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Nuclear Norm Regularized SENSE Reconstruction

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Abstract –

SENSitivity Encoding (SENSE) is a mathematically optimal parallel MRI technique when the coil sensitivities are known. In recent times, Compressed Sensing (CS) based techniques are incorporated within the SENSE reconstruction framework to recover the underlying MR image. CS based techniques exploit the fact that the MR images are sparse in a transform domain (e.g. wavelets). Mathematically this leads to an l_1-norm regularized SENSE reconstruction.

In this work, we show that instead of reconstructing the image by exploiting its transform domain sparsity, we can exploit its rank deficiency to reconstruct it. This leads to a nuclear norm regularized SENSE problem. The reconstruction accuracy from our proposed method is the same as the l_1-norm regularized SENSE, but the advantage of our method is that it is about an order of magnitude faster.

1. INTRODUCTION

Multi-coil parallel MRI [1] is a hardware based acceleration technique for fast acquisition of K-space samples. Instead of employing a single receiver coil for acquiring the full K-space data, multiple coils partially sample the K-space. The ratio at which is the coils under-sample the K-space is called the acceleration factor. Depending on the position of the coils, each coil has a different field-of-view. Thus each of the coils has a different sensitivity profile. All parallel MR image reconstruction techniques, either explicitly or implicitly require this sensitivity profile for recovering the underlying image.

The physical data acquisition model for multi-coil MRI can be expressed as,

$$y_i = RFS_i x + \eta_i, i = 1...C \tag{1}$$

where y_i is the acquired K-space data from the i^{th} coil, R is the under-sampling mask, F is the Fourier transform matrix, S_i is the sensitivity profile and η_i is the noise for the i^{th} coil, x is the vectorised image (formed by row concatenation) to be recovered and C is the total number of coils.

Recovering the image from the acquired K-space samples from multiple coils is a not a well-defined problem. There are two broad approaches to address this problem – Image domain methods directly estimate the image; Frequency domain methods interpolate the missing frequency samples to have the full K-space data from which the image is reconstructed by simple 2D inverse Fast Fourier transform (FFT). Over the years researchers have proposed a plethora of multi-coil parallel MR image reconstruction methods. In the limited scope of this work, it is not possible to discuss them even briefly. SENSitivity Encoding (SENSE) [2] is physically and mathematically the most optimal parallel image reconstruction technique when the sensitivity profiles of the different coils are known. SENSE is by far the most widely used parallel MRI technique. All commercial scanners use modified versions of the basic SENSE method [3] – Philips (SENSE), Siemens (mSENSE), GE (ASSET), Toshiba (SPEEDER).

SENSE combines all the K-space data in the following form,

$$y = Ex + \eta \tag{2}$$

where $y = \begin{bmatrix} y_1 \\ \dots \\ y_C \end{bmatrix}$, $E = \begin{bmatrix} RFS_1 \\ \dots \\ RFS_C \end{bmatrix}$ and $\eta = \begin{bmatrix} \eta_1 \\ \dots \\ \eta_C \end{bmatrix}$.

In traditional SENSE reconstruction, the image is recovered by solving the following least-squares problem,

$$\hat{x} = \min_x \|y - Ex\|_2 \tag{3}$$

The least squares solution (3), yields an image with blurred edges. To alleviate this problem, the basic SENSE reconstruction is modified by adding a Total Variation (TV) or wavelet regularization term,

$$\hat{x} = \min_x \|y - Ex\|_2 + \lambda R(x) \tag{4}$$

where *R(x)* is either *TV* or l_1-norm of wavelet coefficients

The parameter λ needs to be fixed by the user. Researchers in MRI fix this value based on some heuristics. However, from the perspective of optimization theory, the meaning of this free parameter is not readily discernible. A more theoretical approach to SENSE regularization is based on Compressed Sensing (CS) [6]. Instead of solving an unconstrained optimization problem, it proposes a constrained problem of the following form,

$$\hat{x} = \min_x \|Wx\|_1 \text{ such that } \|y - Ex\|_2 \le \varepsilon \tag{5}$$

where *W* represents the wavelet transform and $\varepsilon = \sqrt{CN}\sigma$, σ being the standard deviation of noise.

Comparing (4) and (5) one sees that for the correct choice of the parameters λ and ε, (4) is the unconstrained Lagrangian version of (5). The advantage of (4) is that it is easy to solve, but in general there is no analytical method to deduce λ by knowing ε. It is easy to estimate ε when the standard deviation of noise is known. Thus, from the perspective of optimization theory, solving (5) is more optimal than solving (4). Fortunately there are state-of-the-art fast algorithms to solve (5) – SPGL1 [7], C-SALSA [8], NESTA [9].

