphism of a compact surface without boundary containing a hyperbolic attractor $\Lambda_a$, a periodic point $q \in \Lambda - \Lambda_a$, a separatrix $c$ of $W^u(q)$ contained in $\Lambda$, and a point $y \in c$ such that $y \in W^s(\Lambda_a)$. Then $c \in W^s(\Lambda_a)$.*

## 3. Hyperbolic Sets with Interior

This section examines hyperbolic sets with nonempty interior. Sections 3.1 and 3.2 give sufficient conditions for a diffeomorphism to be Anosov. First, we show that if a hyperbolic set with nonempty interior has a transitive point, then the diffeomorphism is Anosov. Next, we show that if a hyperbolic set in a compact surface has interior and is locally maximal, then the diffeomorphism is Anosov.

In Section 3.3 we give examples of hyperbolic sets with nonempty interior that are not Anosov. These examples are on surfaces and are robust under perturbations.

### 3.1. **Proof of Theorem 1.**

The following Lemma is the key ingredient to the proof and will be useful throughout the section.

**Lemma 4.** *If $\Lambda$ is a hyperbolic set containing points $z$ and $y$ such that $W^u_\epsilon(z) \subset \Lambda$ and $y \in \omega(z, f)$, then $W^u(y) \subset \Lambda$.*

**Proof of Lemma.** Fix a sequence $\{n_j\}$ of natural numbers such that

$$\lim_{j\to\infty} f^{n_j}(z) = y.$$

It follows from the continuity and expansion of the unstable distribution that

$$W^u(y) \subset \mathrm{cl}\left(\bigcup_{j\to\infty} f^{n_j}(W^u_\epsilon(z))\right).$$

Hence, $W^u(y) \subset \Lambda$. □

We now proceed with the proof of the Theorem. By definition $\Lambda$ is a closed set; we proceed to show $\Lambda$ is also open. Fix a transitive point $z \in \Lambda$ such that $z \in \text{int}(\Lambda)$. Then there exists an $\epsilon > 0$ such that $W^s_\epsilon(z) \subset \text{int}(\Lambda)$ and $W^u_\epsilon(z) \subset \text{int}(\Lambda)$. For any $y \in \Lambda$ there exists a bi-infinite subsequence $\{f^{n_k}(z)\}_{k=-\infty}^{\infty}$ of $\mathcal{O}(z)$ such that $\lim_{k\to\infty} f^{n_k}(z) = y$ and $\lim_{k\to-\infty} f^{n_k}(z) = y$.

Lemma 4 shows $W^u(y) \subset \Lambda$. Similarly, one can show $W^s(y) \subset \Lambda$. The continuity of the stable and unstable distributions implies there exists an $r > 0$ such that

$$B_r(y) \subset \bigcup_{x \in W^u(y)} W^s(x) \subset \Lambda.$$

Therefore, $y \in \text{int}(\Lambda)$ and $\Lambda$ is open. □

3.2. **Proof of Theorem 2.** To prove Theorem 2, we show that there exist a nontrivial hyperbolic attractor $\Lambda_a \subset \Lambda$ and a nontrivial hyperbolic repeller $\Lambda_r \subset \Lambda$ such that

$$\text{int}\,(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

We will then show this implies that $\Lambda_a = \Lambda_r = M$.

Let $\Lambda$ be a locally maximal hyperbolic set for a diffeomorphism $f$ of a compact manifold and $\Lambda_1, ..., \Lambda_m$ be given by the Spectral Decomposition Theorem. Additionally, fix $k \in \mathbb{N}$ such that $f^k(\Lambda_i) = \Lambda_i$ for each $i \in \{1, ...m\}$.

**Proposition 5.** *Suppose there exists a point $x \in \Lambda$ and constant $\eta > 0$ such that $W^u_\eta(x) \subset \Lambda$. Then:*

1. *If $\Lambda_i \cap \omega(W^u_\eta(x), f^k) \neq \emptyset$, then $\Lambda_i \subset \omega(W^u_\eta(x), f^k)$.*
2. *If $\Lambda_i \cap \omega(W^u_\eta(x), f^k) \neq \emptyset$ and $\Lambda_i \ll \Lambda_j$, then $\Lambda_j \subset \omega(W^u_\eta(x), f^k)$.*

