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Title: OPTIMAL INSPECTION POLICIES FOR A PARTIALLY OBSERVABLE  
PROCESS WITH HAZARDOUS INSPECTIONS

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An inspection model is considered in which inspections are hazardous to the device being monitored. The model is formulated as a partially observable Markov decision process. Previous research has determined the optimal ongoing inspection policies for a previously inspected device. In this thesis, the optimal inspection initiation policies are investigated for a device which has never been inspected previously. The basic issue is one of the tradeoffs between the benefits of information revealed by inspections versus the risk to the system imposed by the inspection procedures. The conditions under which the benefits outweigh the risk are established and the optimal initiation policies are shown to be periodic regardless of the values of the transition parameters. For both initial and subsequent inspections, efficient computational procedures for calculating the optimal timing of inspections and survival distributions are derived. Also, the coordination of initiation and ongoing policies is discussed.

OPTIMAL INSPECTION POLICIES FOR A PARTIALLY  
OBSERVABLE PROCESS WITH HAZARDOUS INSPECTIONS

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# OPTIMAL INSPECTION POLICIES FOR A PARTIALLY OBSERVABLE PROCESS WITH HAZARDOUS INSPECTIONS

## I. Introduction

### 1.1. Inspection Models

Consider a device which has associated with it a number of states which describe its state of repair. When the state of the device is not fully and directly observable, valuable information may be gained by performing inspections. Usually a cost is incurred by an inspection, and the tradeoff is between the information obtained and the inspection cost. In general, the problem of determining optimal inspection and/or repair policies can be formulated as a partially observable Markov decision process with costly information.

Completely observable Markov decision processes have been studied quite thoroughly. A main result of the theory of partially observable Markov decision processes is that they can be transformed to completely observable processes by suitably enlarging the state space. For each state in the expanded process there is a probability distribution over the states of the original process.

Many authors have studied inspection models. Ross (1971) developed a production process model in which the quality of the item produced was a function of the state of the machine, and the state could be determined only by sampling the item produced. He discussed the sampling schedules when inspection costs, production costs, and process-revision costs were considered.

Rosenfield (1976) presented a model of a deteriorating process with imperfect information. Under the assumption that the state-transition matrix was upper-triangular and totally positive of order two, he showed that the state space could be broken up into at most four regions of actions.

Barlow, Hunter and Proschan (1963) developed and solved the problem of finding the optimal policies for single-cycle systems in which system failure is known only through inspection. Luss and Kander (1974) extended this model to the case where the duration of inspection is nonnegligible and uncertainty probabilities are associated with inspections. Sengupta (1980) further extended this to a system for which failure symptoms are delayed.

Smallwood and Sondik (1973) examined the convexity and piecewise-linear properties of the optimal reward function for the finite-horizon case, and derived a finite algorithm for computing those optimal functions. Sondik (1978) extended this paper to the infinite planning horizon for the case of discounted costs. White (1979) generalized some of these results to a semi-Markov model. Other general approaches to the problem have been given by Aoki (1965), Astrom (1965, 1969), Sawaragi and Yoshikawa (1970), and Albright (1979).

In none of these papers is inspection assumed to have any effect on the true state of the process.

## 1.2. Hazardous Inspections

Inspections or tests which may hasten failure do arise on many occasions. For example, the mechanical and electrical stresses of



a test might destroy or damage the system. Sometimes, tests are nonetheless valuable because they may yield information which would not otherwise be available; of course, there are occasions where they are more harmful than beneficial.

An important instance of hazardous inspection is the use of X-rays to detect cancer. Recently, there has been a considerable controversy as to whether X-rays may themselves cause cancer and, granting a cancer-causing potential, whether or not they should be employed. The National Cancer Institute and the American Cancer Society have even issued a joint advisory: women under the age of 50 should avoid the routine use of mammography (Newsweek, Sept. 1976). Another example is the inspection of nuclear reactors. Since the largest single cause of malfunctions may be human error, a fundamental question is do inspections create more problems than they solve. In 1975, a technician using a lighted candle to check whether air was leaking into an area under the control room of the Tennessee Valley Authority's two nuclear reactors caused a fire which brought the reactors to a halt (New York Times, March 1975). At times routine maintenance of military and commercial aircraft may also be hazardous. In 1979, an American Airlines DC-10 crashed in Chicago and a questionable maintenance procedure was identified by the National Transportation Safety Board as a possible primary cause (New York Times, June 1979).

Wattanapanom and Shaw (1979) considered a hazardous-inspection model in which the device had an exponential or uniform failure

distribution in the absence of inspection. An inspection did not change the form of the conditional lifetime distribution but either caused immediate failure or else increased the failure rate. The system could be in either a good or a failed state, but the failed state was not observable. (For example, a radar system or an alarm system might require a special test to determine if it is not functioning.) They derived algorithms for finding inspection policies which minimize the overall operating cost. Butler (1979) formulated a Markov-decision-process model of hazardous inspection and determined the form of the optimal inspection policy for ongoing inspection situations (i.e., for previously inspected devices), but not for devices which have never been previously inspected. His results showed that there are conditions under which once a device is inspected it becomes optimal to inspect it every period, yet for which it is optimal to never again inspect the device if it has not been inspected within a given number of periods. Such a policy will be called an "all-or-none" policy. This unusual form of the optimal inspection policy for previously inspected devices indicates the importance of the initial decision to inspect.

This thesis investigates the optimal initiation procedures for hazardous inspections. In addition to the optimal initiation policies, how long a device is expected to be working is very important. Thus, an efficient method to compute the infinite-horizon maximal remaining lifetime of a device is introduced. Furthermore, the reliability (or survival probability) for a device is obtained and some examples of interesting cases are presented. Finally, some generalizations are discussed.

## II. Markov Decision Processes

This chapter is devoted to a definition and review of some of the properties of Markov decision processes (Ross, 1970; Veinott, 1969).

### 2.1. Description of Markov Decision Processes

A Markov decision process combines some of the features of probabilistic dynamic programming and Markov chains. It is a mathematical technique for making a sequence of interrelated decisions and provides a systematic procedure for determining the combination of decisions that optimizes some measure of overall effectiveness.

The basic characteristics of a Markov decision process are

- 1) The process is observed at time points (stages)  $t = 0, 1, 2, \dots$ .
- 2) At each stage, the process is observed to be in one of a number of possible states. The number of possible states associated with each stage is finite or countable.
- 3) After observing the state of the process, an action must be taken; the number of possible actions is finite.
- 4) At each state a reward is produced which depends upon the state of the process and the action taken.
- 5) The next state of the process (i.e., the state at the next stage) follows transition probabilities which are functions only of the current state and the action chosen.

This last characteristic is the Markov property; given the present state and action, the past states and actions have no effect on the future.

A policy is defined to be any rule for choosing actions which depends only upon the state and the stage. (The action chosen by a policy may be randomized in the sense that it will be chosen with some probability. However, such policies may be neglected in the search for an optimum. Thus in this thesis attention will be restricted to nonrandomized policies.) A policy is stationary if it depends only upon the state of the process and not on the stage.

The objective of a Markov decision process is to find an "optimal" policy in the sense that optimality under some criterion is achieved by following this policy. Maximization of expected total discounted reward and maximization of expected long-run average reward are the most commonly used optimality criteria.

## 2.2. Notation for Markov Decision Processes

Let  $S$  be the set of all possible states;  $S$  is called the state space and is finite or countable. Let  $A_i$  be the set of possible actions when the process is in state  $i$  and let  $A = \bigcup_{i \in S} A_i$ ;  $A$  is the set of all possible actions.

A function  $f : S \rightarrow A$  for which  $f(i) \in A_i$ , for all  $i \in S$  is called a decision rule. Let  $F$  denote the set of all decision rules; then  $F^\infty = \prod_{t=0}^{\infty} F = \{(f_0, f_1, f_2, \dots) : f_t \in F, t=0,1,2,\dots\}$  is the set of all nonrandomized policies. The generic policy is denoted by  $\pi = (f_0, f_1, f_2, \dots)$ . Using policy  $\pi$  means that if the process is in state  $i$  at time  $t$ , then the action  $f_t(i)$  is chosen. A stationary policy uses a single decision rule  $f$  repeatedly and is denoted by  $f^\infty = (f, f, f, \dots)$ .

If the process is in state  $i$  at time  $t$  and action  $a$  is chosen, a reward  $r(i, a)$  is produced and the next state is determined by transition probabilities  $P_{ij}(a)$ , where  $P_{ij}(a)$  is the probability that the process will be in state  $j$  at time  $t + 1$ .

The criterion of maximizing total discounted reward takes into account the "time value" of the rewards (i.e., a reward received today is worth more than the same reward received a year from now). In general, the total discounted reward =  $\sum_{t=0}^{\infty} \alpha^t \cdot$  (reward received  $t$  periods in future). The discount factor,  $\alpha$ , can be related to a periodic interest rate  $\rho$  by the formula  $\alpha = 1/(1+\rho)$ . For example, if the interest rate is six percent per year, a dollar invested today would be worth \$1.06 after a year and  $\alpha = 1/1.06$ . When problems with a long-time horizon are considered, the use of a discount factor is one way to compare different "streams" of rewards accruing over time. (For the short-run horizon,  $\alpha$  may be assumed to be one and thus neglected.)

Let the random variable  $X_t$  be the state of the process at time  $t$ . Given that the discount factor is  $\alpha \in (0,1)$ , that the policy  $\pi = (f_0, f_1, f_2, \dots)$  is used, and that the initial state is  $i$ , the total expected discounted reward is

$$V_i(\pi) = E_{\pi} \left[ \sum_{t=0}^{\infty} \alpha^t r(X_t, f_t(X_t)) \mid X_0 = i \right].$$

Since the rewards are assumed to be bounded, say by  $M$ , this reward function is bounded by  $\sum_{t=0}^{\infty} \alpha^t M = M/(1-\alpha)$  and is well defined. The policy  $\pi^*$  is optimal if and only if  $\underline{V}(\pi^*) \geq \underline{V}(\pi)$ , for all policies  $\pi$ , where  $\underline{V}(\pi)$  is the vector with elements  $V_i(\pi)$  for all  $i \in S$ .

### 2.3 Determining the Optimal Policy

Let  $m$  be the cardinality of  $S$  ( $m$  could be infinite),  $\underline{r}(f)$  be the vector with elements  $r(i, f(i))$  for all  $i \in S$ , and  $P(f)$  be the matrix with elements  $P_{ij}(f(i))$  for all  $i, j \in S$ . For example, if  $S = \{1, 2, 3, \dots\}$ , then  $\underline{r}(f) = (r(1, f(1)), r(2, f(2)), \dots)^T$  and

$$P(f) = \begin{bmatrix} P_{11}(f(1)) & P_{12}(f(1)) & P_{13}(f(1)) & \dots \\ P_{21}(f(2)) & P_{22}(f(2)) & P_{23}(f(2)) & \dots \\ P_{31}(f(3)) & P_{32}(f(3)) & P_{33}(f(3)) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let  $T_f$  be a function mapping  $\mathbb{R}^m$  to  $\mathbb{R}^m$   $\frac{1}{\alpha}$  defined by

$$T_f(\underline{u}) = \underline{r}(f) + \alpha P(f) \underline{u}$$

then  $T_f$  is a contraction mapping with modulus  $\alpha$ . (A mapping  $T: \mathbb{R}^m \rightarrow \mathbb{R}^m$  is said to be a contraction mapping if and only if  $\|T(\underline{u}) - T(\underline{v})\| \leq \beta \|\underline{u} - \underline{v}\|$  for some  $\beta < 1$  and for all  $\underline{u}, \underline{v} \in \mathbb{R}^m$ , where  $\|\underline{x}\|$  denotes the sup norm of  $\underline{x}$ , i.e.,  $\|\underline{x}\| = \sup_{1 \leq i \leq m} |x_i|$ . The smallest value of  $\beta$  which satisfies the conditions of the definition is called the modulus of  $T$ .)

The vector  $\underline{x}$  is a fixed point of the function  $T$  if and only if  $T(\underline{x}) = \underline{x}$ . A contraction mapping always has a unique fixed point. The unique fixed point of  $T_f$  is  $\underline{V}(f^\infty)$ , i.e.,  $T_f(\underline{V}(f^\infty)) = \underline{V}(f^\infty)$ .

Let  $T_\alpha: \mathbb{R}^m \rightarrow \mathbb{R}^m$  be the function

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$\frac{1}{\alpha} \mathbb{R}^\infty$  is taken to be the set of all bounded sequences from  $\mathbb{R}$ .

$$T_{\alpha}(\underline{u}) = \max_{f \in F} \{ \underline{r}(f) + \alpha P(f) \underline{u} \}$$

where the maximization is over all possible decision rules. The function  $T_{\alpha}$  is also a contraction mapping with modulus  $\alpha$ .

If  $\pi^*$  is an optimal policy, then  $T_{\alpha}(\underline{V}(\pi^*)) = \underline{V}(\pi^*)$ . This is the functional equation satisfied by the optimal reward function. In other words,  $\underline{V}(\pi^*)$  is the unique fixed point of  $T_{\alpha}$ . Let  $f^*$  be the decision rule that achieves the maximum in  $\{ \underline{r}(f) + \alpha P(f) \underline{V}(\pi^*) \}$ , i.e., satisfies  $\underline{r}(f^*) + \alpha P(f^*) \underline{V}(\pi^*) = T_{\alpha}(\underline{V}(\pi^*))$ . Then  $\pi' = (f^*, f^*, \dots)$  is a stationary optimal policy. Thus, a stationary policy can always be found which is optimal and so it is sufficient to search for optimal policies in the class of stationary policies.

