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THE BEST RANK-ONE APPROXIMATION RATIO OF A TENSOR SPACE*

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Abstract. In this paper we define the best rank-one approximation ratio of a tensor space. It turns out that in the finite dimensional case this provides an upper bound for the quotient of the residual of the best rank-one approximation of any tensor in that tensor space and the norm of that tensor. This upper bound is strictly less than one, and it gives a convergence rate for the greedy rank-one update algorithm. For finite dimensional general tensor spaces, third order finite dimensional symmetric tensor spaces, and finite biquadratic tensor spaces, we give positive lower bounds for the best rank-one approximation ratio. For finite symmetric tensor spaces and finite dimensional biquadratic tensor spaces, we give upper bounds for this ratio.

Key words. tensors, best rank-one approximation ratio, bounds

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1. Introduction.

The best rank-one approximation problem for higher-order tensors has wide applications in wireless communication systems, magnetic resonance imaging, signal and image processing, data analysis, higher order statistics, as well as independent component analysis [2], [3], [4], [6], [7], [10], [12], [14], [15], [17], [19], [21], [23], [26].

A basic question for the best rank-one approximation problem is whether there exists a positive lower bound for the quotient of the best rank-one approximation of a tensor and the norm of that tensor such that this lower bound only depends upon the order and dimensions of that tensor. If such a positive lower bound exists, then it will provide an upper bound for the quotient of the residual of the best rank-one approximation of any tensor in that tensor space and the norm of that tensor. This upper bound is strictly less than one, and it gives a convergence rate for the greedy rank-one update algorithm [1], [9], [8], [24]. In the next section, we show that such a positive lower bound exists. We call it the best rank-one approximation ratio of that tensor space.

In section 3, we give a positive lower bound for the best rank-one approximation ratio of a general finite dimensional tensor space. In section 4, we give a positive lower bound for the best rank-one approximation ratio of a third order finite dimensional symmetric tensor space, and an upper bound of this ratio of a finite dimensional symmetric tensor space. In section 5, we give a positive lower bound and an upper bound for the best rank-one approximation ratio of a finite dimensional biquadratic tensor space. Some numerical results are given in section 6. Four open questions are raised in section 7.

2. General discussion.

The following discussion is borrowed from [9] and was suggested by a referee. Let V_j be separable Hilbert spaces with inner product $\langle \cdot, \cdot \rangle_j$ for $j = 1, \ldots, m$. Consider the tensor product Hilbert space $\mathbb{V} = \bigotimes_{j=1}^m V_j$ (or the subspace of symmetric tensors $\mathrm{Sym}^m(V) \subset V^{\otimes m}$, here $V^{\otimes m} = \mathbb{V}$ with $V_i = V$ for $i = 1, \ldots, m$) with norm $\|\cdot\|$ induced by the inner product $\langle \cdot, \cdot \rangle = \Pi_{j=1}^m \langle \cdot, \cdot \rangle_j$. Denote

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the set of rank-one tensors by

$$S_1 = \{\mathcal{B} \in \mathbb{V} : \mathcal{B} = \otimes_{j=1}^m v^{(j)},\ v^{(j)} \in V_j\}.$$

For $\mathrm{Sym}^m(V)$, S_1 should be replaced by the set of symmetric rank-one tensors

$$S_1^{\mathrm{Sym}} = \{\mathcal{B} \in \mathrm{Sym}^m(V) : \mathcal{B} = v^{\otimes m},\ v \in V\}.$$

Denote the zero tensor in $\mathbb{V}$ by $\mathcal{O}$. Since S_1 is weakly closed (see Lemma 1 of [9] and its proof), for $\mathcal{A} \in \mathbb{V} \setminus \{\mathcal{O}\}$, it can be shown (see Lemma 6 of [9]) that

$$\|\mathcal{A} - \mathcal{B}^*\|^2 = \min_{\mathcal{B}\in S_1}\|\mathcal{A} - \mathcal{B}\|^2 = \|\mathcal{A}\|^2 - \sigma(\mathcal{A})^2 = \|\mathcal{A}\|^2\left(1 - \frac{\sigma(\mathcal{A})^2}{\|\mathcal{A}\|^2}\right), \tag{2.1}$$

where

$$\sigma(\mathcal{A}) = \max_{\mathcal{B}\in S_1,\, \|\mathcal{B}\|=1} |\langle \mathcal{A}, \mathcal{B}\rangle|. \tag{2.2}$$

The value $\sigma(\mathcal{A})$ is called the first singular value of $\mathcal{A} \in \mathbb{V}$ in [13]. In the finite dimensional case, it is actually the largest absolute value of the singular values of such a tensor in the sense of [14]. It itself may not be a singular value.

In the symmetric case, we may replace $\sigma(\mathcal{A})$ by

$$\rho(\mathcal{A}) = \max_{\mathcal{B}\in S_1^{\mathrm{Sym}},\, \|\mathcal{B}\|=1} |\langle \mathcal{A}, \mathcal{B}\rangle|. \tag{2.3}$$

In the finite dimensional case, $\rho(\mathcal{A})$ is actually the largest absolute value of the Z-eigenvalues of such a tensor $\mathcal{A}$ in the sense of [19]. It itself may not be a Z-eigenvalue of that tensor. Hence, we call it the *spectral radius* of that tensor in this paper. In section 4, we will give the definition of Z-eigenvalues.

Define

$$\mathrm{App}(\mathbb{V}) = \max\left\{\mu : \mu \le \frac{\sigma(\mathcal{A})}{\|\mathcal{A}\|} \forall \mathcal{A} \in \mathbb{V}, \mathcal{A} \neq \mathcal{O}\right\}. \tag{2.4}$$

We call $\mathrm{App}(\mathbb{V})$ the *best rank-one approximation ratio* of $\mathbb{V}$, or simply the approximation ratio of $\mathbb{V}$. It is independent from a particular tensor; rather, it is an important index of the tensor space $\mathbb{V}$.