**Proof.** Suppose that $\Lambda_i \cap \omega(W^u_\eta(x), f^k) \neq \emptyset$ and let $y$ be a point in this intersection. Fix $x_1 \in W^u_\eta(x)$ such that there exists a subsequence $\{(f^k)^{n_j}(x_1)\}$ converging to $y$. The stable manifolds of points of $(f^k)^{n_j}(W^u_\eta(x))$ near $(f^k)^{n_j}(x_1)$ foliate a small neighborhood of $(f^k)^{n_j}(x_1)$. Hence, if $z$ is a transitive point of $\Lambda_i$ sufficiently close to $y$, then there exists a point $x_2 \in W^u_\eta(x)$ such that $z \in W^s((f^k)^{n_j}(x_2))$ for some $j$. From this it follows that the forward orbit of $x_2$ is dense in $\Lambda_i$. Hence $\Lambda_i \subset \omega(X, f)$. The second part of the proposition follows from a similar argument. □

**Proposition 6.** *Let $\Lambda$ be a locally maximal hyperbolic set and let $x$ and $\eta$ be as in the previous proposition. Then there is a hyperbolic attractor $\Lambda_a$ for $f^n$ for some $n \in \mathbb{N}$ such that $W^u_\eta(x)$ intersects the basin of $\Lambda_a$.*

**Proof.** Let $\Lambda_i, ..., \Lambda_m$ be a spectral decomposition of $\Lambda$, and let $k \in \mathbb{N}$ such that each $\Lambda_i$ is fixed under $f^k$. Theorem 6 and Proposition 5 show there is some maximal element $\Lambda_a$ contained in $\omega(W^u_\eta(x), f^k)$ under the relation $\ll$. We will show that $\Lambda_a$ is an attractor for $f^n$ for some $n \in \mathbb{N}$.

Fix an adapted metric of $\Lambda$ and extend the metric to a neighborhood $V_0$ of $\Lambda$. Let $\lambda \in (0,1)$ be the hyperbolic constant for $\Lambda$ and fix $\lambda' \in (0,1)$ such that $\lambda' > \lambda$. Additionally, fix $V$ a neighborhood of $\Lambda$ such that $\bigcap_{i\in\mathbb{Z}} f^i(V) = \Lambda$, fix a periodic point $p \in \Lambda_a$ of period $n_0$, and let $n = kn_0$. Since $p \in \omega(W^u_\eta(x), f^k)$ it follows that $W^u(p) \subset \Lambda$. For each $y \in W^u(p, f)$ there is an $\epsilon_0 > 0$ satisfying the following:

(1) $W^s_{\epsilon_0}(y) \subset (V_0 \cap V)$,
(2) $\epsilon_0 < d(\Lambda_i, \Lambda_j)$ for any $i, j \in \{1, ..., m\}$, and
(3) under $Df^n$ each vector in $T_q(W^s_{\epsilon_0}(y))$ contracts by a factor of at least $\lambda'$ for each point $q \in W^s_{\epsilon_0}(y)$,

Define $\epsilon(y)$ to be the supremum of all such $\epsilon_0$. The compactness of $\Lambda$ and $M$ implies that $\epsilon > 0$, where

$$\epsilon := \min_{y\in W^u(p)} (\epsilon(y)/2).$$

We will show that the set $\Lambda' = \bigcap_{i\geq 0}(f^n)^i(U)$, where

$$U = \bigcup_{y\in W^u(p)} W^s_{\epsilon/2}(y),$$

is a hyperbolic attractor for $f^n$ and equals $\Lambda_a$. First, we show that $U$ is an attracting neighborhood of $\Lambda'$. Fix $y \in W^u(p)$ and $y'$ in the boundary of $W^s_{\epsilon/2}(y)$, then the uniform contraction along $W^s_{\epsilon/2}(y)$ implies that $f^n(y') \in W^s_{\epsilon/2}(f^n(y))$. Therefore, $f^n(\mathrm{cl}(U)) \subset U$.

We now establish that $\Lambda'$ is a hyperbolic set. Clearly, $\Lambda'$ is an invariant subset of $\Lambda$ and so inherits a hyperbolic structure. From above it is clear that

$$\bigcap_{i\geq 0}(f^n)^i(U) = \bigcap_{i\geq 0}(f^n)^i(\mathrm{cl}(U)),$$

so $\Lambda'$ is closed. Therefore, $\Lambda'$ is a hyperbolic set.