For the time being, assume that the state space is finite. The optimal policy can be obtained by either the policy improvement algorithm, successive approximations, or linear programming.

The policy improvement algorithm is based on the following result, known as the comparison lemma. Write  $(g; f^{\infty})$  for the policy  $(g, f, f, f, \dots)$  which uses decision rule  $g$  at time zero and  $f$  thereafter. If  $g$  is a decision rule such that  $\underline{r}(g) + \alpha P(g) \underline{V}(f^{\infty}) = \max_{h \in F} \{ \underline{r}(h) + \alpha P(h) \underline{V}(f^{\infty}) \}$ , then  $\underline{V}(g; f^{\infty}) \geq \underline{V}(f^{\infty})$ , and  $\underline{V}(g^{\infty}) \geq \underline{V}(f^{\infty})$ , with at least one element strictly larger than if  $\underline{V}(g; f^{\infty}) \neq \underline{V}(f^{\infty})$ .

Thus, for any decision rule  $f$ , exactly one of the following holds.

- 1)  $T_{\alpha} \underline{V}(f^{\infty}) = \underline{V}(f^{\infty})$ ; in this case,  $\underline{V}(f^{\infty})$  is optimal and  $\underline{V}(f^{\infty}) = (I - \alpha P(f))^{-1} \underline{r}(f)$ .

- 2)  $T_\alpha \underline{V}(f^\infty) \geq \underline{V}(f^\infty)$  and at least one element of  $T_\alpha \underline{V}(f^\infty)$  is strictly larger than that of  $\underline{V}(f^\infty)$ . In this case, if  $T_\alpha \underline{V}(f^\infty) = \underline{V}(g; f^\infty)$ , then  $\underline{V}(g^\infty) \geq \underline{V}(f^\infty)$  with at least one element strictly larger.

The policy improvement algorithm is an immediate consequence: starting with any stationary policy  $f^\infty$ , an improved policy  $g^\infty$  can be obtained by selecting  $g$  such that  $\underline{V}(g; f^\infty) = T_\alpha \underline{V}(f^\infty)$ , repeating until  $\underline{V}(g; f^\infty) = \underline{V}(f^\infty)$ . Because there are a finite number of stationary policies, this algorithm will converge in finite time. Also, a better solution is achieved after each iteration.

Successive approximations seek to find the fixed point of the contraction mapping  $T_\alpha$ . Assume the process stops after  $n$  periods and that a terminal reward  $w_i$  is received at termination if the terminal state is  $i$ . The vector  $T_\alpha^n(\underline{w})$ <sup>2/</sup> is the  $n$ -period optimal reward vector and  $T_\alpha^n(\underline{w})$  will approach  $\underline{V}(\pi^*)$  as  $n$  approaches infinity for any terminal reward  $\underline{w}$ . Because  $T_\alpha$  is a contraction mapping with modulus  $\alpha$ , the rate of convergence is geometric:  $\|T_\alpha^n \underline{w} - \underline{V}(\pi^*)\| \leq \frac{\alpha^n}{1-\alpha} \|T_\alpha \underline{w} - \underline{w}\|$ . However, only rarely will  $T_\alpha^n \underline{w}$  actually achieve  $\underline{V}(\pi^*)$ ; furthermore, successive approximations do not guarantee to find the optimal policy  $\pi^*$ .

The following problem can be solved by the standard techniques of linear programming, and has as its unique optimal solution the optimal reward vector.

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<sup>2/</sup>  $T_\alpha^n \underline{w} = T_\alpha(T_\alpha^{n-1} \underline{w})$  for  $n = 2, 3, 4, \dots$ .



$$\text{minimize} \quad \sum_{i=1}^m x_i$$

subject to

$$(I - \alpha P(f)) \underline{x} \geq \underline{r}(f) \quad \text{for all } f \in F$$

where  $\underline{x} = (x_1, x_2, \dots, x_m)^T$ .

This linear programming problem has  $\sum_{i \in S} |A_i|$  constraints. (There are  $|S| \times |F|$  constraints in the above formulation, but most are repetitions.) In the optimal solution, those decision rules ( $f$ 's) such that  $(I - \alpha P(f)) \underline{x} = \underline{r}(f)$  are optimal decision rules.

When the state space is countable, the number of constraints is infinite for the linear programming approach, so the linear program cannot be solved by standard techniques. Theoretically, the results given by policy improvement algorithm and successive approximations can be generalized even to the uncountable state space (Ross, 1970). But problems may be encountered in trying to carry out the computations. Some examples for the discounted costs and applications to optimal stopping and sequential analysis were given by Ross. In these examples, the state spaces can always be divided into a finite number of regions such that within a region a particular action is optimal. This special technique is another efficient approach to countable state space problems.

### III. The Basic Hazardous Inspection Model

#### 3.1. The Basic Model

Consider a device whose operation can be classified into one of three categories: fully functional; functional, but impaired; and failed. The failed state is directly observable, but one can distinguish the partially functional state from the fully functional state only by performing an inspection. Inspection is perfect (i.e., the true state is always revealed) and instantaneous. Once the device is known to be in the impaired state, appropriate action may be taken to prolong its remaining life. In this respect, inspection is valuable. However, the act of inspecting the device when it is fully functional (not impaired) may cause it to become impaired. In this respect, inspection is hazardous. In this thesis, the inspection policy which maximizes the expected lifetime of the device will be determined; in other words, the situations for which the benefits of the information revealed by the inspection outweigh the risk imposed by the inspection will be determined.

The inspection model will be formulated as a Markov decision process.<sup>3/</sup> Let  $X_n$  denote the true state of the device at the start of period  $n$  (before inspection), where the possible states are

<u>True state</u>	<u>Description</u>
1	fully functional (OK)
2	undetected partial failure (UPF)
3	detected partial failure (DPF)
4	failed (F).

---

<sup>3/</sup> In the finite-horizon case, the model is actually a probabilistic dynamic programming problem.

In both state 2 and state 3 the device is functional but impaired. Because the partially failed state cannot be distinguished from the fully functional state except by performing an inspection, a device which has just partially failed would be in state 2 until the next inspection and state 3 thereafter (until it failed completely).

The information available to an observer is whether or not the system is in one of the states  $\{1,2\}$ , or state 3, or state 4, and if in state 1 or 2, how many periods have elapsed since the device was put into operation or last inspected. More specifically, if the device is observed to be failed (directly observable), then it is in state 4; if the device was inspected and found to be partially failed some number of periods ago and has not failed since, then it is in state 3; if the device was put into operation or inspected and found to be OK  $n$  periods ago and has not failed since, then it is in one of the states  $\{1,2\}$ .

Let  $a_n$  denote the action taken at time  $n$ , where the possible actions are  $a_n = 0$  (do not inspect) and  $a_n = 1$  (inspect). The Markov assumption for true states is

$$\begin{aligned} & \Pr\{X_{n+1} = j \mid X_k, a_k, k = 0,1,2, \dots, n\} \\ &= \Pr\{X_{n+1} = j \mid X_n, a_n\} \quad \text{for all } j, n. \end{aligned}$$

Let  $A$  be the set of all possible actions, i.e.,  $A = \{0,1\}$ .

Define

$$Q_{ij}(a) = \Pr\{X_{n+1} = j \mid X_n = i, a_n = a\} \quad i, j = 1, \dots, 4, \quad a \in A.$$

The one-step transition matrices  $Q(0)$  and  $Q(1)$  are taken to be as follows.

		$Q(0)$ (do not inspect)						$Q(1)$ (inspect)			
		(OK)	(UPF)	(DPF)	(F)			(OK)	(UPF)	(DPF)	(F)
(OK)		$1-\alpha_0$	$\alpha_0$	0	0	(OK)		$1-\alpha_1$	$\alpha_1$	0	0
(UPF)		0	$1-\beta$	0	$\beta$	(UPF)		0	0	$1-\beta$	$\beta$
(DPF)		0	0	$1-\gamma$	$\gamma$	(DPF)		0	0	$1-\gamma$	$\gamma$
(F)		0	0	0	1	(F)		0	0	0	1

The parameters  $\alpha_0$ ,  $\alpha_1$ ,  $\beta$ ,  $\gamma$  are the probabilities that the process will go to an inferior state, so they tend to be small. Some other assumptions are  $\alpha_1 > \alpha_0$ ; and  $\beta > \gamma$ . The first assumption implies that inspection is more hazardous to a fully functional device than is no inspection; the second assumption implies that once partial failure is detected, measures can be taken which reduce the periodic probability of full failure of the device. Some features are:

1. The device cannot go to a lower-numbered state.
2. When the device is in state 3 or 4, then either action has the same effect and thus the decision process is essentially completed.
3. The device cannot go to the failed state directly from the OK state.

4. The device will start in the OK state either having been just inspected or else never before operated or inspected.

Because there is a finite state space (four states), a finite action space, and consequently a finite number of decision rules, if the true state of the device could be determined at each time period (stage) then the optimal policy could be determined in finite time by the policy improvement algorithm or by linear programming. But this is not the case, because true states 1 and 2 cannot be distinguished from one another. As mentioned earlier, an observer can only tell that the device is in one of the states  $\{1,2\}$ , or state 3, or state 4, and if in state 1 or 2, how many periods have elapsed since the device was put into operation or last inspected. In accordance with this information, an "observed" state space and the relationship between the true states and the observed states will be established.

The observed states are based upon when the device was last inspected or put into operation, and what the true state of the device was at that time. Policies based upon these observed states which achieve the maximum expected remaining lifetime will be determined. The observed states are defined as follows:

<u>Observed State</u>	<u>Description</u>
-1	device failed
0	detected partial failure
$j \geq 1$	device not failed; last inspection OK; last inspection $j$ periods ago
$j^* \geq 1$	device not failed; never inspected before; put into operation $j$ periods ago.

A device which has never been inspected before will be called a "new" device and a device which has been inspected one or more times will be called an "old" device. The set of states  $\{1^*, 2^*, \dots\}$  will be denoted by  $S^*$ ;  $S^*$  is the set of observed states for a new device which is in one of the true states  $\{1, 2\}$ . Elements of  $S^*$  will be denoted generically by  $i^*, j^*, s^*, (s+1)^*$ , etc. Similarly, the set of states  $\{-1, 0, 1, 2, 3, \dots\}$  will be denoted by  $S$ ;  $S$  is the set of observed states for a failed device ( $-1$ ), a device with a detected partial failure ( $0$ ), and an old device in one of the true states  $\{1, 2\}$ . Elements of  $S$  are generically denoted by  $i, j, s$ , etc. Let  $S = S^* \cup S$  and let elements of  $S$  be denoted generically by lower-case script letters  $i, j, s, s+1$ , etc. ( $s+1$  denotes the state obtained by adding one to the numerical portion of the state descriptor  $s$ .) Relations such as  $s \geq 1$  or  $s^* \geq 2$  will be occasionally used with the intended meaning being that the numerical portion of the state descriptor is in the indicated relation to the right-hand side. With this somewhat complex notation, the results which are common to initial and ongoing inspection situations can be more compactly written.

The observed state transitions also have the Markov property. Let  $Z_n$  denote the observed state at period  $n$ . Let

$$P_{ij}(a) = \Pr\{Z_{n+1} = j | Z_n = i, a_n = a\} \text{ for all } i, j \in S, \text{ for all } a, n.$$

For an old device (which has been inspected some time in the past),

let

$$K_j = \Pr\{X_{n+j} = 2 \mid X_n = 1, a_n = 1, a_{n+k} = 0, 0 < k < j\} \text{ for all } j \in S,$$

$$L_j = \Pr\{X_{n+j} = 1 \mid X_n = 1, a_n = 1, a_{n+k} = 0, 0 < k < j\} \text{ for all } j \in S,$$

and

$$N_j = K_j + L_j.$$

For a new device (which has never been inspected), let

$$K_{j^*} = \Pr\{X_j = 2 \mid X_0 = 1, a_k = 0, 0 \leq k < j\} \text{ for all } j^* \in S^*,$$

$$L_{j^*} = \Pr\{X_j = 1 \mid X_0 = 1, a_k = 0, 0 \leq k < j\} \text{ for all } j^* \in S^*,$$

and

$$N_{j^*} = K_{j^*} + L_{j^*}.$$

Note that  $K_j/N_j$  is the probability that the true state is 2 (UPF) given the observed state is  $j$  for all  $j \geq 1$ , and  $L_j/N_j$  is the probability that the true state is 1 (OK) given the observed state is  $j \geq 1$ .

The observed state space is  $S$ . The transition probabilities among the observed states will be computed via the following propositions.