Similarly, we may define the best rank-one approximation ratio of $\mathrm{Sym}^m(V)$ as

$$\mathrm{App}(\mathrm{Sym}^m(V)) = \max\left\{\mu : \mu \le \frac{\rho(\mathcal{A})}{\|\mathcal{A}\|} \forall \mathcal{A} \in \mathrm{Sym}^m(V),\ \mathcal{A} \neq \mathcal{O}\right\}. \tag{2.5}$$

By (2.1), for $\mathcal{A} \in \mathbb{V} \setminus \{\mathcal{O}\}$, we have

$$\frac{\|\mathcal{A} - \mathcal{B}^*\|^2}{\|\mathcal{A}\|^2} \le 1 - \mathrm{App}(\mathbb{V})^2, \tag{2.6}$$

where $\mathcal{B}^*$ is the best rank-one approximation of $\mathcal{A}$. Hence, the approximation ratio of $\mathbb{V}$ gives an upper bound for the quotient of the residual of the best rank-one approximation of any tensor in $\mathbb{V}$ and the norm of that tensor.

In the finite dimensional case, S_1 is closed. Then, by (2.2), we see that $\sigma(\cdot)$ is also a norm of $\mathbb{V}$. By (2.4) and the norm equivalence theorem [18], we have

$$\text{App}(\mathbb{V}) > 0. \tag{2.7}$$

Thus, in the finite dimensional case, (2.6) provides an upper bound for the quotient of the residual of the best rank-one approximation of any tensor $\mathcal{A}$ in $\mathbb{V}$ and the norm of $\mathcal{A}$. This upper bound is also strictly less than one.

We now consider the following greedy rank-one update algorithm [8], [13] (called progressive separated representation in [9] and, in the symmetric case, called successive symmetric rank-one decomposition in [24]). For $\mathcal{A} \in \mathbb{V} \setminus \{\mathcal{O}\}$, let $\mathcal{A}^{(0)} = \mathcal{A}$. For $k \geq 0$, let $\mathcal{B}^{(k)}$ be the best rank-one approximation of $\mathcal{A}^{(k)}$, and let $\mathcal{A}^{(k+1)} = \mathcal{A}^{(k)} - \mathcal{B}^{(k)}$. Then by (2.1) and (2.6), we have

$$\|\mathcal{A}^{(k+1)}\|^2 \leq \|\mathcal{A}^{(k)}\|^2[1 - \text{App}(\mathbb{V})^2] \leq \cdots \leq \|\mathcal{A}\|^2[1 - \text{App}(\mathbb{V})^2]^{k+1}.$$

This shows that $\mathcal{A} = \sum_{k=0}^{\infty} \mathcal{B}^{(k)}$ and gives a convergence rate for this algorithm. Numerical examples of this algorithm can be found in section 6. More discussion on this algorithm can be found in [1], [8], [9], [13], [24]. The symmetric case can be treated similarly. We also have

$$\text{App}(\text{Sym}^m(\mathfrak{R}^n)) > 0. \tag{2.8}$$

3. A general finite dimensional tensor space. Let $2 \leq n_1 \leq \cdots \leq n_m$. Consider $\mathbb{V} \equiv \mathbb{V}(m; n_1, \ldots, n_m) = \bigotimes_{j=1}^m \mathfrak{R}^{n_j}$ in this section. In this case, for $\mathcal{A} \in \mathbb{V}$, we may denote $\mathcal{A} = (a_{i_1 \cdots i_m})$, where $i_j = 1, \ldots, n_j$. The norm $\|\cdot\|$ induced by the inner product $\langle x, y \rangle \equiv x^\top y$ in $\mathfrak{R}^n$ is actually the Frobenius norm. For $\mathcal{A} \in \mathbb{V}$, it has the form

$$\|\mathcal{A}\| \equiv \sqrt{\sum_{i_1}^{n_1} \cdots \sum_{i_m=1}^{n_m} a_{i_1 \cdots i_m}^2}.$$

For $x^{(j)} \in \mathfrak{R}^{n_j}$, we call it a unit vector if $(x^{(j)})^\top x^{(j)} = 1$. The best rank-one approximation of $\mathcal{A}$ is a rank-one tensor $\lambda x^{(1)} \cdots x^{(m)} \equiv \lambda \otimes_{j=1}^m x^{(j)} \equiv (\lambda x_{i_1}^{(1)} \cdots x_{i_m}^{(m)})$, where $\lambda \in \mathfrak{R}$, $x^{(j)} \in \mathfrak{R}^{n_j}$ are unit vectors such that the Frobenius norm $\|\mathcal{A} - \lambda x^{(1)} \cdots x^{(m)}\|$ is minimized.