In a recent work [10], we showed that instead of reconstructing single coil MR images from partially sampled K-space data by CS based methods such as [11, 12], one could reconstruct them by nuclear norm minimization. The main difference between our previous work and CS based methods is that, while the CS based methods utilise the fact that the MR image is sparse in a transform domain, we exploit the rank deficiency of the MR image to reconstruct it.

We showed that, while the reconstruction accuracy from our proposed method [10] (based on exploiting rank deficiency of the MR image) is the same as CS based methods (based on exploiting transform domain sparsity), the reconstruction time taken by our method is about an order of magnitude (10 times) less.

In this work, we propose to incorporate the information that the MR image is rank deficient, into the SENSE reconstruction framework. This is an extension of our previous work [10] to the multi-coil setting. Mathematically, it leads to the following optimization problem,

$$\hat{x} = \min_{x} \|X\|_* \text{ such that } \|y - Ex\|_2 \le \varepsilon \quad (6)$$

where x is the vectorised image and X is the image in matrix form. The nuclear norm $||X||_*$ is defined as the sum of the singular values of X.

We have carried out experiments with both real and synthetic data. The results show that the reconstruction accuracy from our proposed method is at par with state-of-the-art SENSE based technique like SparSENSE [6]. The main advantage of our method is that, it is an order of magnitude faster.

The rest of the paper is organised into several section. In section 2, the background behind this work is discussed. Section 3, describes our proposed parallel MRI technique. In section 4, the experimental results are shown. Finally in section 5 the conclusions of the work are discussed.

2. BACKGROUND

In this section we will first discuss the Compressed Sensing based techniques for MR image reconstruction from partially sampled K-space data from a single coil by exploiting their transform domain sparsity. Next we will discuss how to reconstruct the image by exploiting their rank deficiency by nuclear norm minimization. This will form the background for our proposed nuclear norm regularization based SENSE reconstruction.

2.1 Compressed Sensing based reconstruction of MR images

The K-space data acquisition model for a single coil MRI scanner is given by,

$$y = RFx + \eta \quad (7)$$

The problem is to recover the image x, given the K-space samples y. The sampling mask R under-samples the K-space and thus (7) is an under-determined inverse problem. In general there is no unique solution to (7) and in order to solve it, some prior information regarding the image is necessary.

MR images are sparse in certain transform domains like wavelets and finite difference. Compressed Sensing (CS) based techniques [11, 12] solve the inverse problem (7) by exploiting this transform domain sparsity. Ideally, one would solve the following l_0 minimization problem to recover the image,

$$\hat{x} = \min_{x} \|Wx\|_0 \text{ such that } \|y - RFx\|_2 \le \varepsilon \quad (8)$$

where the l_0-norm counts the number of non-zeros in the vector.

Unfortunately this is a NP hard problem and thus can not be solved efficiently. Theoretical studies in CS showed that instead of solving the NP hard l_0-norm, its convex envelope, the l_1-norm could be used [13, 14],

$$\hat{x} = \min_{x} \|Wx\|_1 \text{ such that } \|y - RFx\|_2 \le \varepsilon \quad (9)$$

This is a convex problem, and can be solved by quadratic programming. As mentioned earlier, there are state-of-the-art algorithms [7-9] to solve the l_1 minimization problem.

In this work we have shown the CS based method using wavelet regularization, but instead one can use finite differences as well. In that case, the l_1-norm of wavelet coefficients is replaced by a Total Variation (TV) term. In practical MR image reconstruction studies uses a combination of TV and wavelet regularization is used [11, 15-17],

$$\hat{x} = \min_{x} \|Wx\|_1 + \gamma TV(x) \text{ such that } \|y - RFx\|_2 \leq \varepsilon \quad (10)$$

The parameter γ controls the relative importance of wavelet regularization and TV regularization.

2.2 Reconstruction of MR images by exploiting rank deficiency

The MR images are sparse in the transform domain, i.e. the transform domain coefficients have only a few (say *s*) non-zero values while most are zeros. Thus effectively the number degrees of freedom in the transform domain is *2s* (*s* values and *s* positions). This is far less than the length of the image vector. CS says that that if the number of K-space measurements is sufficiently large compared to the number of non-zero transform domain coefficients, the image can be recovered accurately.