Proposition 3.1.

i) For all  $\delta \in S$  and  $a \in A$ ,

$$P_{-1,\delta}(a) = \begin{cases} 1, & \delta = -1 \\ 0, & \text{otherwise.} \end{cases}$$

ii) For all  $\delta \in S$  and  $a \in A$ ,

$$P_{0,\delta}(a) = \begin{cases} \gamma, & \delta = -1 \\ 1 - \gamma, & \delta = 0 \\ 0, & \text{otherwise.} \end{cases}$$

iii) For all  $i, j \in S$  such that  $i \geq 1$ ,

$$P_{ij}(0) = \begin{cases} \beta K_i / N_i, & j = -1 \\ ((1-\beta)K_i + L_i) / N_i, & j = i+1 \\ 0, & \text{otherwise,} \end{cases}$$

$$P_{ij}(1) = \begin{cases} \beta K_i / N_i, & j = -1 \\ (1-\beta)K_i / N_i, & j = 0 \\ L_i / N_i, & j = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof: i) It is obvious that when the device enters failed state, it will stay there forever.



- ii) When the device is in the detected partial failure state, inspection has no influence on the process and

$$P_{0,-1}(a) = Q_{34}(a) = \gamma ,$$

$$P_{0,0}(a) = Q_{33}(a) = 1 - \gamma ,$$

and

$$P_{0,j}(a) = 0 \quad \text{for all other } j \in S .$$

- iii)  $P_{i,-1}(0) = \Pr\{X_{n+1} = 4, X_n = 2 \mid Z_n = i\}$   
 $= \Pr\{X_{n+1} = 4 \mid X_n = 2\} \Pr\{X_n = 2 \mid Z_n = i\}$   
 $= \beta K_i / N_i .$

Also, it is clear that  $P_{i,\delta}(0) = 0$  for all  $\delta \neq -1, i+1$ , so  $P_{i,-1}(0) + P_{i,i+1}(0) = 1$ . To establish the formula for  $P_{i,\delta}(1)$ , note that

$$\begin{aligned} P_{i,-1}(1) &= \Pr\{X_{n+1} = 4, X_n = 2 \mid Z_n = i\} \\ &= \Pr\{X_{n+1} = 4 \mid X_n = 2\} \Pr\{X_n = 2 \mid Z_n = i\} \\ &= \beta K_i / N_i , \end{aligned}$$

and

$$\begin{aligned} P_{i,0}(1) &= \Pr\{X_{n+1} = 3, X_n = 2 \mid Z_n = i, a_n = 1\} \\ &= \Pr\{X_{n+1} = 3 \mid X_n = 2, a_n = 1\} \Pr\{X_n = 2 \mid Z_n = i\} \\ &= (1-\beta) K_i / N_i . \end{aligned}$$

Since  $P_{i,\delta}(1) = 0$  for all other  $\delta \in S$  except  $\delta = 1$ , the proof is completed. □

The transition probabilities are computed as functions of the  $K_i$ 's,  $L_i$ 's, and  $N_i$ 's. The following proposition gives explicit

formulas for these quantities.

Proposition 3.2.

A. i)  $L_i = (1-\alpha_1)(1-\alpha_0)^{i-1}$  for all  $i \geq 1$ .

ii)  $L_{i^*} = (1-\alpha_0)^i$  for all  $i^* \in S^*$ .

B. i)  $K_i = \alpha_1(1-\beta)^{i-1} + \alpha_0(1-\alpha_1)[(1-\beta)^{i-1} - (1-\alpha_0)^{i-1}] / (\alpha_0 - \beta)$

$$i \geq 1, \alpha_0 \neq \beta.$$

ii)  $K_i = \alpha_1(1-\alpha_0)^{i-1} + (i-1)\alpha_0(1-\alpha_1)(1-\alpha_0)^{i-2}$   $i \geq 1, \alpha_0 = \beta.$

iii)  $K_{i^*} = \alpha_0[(1-\beta)^i - (1-\alpha_0)^i] / (\alpha_0 - \beta)$   $i^* \in S^*, \alpha_0 \neq \beta.$

iv)  $K_{i^*} = i \alpha_0(1-\alpha_0)^{i-1}$   $i^* \in S^*, \alpha_0 = \beta.$

Proof:

A. i)  $L_i = \Pr\{X_{n+j} = 1, 0 \leq j \leq i \mid X_n = 1, a_n = 1, a_{n+j} = 0, 0 < j < i\}$   
 $= (1-\alpha_1)(1-\alpha_0)^i.$

ii)  $L_{i^*} = \Pr\{X_k = 1, 0 \leq k \leq i \mid X_0 = 1, a_k = 0, 0 \leq k < i\}$   
 $= (1-\alpha_0)^i.$

B. i) and ii)

$$\begin{aligned}
K_i &= \sum_{\ell=1}^i \Pr\{X_{n+j} = 1, 0 \leq j < \ell, X_{n+j} = 2, \ell \leq j < i \mid X_n = 1, a_n = 1, \\
&\quad a_{n+j} = 0, 1 \leq j < i\} \\
&= \alpha_1 (1-\beta)^{i-1} + \sum_{\ell=2}^i (1-\alpha_1) (1-\alpha_0)^{\ell-2} \alpha_0 (1-\beta)^{i-\ell} \\
&= \alpha_1 (1-\beta)^{i-1} + \alpha_0 (1-\alpha_1) (1-\beta)^{i-2} \sum_{k=0}^{i-2} \left( \frac{1-\alpha_0}{1-\beta} \right)^k. \tag{3.1}
\end{aligned}$$

If  $\alpha_0 \neq \beta$ , by Equation (3.1),

$$\begin{aligned}
K_i &= \alpha_1 (1-\beta)^{i-1} + \alpha_0 (1-\alpha_1) (1-\beta)^{i-2} \left[ \frac{1 - \left( \frac{1-\alpha_0}{1-\beta} \right)^{i-1}}{1 - \frac{1-\alpha_0}{1-\beta}} \right] \\
&= \alpha_1 (1-\beta)^{i-1} + \alpha_0 (1-\alpha_1) \left[ \frac{(1-\beta)^{i-1} - (1-\alpha_0)^{i-1}}{\alpha_0 - \beta} \right].
\end{aligned}$$

If  $\alpha_0 = \beta$ , by Equation (3.1),

$$K_i = \alpha_1 (1-\alpha_0)^{i-1} + (i-1) \alpha_0 (1-\alpha_1) (1-\alpha_0)^{i-2}.$$

$$\text{iii) } K_{i^*} = \sum_{\ell=1}^i \Pr\{X_k = 1, 0 \leq k < \ell, X_j = 2, \ell \leq j \leq i \mid X_0 = 1, a_k = 0, 0 \leq k < i\}$$

$$= \sum_{\ell=1}^i (1-\alpha_0)^{\ell-1} \alpha_0 (1-\beta)^{i-\ell}$$

$$= [\alpha_0 (1-\beta)^i / (1-\alpha_0)] \sum_{\ell=1}^i \left( \frac{1-\alpha_0}{1-\beta} \right)^\ell$$

$$= \alpha_0 [(1-\beta)^i - (1-\alpha_0)^i] / (\alpha_0 - \beta).$$

$$\begin{aligned}
\text{iv) } K_{i,*} &= \sum_{\ell=1}^i \Pr\{X_k=1, 0 \leq k < \ell, X_j=2, \ell \leq j \leq i | X_0=1, a_k=0, 0 \leq k < i\} \\
&= \sum_{\ell=1}^i (1-\alpha_0)^{\ell-1} \alpha_0 (1-\alpha_0)^{i-\ell} \\
&= i \alpha_0 (1-\alpha_0)^{i-1} . \quad \square
\end{aligned}$$

In addition to failures influenced by inspection, there is a possibility of failure through secondary causes each period with probability  $1 - \delta$ . Because the expected remaining lifetime of the device can be written as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Pr\{\text{survive } n \text{ periods}\} \\
&= \sum_{n=0}^{\infty} \delta^n \Pr\{\text{no primary failures in } n \text{ periods}\} ,
\end{aligned}$$

the parameter  $\delta$  acts like an ordinary discount factor.

The main objective of this thesis is to efficiently determine an inspection policy which yields the maximal expected life of the device. Let  $V(\delta, n)$  denote the maximal expected remaining time until failure of a device which is in observed state  $\delta$  and is to be destroyed or taken out of service  $n$  periods in the future. Also, let  $V(\delta)$  denote the maximal remaining time until failure given the device is in observed state  $\delta$  and is facing an infinite time horizon. The recursive relations among the  $V$ 's are given below.

Proposition 3.3.A. Finite Horizon

- i)  $V(s,0) = 0$  for all  $s \in S$ .
- ii)  $V(-1,n) = 0$ ;  $V(0,n) = (1 - (\delta(1-\gamma))^n) / (1 - \delta(1-\gamma))$   $n=1,2,\dots$
- iii)  $V(s,n) = 1 + \delta \max\{K_\delta (1-\beta)V(0,n-1)/N_\delta + L_\delta V(1,n-1)/N_\delta, (K_\delta (1-\beta) + L_\delta)V(s+1,n-1)/N_\delta\}$   $s \geq 1, n \geq 1$ .

B. Infinite Horizon

- i)  $V(-1) = 0$ ;  $V(0) = 1 / (1 - \delta(1-\gamma))$ .
- ii)  $V(s) = 1 + \delta \max\{K_\delta (1-\beta)V(0)/N_\delta + L_\delta V(1)/N_\delta, (K_\delta (1-\beta) + L_\delta)V(s+1)/N_\delta\}$   $s \geq 1$ .

Proof:

$$V(0,n) = \sum_{k=0}^{\infty} \Pr\{\text{remaining life of device exceeds } k \mid \text{device currently in true state } 3\}$$

$$= \sum_{k=0}^{n-1} (\delta(1-\gamma))^k$$

$$= (1 - (\delta(1-\gamma))^n) / (1 - \delta(1-\gamma)).$$

Similarly,

$$V(0) = \sum_{k=0}^{\infty} (\delta(1-\gamma))^k = 1 / (1 - \delta(1-\gamma)) .$$

By the principle of optimality of dynamic programming (Hillier and Lieberman, 1980),

$$V(\delta, n) = \max_{a \in \{0,1\}} \left\{ 1 + \delta \sum_{t \in S} P_{\delta t}(a) V(t, n-1) \right\}$$

Recalling Proposition 3.1, the recursive formula for  $V(\delta, n)$  follows immediately.

Because  $\{V(\delta, 1), V(\delta, 2), V(\delta, 3), \dots\}$  is an increasing sequence and  $V(\delta, n)$  is bounded for all  $\delta, n$ , the finite-horizon optimal values converge to the infinite-horizon optimal value as  $n$  approaches infinity. That is

$$V(\delta) = \lim_{n \rightarrow \infty} V(\delta, n) \quad \text{for all } \delta \in S .$$

The recursive formula for  $V(\delta)$  follows from taking limits in the above. □

### 3.2 Preliminary Analysis and Review of Previous Results

The observed state space is countable, so some other solution approaches besides the three basic methods (policy improvement algorithm, linear programming, and successive approximations) may offer more efficient ways to find the optimal stationary policies. In this

thesis, a particular analysis is discussed, which is quicker and more efficient than these three methods and provides the forms of the optimal policies simultaneously for broad ranges of parameters.

Previous research (Butler, 1979) has established the optimal inspection policy for devices which have been inspected some time in the past.

First, the behavior of the quantities  $K_{\delta}/N_{\delta}$  and  $L_{\delta}/N_{\delta}$  will be investigated. (Recall that  $K_{\delta}/N_{\delta}$  is the probability that the true state is 2 (UPF) given the observed state is  $\delta$ , and  $L_{\delta}/N_{\delta}$  is the probability that the true state is 1 (OK) given the observed state is  $\delta$ .) Consider a new device which is still working. As one would expect, the longer it has been since it was put into operation, the more likely it is to be partially failed. But, since the inspection is hazardous, an old device does not necessarily have this property. In particular, when  $\alpha_0 > \alpha_1\beta$ , the longer since the last inspection, the more likely the old device is to be partially failed; but when  $\alpha_0 < \alpha_1\beta$  the opposite is true. Essentially, most of the results to be obtained depend upon the relationships among  $\alpha_0$ ,  $\gamma$ , and  $\alpha_1\beta$ . The quantity  $\alpha_0$  is the probability of a transition from the OK state to the partially failed state if no inspection is performed, the quantity  $\gamma$  is the probability of a transition from the partially failed state to the failed state, and the quantity  $\alpha_1\beta$  can be interpreted as the probability that a device which was inspected and found to be OK one period ago will be failed at the start of the next period.

Proposition 3.4.

- A. i) If  $\alpha_0 > \alpha_1 \beta$ , then  $K_i/N_i$  is increasing in  $i$ , and  $L_i/N_i$  and  $(K_i(1-\beta) + L_i)/N_i$  are decreasing in  $i$ .
- ii) If  $\alpha_0 = \alpha_1 \beta$ , then  $K_i/N_i$ ,  $L_i/N_i$ , and  $(K_i(1-\beta) + L_i)/N_i$  are constant in  $i$ .
- iii) If  $\alpha_0 < \alpha_1 \beta$ , then  $K_i/N_i$  is decreasing in  $i$ , and  $L_i/N_i$  and  $(K_i(1-\beta) + L_i)/N_i$  are increasing in  $i$ .
- B.  $K_{i^*}/N_{i^*}$  is increasing in  $i^*$ , and  $L_{i^*}/N_{i^*}$  and  $(K_{i^*}(1-\beta) + L_{i^*})/N_{i^*}$  are decreasing in  $i^*$ .