Let $\mathcal{A} \in \mathbb{V}$. For $x^{(j)} \in \mathfrak{R}^{n_j}, j = 1, \ldots, m$, denote

$$\mathcal{A}x^{(1)} \cdots x^{(m)} \equiv \langle \mathcal{A}, \otimes_{j=1}^m x^{(j)} \rangle = \sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} a_{i_1 \cdots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}.$$

Then we have

$$\sigma(\mathcal{A}) = \max\{|\mathcal{A}x^{(1)} \cdots x^{(m)}| : x^{(j)} \in \mathfrak{R}^{n_j},\ (x^{(j)})^\top x^{(j)} = 1 \text{ for } j = 1, \ldots, m\}. \tag{3.1}$$

We may see that $\sigma(\mathcal{A})$ is the largest absolute value of the singular values of $\mathcal{A}$ in the sense of [14]. By (3.1), for any $\mathcal{A} \in \mathbb{V}$ and any unit vectors $x^{(j)} \in \mathfrak{R}^{n_j}$ for $j = 1, \ldots, m$, we have

$$\sigma(\mathcal{A}) \geq |\mathcal{A}x^{(1)} \cdots x^{(m)}| = \left| \sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} a_{i_1 \cdots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)} \right|. \tag{3.2}$$

Clearly, for any $\mathcal{A} \in \mathbb{V}$ and $\mathcal{A} \neq \mathcal{O}$, we have

$$0 < \frac{\sigma(\mathcal{A})}{\|\mathcal{A}\|} \leq 1.$$

Then we have

$$0 < \mathrm{App}(\mathbb{V}) \leq 1.$$

For a matrix space, we have $m = 2$. It is not difficult to see that in that case

$$\mathrm{App}(\mathbb{V}(m; n_1, n_2)) = \frac{1}{\sqrt{n_1}}.$$

THEOREM 3.1. *Let*

$$\mu = \frac{1}{\sqrt{n_1 \cdots n_{m-1}}}.$$

Then μ *is a positive lower bound for* $\mathrm{App}(\mathbb{V}(m; n_1, \ldots, n_m))$.

Proof. Suppose that $\mathcal{A} \in \mathbb{V}(m; n_1, \ldots, n_m)$. For each $(i_1, \ldots, i_{m-2})$, satisfying that $1 \leq i_1 \leq n_1, \ldots, 1 \leq i_{m-2} \leq n_{m-2}$, let $K_{i_1 \cdots i_{m-2}}$ be an $n_{m-1} \times n_m$ matrix with its (i, j)th element as $a_{i_1 \cdots i_{m-2} ij}$. Then by (3.1), we have

$$\sigma(K_{i_1 \cdots i_{m-2}}) \leq \sigma(\mathcal{A}).$$

We have

$$\begin{aligned} \|\mathcal{A}\|^2 &= \sum_{i_1=1}^{n_1} \cdots \sum_{i_{m-2}=1}^{n_{m-2}} \|K_{i_1 \cdots i_{m-2}}\|^2 \leq \sum_{i_1=1}^{n_1} \cdots \sum_{i_{m-2}=1}^{n_{m-2}} n_{m-1} \sigma(K_{i_1 \cdots i_{m-2}})^2 \\ &\leq \sum_{i_1=1}^{n_1} \cdots \sum_{i_{m-2}=1}^{n_{m-2}} n_{m-1} \sigma(\mathcal{A})^2 = n_1 \cdots n_{m-1} \sigma(\mathcal{A})^2. \end{aligned}$$

Now the conclusion follows. □

The above bound is tight when $m = 2$. The question is if it is the exact value of $\mathrm{App}(\mathbb{V}(m; n_1, \ldots, n_m))$ for $m \geq 3$.

4. A finite dimensional symmetric tensor space. We now consider $\mathrm{Sym}^m(\mathfrak{R}^n)$. For $\mathcal{A} \in \mathrm{Sym}^m(\mathfrak{R}^n)$, we can denote $\mathcal{A} = (a_{i_1 \cdots i_m})$, where $i_1, \ldots, i_m = 1, \ldots, n$ and the entries $a_{i_1 \cdots i_m}$ are invariant under any permutation of its indices. Let $\lambda \in \mathfrak{R}$ and $x \in \mathfrak{R}^n$ be a unit vector. Then $\lambda x^m \equiv \lambda x^{\otimes m}$ denotes the rank-one mth order n-dimensional real symmetric tensor, whose $(i_1 \cdots i_m)$th element is $\lambda x_{i_1} \cdots x_{i_m}$. The best rank-one approximation of $\mathcal{A}$ is a rank-one tensor λx^m such that the Frobenius norm $\|\mathcal{A} - \lambda x^m\|$ is minimized. The Frobenius norm of tensor $\mathcal{A}$ has the form

$$\|\mathcal{A}\| \equiv \sqrt{\sum_{i_1,\ldots,i_m=1}^{n} a^2_{i_1\cdots i_m}}.$$

According to [19], λx^m is the best rank-one approximation of $\mathcal{A}$ if and only if λ is a Z-eigenvalue of $\mathcal{A}$ with the largest absolute value, while x is a Z-eigenvector of $\mathcal{A}$, associated with the Z-eigenvalue λ.

Denote $\mathcal{A}x^{m-1}$ as an n-dimensional vector whose ith component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2\cdots i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Suppose $\lambda \in \mathfrak{R}$ and $x \in \mathfrak{R}^n$ satisfy the system

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^\top x = 1. \end{cases} \tag{4.1}$$

Then we call λ a Z-eigenvalue of $\mathcal{A}$, and we call x a Z-eigenvector of $\mathcal{A}$, associated with the Z-eigenvalue λ. Then the spectral radius $\rho(\mathcal{A})$ is the largest absolute value of the Z-eigenvalues of $\mathcal{A}$.

For $\mathcal{A} \in \mathrm{Sym}^m(\mathfrak{R}^n)$ and $x \in \mathfrak{R}^n$, we have

$$\mathcal{A}x^m \equiv \langle \mathcal{A}, x^m \rangle = \sum_{i_1\cdots i_m=1}^{n} a_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m}.$$

By (2.3), we have

$$\rho(\mathcal{A}) = \max_{x\in\mathfrak{R}^n,\, x^\top x=1} |\mathcal{A}x^m|. \tag{4.2}$$

Thus, we have

$$0 < \frac{\rho(\mathcal{A})}{\|\mathcal{A}\|} \le 1$$

for any $\mathcal{A} \in \mathrm{Sym}^m(\mathfrak{R}^n)$, $\mathcal{A} \neq \mathcal{O}$.