In a recent work we have shown that it is possible to recover the MR image by exploiting its rank deficiency [10]. If we assume that the image is of size n^2 but has a rank *r*, then the number of degrees of freedom in the image matrix is *r(2n-r)*. Theoretical studies in nuclear norm minimization showed that as long as the number of samples is sufficiently large compared to the number of degrees of freedom of the image matrix, it is possible to recover the image matrix accurately [18, 19].

In order to recover a matrix by exploiting its rank deficiency, the following optimization problem needs to be solved,

$$\hat{x} = \min_{x} \; rank(X) \text{ such that } \|y - RFx\|_2 \leq \varepsilon \quad (11)$$

Here *X* is the image in matrix form and *x* is the vectorized version of the image obtained by row concatenation.

This (11) is an NP hard problem similar to (8). Theoretical research in nuclear norm minimization [18, 19] has shown that, it is possible to recover the image matrix by using the Nuclear Norm which is the tightest convex surrogate for the rank of the matrix.

$$\hat{x} = \min_{x} \|X\|_* \text{ such that } \|y - RFx\|_2 \leq \varepsilon \quad (12)$$

Unlike Compressed Sensing, not many algorithms exist that solves the nuclear norm minimization problem (12). In [20], we have developed a fast shrinkage algorithm to solve it based on the majorization minimization approach.

3. NUCLEAR NORM REGULARISED SENSE

SENSE accurately depicts the physical data acquisition model for multi-coil parallel MRI (2). However, traditional SENSE reconstruction is based on solving a least-squares problem (3). It is easy to see that the least-squares problem does not have any inherent denoising capability. Images obtained by the traditional SENSE method were noisy [21]. Regularised SENSE reconstruction (4) yields better reconstruction results. This is because the regularization term incorporates some prior information regarding the image, e.g. in TV regularization it is assumed that the image is piece-wise smooth. Unfortunately, the regularization parameter was chosen arbitrarily; even though it was known that regularization improves the image, it was not possible to determine for what value of the parameter the best reconstruction result could be obtained.

In order to alleviate the problem of parameter tuning, a CS based approach called SparSENSE [6] was introduced which converted the unconstrained optimization problem (4) to one of constrained optimization (5). The constrained optimization problem required the noise variance in the data to be specified; fortunately in most cases this is easy to be determined. The SparSENSE yields very good reconstruction results, but it is not actually a CS problem. CS solves an under-determined inverse problem. In most cases the acceleration factor is less than the number of coils in the scanner. Thus the inverse problem (2) is over-determined. Therefore in SparSENSE the l_1-norm on the wavelet coefficients acts as a regularization term.

In CS based reconstruction techniques like SparSENSE, the transform domain sparsity of the image is used to reconstruct the image. Mathematically, this information appears as the l_1-norm (over the wavelet coefficients) regularization term for the SENSE method. In this study we propose to show that instead of exploiting the wavelet domain sparsity of the image, one can get equally good results (compared to CS based methods) by exploiting the rank deficiency of the MR image. This is a direct extension of our previous work [10] to the multi-coil setting.

Mathematically, this appears as a nuclear norm (NN) regularization term for the SENSE method (6). Deriving the algorithm for solving (6) is beyond the scope of this paper. For the sake of completeness we provide the algorithm for solving it.

Initialize: $x_0 = 0$; $\lambda < \max(E^T x)$
Choose a decrease factor (*DecFac*) for cooling λ
Outer Loop: While[1] $\| y - Ex \|_2 > \varepsilon$
Inner Loop: While[2] $\frac{J_k - J_{k+1}}{J_k + J_{k+1}} \geq Tol$

1. Compute objective function for current iterate: $J_k = \| y - Ex_k \|_2^2 + \lambda \| X_k \|_*$
2. Decrease λ by $\lambda = \lambda * DecFac$
   i. $x_k = x_{k-1} + E^T (y - Ex_{k-1})$
   ii. Form the matrix X_k by reshaping x_k.
   iii. SVD: $X_k = U \Sigma V^T$.
   iv. Soft threshold the singular values: $\hat{\Sigma} = soft(diag(\Sigma), \frac{\lambda}{2})$
   v. $X_{k+1} = U \hat{\Sigma} V^T$. Form x_{k+1} by vectorizing X_{k+1}.
   vi. Update: $k=k+1$ and return to step i.
3. Compute objective function for next iterate: $J_{k+1} = \| y - Ex_{k+1} \|_2^2 + \lambda \| X_{k+1} \|_*$

End While[2] (inner loop ends)