Proof: A. Because

$$\begin{aligned} \frac{K_i}{N_i} - \frac{K_{i-1}}{N_{i-1}} &= \frac{K_i N_{i-1} - K_{i-1} N_i}{N_i N_{i-1}} \\ &= \frac{1}{N_i N_{i-1}} [K_i (K_{i-1} + L_{i-1}) - K_{i-1} (K_i + L_i)] \\ &= \frac{K_i L_{i-1} - K_{i-1} L_i}{N_i N_{i-1}}, \end{aligned}$$

$K_i/N_i$  is increasing or decreasing according to whether  $K_i L_{i-1} - K_{i-1} L_i$  is positive or negative.

By Equation (3.1) in Proposition 3.2,

$$K_i = \alpha_1 (1-\beta)^{i-1} + \alpha_0 (1-\alpha_1) (1-\beta)^{i-2} \sum_{\ell=0}^{i-2} \left( \frac{1-\alpha_0}{1-\beta} \right)^\ell .$$



Thus,

$$\begin{aligned}
& K_i L_{i-1} - K_{i-1} L_i \\
&= (1-\alpha_1)(1-\alpha_0)^{i-2} \left[ \alpha_1 (1-\beta)^{i-1} + \alpha_0 (1-\alpha_1) (1-\beta)^{i-2} \sum_{k=0}^{i-2} \left( \frac{1-\alpha_0}{1-\beta} \right)^k \right] \\
&\quad - (1-\alpha_1)(1-\alpha_0)^{i-1} \left[ \alpha_1 (1-\beta)^{i-2} + \alpha_0 (1-\alpha_1) (1-\beta)^{i-3} \sum_{k=0}^{i-3} \left( \frac{1-\alpha_0}{1-\beta} \right)^k \right] \\
&= (1-\alpha_1)(1-\alpha_0)^{i-2} (1-\beta)^{i-2} \left[ \alpha_1 (1-\beta) - \alpha_1 (1-\alpha_0) \right. \\
&\quad \left. + \alpha_0 (1-\alpha_1) \left[ \sum_{k=0}^{i-2} \left( \frac{1-\alpha_0}{1-\beta} \right)^k - \frac{1-\alpha_0}{1-\beta} \sum_{k=0}^{i-3} \left( \frac{1-\alpha_0}{1-\beta} \right)^{k+1} \right] \right] \\
&= (1-\alpha_1)(1-\alpha_0)^{i-2} (1-\beta)^{i-2} (\alpha_0 - \alpha_1 \beta) .
\end{aligned}$$

This proves the assertions regarding  $K_i/N_i$ . Since  $L_i/N_i = 1 - K_i/N_i$  and  $(K_i(1-\beta)+L_i)/N_i = 1 - \beta K_i/N_i$ , the proof of part A is completed. To prove part B, note that

$$K_{i^*}/N_{i^*} - K_{(i-1)^*}/N_{(i-1)^*} = \frac{K_{i^*} L_{(i-1)^*} - K_{(i-1)^*} L_{i^*}}{N_{i^*} N_{(i-1)^*}} ,$$

so  $K_{i^*}/N_{i^*}$  is increasing if and only if

$$K_{i^*} L_{(i-1)^*} - K_{(i-1)^*} L_{i^*} \text{ is positive.}$$

If  $\alpha_0 \neq \beta$ ,

$$K_{i*}L_{(i-1)*} - K_{(i-1)*}L_{i*} = \alpha_0(1-\alpha_0)^{i-1}(1-\beta)^{i-1} > 0.$$

If  $\alpha_0 = \beta$ ,

$$K_{i*}L_{(i-1)*} - K_{(i-1)*}L_{i*} = \alpha_0(1-\alpha_0)^{2i-2} > 0.$$

This proves the assertion regarding  $K_{i*}/N_{i*}$ . The other assertions follow directly from the formulas

$$L_{i*}/N_{i*} = 1 - K_{i*}/N_{i*}$$

$$(K_{i*}(1-\beta) + L_{i*})/N_{i*} = 1 - \beta K_{i*}/N_{i*}.$$

□

The relationship between  $V(1,n)$  and  $V(0,n)$  has some effect on the behavior of  $V(s,n)$ . (For example, it influences whether or not  $V(s,n)$  is increasing in  $s$ .) The quantity  $V(1,n)$  is the expected remaining life of a device which was inspected and found to be OK one period ago and the quantity  $V(0,n)$  is the expected remaining life of a device which is known to be partially failed. In other words, the quantities  $V(1,n)$  and  $V(0,n)$  give a measure of the risk and the benefit of the inspection. The following lemma gives recursive formulas for  $V(0,n)$  and  $V(0)$ .

Lemma 3.1. i)  $V(0,n) = 1 + \delta(1-\gamma)V(0,n-1)$  for all  $n \geq 1$ .

ii)  $V(0) = 1 + \delta(1-\gamma)V(0)$ .

Proof: The results follow immediately from Proposition 3.3.

□

The following lemma gives an inequality between  $V(1,n)$  and  $(1-\beta)V(0,n)$ ; these two quantities can be interpreted as the maximum expected remaining lifetimes of a device which was inspected and found to be OK, or partially failed, respectively, one period ago. Thus, intuitively one would expect  $V(1,n) \geq (1-\beta)V(0,n)$ .

Lemma 3.2.  $V(1,n) \geq (1-\beta)V(0,n)$  for all  $n$ .

Proof: The proof is by induction. For  $n = 0,1$ , the result is clearly true. Assume the result holds for  $n-1$ . By Proposition 3.3 and Lemma 3.1,

$$\begin{aligned}
 V(1,n) &\geq 1 + \delta((1-\beta)V(0,n-1)K_1/N_1 + V(1,n-1)L_1/N_1) \\
 &\geq 1 + \delta((1-\beta)V(0,n-1)K_1/N_1 + (1-\beta)V(0,n-1)L_1/N_1) \\
 &= 1 + \delta(1-\beta)V(0,n-1) \\
 &\geq (1-\beta)[1 + \delta(1-\gamma)V(0,n-1)] \\
 &= (1-\beta)V(0,n).
 \end{aligned}$$

□

For an old device, whether or not  $V(s,n)$  and  $V(s)$  are increasing in  $s$  depends upon the relationship between  $\alpha_0$  and  $\alpha_1\beta$ ; but for a new device,  $V(s^*,n)$  and  $V(s^*)$  are always nonincreasing in  $s^*$ .

Lemma 3.3. A. i) If  $\alpha_0 \geq \alpha_1\beta$ , then  $V(s,n)$  and  $V(s)$  are nonincreasing in  $s \geq 1$ .  
 ii) If  $\alpha_0 \leq \alpha_1\beta$ , then  $V(s,n)$  and  $V(s)$  are nondecreasing in  $s \geq 1$ .

B.  $V(s^*, n)$  and  $V(s^*)$  are nonincreasing in  $s^* \geq 1$ .

Proof: The proof is by induction.

A. i) and B.

For  $n = 1$ ,  $V(\delta, n)$  is constant for  $\delta \geq 1$  and therefore nonincreasing. Now assume  $V(\delta, n-1)$  is nonincreasing in  $\delta \geq 1$ . The two arguments of the maximization operator in Proposition 3.3 are

$$V(\delta+1, n-1)((1-\beta)K_\delta + L_\delta)/N_\delta \quad (3.2)$$

and

$$\begin{aligned} & V(0, n-1)(1-\beta)K_\delta/N_\delta + V(1, n-1)L_\delta/N_\delta \\ &= (1-\beta)V(0, n-1) + [V(1, n-1) - (1-\beta)V(0, n-1)]L_\delta/N_\delta. \end{aligned} \quad (3.3)$$

By Proposition 3.4,  $((1-\beta)K_s + L_s)/N_s$  is decreasing in  $s$  when  $\alpha_0 \geq \alpha_1\beta$ ,  $((1-\beta)K_{s^*} + L_{s^*})/N_{s^*}$  is decreasing in  $s^*$ , and by inductive hypothesis  $V(\delta+1, n-1)$  is nonincreasing in  $\delta$ . Therefore (3.2) is nonincreasing. Similarly, by Proposition 3.4,  $L_s/N_s$  is decreasing in  $s$  when  $\alpha_0 \geq \alpha_1\beta$ ,  $L_{s^*}/N_{s^*}$  is decreasing, and by Lemma 3.2,  $V(1, n-1) \geq (1-\beta)V(0, n-1)$ . Thus, (3.3) is nonincreasing and so  $V(\delta, n)$  is nonincreasing in  $\delta \geq 1$ . Since  $V(\delta) = \lim_{n \rightarrow \infty} V(\delta, n)$ ,  $V(\delta)$  is nonincreasing in  $\delta \geq 1$ .

A. ii) Similarly, for  $n = 1$ ,  $V(s, n)$  is constant for  $s \geq 1$ , thus nondecreasing. The two arguments of the maximization operator again appear in (3.2) and (3.3). By Proposition 3.4,

$((1-\beta)K_s + L_s)/N_s$  is increasing in  $s$  when  $\alpha_0 \leq \alpha_1\beta$  and by inductive hypothesis  $V(s,n)$  is nondecreasing in  $s$ , so (3.2) is nondecreasing. By Proposition 3.4,  $L_s/N_s$  is increasing in  $s$  when  $\alpha_0 < \alpha_1\beta$  and by Lemma 3.2  $V(1,n-1) \geq (1-\beta)V(0,n-1)$ . Therefore (3.3) is nondecreasing in  $s \geq 1$ , and so  $V(s,n)$  is nondecreasing. The assertion regarding  $V(s)$  follows by letting  $n$  approach infinity.  $\square$

In order to compare the merits of the two possible actions, an explicit inequality between  $V(0,n)$  and  $V(1,n)$  is required. The following lemma states that when  $\gamma \geq \alpha_1\beta$ , i.e., the failure probability of a device which is in the detected partially failed state is at least that of a device which was inspected one period ago and found to be OK, a device in observed state 0 is worse than a device in observed state 1 (using a criterion of maximum expected remaining life); when  $\gamma \leq \alpha_1\beta$  and  $\alpha_0 \geq \alpha_1\beta$ , the opposite is true.

Lemma 3.4. i) If  $\gamma \leq \alpha_1\beta$  and  $\alpha_0 \geq \alpha_1\beta$ , then  

$$V(0,n) \geq V(1,n) \text{ and } V(0) \geq V(1) .$$

ii) If  $\gamma \geq \alpha_1\beta$ , then  $V(0,n) \leq V(1,n)$   
and  $V(0) \leq V(1) .$

Proof: The proof is by induction. The results are obvious for  $n=1$ .

i) Assume that  $V(1,n-1) \leq V(0,n-1)$ . By Lemma 3.3  

$$V(2,n-1) \leq V(1,n-1) , \text{ so}$$

$$\begin{aligned}
V(1,n) &= 1 + \delta \max \{V(0,n-1)(1-\beta)K_1/N_1 + V(1,n-1)L_1/N_1, \\
&\quad V(2,n-1)((1-\beta)K_1 + L_1)/N_1\} \\
&= 1 + \delta(V(0,n-1)(1-\beta)K_1/N_1 + V(1,n-1)L_1/N_1) \\
&\leq 1 + \delta V(0,n-1)((1-\beta)K_1 + L_1)/N_1 \\
&\leq 1 + \delta(1-\alpha_1\beta)V(0,n-1) \\
&\leq 1 + \delta(1-\gamma)V(0,n-1) \\
&= V(0,n) .
\end{aligned}$$

ii) Assume that  $V(0,n-1) \leq V(1,n-1)$ . Similarly, by Lemma 3.3 and Proposition 3.3,

$$\begin{aligned}
V(1,n) &\geq 1 + \delta V(0,n-1)((1-\beta)K_1 + L_1)/N_1 \\
&= 1 + \delta(1-\alpha_1\beta)V(0,n-1) \\
&\geq 1 + \delta(1-\gamma)V(0,n-1) \\
&= V(0,n) .
\end{aligned}$$

The results regarding the infinite-horizon case follow by taking limits. □

Since  $V(0,n)$  and  $V(s+1,n)$  are the crucial elements in the recursive equation of  $V(s,n+1)$ , it is very important to identify the relationships among  $V(0,n)$  and  $V(1,n)$ ,  $V(2,n)$ ,  $\dots$ . The optimal

policy depends heavily on those factors which determine these relationships. The following proposition and lemma give a comparison between  $V(0,n)$  and the limiting behavior of  $V(s,n)$  as  $s$  approaches infinity.

Proposition 3.5.

- i) If  $\alpha_0 \leq \beta$ , then  $\lim_{\delta \rightarrow \infty} K_\delta / N_\delta = \alpha_0 / \beta$ ,  $\lim_{\delta \rightarrow \infty} L_\delta / N_\delta = (\beta - \alpha_0) / \beta$ , and  $\lim_{\delta \rightarrow \infty} (K_\delta(1-\beta) + L_\delta) / N_\delta = 1 - \alpha_0$ .
- ii) If  $\alpha_0 > \beta$ , then  $\lim_{\delta \rightarrow \infty} K_\delta / N_\delta = 1$ ,  $\lim_{\delta \rightarrow \infty} L_\delta / N_\delta = 0$ , and  $\lim_{\delta \rightarrow \infty} (K_\delta(1-\beta) + L_\delta) / N_\delta = 1 - \beta$ .