Clearly,

$$0 < \mathrm{App}(\mathrm{Sym}^m(\mathfrak{R}^n)) \le 1$$

for all $m, n \ge 2$. In the case of a symmetric matrix space, we have $m = 2$. It is not difficult to see that

$$\mathrm{App}(\mathrm{Sym}^2(\mathfrak{R}^n)) = \frac{1}{\sqrt{n}}.$$

Again, it is an open question to find the exact values of $\mathrm{App}(\mathrm{Sym}^m(\mathfrak{R}^n))$ for $m \ge 3$. By Theorem 2.2 of [28], we have the following theorem.

THEOREM 4.1. *For any $\mathcal{A} \in \mathrm{Sym}^3(\mathfrak{R}^n)$, we have $\rho(\mathcal{A}) = \sigma(\mathcal{A})$.*

CONJECTURE 1. *For any $\mathcal{A} \in \mathrm{Sym}^m(\mathfrak{R}^n)$ with $m \ge 4$, we still have $\rho(\mathcal{A}) = \sigma(\mathcal{A})$.*

PROPOSITION 4.2.

$$\max\left\{\left|\sum_{i=1}^{n} x_i\right| : x \in \mathfrak{R}^n,\ x^\top x = 1\right\} = \sqrt{n}.$$

Proof. We have

$$\max\left\{\left|\sum_{i=1}^n x_i\right| : x \in \Re^n,\ x^\top x = 1\right\} = \max\left\{\sum_{i=1}^n x_i : x \in \Re^n,\ x^\top x = 1\right\}.$$

Following the optimization theory, we have the conclusion. □

Let

$$\mu_{m,n} = \frac{1}{\sqrt{n^{m-1}}}.$$

If $m = 2k$ is even, then let $\mathcal{A}^{(m,n)} \in \mathrm{Sym}^m(\Re^n)$, and let $\bar{\mu}_{m,n}$ be defined by

$$\mathcal{A}^{(m,n)} x^m = (x^\top x)^k \tag{4.3}$$

and

$$\bar{\mu}_{m,n} = \frac{1}{\|\mathcal{A}^{(m,n)}\|}.$$

If $m = 2k+1$ is odd, then let $\mathcal{A}^{(m,n)} \in \mathrm{Sym}^m(\Re^n)$, and let $\bar{\mu}_{m,n}$ be defined by

$$\mathcal{A}^{(m,n)} x^m = (x^\top x)^k \left(\sum_{i=1}^n x_i\right) \tag{4.4}$$

and

$$\bar{\mu}_{m,n} = \frac{\sqrt{n}}{\|\mathcal{A}^{(m,n)}\|}.$$

THEOREM 4.3. *The value $\mu_{3,n}$ is a positive lower bound for* App(Sym$^3(\Re^n)$). *On the other hand, the value $\bar{\mu}_{m,n}$ is an upper bound for* App(Sym$^m(\Re^n)$) *for $m = 2, 3, \ldots$. We have*

$$\frac{1}{\sqrt{n}} = \mu_{2,n} = \mathrm{App}(\mathrm{Sym}^2(\Re^n)) = \bar{\mu}_{2,n} = \frac{1}{\sqrt{n}}, \tag{4.5}$$

$$\frac{1}{n} = \mu_{3,n} \le \mathrm{App}(\mathrm{Sym}^3(\Re^n)) \le \bar{\mu}_{3,n} = \sqrt{\frac{6}{n+5}}, \tag{4.6}$$

and

$$\mathrm{App}(\mathrm{Sym}^4(\Re^n)) \le \bar{\mu}_{4,n} = \sqrt{\frac{3}{n^2+2n}}. \tag{4.7}$$

Proof. By Theorems 3.1 and 4.1, (2.4), and (2.5), we have the first conclusion. If m is even, by (4.2) and (4.3), we have $\rho(\mathcal{A}^{(m,n)}) = 1$. If m is odd, by (4.2), (4.4), and Proposition 4.2, we have $\rho(\mathcal{A}^{(m,n)}) = \sqrt{n}$. By (2.5), we have the second conclusion.

The equalities (4.5) are basic knowledge of linear algebra.

For (4.6), we need only prove the last equality. The other equality and inequalities of (4.6) follow from the first two conclusions. Let $\mathcal{A}^{(3,n)} = (a_{ijk})$ be defined by (4.4). Then

$$a_{iii} = 1$$

for $i = 1, \ldots, n$,

$$a_{iij} = a_{iji} = a_{jii} = \frac{1}{3}$$

for $i, j = 1, \ldots, n$, $i \neq j$, and the other elements of $\mathcal{A}^{(3,n)}$ are zero. Then,

$$\|\mathcal{A}^{(3,n)}\|^2 = \sum_{i=1}^{n} a_{iii}^2 + \sum_{\substack{1 \le i,j \le n \\ i \neq j}} [a_{iij}^2 + a_{iji}^2 + a_{jii}^2] = \frac{n^2 + 5n}{6}.$$

Hence, the last equality of (4.6) holds.