Next, we show that $\Lambda_a = \Lambda'$. Fix $0 < \epsilon' < \epsilon/4$ and a transitive point $z$ of $\Lambda_a$ within $\epsilon'/2$ of the periodic point $p$. Then $\Lambda_a \subset \Lambda'$ since $(f^n)^i(z) \in U$ for all $i \in \mathbb{N}$. We now show that $\Lambda' \subset \Lambda_a$. By the way $U$ was constructed we have that $\Lambda' \subset \mathrm{cl}(W^u(p))$. Hence, it is sufficient to show that $W^u(p) \subset \Lambda_a$. Fix $y \in W^u(p)$, $\epsilon' \in (0, \epsilon/4)$, and a transitive point $z$ of $\Lambda_a$. We now construct an $\epsilon'$-chain from $y$ to itself. First, take $y_0 = y$ and follow the orbit of $y$ until it is within $\epsilon'/2$ of a point $x \in \omega(y, f^n)$. Then, $x$ is contained in a set $\Lambda_i$ for some $i \in \{1, ..., m\}$. It follows from the continuity of the stable and unstable distributions that for some $n_j \in \mathbb{N}$ that $W^u((f^k)^{n_j}(y)) \pitchfork W^s(x) \neq \emptyset$. Hence, $x \in \Lambda_a$ since $\Lambda_a$ is a maximal element under the relation $\ll$. This implies there is a point $(f^n)^{-k_1}(z)$ in the backward orbit of $z$ within $\epsilon'/2$ of $x$. Next, follow the orbit of $(f^n)^{-k_1}(z)$ until it is within $\epsilon'/2$ of $p$. Lastly, fix a point $(f^n)^{-k_2}(y)$ in the backward orbit of $y$ within $\epsilon'/2$ of $p$. Then, follow the orbit of $(f^n)^{-k_2}(y)$ back to $y$ completing the $\epsilon$-chain. Hence, $y \in \mathcal{R}(f^n|\Lambda)$, which implies $y \in \Lambda_a$.

The proposition now follows since the set $\Lambda_a$ is a transitive hyperbolic set with attracting neighborhood $U$ under the action of $f^n$. Therefore, $\Lambda_a$ is a nontrivial hyperbolic attractor for $f^n$. □

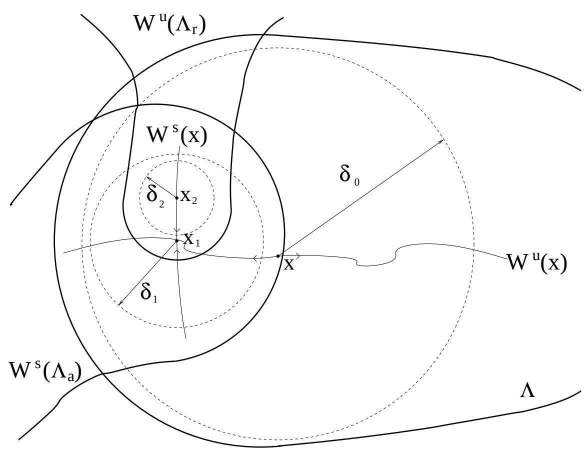


FIGURE 2. A point in $W^s(\Lambda_a) \cap W^u(\Lambda_r)$

**Proposition 7.** *Let $\Lambda$ be a locally maximal hyperbolic set with nonempty interior for a diffeomorphism $f$. Then there exist $n \in \mathbb{N}$, a nontrivial hyperbolic attractor $\Lambda_a$ for $f^n$, and a nontrivial hyperbolic repeller $\Lambda_r$ for $f^n$, such that*

$$\mathrm{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

*Furthermore, there exist periodic points $q \in \Lambda_a$ and $p \in \Lambda_r$ such that*

$$(W^s(q) \pitchfork W^u(p)) \cap \mathrm{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

**Proof.** Let $x \in \mathrm{int}\Lambda$. Fix $\delta_0 > 0$ such that $B_{\delta_0}(x) \subset \mathrm{int}(\Lambda)$, see Figure 2. Proposition 6 shows there exist a $x_1 \in W^u_{\delta_0}(x)$ and $m_1 \in \mathbb{N}$ such that $x_1$ is in the basin of attraction of a nontrivial hyperbolic attractor $\Lambda_a$ for $f^{m_1}$ such that $\Lambda_a$ is contained in $\omega(W^u_{\delta_0}(x), f^{m_1})$. It immediately follows that there is a $\delta_1 < \delta_0$ such that $B_{\delta_1}(x_1) \subset B_{\delta_0}(x)$ and $W^u_{\delta_1}(x_1) \subset W^s(\Lambda_a)$.