Proof: By Proposition 3.2, if  $\alpha_0 \neq \beta$ ,

$$K_s / N_s = \frac{(\alpha_0 - \beta) \alpha_1 \left( \frac{1-\beta}{1-\alpha_0} \right)^{s-1} + \alpha_0 (1-\alpha_1) \left[ \left( \frac{1-\beta}{1-\alpha_0} \right)^{s-1} - 1 \right]}{(\alpha_0 - \beta) \alpha_1 \left( \frac{1-\beta}{1-\alpha_0} \right)^{s-1} + \alpha_0 (1-\alpha_1) \left[ \left( \frac{1-\beta}{1-\alpha_0} \right)^{s-1} - 1 \right] + (1-\alpha_1) (\alpha_0 - \beta)} .$$

Thus, if  $\alpha_0 < \beta$ , then  $\frac{1-\beta}{1-\alpha_0} < 1$  and

$$\lim_{s \rightarrow \infty} \frac{K_s}{N_s} = \frac{\alpha_0 (1-\alpha_1)}{\alpha_0 (1-\alpha_1) - (1-\alpha_1) (\alpha_0 - \beta)} = \frac{\alpha_0}{\beta} ,$$

if  $\alpha_0 > \beta$ , then

$$\begin{aligned} \lim_{s \rightarrow \infty} \frac{K_s}{N_s} &= \lim_{s \rightarrow \infty} \frac{(\alpha_0 - \beta) \alpha_1 + \alpha_0 (1-\alpha_1) \left[ 1 - \left( \frac{1-\alpha_0}{1-\beta} \right)^{s-1} \right]}{(\alpha_0 - \beta) \alpha_1 + \alpha_0 (1-\alpha_1) \left[ 1 - \left( \frac{1-\alpha_0}{1-\beta} \right)^{s-1} \right] + (1-\alpha_1) (\alpha_0 - \beta) \left( \frac{1-\alpha_0}{1-\beta} \right)^{s-1}} \\ &= 1 . \end{aligned}$$

If  $\alpha_0 = \beta$ ,

$$\frac{K_s}{N_s} = \frac{\alpha_1(1-\alpha_0)^{s-1} + (s-1)\alpha_0(1-\alpha_1)(1-\alpha_0)^{s-2}}{\alpha_1(1-\alpha_0)^{s-1} + (s-1)\alpha_0(1-\alpha_1)(1-\alpha_0)^{s-2} + (1-\alpha_1)(1-\alpha_0)^{s-1}}$$

and

$$\lim_{s \rightarrow \infty} \frac{K_s}{N_s} = 1.$$

Again, by Proposition 3.2, if  $\alpha_0 \neq \beta$ ,

$$\frac{K_{s^*}}{N_{s^*}} = \frac{\alpha_0 \left[ (1-\beta)^s - (1-\alpha_0)^s \right] / (\alpha_0 - \beta)}{\alpha_0 \left[ (1-\beta)^s - (1-\alpha_0)^s \right] / (\alpha_0 - \beta) + (1-\alpha_0)^s}.$$

If  $\alpha_0 < \beta$ ,

$$\lim_{s^* \rightarrow \infty} \frac{K_{s^*}}{N_{s^*}} = \lim_{s \rightarrow \infty} \frac{\alpha_0 \left[ \left( \frac{1-\beta}{1-\alpha_0} \right)^s - 1 \right]}{\alpha_0 \left[ \left( \frac{1-\beta}{1-\alpha_0} \right)^s - 1 \right] + (\alpha_0 - \beta)} = \frac{\alpha_0}{\beta},$$

if  $\alpha_0 > \beta$ ,

$$\lim_{s^* \rightarrow \infty} \frac{K_{s^*}}{N_{s^*}} = \lim_{s \rightarrow \infty} \frac{\alpha_0 \left[ 1 - \left( \frac{1-\alpha_0}{1-\beta} \right)^s \right]}{\alpha_0 \left[ 1 - \left( \frac{1-\alpha_0}{1-\beta} \right)^s \right] + (\alpha_0 - \beta) \left( \frac{1-\alpha_0}{1-\beta} \right)^s} = 1.$$



if  $\alpha_0 = \beta$ ,

$$\lim_{s^* \rightarrow \infty} \frac{K_{s^*}}{N_{s^*}} = \lim_{s \rightarrow \infty} \frac{s \alpha_0 (1-\alpha_0)^{s-1}}{s \alpha_0 (1-\alpha_0)^{s-1} + (1-\alpha_0)^s} = 1 .$$

The other assertions follow directly from the formula

$$K_\delta + L_\delta = N_\delta .$$

□

The relationship between  $V(0,n)$  and  $V(1,n)$  has been determined except when  $\gamma < \alpha_1 \beta$  and  $\alpha_0 < \alpha_1 \beta$ . When  $\alpha_0 \leq \gamma < \alpha_1 \beta$ , it can be shown that neither inequality regarding  $V(0,n)$  and  $V(1,n)$  holds in general, but the limit of  $V(s,n)$  as  $s$  approaches infinity will be greater than or equal to  $V(0,n)$  (as shown in the next lemma). When  $\gamma \leq \alpha_0 \leq \alpha_1 \beta$ , it will be shown that  $V(s,n) \leq V(0,n)$  for all  $s$ .

Lemma 3.5.

- i) If  $\alpha_0 \leq \gamma$ , then  $V(0,n) \leq \lim_{s \rightarrow \infty} V(s,n)$  for all  $n \geq 1$ ,  
and  $V(0) \leq \lim_{s \rightarrow \infty} V(s)$ .
- ii) If  $\gamma \leq \alpha_0 \leq \alpha_1 \beta$ , then  $V(s,n) \leq V(0,n)$  for all  $s \geq 0$ ,  
 $n \geq 1$ , and  $V(s) \leq V(0)$  for all  $s \geq 0$ .

Proof:

- i) The proof is by induction. For  $n=1$ ,  $V(0,n) = V(s,n) = 1$  for all  $s \geq 0$  and therefore the result holds. Now, assume the result holds for  $n - 1$ .

$$V(s,n) = 1 + \delta \max\{K_s(1-\beta)V(0,n-1)/N_s + L_s V(1,n-1)/N_s ,$$

$$(K_s(1-\beta) + L_s)V(s+1, n-1) / N_s \}$$

$$\lim_{s \rightarrow \infty} V(s,n) = 1 + \delta \max\{V(0,n-1) \lim_{s \rightarrow \infty} \frac{K_s(1-\beta)}{N_s} + V(1,n-1) \lim_{s \rightarrow \infty} \frac{L_s}{N_s} ,$$

$$\lim_{s \rightarrow \infty} [V(s+1,n-1)(K_s(1-\beta) + L_s)/N_s]\}$$

$$\geq 1 + \delta [\lim_{s \rightarrow \infty} V(s+1, n-1)] [\lim_{s \rightarrow \infty} (K_s(1-\beta) + L_s)/N_s]$$

By Proposition 3.5, Lemma 3.1, and the inductive hypothesis,

$$\lim_{s \rightarrow \infty} V(s,n) \geq 1 + \delta V(0,n-1)(1-\alpha_0)$$

$$\geq 1 + \delta(1-\gamma)V(0,n-1)$$

$$= V(0,n) .$$

ii) For  $n = 1$  the result is clearly true. Now assume

$V(s,n-1) \leq V(0,n-1)$  for all  $s \geq 0$ . Since  $\alpha_0 \leq \alpha_1\beta$ , by Lemma 3.4

$V(s,n)$  is nondecreasing and by the induction hypothesis

$V(1,n-1) \leq \lim_{s \rightarrow \infty} V(s,n-1) \leq V(0,n-1)$ . Thus

$$\lim_{s \rightarrow \infty} V(s,n) = 1 + \delta \max\{V(0,n-1) \lim_{s \rightarrow \infty} [K_s(1-\beta)/N_s] + V(1,n-1) \lim_{s \rightarrow \infty} L_s/N_s ,$$

$$\lim_{s \rightarrow \infty} [V(s+1,n-1)(K_s(1-\beta) + L_s)/N_s]\}$$

$$\leq 1 + \delta \max\{V(0,n-1) \lim_{s \rightarrow \infty} [K_s(1-\beta)/N_s] + V(0,n-1) \lim_{s \rightarrow \infty} L_s/N_s ,$$

$$\begin{aligned}
& V(0,n-1) \lim_{s \rightarrow \infty} [(K_s(1-\beta) + L_s)/N_s] \\
& \leq 1 + \delta(1-\alpha_0)V(0,n-1) \\
& \leq 1 + \delta(1-\gamma)V(0,n-1) = V(0,n) .
\end{aligned}$$

The result then follows since  $V(s,n) \leq \lim_{t \rightarrow \infty} V(t,n)$  for all  $s \geq 1$ .

The assertions regarding the infinite-horizon optimal values follow by taking limits. □

For a device which is in observed state  $\delta$ , the risk of inspection is the increased hazard of partial failure incurred when the device is fully functional, and the benefit is to find out the true state of the device. So, when  $V(0,n) \leq V(\delta,n) \leq V(1,n)$ , the benefit is to know that the device is in the OK state and no benefit will be realized when the device is partially failed; when  $V(1,n) \leq V(\delta,n) \leq V(0,n)$ , the opposite is true; when  $V(1,n) \leq V(\delta,n)$  and  $V(0,n) \leq V(\delta,n)$ , there is no benefit for inspection at all; when  $V(\delta,n) \leq V(1,n)$  and  $V(\delta,n) \leq V(0,n)$ , it is always beneficial to know the true state of the device. These relationships will be verified in Chapter 4.

#### IV. Determination of Optimal Inspection Policies

A new device (one which has never been inspected before) and an old device (one which has been inspected at least once) may have different optimal inspection policies. Basically, the forms of the optimal inspection policies depend upon the relationships among  $\alpha_0$ ,  $\gamma$ , and  $\alpha_1\beta$ . For an infinite-horizon problem, when  $\alpha_0 \neq \alpha_1\beta$  and  $\gamma \neq \alpha_1\beta$ , only one stationary policy will be optimal and the optimal policy will be either to inspect if and only if at least some given minimum number of periods have elapsed since the true state of the device was last known (periodic inspection), or else to inspect if and only if at most some given maximum number of periods have elapsed (all-or-none inspection).

##### 4.1. Review of the Optimal Inspection Policies for Old Devices

Most of the following results about old devices were obtained by Butler (1979). The first theorem states that for an old device, when the failure probability of a known partially failed device is smaller than the failure probability of a device which was inspected and found to be OK one period ago, and the latter probability is in turn smaller than the partial failure probability of an OK device which was not inspected, then it is optimal to inspect every period. When the opposite relations hold, it is best to never inspect.

Theorem 4.1. (Butler, 1979). For an old device,

- i) If  $\gamma \leq \alpha_1 \beta \leq \alpha_0$ , then the optimal inspection policy for both the finite- and infinite-horizon problems is to inspect every period.
- ii) If  $\alpha_0 \leq \alpha_1 \beta \leq \gamma$ , then the optimal inspection policy for both the finite- and infinite-horizon problems is to never inspect.

Proof:

Define

$$D(s,n) = V(0,n-1) K_s (1-\beta)/N_s + V(1,n-1) L_s/N_s \\ - V(s+1,n-1)(K_s (1-\beta) + L_s)/N_s .$$

Then for an old device which is in state  $s$  facing an  $n$ -period horizon, it is optimal to inspect if  $D(s,n) \geq 0$ , and to not inspect if  $D(s,n) \leq 0$ .

By Lemma 3.3 and Lemma 3.4, when  $\gamma \leq \alpha_1 \beta$  and  $\alpha_0 \geq \alpha_1 \beta$ ,  $V(s,n)$  is nonincreasing in  $s \geq 0$  for all  $n$ . Therefore,  $D(s,n) \geq 0$  for all  $s,n$ , and it is optimal to inspect every period.

Again, by Lemma 3.3 and 3.4, when  $\gamma \geq \alpha_1 \beta$  and  $\alpha_0 \leq \alpha_1 \beta$ ,  $V(s,n)$  is nondecreasing in  $s \geq 0$  for all  $s,n$ . Thus,  $D(s,n) \leq 0$  for all  $s,n$ , and it is optimal to never inspect.

$$\text{Define } D(s) = V(0)K_s (1-\beta)/N_s + V(1)L_s/N_s - V(s+1)(K_s (1-\beta) + L_s)/N_s .$$

The results for the infinite-horizon case then follow similarly.  $\square$

The following lemma provides the basis for determination of an optimal inspection policy for an old device when  $\alpha_0 > \alpha_1\beta$  and  $\gamma > \alpha_1\beta$ . Let

$$G(n) = \alpha_0(1-\beta)V(0,n-1) + (1-\alpha_0)V(1,n-1) - (1-\alpha_1\beta)V(2,n-1).$$

Lemma 4.1. (Butler, 1979). If  $\alpha_0 > \alpha_1\beta$  and  $\gamma > \alpha_1\beta$ , then  $G(n) \geq 0$  for all  $n=1,2,\dots$ , and  $D(s,n)$  and  $D(s)$  are nondecreasing in  $s$ .