We will now describe in brief, what the algorithm does. Our problem is to solve the constrained optimization problem (12)

$$\hat{x} = \min_x \|X\|_* \text{ such that } \|y - Ex\|_2 \leq \varepsilon$$

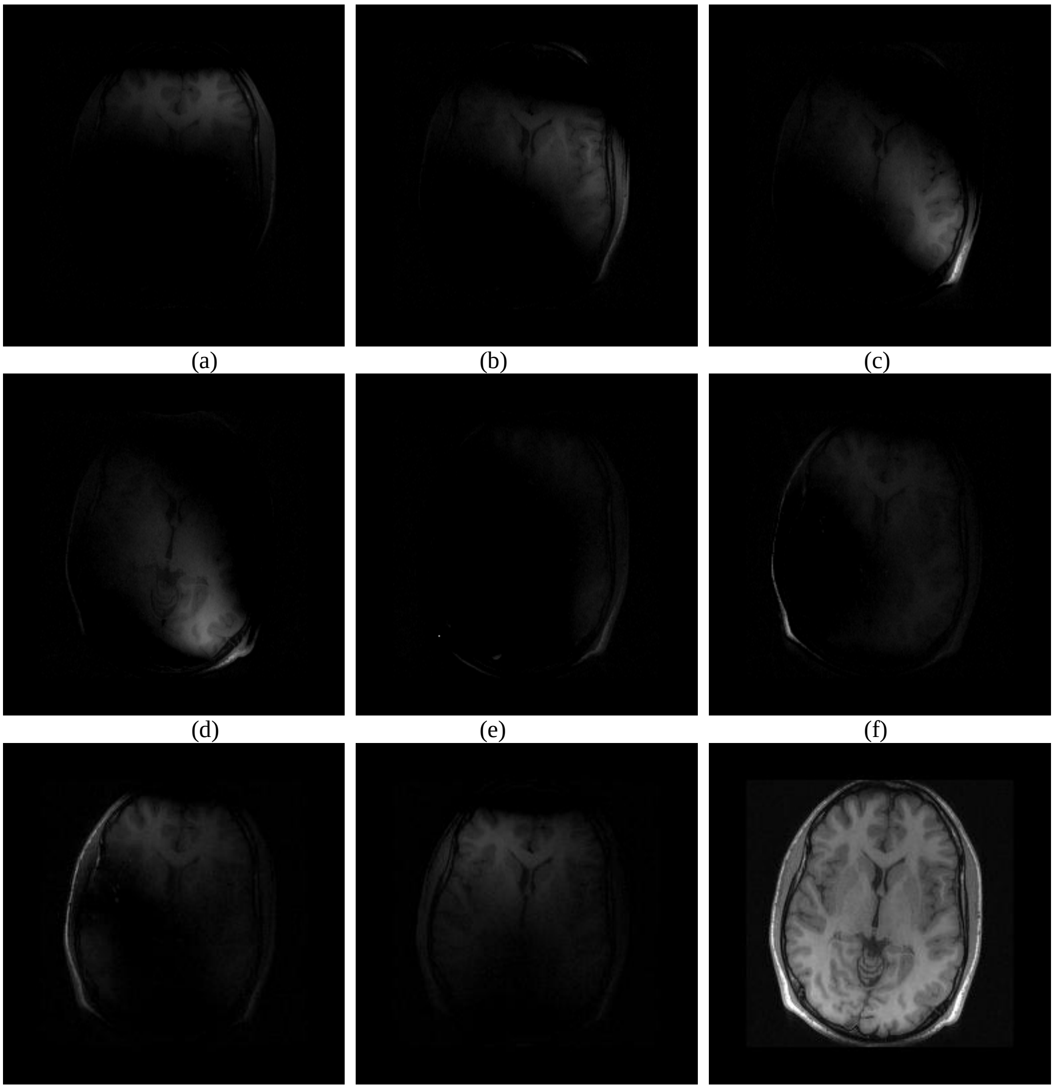
Solving the constrained problem (12) directly is not possible. Our algorithm solves (12) via a cooling method. The inner-loop (steps i-vi) solves an unconstrained Lagrangian version of (12),

$$\hat{x} = \min_{x} \ \|y - Ex\|_2 + \lambda \|X\|_* \tag{13}$$

However, it is not possible to determine the value of Lagrangian multiplier λ analytically. Thus the outer loop (steps 1-3) starts from a high value of λ. For each value of λ, the inner loop solves (13). The outer loop progressively reduces the value of λ till the solution converges to $\|y - Ex\|_2 \leq \varepsilon$. Such a cooling technique is guaranteed to converge and has been used previously [22] to solve l_1 minimization in geophysical CS problems.