For (4.7), we need only prove the equality. The inequality of (4.7) follows from the second conclusion. Let $\mathcal{A}^{(4,n)} = (a_{ijkl})$ be defined by (4.3). Then

$$a_{iiii} = 1$$

for $i = 1, \ldots, n$,

$$a_{iijj} = a_{ijij} = a_{ijji} = a_{jiji} = a_{jiij} = a_{jjii} = \frac{1}{3}$$

for $i, j = 1, \ldots, n$, $i \neq j$, and the other elements of $\mathcal{A}^{(4,n)}$ are zero. Then,

$$\begin{aligned} &\|\mathcal{A}^{(4,n)}\|^2 \\ &= \sum_{i=1}^{n} a_{iiii}^2 + \sum_{1 \le i < j \le n} [a_{iijj}^2 + a_{ijij}^2 + a_{ijji}^2 + a_{jiji}^2 + a_{jiij}^2 + a_{jjii}^2] \\ &= \frac{n^2 + 2n}{3}. \end{aligned}$$

Hence, the equality of (4.7) also holds. □

CONJECTURE 2. *For* $m \ge 4$, $\mu_{m,n} = \frac{1}{\sqrt{n^{m-1}}}$ *is also a positive lower bound for* $\mathrm{App}(\mathrm{Sym}^m(\Re^n))$.

In the previous version of this paper, we got a positive lower bound

$$\mu = \frac{3}{\sqrt{4n^4 + 12\sqrt{3}n^3 + (35 - 36\sqrt{3})n^2 + (24\sqrt{3} - 30)n}}$$

for $\mathrm{App}(\mathrm{Sym}^4(\Re^n))$. The proof is tedious, and the bound is much smaller than $\mu_{4,n} = \frac{1}{\sqrt{n^3}}$. Hence, we do not include that result here.

Now, (4.5) gives the exact values of $\mathrm{App}(\mathrm{Sym}^m(\Re^n))$ for $m = 2$. What are the exact values of $\mathrm{App}(\mathrm{Sym}^m(\Re^n))$ for $m \ge 3$? Does an equality hold for one of the two inequalities of (4.6), or are both the inequalities of (4.6) strict? What is the exact value of $\mathrm{App}(\mathrm{Sym}^3(\Re^n))$? What is the exact value of $\mathrm{App}(\mathrm{Sym}^4(\Re^n))$?

5. A finite dimensional biquadratic tensor space. Beside symmetric and general tensors, there are also various partially symmetric tensors. Among partially symmetric tensors, biquadratic tensors have received much attention in recent years [5], [11], [16], [20], [22], [25], [27].

An $(n \times p)$-dimensional biquadratic tensor $\mathcal{A}$ has the form $\mathcal{A} = (a_{ijkl})$, where $i, j = 1, \ldots, n;\ k, l = 1, \ldots, p;\ 2 \le n \le p$, with symmetric property $a_{ijkl} = a_{jikl} = a_{ijlk}$ for any i, j, k, and l. We use $\mathbb{B}_{n,p}$ to denote the set of all $(n \times p)$-dimensional biquadratic tensors. Then $\mathbb{B}_{n,p} = \mathrm{Sym}^2(\Re^n)\bigotimes\mathrm{Sym}^2(\Re^p)$ is a tensor space.

The best rank-one approximation of $\mathcal{A} \in \mathbb{B}_{n,p}$ is a rank-one tensor $\lambda x^2y^2 \equiv \lambda x \otimes x \otimes y \otimes y \equiv (\lambda x_i x_j y_k y_l)$, where $\lambda \in \Re$, $x \in \Re^n$, and $y \in \Re^p$ are unit vectors with $x^\top x = y^\top y = 1$ such that the Frobenius norm $\|\mathcal{A} - \lambda x^2 y^2\|$ is minimized.

Let $\mathcal{A} \in \mathbb{B}_{n,p}$. For $x \in \Re^n$ and $y \in \Re^p$, denote

$$\mathcal{A}x^2y^2 \equiv \langle \mathcal{A}, x^2y^2 \rangle = \sum_{i,j=1}^{n} \sum_{k,l=1}^{p} a_{ijkl} x_i x_j y_k y_l.$$

For $\mathcal{A} \in \mathbb{B}_{n,p}$, define

$$\rho_B(\mathcal{A}) = \max\{|\mathcal{A}x^2y^2| : x \in \Re^n,\ x^\top x = 1,\ y \in \Re^p,\ y^\top y = 1\}. \tag{5.1}$$

Again, we see that $\rho_B(\cdot)$ is a norm of $\mathbb{B}_{n,p}$. We may also see that $\rho_B(\mathcal{A})$ is the largest absolute value of the M-eigenvalues of $\mathcal{A}$, defined as below [20], [25]. Denote $\mathcal{A}\cdot\, xyy$ as a vector in $\Re^n$, whose ith component is $\sum_{j=1}^n \sum_{k,l=1}^p a_{ijkl} x_j y_k y_l$, and denote $\mathcal{A}xxy\,\cdot$ as a vector in $\Re^p$, whose lth component is $\sum_{i,j=1}^n \sum_{k=1}^p a_{ijkl} x_i x_j y_k$. If $\lambda \in \Re$, $x \in \Re^n$, and $y \in \Re^p$ satisfy the system

$$\begin{cases} \mathcal{A}\cdot\, xyy = \lambda x, \\ \mathcal{A}xxy\,\cdot = \lambda y, \\ x^\top x = 1, \\ y^\top y = 1, \end{cases}$$

then we call λ an M-eigenvalue of $\mathcal{A}$, and we call x and y left and right M-eigenvectors of $\mathcal{A}$, associated with the M-eigenvalue λ. We call $\rho_B(\mathcal{A})$ the *bispectral radius* of $\mathcal{A}$.

CONJECTURE 3. *If $n = p$ and $\mathcal{A} \in \mathrm{Sym}^4(\Re^n)$, then $\rho_B(\mathcal{A}) = \rho(\mathcal{A})$.*

Similarly, for any $\mathcal{A} \in \mathbb{B}_{n,p}$ and $\mathcal{A} \neq \mathcal{O}$, we have

$$0 < \frac{\rho_B(\mathcal{A})}{\|\mathcal{A}\|} \le 1.$$

Define the *best rank-one approximation ratio* of $\mathbb{B}_{n,p}$ as

$$\mathrm{App}(\mathbb{B}_{n,p}) = \max\left\{\mu : \mu \le \frac{\rho_B(\mathcal{A})}{\|\mathcal{A}\|}\ \forall \mathcal{A} \in \mathbb{B}_{n,p},\ \mathcal{A} \neq \mathcal{O}\right\}.$$

Then,

$$0 < \mathrm{App}(\mathbb{B}_{n,p}) \le 1.$$

We now have the following theorem.