Define  $s(n) = \min\{s:D(s,n) \geq 0\}$  and  $s(\infty) = \min\{s:D(s) \geq 0\}$ , where the minimum over an empty set is taken to be  $\infty$ .

Theorem 4.2. (Butler, 1979). If  $\alpha_0 > \alpha_1\beta$  and  $\gamma > \alpha_1\beta$  and the device is in state  $s$  facing an  $n$ -period horizon, it is optimal to inspect if and only if  $s \geq s(n)$  ( $n \leq \infty$ ).

A policy having a form as given in Theorem 4.2 will be called periodic. A policy which never inspects is periodic ( $s(n) = +\infty$ ); a policy which inspects every period is also periodic ( $s(n) = 1$ ). It will be shown in Chapter V that when  $\alpha_0 > \gamma$ ,  $s(n) < \infty$  and it is optimal to inspect when  $s$  is big enough; when  $\alpha_0 < \gamma$ , it can be shown that either to never inspect or to periodically inspect can be optimal.

In the case where  $\alpha_0 < \alpha_1\beta$  and  $\gamma < \alpha_1\beta$ , it turns out that  $D(s,n)$  crosses zero at most once and from above as  $s$  increases. Thus, it is optimal to inspect if and only if  $s$  is at most some value  $z(n)$ . Define  $z(n) = \max\{s:D(s,n) \geq 0\}$  and  $z(\infty) = \max\{s:D(s) \geq 0\}$ , where the maximum over an empty set is

assume  $-\infty$ .

Theorem 4.3. (Butler, 1979). If  $\alpha_0 < \alpha_1\beta$  and  $\gamma < \alpha_1\beta$ , then when the device is in state  $s$  facing an  $n$ -period horizon, it is optimal to inspect if and only if  $s \leq z(n)$  ( $1 \leq n \leq \infty$ ).

Proof:  $D(s,n) = [V(0,n-1) - V(s+1,n-1)] (1-\beta)K_s/N_s$   
 $+ [V(1,n-1) - V(s+1,n-1)] L_s/N_s$ .

By Lemma 3.3,  $V(s+1,n-1)$  is nondecreasing in  $s$ , and by Proposition 3.4,  $K_s/N_s$  is nonincreasing and  $L_s/N_s$  is nondecreasing. In the range of  $s$  such that  $V(0,n-1) \geq V(s+1,n-1)$ , both terms in  $D(s,n)$  are nonincreasing. For the range of  $s$  for which  $V(0,n-1) < V(s+1,n-1)$ , both terms are negative. Thus  $D(s,n)$  crosses zero at most once and from above.

The infinite-horizon properties follow by taking limits. □

This is a most unusual policy, because it says to inspect the device every period unless the device has survived more than  $z(n)$  periods without inspection or failure; in this case, never again inspect the device. This kind of policy will be called an all-or-none policy. All-or-none policies also include never inspecting and always inspecting as special cases. This strange optimal inspection policy could happen when  $\alpha_1$  and  $\beta$  are quite large compared to  $\alpha_0$  and  $\gamma$ . In this case, if the device is in observed state 1 (i.e., it was inspected and found to be OK one period ago), the probability that the true state is partially failed is large because the inspection

tends to cause the device to fail. On the other hand, if the device is in observed state 10, i.e., it has survived ten periods since the last inspection, the posterior probability that the device is not partially failed is high. This is so because if the last inspection had caused partial failure, the device would probably have failed completely within ten periods. For example, let  $\alpha_0 = .15$ ,  $\alpha_1 = .8$ ,  $\gamma = .1$ ,  $\beta = .7$ ; if the device is in observed state 1 or 10, then the true state is partially failed with probability  $K_1/N_1 = .8$  or  $K_{10}/N_{10} = .214$ , respectively.

When  $\alpha_0 = \gamma = \alpha_1\beta$ , by Theorem 4.1, it is both optimal to inspect every period and optimal to never inspect. Actually,  $D(s,n) = 0$  for all  $s,n$  and any policy is optimal.

Table 1 summarizes the optimal policy forms for old devices.

Table 1. Forms of the optimal inspection policy for old devices based upon the relationships among  $\alpha_0$ ,  $\gamma$ , and  $\alpha_1\beta$

	$\alpha_0 < \alpha_1\beta$	$\alpha_0 = \alpha_1\beta$	$\alpha_0 > \alpha_1\beta$
$\gamma < \alpha_1\beta$	all-or-none	always	always
$\gamma = \alpha_1\beta$	never	any policy is optimal	always
$\gamma > \alpha_1\beta$	never	never	periodic



#### 4.2. Determination of the Optimal Initiation Policies

The optimal inspection policies for an old device have been analyzed. The next step will be to determine the possible forms of the optimal initiation policy for devices which have never been inspected since put into service. Assume that the device was OK with probability one when it was put into service.

By Proposition 3.4,  $K_s^*/N_s^*$  is increasing in  $s$ ; this simply says that the longer a new device has been in operation, the more likely it is to be partially failed. Lemma 3.3 gives a more explicit statement which says the optimal expected remaining lifetime is nonincreasing in the number of time periods since the device was put into operation. So, it seems to be that the longer a new device has been operating, the more attractive inspection is and the optimal initiation policy should be to periodically inspect,<sup>4/</sup> i.e., inspect if and only if a given minimum number of periods have elapsed since the device was put into service. Indeed, this is so.

Before proceeding to analyze the initiation policies, the relationship between the optimal expected remaining lifetimes of an old and a new device of the same "age" (time since true state last known) will be established. Because of the hazard of the inspection, the remaining life of an old device depends upon the relationship between  $\alpha_0$  and  $\alpha_1\beta$ , the partial failure probability of an OK device and the failure probability of a device which was inspected and found to be OK one period ago. But for a new device, it does not

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<sup>4/</sup> The term periodic inspection will be used even though the initial inspection is of course never repeated.

depend on  $\alpha_1\beta$ . The following proposition states that a new device which has been operating for  $s^*$  periods is better than a device which was inspected and found to be OK  $s$  periods ago and has been operating since then, no matter what the values of the parameters are. (Here  $s^*$  and  $s$  have the same numerical value.)

Proposition 4.1.  $V(s,n) \leq V(s^*,n)$  for all  $s \geq 1$   $n \geq 1$ ,  
and  $V(s) \leq V(s^*)$  for all  $s \geq 1$ .

Proof: By Proposition 3.3,

$$V(s,n) = 1 + \delta \max\{(1-\beta)V(0,n-1) + [V(1,n-1) - (1-\beta)V(0,n-1)]L_s/N_s, \\ (K_s(1-\beta) + L_s)V(s+1,n-1)/N_s\}$$

and

$$V(s^*,n) = 1 + \delta \max\{(1-\beta)V(0,n-1) + [V(1,n-1) - (1-\beta)V(0,n-1)]L_{s^*}/N_{s^*}, \\ (K_{s^*}(1-\beta) + L_{s^*})V((s+1)^*, n-1) / N_{s^*}\} .$$

For  $n=1$ ,  $V(s,n) = V(s^*,n) = 1$ , hence, the result is true.

Assume the result holds for  $n-1$ . The difference between the first terms of the maximization operators is

$$[V(1,n-1) - (1-\beta)V(0,n-1)](K_{s^*}L_s - K_sL_{s^*})/(N_sN_{s^*}) .$$

By Proposition 3.2, if  $\alpha_0 \neq \beta$

$$L_sK_{s^*} - L_{s^*}K_s = (1-\alpha_1)(1-\alpha_0)^{s-1} \alpha_0((1-\beta)^s - (1-\alpha_0)^s)/(\alpha_0-\beta) \\ - (1-\alpha_0)^s [\alpha_1(1-\beta)^{s-1} + \alpha_0(1-\alpha_1)((1-\beta)^{s-1} - (1-\alpha_0)^{s-1})/(\alpha_0-\beta)]$$

$$\begin{aligned}
&= \{\alpha_0(1-\alpha_1)(1-\alpha_0)^{s-1}(1-\beta)^{s-1}[(1-\beta)-(1-\alpha_0)]-\alpha_1(\alpha_0-\beta)(1-\alpha_0)^s(1-\beta)^{s-1}\}/(\alpha_0-\beta) \\
&= (1-\alpha_0)^{s-1}(1-\beta)^{s-1}(\alpha_0-\alpha_1) < 0.
\end{aligned} \tag{4.1}$$

If  $\alpha_0 = \beta$ ,

$$\begin{aligned}
L_s K_{s^*} - L_{s^*} K_s &= (1-\alpha_1)(1-\alpha_0)^{s-1} s(1-\alpha_0)^{s-1} \alpha_0 \\
&\quad - [\alpha_1(1-\alpha_0)^{s-1} + (s-1)\alpha_0(1-\alpha_1)(1-\alpha_0)^{s-2}](1-\alpha_0)^s \\
&= (1-\alpha_0)^{2s-2} [\alpha_0(1-\alpha_1) - \alpha_1(1-\alpha_0)] \\
&= (1-\alpha_0)^{2s-2} (\alpha_0 - \alpha_1) < 0.
\end{aligned} \tag{4.2}$$

Thus the first term in the maximization operator of  $V(s, n)$  is less than or equal to that quantity of  $V(s^*, n)$ .

The difference of the second terms is

$$(K_s(1-\beta) + L_s)V(s+1, n-1)/N_s - (K_{s^*}(1-\beta) + L_{s^*})V((s+1)^*, n-1)/N_{s^*}$$

By the inductive hypothesis,  $V((s+1)^*, n-1) \geq V(s+1, n-1)$ . Also

$$\begin{aligned}
&(K_{s^*}(1-\beta) + L_{s^*})/N_{s^*} - (K_s(1-\beta) + L_s)/N_s \\
&= (1-\beta K_{s^*}/N_{s^*}) - (1-\beta K_s/N_s) \\
&= \beta(K_s L_{s^*} - L_s K_{s^*})/(N_s N_{s^*}).
\end{aligned}$$

This quantity is positive. Thus, the second term in the maximum operator of  $V(s,n)$  is less than or equal to that of  $V(s^*,n)$ .

The assertion regarding the infinite-horizon case follows by taking limits. □

In addition to  $\alpha_0$  and  $\alpha_1\beta$ , the parameter  $\gamma$  plays an important role in the determination of the form of the optimal initiation policy, because it determines the relationship between  $V(0,n)$  and  $V(1,n)$ . The form of the optimal initiation policy is directly determined by the relative magnitudes of  $\alpha_0$ ,  $\gamma$ , and  $\alpha_1\beta$ .

Theorem 4.4. If  $\alpha_0 \leq \alpha_1\beta$  and  $\alpha_0 \leq \gamma$ , then the optimal initiation policy for both the finite- and infinite-horizon problems is to never initiate inspection.

Proof: The optimal action when the device is in state  $s^*$  facing an  $n$ -period horizon is that action which achieves the maximum in Proposition 3.3.A.iii.

$$\begin{aligned} \text{Let } D(s^*,n) = & V(0,n-1) K_{s^*}(1-\beta)/N_{s^*} + V(1,n-1)L_{s^*}/N_{s^*} \\ & - V((s+1)^*, n-1) (K_{s^*}(1-\beta) + L_{s^*})/N_{s^*} \end{aligned}$$

When the device is in state  $s^*$  facing an  $n$ -period horizon it is optimal not to inspect if and only if  $D(s^*,n) \leq 0$ .

Case 1:  $\gamma \geq \alpha_1\beta$ : By Lemma 3.3, Lemma 3.4, and Proposition 4.1,

$$V((s+1)^*, n-1) \geq V(1, n-1) \geq V(0, n-1).$$

Thus  $D(s^*, n) \leq 0$  for  $s^*$ ,  $n \geq 1$  and the optimal policy is to never inspect.

For the infinite-horizon case, let

$$D(s^*) = K_{s^*}(1-\beta)V(0)/N_{s^*} + L_{s^*}V(1)/N_{s^*} - (K_{s^*}(1-\beta) + L_{s^*})V((s+1)^*)/N_{s^*}.$$

When in state  $s^*$ , it is optimal not to inspect if and only if  $D(s^*) \leq 0$ . The result then follows since  $D(s^*) = \lim_{n \rightarrow \infty} D(s^*, n) \leq 0$ .

Case 2.  $\gamma \leq \alpha_1 \beta$ : By Lemmas 3.3 and 3.5 and Proposition 4.1,

$$V(0, n) \leq \lim_{s \rightarrow \infty} V(s, n) \leq V(t^*, n) \quad \text{for all } t^* \in S^*.$$

Thus  $V(0, n-1) \leq V((s+1)^*, n-1)$  and by Lemma 3.3,  $V(1, n-1) \leq V((s+1)^*, n-1)$

so  $D(s^*, n) = (V(0, n-1) - V((s+1)^*, n-1)) K_{s^*}(1-\beta)/N_{s^*}$

$$+ (V(1, n-1) - V((s+1)^*, n-1)) L_{s^*}/N_{s^*} \leq 0$$

and the optimal inspection policy for a finite horizon is to never inspect. The infinite-horizon result follows as before.  $\square$

When  $\gamma \leq \alpha_1 \beta$  and  $\gamma \leq \alpha_0$ , it will be shown that  $D(s^*, n)$  and  $D(s^*)$  cross zero at most once and from below (as  $s^*$  increases) and therefore it is optimal to inspect in state  $s^*$  whenever  $s^*$  is greater than or equal to some critical number  $t(n)$ .