Apply Proposition 6 to $f^{-1}$, $x_1$, and $\delta_1$. We obtain a point $x_2 \in W^s_{\delta_1}(x_1)$ and $m_2 \in \mathbb{N}$ such that $x_2$ is in the basin of attraction of a nontrivial hyperbolic repeller $\Lambda_r$ for $f^{m_1 m_2}$. It follows that there is $\delta_2 < \delta_1$ such that $B_{\delta_2}(x_2) \subset B_{\delta_1}(x_1)$ and $W^s_{\delta_2}(x_2) \subset W^u(\Lambda_r)$.

Since $x_1 \in W^s(\Lambda_a)$ we also have $W^s(x_1) \subset W^s(\Lambda_a)$. Similarly, we have $W^u(x_2) \subset W^u(\Lambda_r)$. Therefore, $x_2 \in \text{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda)$.

To complete the proof pick periodic points $q \in \Lambda_a$ and $p \in \Lambda_r$ and let $n = m_1 m_2$. Since $\Lambda_a$ and $\Lambda_r$ are topologically transitive locally maximal sets with periodic points dense and the closure of heteroclinic classes under $f^n$, we have that $\overline{W^s(q)} = W^s(\Lambda_a)$ and $\overline{W^u(p)} = W^u(\Lambda_r)$. Hence,

$$(W^s(q) \pitchfork W^u(p)) \cap \text{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

□

We now use results of Bonatti and Langevin to show that the final conclusion of Proposition 7 implies that $\Lambda_a = \Lambda_r = M$, and so $f$ is Anosov.

**Proof of Theorem 2.** Fix $n$ such that $g = f^n$ has a nontrivial hyperbolic attractor $\Lambda_a \subset \Lambda$, a nontrivial hyperbolic repeller $\Lambda_r \subset \Lambda$, and fixed points $q \in \Lambda_a$ and $p \in \Lambda_r$ satisfying:

$$(W^s(q) \pitchfork W^u(p)) \cap \text{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

The intuitive idea is the following: If there is a component $I$ of $\partial(\text{int}(\Lambda))$ that is contained in $W^s(\Lambda_a)$, then the local product structure of $\Lambda$ implies that $I$ is contained in the unstable direction. Similarly, if there is a component $J$ of $\partial(\text{int}(\Lambda))$ contained in $W^u(\Lambda_r)$, then the local product structure of $\Lambda$ implies that $J$ is contained in the stable direction. Proposition 7 then appears to show that $\partial\text{int}(\Lambda) = \emptyset$.

This approach although intuitive does not yield the most straight forward proof. Instead, we use the results of Bonatti and Langevin on the structure of hyperbolic attractors and repellers to show that $f$ is Anosov.

The first step is to find a separatrix of $W^u(p)$ contained in $\Lambda$. Let

$$y \in (W^s(q) \pitchfork W^u(p)) \cap \text{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda)$$

For $x_1, x_2 \in W^u(y)$, denote the segment of $W^u(y)$ between $x_1$ and $x_2$ by $[x_1, x_2]_u$. We now show that $[y, g(y)]_u \subset \Lambda$. For $m$ sufficiently large the interval $g^m([y, g(y)]_u)$ is within $\delta/2$ of $\Lambda_a$, where $\delta$ is given by the local product structure of $\Lambda$. The set $g^m([y, g(y)]_u) \cap \Lambda$ is nonempty and closed. On the other hand, the local product structure on $\Lambda$ implies that $g^m([y, g(y)]_u) \cap \Lambda$ is also open and nonempty in $g^m([y, g(y)]_u)$. It follows that $[y, g(y)]_u \subset \Lambda$. Then $[y, g(y)]_u$ is a fundamental domain of a separatrix $c$ of $W^u(q)$, and the invariance of $\Lambda$ implies that $c$ is contained in $\Lambda$. Now Corollary 2 implies that $c \subset W^s(\Lambda_a)$.