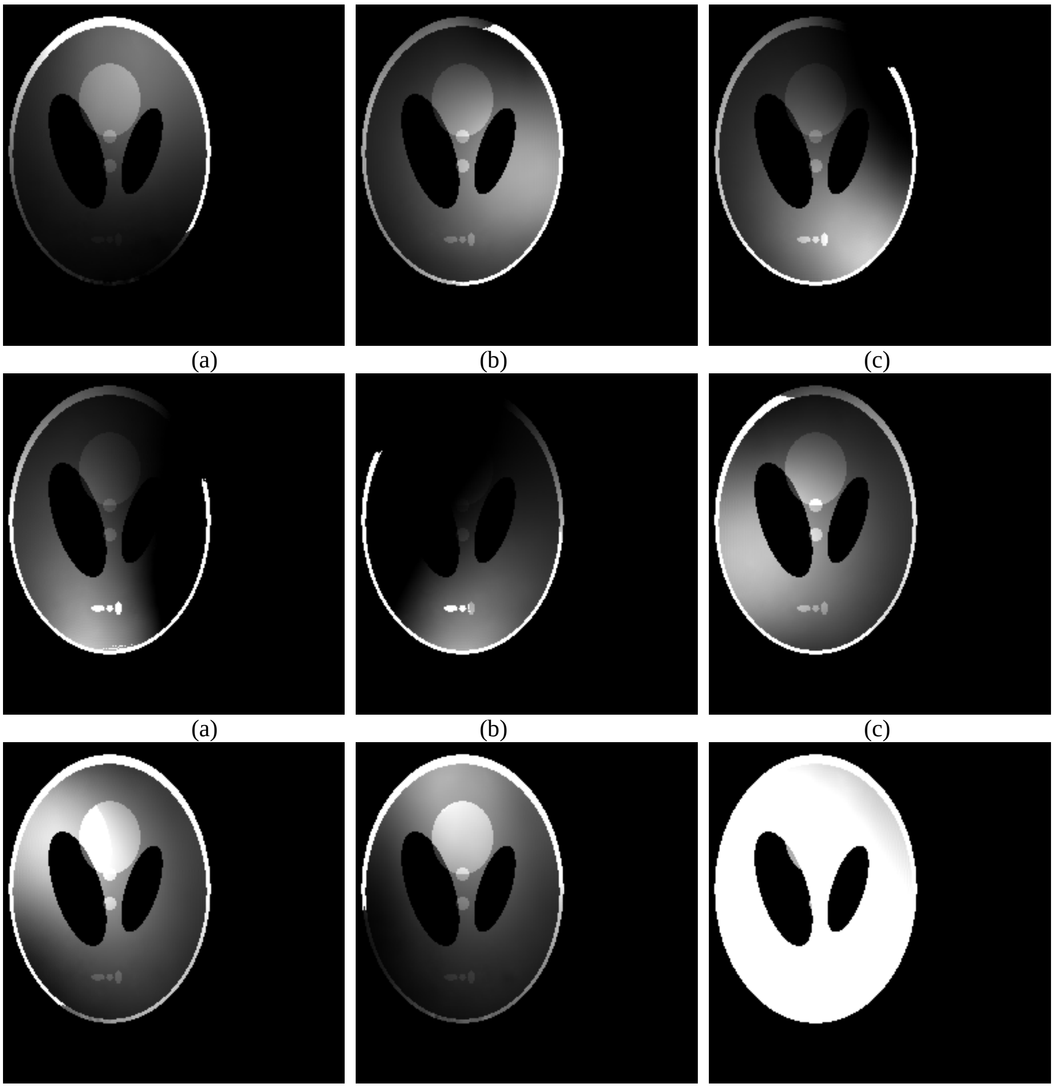
4. EXPERIMENTAL EVALUATION

The experiments were carried out on real 8 coil brain data (Fig. 1) and synthesized phantom data (Fig. 2). The data had been obtained from [23]. To find out the parameters used by the scanner, we request the reader to refer to [24]. The complete data consists of fully sampled K-space for all the 8 coils from which the corresponding coil images can be reconstructed by inverse 2D FFT. The images from all the coils were combined together by the sum-of-squares method (which is the standard procedure in GRAPPA [3] and SPIRiT [24] reconstruction) to obtain the ground-truth.