Lemma 4.2. If  $\gamma \leq \alpha_1 \beta$  and  $\gamma \leq \alpha_0$ , then as  $s^*$  increases  $D(s^*, n)$  and  $D(s^*)$  cross zero at most once and from below.

Proof: 
$$D(s^*,n) = [V(0,n-1) - V((s+1)^*, n-1)] (1-\beta) K_{s^*}/N_{s^*} \\ + [V(1,n-1) - V((s+1)^*, n-1)] L_{s^*}/N_{s^*} \quad (4.3)$$

By Lemma 3.3  $V((s+1)^*,n-1)$  is nonincreasing, and by Lemmas 3.4 and 3.5  $V(0,n-1) \geq V(1,n-1)$ . Consider the interval of  $s^*$  for which  $V((s+1)^*,n-1) > V(0,n-1) \geq V(1,n-1)$ . In this range, both terms in Equation (4.3) are negative, thus  $D(s^*,n) < 0$ . For the interval of  $s^*$  for which  $V(0,n-1) \geq V((s+1)^*,n-1) \geq V(1,n-1)$ ,  $V(0,n-1) - V((s+1)^*,n-1)$  is nonnegative and nondecreasing, and also  $V(1,n-1) - V((s+1)^*,n-1)$  is nonpositive and nondecreasing. Thus both terms in (4.3) are nondecreasing since  $L_{s^*}/N_{s^*}$  is decreasing and  $K_{s^*}/N_{s^*}$  is increasing. Therefore  $D(s^*,n)$  is nondecreasing for  $s^*$  in this range. For the interval of  $s^*$  for which  $V(0,n-1) \geq V(1,n-1) > V((s+1)^*,n-1)$  both terms in Equation (4.3) are positive and  $D(s^*,n) > 0$ . So, as  $s^*$  increases  $D(s^*,n)$  crosses zero at most once and from below.

Since  $D(s^*) = \lim_{n \rightarrow \infty} D(s^*,n)$ , it is easy to verify that  $D(s^*)$  must also have this property. □

Define 
$$t(n) = \min \{ s^* : D(s^*,n) \geq 0 \} \text{ and} \\ t(\infty) = \min \{ s^* : D(s^*) \geq 0 \},$$

where the minimum over an empty set is taken to be  $\infty$ .

The following theorem is an immediate consequence of Lemma 4.2.

Theorem 4.5. If  $\gamma \leq \alpha_1 \beta$  and  $\gamma \leq \alpha_0$ , then if the device is in state  $s^*$  facing an  $n$ -period horizon, it is optimal to inspect if and only if  $s^* \geq t(n)$  ( $1 \leq n \leq \infty$ ).

The only case for which the form of optimal initiation policy has not been analyzed is when  $\alpha_0 > \alpha_1\beta$  and  $\gamma > \alpha_1\beta$ . It will be shown that in this case  $D(s^*,n)$  and  $D(s^*)$  are nondecreasing in  $s^*$  and so it is optimal to inspect when in state  $s^*$  if and only if  $s^* \geq t(n)$ . Thus, the optimal initiation policy again belongs to the class of periodic policies.

Lemma 4.3. If  $\alpha_0 > \alpha_1\beta$  and  $\gamma > \alpha_1\beta$  then for all  $n = 1, 2, \dots$ ,  $D(s^*,n)$  and  $D(s^*)$  are nondecreasing in  $s^*$ .

Proof: For  $n = 1, 2$ ,  $D(s^*,n)$  is constant, thus nondecreasing. Assume that for  $n = N - 1$ ,  $D(s^*,n)$  is nondecreasing. Therefore,  $D(s^*,N-1)$  is nonnegative if and only if  $s^* \geq t(N-1)$ . Thus, for  $s^* < t(N-1)$ ,

$$V(s^*,N-1) = 1 + \delta (K_{s^*}(1-\beta) + L_{s^*})V((s+1)^*,n-2) / N_{s^*} \quad (4.4)$$

and for  $s^* \geq t(N-1)$ ,

$$V(s^*,N-1) = 1 + \delta K_{s^*}(1-\beta) V(0,N-2) / N_{s^*} + \delta L_{s^*} V(1,N-2) / N_{s^*}. \quad (4.5)$$

By Theorem 4.2, for  $s < s(n)$ ,

$$V(s,n) = 1 + \delta (K_s(1-\beta) + L_s) V(s+1,n-1) / N_s \quad (4.6)$$

and for  $s \geq s(n)$ ,

$$V(s,n) = 1 + \delta K_s(1-\beta) V(0,n-1) / N_s + \delta L_s V(1,n-1) / N_s. \quad (4.7)$$

In the remainder of the proof, the following easily verified formulas for  $K_{s^*}$ ,  $L_{s^*}$ , and  $N_{s^*}$  will be used without mention:

$$\begin{aligned}
K_{(s+1)}^* &= (1-\beta)K_s^* + \alpha_0 L_s^* \\
L_{(s+1)}^* &= (1-\alpha_0)L_s^* \\
N_{(s+1)}^* &= (1-\beta)K_s^* + L_s^* .
\end{aligned} \tag{4.8}$$

Case 1:  $\underline{t(N-1) = 1}$ : The cases  $s(N-1) = 1$  and  $s(N-1) > 1$  will be considered separately.

i)  $\underline{s(N-1) = 1}$ : By Equations (4.4) - (4.8), and Lemma 3.1,

$$\begin{aligned}
D(s^*, N) &= K_s^*(1-\beta)V(0, N-1)/N_s^* + L_s^* V(1, N-1)/N_s^* \\
&\quad - (K_s^*(1-\beta) + L_s^*)V((s+1)^*, N-1)/N_s^* \\
&= K_s^*(1-\beta)(1+\delta(1-\gamma)V(0, N-2))/N_s^* \\
&\quad + L_s^*[1+\delta K_1(1-\beta)V(0, N-2)/N_1 + \delta L_1 V(1, N-2)/N_1]/N_s^* \\
&\quad - (K_s^*(1-\beta) + L_s^*)[1+\delta K_{(s+1)}^*(1-\beta)V(0, N-2)/N_{(s+1)}^* \\
&\quad + \delta L_{(s+1)}^* V(1, N-2)/N_{(s+1)}^*]/N_s^* \\
&= \delta(1-\beta)(\beta-\gamma)V(0, N-2)K_s^*/N_s^* \\
&\quad - \delta(\alpha_1 - \alpha_0)[V(1, N-2) - (1-\beta)V(0, N-2)]L_s^*/N_s^* .
\end{aligned} \tag{4.9}$$

By Proposition 3.4 and Lemma 3.2,  $K_s^*/N_s^*$  is increasing in  $s^*$ ,  $L_s^*/N_s^*$  is decreasing in  $s^*$ , and  $V(1, N-2) - (1-\beta)V(0, N-2) \geq 0$ ; also,  $\beta > \gamma$  and  $\alpha_1 > \alpha_0$  by assumption. Thus  $D(s^*, N)$  is nondecreasing in  $s^*$ .



ii)  $s(N-1) > 1$  :

$$\begin{aligned}
D(s^*, N) &= \{K_{s^*}(1-\beta) [1+\delta(1-\gamma)V(0, N-2)] + L_{s^*} [1+\delta(1-\alpha_1\beta)V(2, N-2)/N_1] \\
&\quad - (K_{s^*}(1-\beta) + L_{s^*}) [1+\delta K_{(s+1)^*}(1-\beta)V(0, N-2)/N_{(s+1)^*} \\
&\quad\quad + \delta L_{(s+1)^*} V(1, N-2)/N_{(s+1)^*}] \} / N_{s^*} \\
&= \delta(1-\beta)(1-\gamma)V(0, N-2)K_{s^*}/N_{s^*} + \delta(1-\alpha_1\beta)V(2, N-2)L_{s^*}/N_{s^*} \\
&\quad - \delta(1-\beta)V(0, N-2)K_{(s+1)^*}/N_{s^*} - \delta V(1, N-2)L_{(s+1)^*}/N_{s^*} \\
&= \delta(1-\beta)(\beta-\gamma)V(0, N-2)K_{s^*}/N_{s^*} - \delta G(N-1) L_{s^*}/N_{s^*}, \quad (4.10)
\end{aligned}$$

where  $G(\cdot)$  is as defined in Lemma 4.1,  $D(s^*, N)$  is nondecreasing.

Case 2:  $t(N-1) > 1$  : It will be shown that  $D(s^*, N)$  is nondecreasing for  $s^*$  in the ranges  $[1, t(N-1)-2]$  and  $[t(N-1)-1, \infty)$ , and also that  $D(t(N-1)-1, N) \geq D(t(N-1)-2, N)$ . The cases  $s(N-1) = 1$  and  $s(N-1) > 1$  will be considered separately.

i)  $s(N-1) = 1$  :

$s^* \in [1, t(N-1) - 2]$  : Using Equations (4.3) - (4.8),

$$\begin{aligned}
D(s^*, N) &= \{K_{s^*}(1-\beta) [1+\delta(1-\gamma)V(0, N-2)] + L_{s^*} [1+\delta K_1(1-\beta)V(0, N-2)/N_1 \\
&\quad + \delta L_1 V(1, N-2)/N_1] - (K_{s^*}(1-\beta) + L_{s^*}) [1+\delta(K_{(s+1)^*}(1-\beta) \\
&\quad\quad + L_{(s+1)^*}) V((s+2)^*, N-2)/N_{(s+1)^*}] \} / N_{s^*} \\
&= \delta(1-\beta)(1-\gamma)V(0, N-2)K_{s^*}/N_{s^*} + \delta \alpha_1 (1-\beta)V(0, N-2)L_{s^*}/N_{s^*}
\end{aligned}$$

$$\begin{aligned}
& + \delta(1-\alpha_1)V(1,N-2)L_{s^*}/N_{s^*} - \delta V((s+2)^*, N-2)N_{(s+2)^*}/N_{s^*} \\
& = \delta D((s+1)^*, N-1)N_{(s+1)^*}/N_{s^*} + \delta(1-\beta)(\beta-\gamma)V(0,N-2)K_{s^*}/N_{s^*} \\
& - \delta(\alpha_1 - \alpha_0)[V(1,N-2) - (1-\beta)V(0,N-2)]L_{s^*}/N_{s^*}. \quad (4.11)
\end{aligned}$$

By Proposition 3.4  $N_{(s+1)^*}/N_{s^*} = 1-\beta K_{s^*}/N_{s^*}$  is decreasing in  $s^*$ ,  $D((s+1)^*, N-1)$  is nondecreasing and negative, and  $V(1,N-2) \geq (1-\beta)V(0,N-2)$ . Thus, the three terms in (4.11) are all nondecreasing and so  $D(s^*,N)$  is nondecreasing over this range of  $s^*$ .

$s^* \in [t(N-1)-1, \infty)$ : In this range Equation (4.9) holds and thus  $D(s^*,N)$  is nondecreasing.

Denote  $t(N-1)$  for the time being by  $t$ . Using Equations (4.9) and (4.11),  $D((t-1)^*, N) - D((t-2)^*, N)$

$$\begin{aligned}
& = \delta(1-\beta)(\beta-\gamma)V(0,N-2)[K_{(t-1)^*}/N_{(t-1)^*} - K_{(t-2)^*}/N_{(t-2)^*}] \\
& + \delta(\alpha_1 - \alpha_0)[V(1,N-2) - (1-\beta)V(0,N-2)][L_{(t-2)^*}/N_{(t-2)^*} \\
& - L_{(t-1)^*}/N_{(t-1)^*}] - \delta D((t-1)^*, N-1)N_{(t-1)^*}/N_{(t-2)^*}.
\end{aligned}$$

By Proposition 3.4,  $K_{s^*}/N_{s^*}$  is increasing and  $L_{s^*}/N_{s^*}$  is decreasing. By Lemma 3.2,  $V(1,N) \geq (1-\beta)V(0,N-2)$ ; also  $D((t-1)^*, N-1) < 0$  and  $\alpha_1 > \alpha_0$  by assumption. Thus  $D((t-1)^*, N) \geq D((t-2)^*, N)$ .

ii)  $s(N-1) > 1$  :

$s^* \in [1, t(N-1)-2]$  : Using Equations (4.3) - (4.8),

$$\begin{aligned}
 D(s^*, N) &= \{K_s^*(1-\beta)[1+\delta(1-\gamma)V(0, N-2)] + L_s^*[1+\delta(1-\alpha_1\beta)V(2, N-2)] \\
 &\quad - (K_s^*(1-\beta) + L_s^*)[1+\delta(K_{(s+1)}^*(1-\beta) \\
 &\quad\quad\quad + L_{(s+1)}^*)V((s+2)^*, N-2)/N_{(s+1)}^*]\}/N_s^* \\
 &= \delta(1-\beta)(1-\gamma)V(0, N-2)K_s^*/N_s^* + \delta(1-\alpha_1\beta)V(2, N-2)L_s^*/N_s^* \\
 &\quad - \delta V((s+2)^*, N-2)N_{(s+2)}^*/N_s^* \\
 &= \delta D((s+1)^*, N-1)N_{(s+1)}^*/N_s^* + \delta(1-\beta)(\beta-\gamma)V(0, N-2)K_s^*/N_s^* \\
 &\quad - \delta G(N-1)L_s^*/N_s^* \tag{4.12}
 \end{aligned}$$

where  $G(\cdot)$  is as defined in Lemma 4.1. Since by Proposition 3.4  $N_{(s+1)}^*/N_s^* = 1-\beta K_s^*/N_s^*$  is decreasing in  $s^*$ , and  $D((s+1)^*, N-1)$  is nondecreasing and negative, the three terms in (4.12) are all nondecreasing, and thus  $D(s^*, n)$  is nondecreasing in this range of  $s^*$ .