THEOREM 5.1. *We have*

$$\underline{\eta}_{n,p} \equiv \frac{1}{\sqrt{n^2 p}} \leq \mathrm{App}(\mathbb{B}_{n,p}) \leq \bar{\eta}_{n,p} \equiv \frac{1}{\sqrt{np}}.$$

Proof. For each (i,j), $1 \leq i,j \leq n$, let K_{ij} be a $p \times p$ symmetric matrix with its (k,l)th element as a_{ijkl}. Then by (5.1), we have

$$\rho(K_{ij}) \leq \rho_B(\mathcal{A}).$$

We have

$$\|\mathcal{A}\|^2 = \sum_{i,j=1}^{n} \|K_{ij}\|^2 \leq \sum_{i,j=1}^{n} p\rho(K_{ij})^2 \leq n^2 p \rho_B(\mathcal{A})^2.$$

The first inequality of (5.3) follows.

Let $\mathcal{A} \in \mathbb{B}_{n,p}$ be defined by

$$\mathcal{A}x^2y^2 = (x^\top x)(y^\top y).$$

By (5.1), $\rho_B(\mathcal{A}) = 1$. It is easy to see that $\|\mathcal{A}\|^2 = np$. By (5.2), we have the second inequality of (5.3). The proof is complete. □

Again, what is the exact value of $\mathrm{App}(\mathbb{B}_{n,p})$?

6. Numerical results. In this section, we present some intuitive numerical results of general third order tensors, symmetric third order tensors, and biquadratic tensors to show the validity of the theoretical results established in this paper. We use the greedy update algorithm to decompose the tensors. In every iteration of the greedy method, we use the higher order power method [12], its symmetric version, and the bisymmetric power method [25] to compute the best rank-one approximation of each of the three kinds of tensors, respectively. Since all the best rank-one approximation problems for higher order tensors are NP-hard, the solution found by the power method is only an approximate value of the best rank-one approximation. Nevertheless, favorable numerical results are achieved for the tested tensors. The experiments were conducted in MATLAB on a personal PC.

Let $\mathcal{A}^{(1)}$ be the tensor given in the following examples for $k \geq 1$; let $\{\mathcal{B}_i^{(k)}\}_{i\geq 1}$ be the sequence of computed rank-one approximations of $\mathcal{A}^{(k)}$ by the power method. The power method is terminated whenever $\|\mathcal{B}_{i+1}^{(k)} - \mathcal{B}_i^{(k)}\| < 1.0 \times 10^{-6}$. Then, let $\mathcal{B}^{(k)} := \mathcal{B}_{i+1}^{(k)}$ be the computed rank-one approximation of $\mathcal{A}^{(k)}$. Let $\mathcal{A}^{(k+1)} := \mathcal{A}^{(k)} - \mathcal{B}^{(k)}$. We terminate the greedy update algorithm whenever $\frac{\|\mathcal{A}^{(k)}\|}{\|\mathcal{A}^{(1)}\|} < 1.0 \times 10^{-6}$. The results are shown in Figures 1–3. In these figures, the horizontal axis represents the iteration k, *rank-one ration* denotes $\frac{\|\mathcal{B}^{(k)}\|}{\|\mathcal{A}^{(k)}\|}$, *computed residual* denotes $\frac{\|\mathcal{A}^{(k)}\|}{\|\mathcal{A}^{(1)}\|}$, and *theoretical residual* denotes $\sqrt{(1-\alpha^2)^k}$ with α the corresponding lower bound for $\mathrm{App}(\mathbb{V})$ established in sections 3, 4, and 5, respectively.