(g) (h) (i)

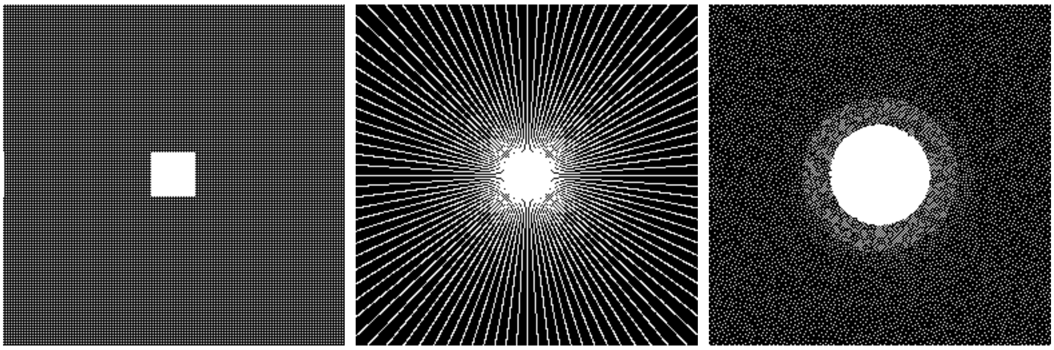
Fig. 1. (a) to (h) – brain images from 8 different coils; (i) – sum-of-squares reconstructed brain image



(g) (h) (i)

Fig. 2. (a) to (h) – simulated phantom images from 8 different coils; (i) – sum-of-squares reconstructed phantom image

Three different sampling schemes were used in this work. The first one is the periodic undersampling (Fig 3a); this is the traditionally the most widely used method in multi-coil data acquisition. Our second method is radial sampling (Fig 3b); it is a non-Cartesian sampling method but is one of the fastest possible [25]. Radial sampling has been used with SENSE like reconstruction for parallel MRI in the past [26]. The third method is sampling from a Gaussian distribution (Fig. 3c). In practice it may not be efficient to sample points from the K-space following a Gaussian distribution, but previous works in CS based MR image reconstruction have reported results based on sampling trajectories based on probability distributions [17, 27, 28].



(a) (b) (c)

Fig. 3. (a) Periodic under-sampling (26%) , (b) Radial sampling (23%), (c) Gaussian under-sampling (20%)

For the CS based reconstruction, it is claimed that the best image reconstruction is obtained if a combination of wavelet and TV regularization (10) is used [11, 6, 17]. When used in the SENSE framework, the optimization takes the following form,

$$\hat{x} = \min_x \|Wx\|_1 + \gamma TV(x) \text{ such that } \|y - Ex\|_2 \leq \varepsilon \quad (14)$$

Unfortunately there is no efficient algorithm to solve the constrained problem (10) in general. The NESTA method [9] is able to solve the constrained formulation (14), only when $EE^T = I$. This condition is not satisfied for non-Cartesian sampling trajectories. Thus all aforementioned works [11, 16, 17] developed algorithms for solving the unconstrained Lagrangian version to reconstruct the MR image. The Lagrangian multiplier was fixed empirically to yield the best results for the data at hand. In this work, we refrain from such parameter guessing and rely on a more theoretically oriented framework. The CS based reconstruction is solved by the SparSENSE method (5) [6]. The Spectral Projected Gradient L1 (SPGL1) algorithm [7] is used to solve the l_1 minimization problem. SPGL1 is the fastest and the most generalized solver for the said task.

We tried several wavelet families (Haar, Daubechies, Complex Dualtree, Fractional Spline) as the sparsifying transform for the CS based reconstruction. The best results were obtained for Complex Dualtree wavelets. For non-Cartesian radial sampling, the mapping from the image space to the Frequency space is performed by Non Uniform FFT [29].

Quantitatively the reconstruction accuracy is determined in terms of Normalized Mean Squared Error (NMSE). The NMSE's for various sampling schemes are shown in Table 1.

Table 1. NMSE from SparSENSE and Proposed method

Sampling Method	Brain		Phantom	
	SparSENSE	Proposed	SparSENSE	Proposed
Uniform	0.17	0.17	0.06	0.07
Radial	0.24	0.17	0.07	0.07
Gaussian	0.19	0.17	0.06	0.06

The numerical results show that the reconstruction results from SparSENSE is the same as our proposed method, except for radial sampling on the brain image, where our method gives significantly better results. For qualitative evaluation, the reconstructed images and the difference images are shown in Fig. 4 and Fig. 5.

