*J. Inst. Maths Applics* (1973) **11**, 329–333

# Two Interpolatory Cubic Splines

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[Received 5 September 1972 and in revised form 13 October 1972]



The third order and the uniform cubic spline are defined and shown to have $O(h^3)$ and $O(h^4)$ convergence respectively when used for interpolation.

## 1. Introduction

A NATURAL CUBIC SPLINE arises as the solution of the following interpolation problem.

"Given numbers $t_0, t, \ldots, t_N$, where $0 = t_0 < t < \cdots t_N = 1$, find $z \in C^2[0, 1]$ such that if $x_0, x_1, \ldots, x_N$ are given numbers then

(i) $z(t_i) = x_i, \quad i = 0, 1, \ldots, N$

(ii) $\int_0^1 |z^{(2)}(t)|^2 \, dt$ is a minimum."

It can be shown (see for example Ahlberg, Nilson & Walsh, 1967, p. 76) that the unique function $y$ which solves this problem is, in each of the intervals $[t_i, t_{i+1}]$, a polynomial of degree at most three, and that $y^{(2)}(0) = y^{(2)}(1) = 0$. If it is assumed that $x_i = x(t_i)$, $i = 0, 1, \ldots, N$ where $x \in C^4[0, 1]$ then, with the uniform norm on [0, 1], $\|x-y\| = O(h^2)$ where $h = \max (t_{i+1}-t_i)$.

A careful analysis reveals that an improvement in this result is possible, and in Kershaw (1971) it is shown that $\max |x(t)-y(t)| = O(h^4)$ on an interval contained in [0, 1] which expands to fill it as $h$ decreases to zero. The $O(h^2)$ of the uniform norm of the error is a direct result of the minimization requirement (ii) which leads to the natural boundary conditions on $y^{(2)}$ at $t = t_0, t_N$ (namely $y^{(2)}(t_0) = y^{(2)}(t_N) = 0$); and if, for example, the correct values of the first derivative are imposed at the points $t = t_0, t_N$ instead of the natural conditions, then the convergence would be $O(h^4)$ uniformly on [0, 1] (further details will be found in Kershaw, 1971). However in an interpolation problem usually only function values are known at the interpolation points and it is the main purpose if this note to describe two interpolatory cubic splines which have more favourable convergence properties than the natural one but which require no additional information. One of these, *the third order cubic spline*, has $O(h^3)$ convergence; the other, the *uniform cubic spline* has $O(h^4)$ convergence.

## 2. Notation and Definitions

$y$ denotes a cubic spline on [0, 1] with the *knots* $t_0, t_1, \ldots, t_N$; that is, $y \in C^2[0, 1]$ and the restriction of $y$ to each interval $[t_i, t_{i+1}]$ is a polynomial of degree at most three.

$x$ is an element of $C^n[0, 1]$ where $n$ is to be taken appropriate to the context, and we define $M_n = \|x^{(n)}\| = \max_{0 \leq t \leq 1} |x^{(n)}(t)|$.

We use the following abbreviations;

$$\lambda_i = y_i^{(1)} = y^{(1)}(t_i), \qquad i = 0, 1, \ldots, N,$$

and we assume that $y_i = y(t_i) = x(t_i)$, $i = 0, 1, \ldots, N$.

For simplicity of presentation we suppose that $t_{i+1} - t_i = h$, consequently $t_i = ih = i/N$, $i = 0, 1, \ldots, N$. The technique which is used here to examine convergence behaviour can be applied when the knots are not uniformly distributed and indeed many of the results in the general case will be found to be substantially the same. However it is felt that the added generality is of little practical importance.

The following set of relations is a direct consequence of $y$ being a cubic spline (see Ahlberg, Nilson & Walsh, 1967, p. 11),

$$h[\lambda_{i-1} + 4\lambda_i + \lambda_{i+1}] = 6\mu\delta y_i = 6\mu\delta x_i, \qquad i = 1, 2, \ldots, N-1. \tag{2.1}$$

When $\lambda_0, \lambda_1, \ldots, \lambda_N$ are known the spline can be constructed in each interval by the use of Hermite's two-point interpolation formula, but clearly the relations of (2.1) are insufficient to determine $\lambda_0, \lambda_1, \ldots, \lambda_N$ when $y_0, y_1, \ldots, y_N$ are given. If the spline is natural the following pair of equations is satisfied in addition to those of (2.1);

$$h[2\lambda_0 + \lambda_1] = 3\Delta x_0, \qquad h[\lambda_{n-1} + 2\lambda_N] = 3\nabla x_N. \tag{2.2}$$

As was stated, these $N+1$ linear algebraic equations are sufficient to determine uniquely $\lambda_0, \lambda_1, \ldots, \lambda_N$. Our aim is to find equations to replace those of (2.2) so that the resulting set of equations will determine cubic splines which have better convergence properties than the natural one.

## 3. Preliminary Results

Three lemmas will be given in this section. The first is a simple consequence of Hermite's two-point interpolation formula.