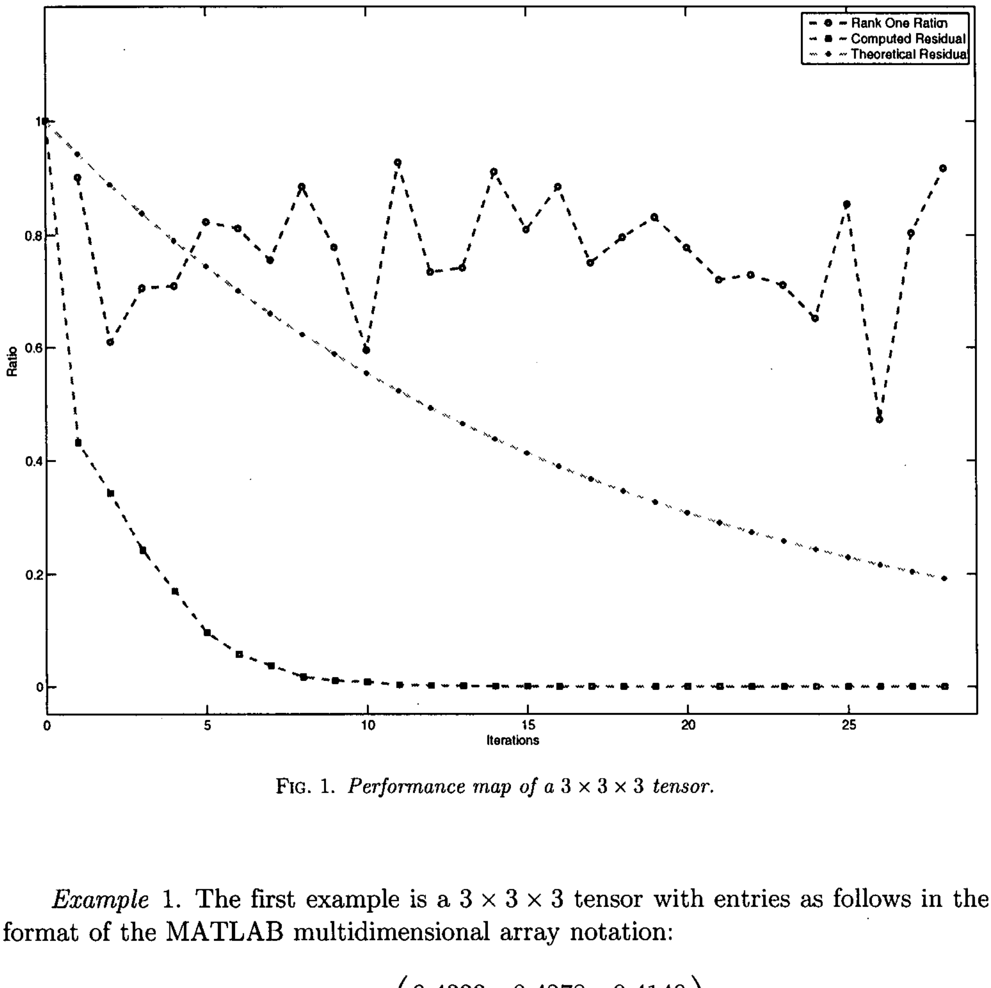

FIG. 1. *Performance map of a* $3 \times 3 \times 3$ *tensor.*

Example 1. The first example is a $3 \times 3 \times 3$ tensor with entries as follows in the format of the MATLAB multidimensional array notation:

$$\mathcal{A}(:,:,1) = \begin{pmatrix} 0.4333 & 0.4278 & 0.4140 \\ 0.8154 & 0.0199 & 0.5598 \\ 0.0643 & 0.3815 & 0.8834 \end{pmatrix},$$

$$\mathcal{A}(:,:,2) = \begin{pmatrix} 0.4866 & 0.8087 & 0.2073 \\ 0.7641 & 0.9924 & 0.8752 \\ 0.6708 & 0.8296 & 0.1325 \end{pmatrix},$$

$$\mathcal{A}(:,:,3) = \begin{pmatrix} 0.3871 & 0.0769 & 0.3151 \\ 0.1355 & 0.7727 & 0.4089 \\ 0.9715 & 0.7726 & 0.5526 \end{pmatrix}.$$

The results are shown in Figure 1. The lower bound for App($\mathbb{V}$) in this case is $\frac{1}{3}$. We observe from Figure 1 that all the computed rank-one rations are above the lower bound, and theoretical residual dominates computed residual as expected.

Example 2. The second example is a $3 \times 3 \times 3$ symmetric tensor with the independent entries as follows in the format of the MATLAB multidimensional array notation:

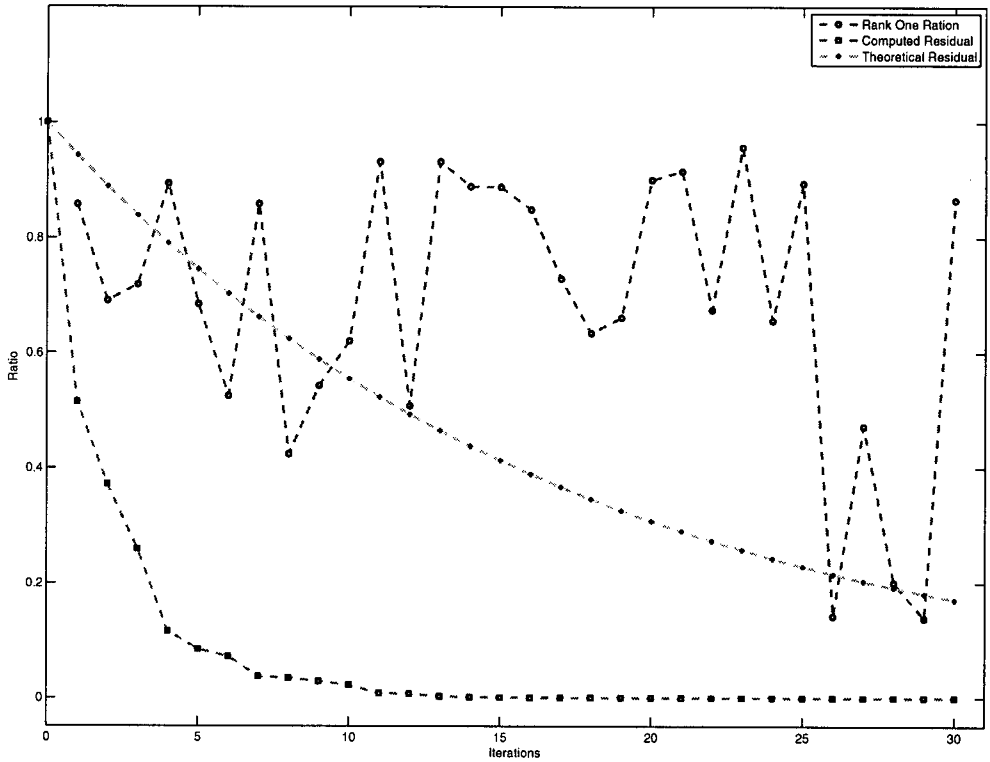

FIG. 2. *Performance map of a* $3 \times 3 \times 3$ *symmetric tensor.*

$$\mathcal{A}(1,1,1) = 0.0517; \quad \mathcal{A}(2,2,2) = 0.3943; \quad \mathcal{A}(3,3,3) = 0.9723;$$
$$\mathcal{A}(1,1,2) = 0.3579; \quad \mathcal{A}(1,1,3) = 0.5298; \quad \mathcal{A}(1,2,2) = 0.7544;$$
$$\mathcal{A}(1,3,3) = 0.3612; \quad \mathcal{A}(1,2,3) = 0.2156; \quad \mathcal{A}(2,2,3) = 0.0146;$$
$$\mathcal{A}(2,3,3) = 0.6718.$$