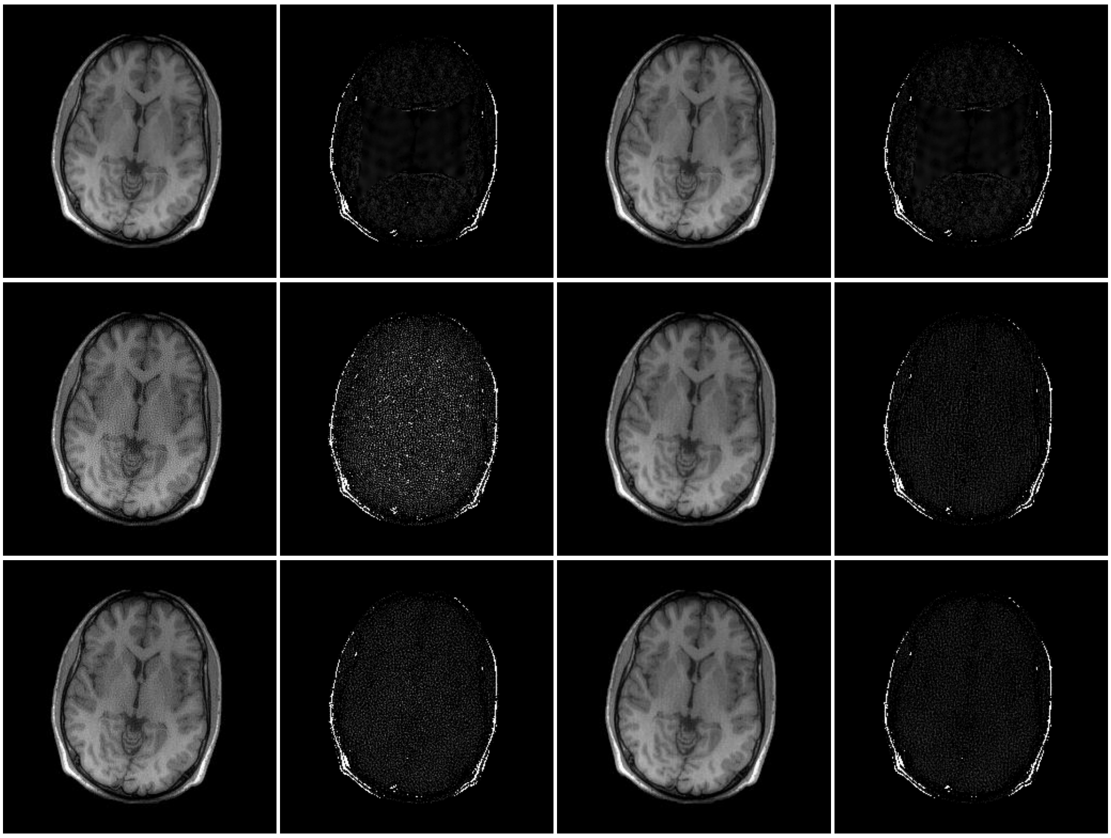


Fig. 4. Reconstruction of Brain image. 1st row – Uniform under-sampling, 2nd row – Radial sampling, 3rd Row – Gaussian under-sampling; 1st Column – SparSENSE reconstructed image, 2nd Column – SparSENSE reconstructed difference image, 3rd Column – proposed NN regularized SENSE reconstructed image; 4th Column - proposed NN regularized SENSE reconstructed difference image.