LEMMA 3.1

$$|x(t) - y(t)| \leqslant \tfrac{1}{4}h \max\{|x_i^{(1)} - \lambda_i|, |x_{i+1}^{(1)} - \lambda_{i+1}|\} + h^4 M_4/288 \tag{3.1}$$

*for* $t_i \leqslant t \leqslant t_{i+1}$.

It is clear from this last result that $O(h^4)$ is the best order of approximation which can be expected of the spline; moreover the precise order will depend on $\max_i |x_i^{(1)} - \lambda_i|$.

LEMMA 3.2

$$[\lambda_{i-1} - x_{i-1}^{(1)}] + 4[\lambda_i - x_i^{(1)}] + [\lambda_{i+1} - x_{i+1}^{(1)}] = -h^4 x^{(5)}/30,$$
$$i = 1, 2, \ldots, N-1. \tag{3.2}$$

The proof follows from the observation that the left hand side of (3.2) can be rewritten with aid of (2.1) as $6\mu\delta x_i/h - [x_{i-1}^{(1)} + 4x_i^{(1)} + x_{i+1}^{(1)}]$, and this expression is easily shown to be equal to the right hand side of (3.2). As we stated in Section 2 it is necessary to adjoin to the equations (2.1) two further equations in order to determine a cubic spline. Two possible pairs are given in Lemma 3.3 below. We present this without proof, however we indicate now the motivation which gave rise to one of them. (It will be clear that splines of higher order can be treated in a similar fashion, but we do not consider this problem here.)

We require one of the extra equations to have the form

$$\lambda_0 + a\lambda_1 = b\Delta x_0$$

where the scalars $a$ and $b$ are to be chosen so that when this is rewritten in the form

$$[\lambda_0 - x_0^{(1)}] + a[\lambda_1 - x_1^{(1)}] = b\Delta x_0 - x_0^{(1)} - ax_1^{(1)}$$

then the right hand side vanishes for any polynomial of degree two. It will be found that this is true if $a = 1$ and $b = 2/h$.

LEMMA 3.3

*If*

$$h[\lambda_0 + \lambda_1] = 2\Delta x_0, \qquad h[\lambda_{N-1} + \lambda_N] = 2\nabla x_N \tag{3.3a}$$

*then*

$$\begin{aligned} [\lambda_0 - x_0^{(1)}] + [\lambda_1 - x_1^{(1)}] &= -h^2 x^{(3)}/6, \\ [\lambda_{N-1} - x_{N-1}^{(1)}] + [\lambda_N - x_N^{(1)}] &= -h^2 x^{(3)}/6. \end{aligned} \tag{3.3b}$$

*If*

$$h[\lambda_0 + 2\lambda_1] = 2\Delta x_0 + \mu x_1, \qquad h[2\lambda_{N-1} + \lambda_N] = 2\nabla x_N + \mu x_{N-1} \tag{3.4a}$$

*then*

$$\begin{aligned} [\lambda_0 - x_0^{(1)}] + 2[\lambda_1 - x_1^{(1)}] &= h^3 x^{(4)}/12, \\ 2[\lambda_{N-1} - x_{N-1}^{(1)}] + [\lambda_N - x_N^{(1)}] &= -h^3 x^{(4)}/12. \end{aligned} \tag{3.4b}$$

## 4. Two Interpolatory Cubic Splines

We now use the results of Lemma 3.3 to define two types of cubic spline.

*Definition* 4.1

A cubic spline for which

$$\begin{aligned} h[\lambda_0 + \lambda_1] &= 2\Delta y_0, \\ h[\lambda_{i-1} + 4\lambda_i + \lambda_{i+1}] &= 6\mu\delta y_i, \qquad i = 1, 2, \ldots, N-1, \\ h[\lambda_{N-1} + \lambda_N] &= 2\nabla y_N, \qquad \text{where } N \geqslant 2, \end{aligned}$$

will be called a *third order interpolatory cubic spline.*

*Definition* 4.2

A cubic spline for which

$$\begin{aligned} h[\lambda_0 + 2\lambda_1] &= 2\Delta y_0 + \mu\delta y_1, \\ h[\lambda_{i-1} + 4\lambda_i + \lambda_{i+1}] &= 6\mu\delta y_i, \qquad i = 1, 2, \ldots, N-1, \\ h[2\lambda_{N-1} + \lambda_N] &= 2\nabla y_N + \mu\delta y_{N-1}, \qquad \text{where } N \geqslant 3, \end{aligned}$$

will be called a *uniform interpolatory cubic spline.*

THEOREM 4.3. (a) *If $y$ is the third order interpolatory cubic spline which agrees with $x \in C^5[0, 1]$ at the knots $t_0, t_1, \ldots, t_N$, where $N \geqslant 2$, then*

$$\|x - y\| = O(h^3).$$

(b) *If $y$ is the uniform interpolatory cubic spline which agrees with $x \in C^5[0, 1]$ at the knots $t_0, t_1, \ldots, t_N$ where $N \geqslant 3$ then*

$$\|x - y\| = O(h^4).$$

*Proof.* (a) We eliminate $\lambda_0 - x_0^{(1)}$ and $\lambda_N - x_N^{(1)}$ between the equations (3.2) and (3.3b). This gives the following system of equations,