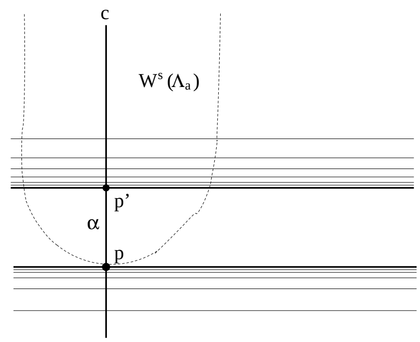


FIGURE 3. $\Lambda_r \cap W^s(\Lambda_a, g) \neq \emptyset$

We conclude the proof by showing that $\Lambda_r \cap W^s(\Lambda_a) \neq \emptyset$ which implies that $\Lambda_a = \Lambda_r = M$. The point $y$ is in $c \cap W^s(q)$ applying Proposition 4 to $g^{-1}$ it follows that there exists a $u$-arch (defined in the preliminary section on the structure of hyperbolic attractors) $\alpha \subset c \subset W^s(\Lambda_a, g)$ such that an endpoint $p'$ of $\alpha$ is contained in $\Lambda_r$, see Figure 3. Hence, $\Lambda_r \cap W^s(\Lambda_a, g) \neq \emptyset$. □

3.3. **Proof of Theorem 3.** In this section we prove Theorem 3 by constructing two examples of hyperbolic sets with nonempty interior for non-Anosov diffeomorphisms.

**Theorem 3.** *There exists a diffeomorphism of a compact smooth surface and hyperbolic set $\Lambda$ such that $\Lambda$ contains nonempty interior and is not contained in any locally maximal hyperbolic set.*

**Proof of Theorem 3.** We will in fact construct two examples satisfying the conclusions of Theorem 3. The first example has interior that is completely wandering. The second example has one fixed point in the interior.

3.4. **Example 1.** The first example is based on the simple diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (\frac{1}{2}x, 2y)$. Denote the closed disk or radius 1/4 centered at the point $(1, 1)$ by $D_{1/4}(1, 1)$. If $\Lambda = \bigcup_{n\in\mathbb{Z}} f^n(D_{1/4}(1, 1))$, then $\Lambda$ has a hyperbolic splitting and has interior. The idea is to compactify the example.

Specifically, take a diffeomorphism $f$ of a compact surface $M$, such that $M$ contains a hyperbolic repeller $\Lambda_r$ containing a fixed point $p$ and a hyperbolic attractor $\Lambda_a$ containing a fixed point $q$ where $\Lambda_a \cap \Lambda_r = \emptyset$

and $W^u(p) \pitchfork W^s(q) \neq \emptyset$. We show for a point $z \in W^u(p) \pitchfork W^s(q)$ and $r$ sufficiently small that the set

$$\Lambda = \Lambda_r \cup \Lambda_a \cup \left( \bigcup_{n \in \mathbb{Z}} f^n(D_r(z)) \right)$$

is a hyperbolic set with nonempty interior. Theorem 2 implies that $\Lambda$ is not contained in a locally maximal hyperbolic set since $f$ is not Anosov.

In [7] a diffeomorphism $g$ is constructed on a compact surface of genus two containing a DA-attractor, $\Lambda_a$, and DA-repeller, $\Lambda_r$, such that

$$\text{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r)) \neq \emptyset.$$

Pick periodic points $p \in \Lambda_r$ and $q \in \Lambda_a$. Since $\Lambda_a$ and $\Lambda_r$ are topologically mixing locally maximal sets with periodic points dense, we have that $\overline{W^s(q)} = W^s(\Lambda_a)$ and $\overline{W^u(p)} = W^u(\Lambda_r)$. Hence,

$$(W^s(q) \pitchfork W^u(p)) \cap \text{int}(W^s(\Lambda_a) \cap W^u(\Lambda_r) \cap \Lambda) \neq \emptyset.$$

Fix $n \in \mathbb{N}$ such that $p$ and $q$ are fixed under $g^n$, let $f = g^n$, and fix $z \in W^u(p) \pitchfork W^s(q)$. The first step is to define a continuous invariant splitting for $\Lambda$. If $r, \epsilon > 0$ are sufficiently small, then for any point $y \in D_r(z)$ there exist points $p' \in W^s_\epsilon(p) \subset \Lambda_r$ and $q' \in W^u_\epsilon(q) \subset \Lambda_a$ such that $y \in W^u(p') \pitchfork W^s(q')$. Define a splitting $T_yM = \mathbb{E}^+_y \oplus \mathbb{E}^-_y$ of the tangent space at $y$ by $\mathbb{E}^-_y = T_yW^u(p')$ and $\mathbb{E}^+_y = T_yW^s(q')$. Extend this splitting to the orbit of $y$ by:

$$\mathbb{E}^-_{f^m(y)} = T_{f^m(y)}W^u(f^m(p'))$$

and

$$\mathbb{E}^+_{f^m(y)} = T_{f^m(y)}W^s(f^m(q')),$$

where $m \in \mathbb{Z}$. For $r$ perhaps smaller this is a well defined splitting. Extend the splitting to points in $\Lambda_a$ and $\Lambda_r$ using the given hyperbolic splitting. Let

$$\Lambda = \Lambda_r \cup \Lambda_a \cup \left( \bigcup_{n \in \mathbb{Z}} f^n(D_r(z)) \right).$$

Then the Inclination Lemma and the continuity of the stable and unstable distributions implies the splitting on $\Lambda$ is continuous and invariant.

We now show that the splitting carries a hyperbolic structure. Let $\lambda_a, \lambda_r \in (0,1)$ be the constants of hyperbolicity for $\Lambda_a$ and $\Lambda_r$, respectively, and fix $\lambda \in (0,1)$ such that $\lambda > \max\{\lambda_a, \lambda_r\}$. Then for points in $D_r(z)$ sufficiently near $\Lambda_a$ and $\Lambda_r$, if $v \in E^\pm_x$, and $n \in \mathbb{N}$, then

$$\|Df^n_x v\| \leq \lambda^n \|v\|, \text{ for } v \in E^+_x$$
$$\|Df^{-n}_x v\| \leq \lambda^n \|v\|, \text{ for } v \in E^-_x.$$

The continuity of the splitting implies there exists a constant $C > 0$ such that for any $x \in \Lambda$, $v \in E^{\pm}_x$, and $n \in \mathbb{N}$ the following hold:

$$\|Df^n_x v\| \leq C\lambda^n \|v\|, \text{ for } v \in E^+_x \text{ and}$$
$$\|Df^{-n}_x v\| \leq C\lambda^n \|v\|, \text{ for } v \in E^-_x.$$

Hence, $\Lambda$ is a hyperbolic set which is not contained in a locally maximal hyperbolic set since $f$ is not Anosov.

3.5. **Example 2.** In Example 1 the interior of $\Lambda$ contains only wandering points. In the next example we show that there exists a diffeomorphism of a compact surface containing a hyperbolic set $\Lambda$ such that $\mathrm{Per}(\Lambda) \cap \mathrm{int}(\Lambda) \neq \emptyset$. In our construction $\mathrm{Per}(\Lambda) \cap \mathrm{int}(\Lambda)$ will consist of one fixed point.

**Remark 1.** While Example 2 shows that the interior of a hyperbolic set can contain nonwandering points it cannot contain an open set of them. This well-known fact is proven in [1], and the proof is similar to the proof of Theorem 1.

This raises the following question:
**Question:** Can a component of the interior of a compact hyperbolic set for a non-Anosov diffeomorphism contain more than 1 periodic point?

We now proceed with the construction of the second example.

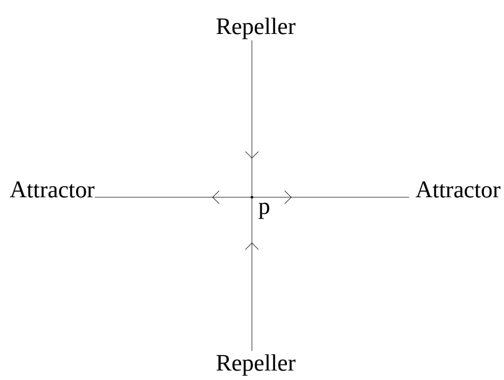


FIGURE 4. Example 2 overview

We will show there exists a diffeomorphism $f$ of a surface containing: two hyperbolic attractors $\Lambda_{a1}$ and $\Lambda_{a2}$, two hyperbolic repellers $\Lambda_{r1}$ and $\Lambda_{r2}$, and a hyperbolic fixed point $p$, as shown in Figure 4. For $r > 0$

sufficiently small, the set

$$\Lambda = \Lambda_{a1} \cup \Lambda_{a2} \cup \Lambda_{r1} \cup \Lambda_{r2} \cup \left( \bigcup_{n\in\mathbb{Z}} f^n(D_r(p)) \right),$$

will be a hyperbolic set. By Theorem 2 the set $\Lambda$ will not be contained in a locally maximal hyperbolic set, since $f$ is not Anosov.