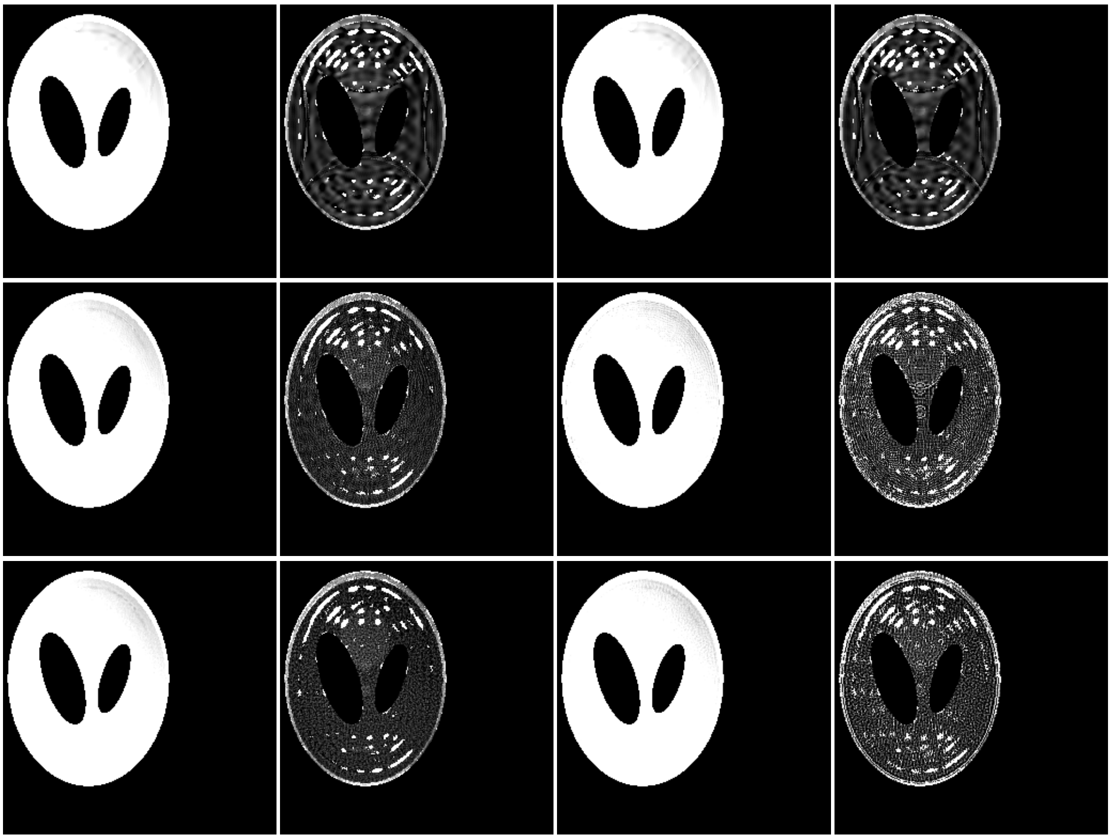


Fig. 5. Reconstruction of Phantom image. 1st row – Uniform under-sampling, 2nd row – Radial sampling, 3rd Row – Gaussian under-sampling; 1st Column – SparSENSE reconstructed image, 2nd Column – SparSENSE reconstructed difference image, 3rd Column – proposed NN regularized SENSE reconstructed image; 4th Column - proposed NN regularized SENSE reconstructed difference image.

The reconstructed and the difference images (between reconstructed image and the ground-truth) for the brain (Fig. 4) and phantom (Fig. 5) are shown above. The difference images are magnified 5 times for better clarity. Apart from the case where radial sampling is being used on the brain image, the results from our proposed method (Nuclear Norm Regularised SENSE) are the same as SparSENSE. For radial sampling on the brain image, our proposed method is better. Thus the conclusions from our quantitative and qualitative evaluations are similar.

The advantage of using our proposed method is that the reconstruction is faster. The reconstruction times (in seconds) for various sampling schemes are shown in Table 2.

Table 2. Reconstruction times (rounded in seconds) from SparSENSE and Proposed method

Sampling Method	Brain		Phantom	
	SparSENSE	Proposed	SparSENSE	Proposed
Uniform	151	10	87	7
Radial	276	23	192	15
Gaussian	171	10	91	7

Reconstruction from radial sampling requires considerably more time than the others because of the NUFFT mapping between the Cartesian image space and the non-Cartesian K-space. NUFFT is considerably slower than the FFT.

The results show that our proposed reconstruction method is more than an order of magnitude faster than SparSENSE. This is a slight improvement over our previous work [10]. There are two reasons behind this improvement. The main reason is that we have improved our reconstruction algorithm since the previous work. The computational load during Nuclear Norm minimization arises from the SVD decomposition of the image matrix, which needs to be computed in each iteration. Since it is known that the matrix is low rank, we use the PROPACK algorithm [30] for computing the partial SVD. This improves the computational speed of our current algorithm compared to the previous one where the full SVD was being computed. The other reason is that, we have used Complex Dualtree wavelets as the sparsifying transform in this work. This yields considerably better reconstruction results but at the cost of slower speed.

5. CONCLUSION

SENSitivity Encoding (SENSE) is a mathematically optimal model for parallel MRI when the sensitivity profiles are available. In recent years Compressed Sensing (CS) is used in conjunction with SENSE to obtain better reconstruction results compared to traditional SENSE. Such CS based methods like SparSENSE [6] exploit the sparsity of the MR image in the wavelet domain to reconstruct the image. In this work, we have shown that instead of exploiting the wavelet domain sparsity of the image, reconstruction results of similar accuracy can be obtained by exploiting the rank deficiency of the MR image.

In this work we regularize the SENSE method by incorporating the nuclear norm (NN) of the image. Nuclear norm is the tightest convex surrogate to the rank of a matrix. By carrying out thorough experimentation, we have shown that the reconstruction accuracy from our proposed method (NN regularized SENSE) is similar to SparSENSE. But the main advantage of using our method is in terms of reconstruction speed. The NN regularized SENSE is more than an order of magnitude faster than the one of the fastest SparSENSE methods (using SPGL1).

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