$$\begin{aligned} 3[\lambda_1 - x_1^{(1)}] + [\lambda_2 - x_2^{(1)}] &= -h^4x^{(5)}/30 + h^2x^{(3)}/6, \\ [\lambda_{i-1} - x_{i-1}^{(1)}] + 4[\lambda_i - x_i^{(1)}] + [\lambda_{i+1} - x_{i+1}^{(1)}] &= -h^4x^{(5)}/30, \qquad i = 2, 3, \ldots, N-2, \\ [\lambda_{N-2} - x_{N-2}^{(1)}] + 3[\lambda_{N-1} - x_{N-1}^{(1)}] &= -h^4x^{(5)}/30 + h^2x^{(3)}/6. \end{aligned}$$

The uniform matrix norm of the inverse of the matrix of these equations is bounded above by $\frac{1}{2}$; consequently,

$$\max_{1 \leqslant i \leqslant N-1} |\lambda_i - x_i^{(1)}| \leqslant h^2M_3/12 + h^4M_5/60. \tag{4.1}$$

Moreover we have

$$\begin{aligned} |\lambda_0 - x_0^{(1)}| &\leqslant |\lambda_1 - x_1^{(1)}| + h^2M_3/6 \\ &\leqslant h^2M_3/4 + h^4M_5/60 \end{aligned} \tag{4.2}$$

with the same bound for $|\lambda_N - x_N^{(1)}|$. It remains only to insert these bounds in (3.1) to obtain the desired result, namely

$$\|x-y\| \leqslant h^3M_3/16 + h^4M_4/288 + h^5M_5/240.$$

The proof of (b) is similar to that of (a); the precise result which we obtain is as follows,

$$\|x-y\| \leqslant 19h^4M_4/288 + h^5M_5/120.$$

*Remark*

The approach used here which lead to these two cubic splines can be used successfully when the knots are not equally spaced. For the third order spline the convergence result is again $O(h^3)$ where $h = \max(t_{i+1} - t_i)$. The result for the uniform cubic spline is as follows,

$$\begin{aligned} |x(t) - y(t)| &= O(h^4) + \frac{h_0^2}{h_1}O(h^3), & t_0 \leqslant t \leqslant t, \\ |x(t) - y(t)| &= O(h^4), & t_1 \leqslant t \leqslant t_{N-1}, \\ |x(t) - y(t)| &= O(h^4) + \frac{h_{N-1}^2}{h_{N-1}}O(h^3), & t_{N-1} \leqslant t \leqslant t_N \end{aligned}$$

where $h_i = t_{i+1} - t_i$ and $h = \max_i h_i$.

## 5. Qualitative Properties

It is easily shown that if $y$ is a cubic polynomial then

$$h^3y^{(3)}(t) = -12\Delta y_0 + 6h[y_0^{(1)} + y_1^{(1)}].$$

The meaning of the first equation of (3.3a) is now clear. It imposes the condition that the cubic spline should be at most quadratic in $[t_0, t_1]$. Consequently we have the following alternative definition of the third order cubic spline.

*Definition* 5.1

A *third order cubic spline* with the knots $t_0, t_1, \ldots, t_N$, where $N \geqslant 2$, is in $C^2[0, 1]$ and is such that

(i) it is a polynomial of degree at most three in $[t_i, t_{i+1}]$, $i = 1, 2, \ldots, N-2$,

(ii) it is polynomial of degree at most two in $[t_0, t_1]$ and $[t_{N-1}, t_N]$.

The uniform cubic spline can be shown to satisfy the following.

*Definition* 5.2

A *uniform cubic spline* with the knots $t_0, t_1, \ldots, t_N$, where $N \geqslant 3$, is in $C^2[0, 1]$ and is such that

(i) it is a polynomial of degree at most three in $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, N-1$,

(ii) $y^{(3)}$ is continuous at $t = t_1, t_{N-1}$.

For completeness we state the final results. The proofs follow familiar lines.

THEOREM 5.3. *The unique* $z \in C^2[0, 1]$ *such that for* $N \geqslant 2$,

(a) $z_i = x_i, \quad i = 1, 2, \ldots, N-1,$

(b) $z_0^{(1)} = x_0^{(1)}, \qquad z_N^{(1)} = x_N^{(1)},$

(c) $\int_0^1 |z^{(2)}(t)|^2 \, dt$ *is a minimum,*

*is a third order cubic spline with the knots* $t_0, t_1, \ldots, t_N$.

THEOREM 5.4. *The unique* $z \in C^2[0, 1]$ *such that for* $N \geqslant 3$

(a) $z_i = x_i, \quad i = 0, 2, \ldots, N-2, N,$

(b) $z_0^{(1)} = x_0^{(1)}, \qquad z_N^{(1)} = x_N^{(1)},$

(c) $\int_0^1 |z^{(2)}(t)|^2 \, dt$ *is a minimum,*

*is a uniform cubic spline with the knots* $t_0, t_1, \ldots, t_N$.

I am grateful to Professor C. W. Clenshaw for his helpful comments.

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