The results are shown in Figure 2. The lower bound for App($\mathbb{V}$) in this case is $\frac{1}{3}$. We observe from Figure 2 that 27 of 30 computed rank-one rations are above the lower bound. The three exception cases are due to the fact that the power method does not guarantee the computed solution is the best rank-one approximation, while theoretical residual dominates computed residual as expected.

Example 3. The third example is a $2 \times 2 \times 3 \times 3$ biquadratic tensor with the independent entries as follows in the format of the MATLAB multidimensional array notation:

$$\mathcal{A}(1,1,1,1) = 0.8728; \quad \mathcal{A}(1,1,1,2) = 0.8932; \quad \mathcal{A}(1,1,1,3) = 0.6199;$$
$$\mathcal{A}(1,1,2,2) = 0.7716; \quad \mathcal{A}(1,1,2,3) = 0.6240; \quad \mathcal{A}(1,1,3,3) = 0.7999;$$
$$\mathcal{A}(1,2,1,1) = 0.7562; \quad \mathcal{A}(1,2,1,2) = 0.7749; \quad \mathcal{A}(1,2,1,3) = 0.5485;$$
$$\mathcal{A}(1,2,2,2) = 0.5406; \quad \mathcal{A}(1,2,2,3) = 0.5487; \quad \mathcal{A}(1,2,3,3) = 0.6386;$$
$$\mathcal{A}(2,2,1,1) = 0.8378; \quad \mathcal{A}(2,2,1,2) = 0.7583; \quad \mathcal{A}(2,2,1,3) = 0.5386;$$
$$\mathcal{A}(2,2,2,2) = 0.6850; \quad \mathcal{A}(2,2,2,3) = 0.6113; \quad \mathcal{A}(2,2,3,3) = 0.5993.$$

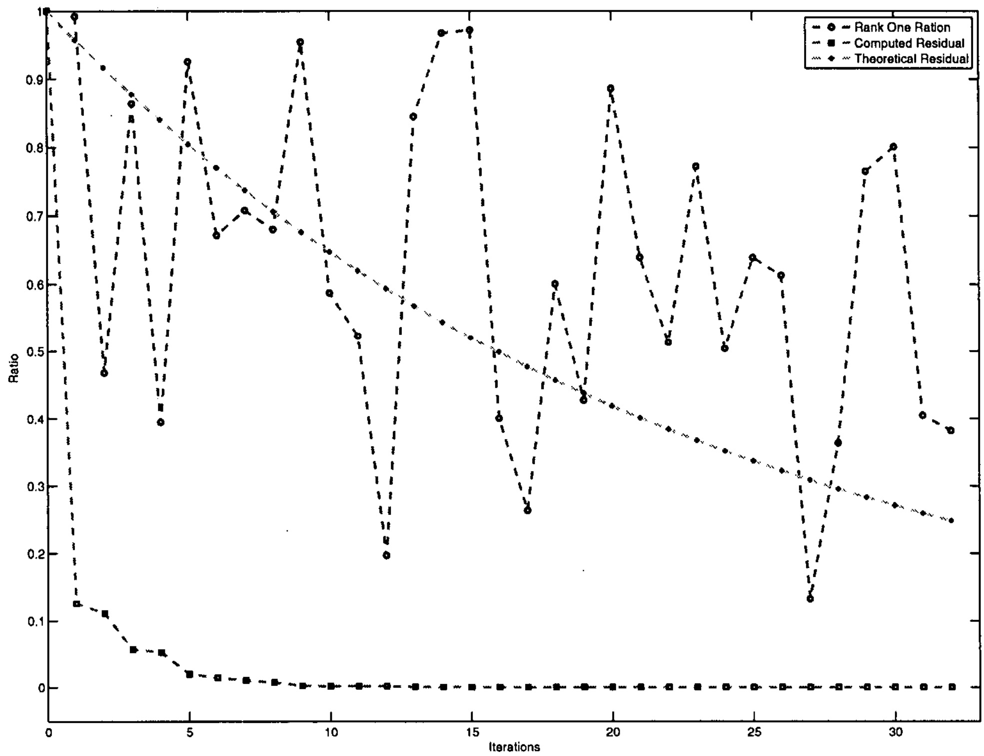

FIG. 3. *Performance map of a* $2 \times 2 \times 3 \times 3$ *biquadratic tensor.*

The results are shown in Figure 3. The lower bound for App($\mathbb{V}$) in this case is $\frac{1}{\sqrt{12}} = 0.2887$. Similar phenomena as that in Figure 2 could be observed.

From the numerical experiments, we see that the results established in this paper do give a convergence rate for the greedy rank-one update algorithm.

7. Four open questions. This paper leaves four outstanding challenging questions.

1. Are Conjectures 1–3 true? By Theorem 3.1, (2.4), and (2.5), we may see that if Conjecture 1 is true, then Conjecture 2 is true.
2. What are the exact values of App($\mathbb{V}(m; n_1, \ldots, n_m)$) for $m \geq 3$?
3. What are the exact values of App($\mathrm{Sym}^m(\mathfrak{R}^n)$) for $m \geq 3$?
4. What are the exact values of App($\mathbb{B}_{n,p}$)?

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