We proceed with this construction. Let $f_0$ be a diffeomorphism of the two sphere $S^2$ containing a Plykin attractor $\Lambda_a$, a repelling period three orbit, and a repelling fixed point $p_0$. Puncture the sphere at $p_0$ and replace $p_0$ with a closed circle, obtaining a closed disk $D$. The homeomorphism induced from $f_0$ of $D$ is not a diffeomorphism, but can be deformed near $\partial D$ to obtain a diffeomorphism $\tilde{f}$ of $D$ such that $f|_{\partial D} = \mathrm{Id}$ and $D\backslash\partial D = W^s(\Lambda_a)$. Next, deform $\tilde{f}$ in a small neighborhood of an interval $S_0 \subset \partial D$ such that $\tilde{f}|_{S_0}$ contains three hyperbolic fixed points $p$, $p_1$, and $p_2$. We can do this so that $p$ is of saddle type, the unstable manifold of $p$ is nowhere tangent to a stable manifold of a point in $\Lambda_a$, and the fixed points $p_1$ and $p_2$ are repellers. We may further carry out this deformation so that the stable manifold $W^s(p)$ intersects both repelling neighborhoods $W^u(p_1)$ and $W^u(p_2)$, as in Figure 5.

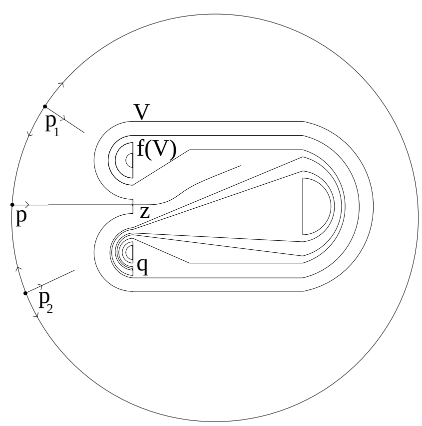


FIGURE 5. Example 2 step 1

Next, construct a diffeomorphism $f_1$ of $S^2$ by gluing two copies of $\tilde{f}$ together along the equator, see Figure 6.

We use the same construction as is used in constructing the example in [7] to attach two repellers by cutting out small disks around $p_1$ and $p_2$, see Figure 7. This construction can be carried out so that each

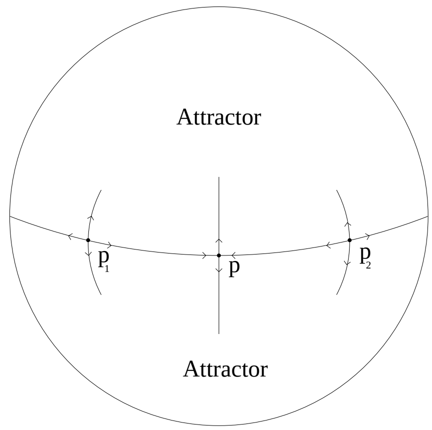


FIGURE 6. Example 2 step 2

separatrix of $W^s(p)$ is not tangent to an unstable manifold of a point in $\Lambda_{r1}$ or $\Lambda_{r2}$. Furthermore, the set

$$\Lambda_1 = W^s(p) \cup W^u(p) \cup \Lambda_{a1} \cup \Lambda_{a2} \cup \Lambda_{r1} \cup \Lambda_{r2}$$

is a hyperbolic invariant set.

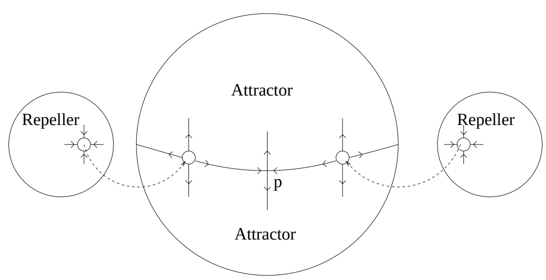


FIGURE 7. Example 2 step 3

For $r > 0$ sufficiently small define

$$\Lambda = \Lambda_{a1} \cup \Lambda_{a2} \cup \Lambda_{r1} \cup \Lambda_{r2} \cup \left( \bigcup_{n \in \mathbb{Z}} f^n(D_r(p)) \right).$$

Using the same techniques as in the previous subsection it is not hard to show that $\Lambda$ is a hyperbolic set.

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