$s^* \in [t(N-1)-1, \infty)$  : In this range Equation (4.10) holds and  $D(s^*, N)$  is nondecreasing.

Using Equations (4.10) and (4.12) and again denoting  $t(N-1)$  for the time being by  $t$ ,

$$\begin{aligned}
& D((t-1)^*, N) - D((t-2)^*, N) \\
&= \delta(1-\beta)(\beta-\gamma)V(0, N-2)K_{(t-1)^*}/N_{(t-1)^*} - \delta G(N-1)L_{(t-1)^*}/N_{(t-1)^*} \\
&- \delta D((t-1)^*, N-1)N_{(t-1)^*}/N_{(t-2)^*} - \delta(1-\beta)(\beta-\gamma)V(0, N-2)K_{(t-2)^*}/N_{(t-2)^*} \\
&+ \delta G(N-1)L_{(t-2)^*}/N_{(t-2)^*} \\
&= \delta(1-\beta)(\beta-\gamma)V(0, N-2)[K_{(t-1)^*}/N_{(t-1)^*} - K_{(t-2)^*}/N_{(t-2)^*}] \\
&+ \delta G(N-1)[L_{(t-2)^*}/N_{(t-2)^*} - L_{(t-1)^*}/N_{(t-1)^*}] \\
&- \delta D((t-1)^*, N-1)N_{(t-1)^*}/N_{(t-2)^*} .
\end{aligned}$$

By Proposition 3.4,  $K_{s^*}/N_{s^*}$  is increasing,  $L_{s^*}/N_{s^*}$  is decreasing, so the first two terms of the above expression are nonnegative. Also,  $t = t(N-1)$ , so  $D((t-1)^*, N-1) < 0$ . Thus  $D((t-1)^*, N) \geq D((t-2)^*, N)$  and the proof is completed.  $\square$

Theorem 4.6. If  $\alpha_0 > \alpha_1\beta$  and  $\gamma > \alpha_1\beta$ , and the device is in state  $s^*$  facing an  $n$ -period horizon ( $1 \leq n \leq \infty$ ), it is optimal to inspect if and only if  $s^* \geq t(n)$ .

Proof: This result follows immediately from Lemma 4.3.  $\square$

#### 4.3. Coordination of Initiation and Ongoing Inspection Policies

One would like to know whether or not  $t(n)$  is finite. The following lemma and theorem will give a partial answer.

Lemma 4.4.  $\lim_{s \rightarrow \infty} V(s, n) = \lim_{s^* \rightarrow \infty} V(s^*, n)$  for all  $n$  and

$$\lim_{s \rightarrow \infty} V(s) = \lim_{s^* \rightarrow \infty} V(s^*).$$

Proof: The proof is by induction. For  $n = 1$ ,  $V(s, 1) = V(s^*, 1) = 1$  for all  $s, s^*$ , therefore  $\lim_{s \rightarrow \infty} V(s, 1) = \lim_{s^* \rightarrow \infty} V(s^*, 1)$ . Now, assume the result holds for  $n - 1$ . By Proposition 3.3,

$$V(\delta, n) = 1 + \delta \max \{ K_\delta (1-\beta) V(0, n-1) / N_\delta + L_\delta V(1, n-1) / N_\delta, \\ (K_\delta (1-\beta) + L_\delta) V(\delta+1, n-1) / N_\delta \}.$$

By Proposition 3.5,  $\lim_{s \rightarrow \infty} K_s / N_s = \lim_{s^* \rightarrow \infty} K_{s^*} / N_{s^*}$  and

$\lim_{s \rightarrow \infty} L_s / N_s = \lim_{s^* \rightarrow \infty} L_{s^*} / N_{s^*}$ , and by the induction hypothesis,

$$\lim_{s \rightarrow \infty} V(s+1, n-1) = \lim_{s^* \rightarrow \infty} V((s+1)^*, n-1). \quad \text{Thus, } \lim_{s \rightarrow \infty} V(s, n) = \lim_{s^* \rightarrow \infty} V(s^*, n).$$

Since  $V(\delta) = \lim_{n \rightarrow \infty} V(\delta, n)$ , the result regarding the infinite-horizon case follows immediately. □

Theorem 4.7. If  $\gamma \leq \alpha_1 \beta \leq \alpha_0$  and  $\gamma < \alpha_0$ , then  $t(n) < \infty$ , for  $n \leq \infty$ .

Proof:  $D(\delta, n) = K_\delta (1-\beta) V(0, n-1) / N_\delta + L_\delta V(1, n-1) / N_\delta \\ - (K_\delta (1-\beta) + L_\delta) V(\delta+1, n-1) / N_\delta.$

If  $\alpha_0 \geq \beta$ , by Proposition 3.5,  $\lim_{s \rightarrow \infty} K_s / N_s = 1$  and

$\lim_{s \rightarrow \infty} L_s / N_s = 0$ . Thus

$$\begin{aligned}
\lim_{s \rightarrow \infty} D(s,n) &= \lim_{s \rightarrow \infty} (K_s(1-\beta)V(0,n-1)/N_s + L_s V(1,n-1)/N_s \\
&\quad - (K_s(1-\beta) + L_s)V(s+1,n-1)/N_s) \\
&= (1-\beta)V(0,n-1) - (1-\beta) \lim_{s \rightarrow \infty} V(s+1,n-1) .
\end{aligned}$$

By Theorem 4.1, it is optimal to inspect every period for an old device. So,

$$\begin{aligned}
\lim_{s \rightarrow \infty} D(s,n) &= (1-\beta)(1+\delta(1-\gamma)V(0,n-2)) - (1-\beta)(1 + \\
&\quad \delta \lim_{s \rightarrow \infty} (K_s(1-\beta)V(0,n-2)/N_s + L_s V(1,n-2)/N_s)) \\
&= \delta(1-\beta)[(1-\gamma)V(0,n-2) - (1-\beta)V(0,n-2)] \\
&= \delta(1-\beta)(\beta-\gamma)V(0,n-2) .
\end{aligned}$$

Since  $V(0,n-2) > 0$  for  $n \geq 3$ ,  $\lim_{s \rightarrow \infty} D(s,n) > 0$  for  $n \geq 3$ .

If  $\alpha_0 < \beta$ , by Proposition 3.5,  $\lim_{s \rightarrow \infty} K_s/N_s = \alpha_0/\beta$  and

$\lim_{s \rightarrow \infty} L_s/N_s = (\beta - \alpha_0)/\beta$ , then

$$\begin{aligned}
\lim_{s \rightarrow \infty} D(s,n) &= \alpha_0(1-\beta)V(0,n-1)/\beta + (\beta-\alpha_0)V(1,n-1)/\beta - (1-\alpha_0) \lim_{s \rightarrow \infty} V(s,n-1) \\
&= \alpha_0(1-\beta)[1+\delta(1-\gamma)V(0,n-2)]/\beta + (\beta-\alpha_0)[1+\delta\alpha_1(1-\beta)V(0,n-2) \\
&\quad + \delta(1-\alpha_1)V(1,n-2)]/\beta \\
&\quad - (1-\alpha_0)[1+\delta(\alpha_0(1-\beta)V(0,n-2) + (\beta-\alpha_0)V(1,n-2))]/\beta
\end{aligned}$$

$$\begin{aligned}
&= \delta(1-\beta)V(0,n-2)[\alpha_0(\alpha_0-\gamma)+\alpha_1(\beta-\alpha_0)]/\beta + \delta(\alpha_0-\alpha_1)(\beta-\alpha_0)V(1,n-2)/\beta \\
&= \delta\alpha_0(1-\beta)(\alpha_0-\gamma)V(0,n-2)/\beta + \delta(\beta-\alpha_0)[\alpha_1(1-\beta)V(0,n-2) - (\alpha_1-\alpha_0)V(1,n-2)].
\end{aligned}$$

By Lemma 3.4,  $V(0,n-2) \geq V(1,n-2)$ . Thus, the second term in  $\lim_{s \rightarrow \infty} D(s,n)$  is

$$\begin{aligned}
&\delta(\beta-\alpha_0)[\alpha_1(1-\beta)V(0,n-2) - (\alpha_1-\alpha_0)V(1,n-2)] \\
&\geq \delta(\beta-\alpha_0)V(1,n-2)[\alpha_1(1-\beta) - (\alpha_1-\alpha_0)] \\
&= \delta(\beta-\alpha_0)V(1,n-2)(\alpha_0-\alpha_1\beta).
\end{aligned}$$

This quantity is nonnegative, and the first term in  $\lim_{s \rightarrow \infty} D(s,n)$  is positive for  $n \geq 3$ , so  $\lim_{s \rightarrow \infty} D(s,n) > 0$  for  $n \geq 3$ .

By Lemma 4.4 and Proposition 3.5,  $\lim_{s \rightarrow \infty} D(s,n) = \lim_{s^* \rightarrow \infty} D(s^*,n)$ .

Thus,  $\lim_{s^* \rightarrow \infty} D(s^*,n) > 0$  for  $n \geq 3$  and  $t(n) < \infty$  for  $3 \leq n < \infty$ .

For  $n = 1, 2$ ,  $D(s,n) = 0$  for all  $s \geq 1$ , so  $t(n) = 1$ .

For the infinite-horizon case, the proof follows similarly.  $\square$

When  $\alpha_0 = \gamma = \alpha_1\beta$ , by Theorems 4.4 and 4.5, it is both optimal to never inspect and to inspect if and only if  $s^* \geq t(n)$  ( $1 \leq n \leq \infty$ ). In this case  $t(n) = \infty$ .

For a previously inspected device, the optimal ongoing inspection policy can be an all-or-none policy: inspect if and only if the last inspection was at most a given number of periods ago. This happens when  $\alpha_0 < \alpha_1\beta$  and  $\gamma < \alpha_1\beta$ . It has been shown that if, in addition,

$\alpha_0 \leq \gamma$ , then it is optimal to never initiate inspection in the first place, and so the all-or-none ongoing inspection policy would never be implemented. However, if  $\alpha_0 > \gamma$  the possibility may exist for it to be optimal to initiate inspection after some number of periods and then to follow an all-or-none policy. The following theorem will show that this cannot in fact happen.

Theorem 4.8. If  $\gamma \leq \alpha_0 \leq \alpha_1 \beta$ ,  $z(n) < \infty$ , and the device is in state  $s^*$  facing an  $n$ -period horizon, it is optimal to not inspect ( $1 \leq n \leq \infty$ ).

Proof:

$$N_\delta D(\delta, n)/N_{\delta+1} = K_\delta (1-\beta)V(0, n-1)/N_{\delta+1} + L_\delta V(1, n-1)/N_{\delta+1} \\ - V(\delta+1, n-1)$$

$$K_\delta (1-\beta)/N_{\delta+1} = K_\delta (1-\beta)/[K_\delta (1-\beta) + L_\delta] = (1-\beta)/(1-\beta + L_\delta/K_\delta).$$

By Equations (4.1) and (4.2),  $L_s/K_s < L_{s^*}/K_{s^*}$ , thus

$$K_{s^*}(1-\beta)/N_{(s+1)^*} < K_s(1-\beta)/N_{s+1}.$$

Since  $K_\delta(1-\beta)/N_{\delta+1} + L_\delta/N_{\delta+1} = 1$ , and by Lemma 3.5

$V(1, n-1) \leq V(0, n-1)$ , and by Proposition 4.1,  $V((s+1)^*, n-1) \geq V(s+1, n-1)$ ,

$N_s D(s, n)/N_{s+1} \geq N_{s^*} D(s^*, n)/N_{(s+1)^*}$ . Thus  $D(s^*, n) < 0$  for

$s^* > z(n)$ . Also, by Lemma 4.2,  $D(s^*, n) < 0$  for  $s^* \leq z(n)$ . Thus

it is optimal to not inspect. □



A new device has been shown to be better than an old device of the same "age": the optimal expected remaining life is longer, and the probability that it is still OK is greater. If it is optimal to inspect a new device which was put into service  $s$  periods ago, then it would seem that it should be optimal to inspect an old device which was inspected and found to be OK  $s$  periods ago, because the inspection is hazardous. This has been shown for all cases except when  $\alpha_0 > \alpha_1\beta$  and  $\gamma > \alpha_1\beta$ . The following lemma and theorem will give an answer for the remaining case.

Lemma 4.5. If  $\alpha_0 > \alpha_1\beta$  and  $\gamma > \alpha_1\beta$ , then  $D(s^*,n) \leq D(s,n)$  for all  $s,n$  and  $D(s^*) \leq D(s)$  for all  $s$ .

Proof: The proof is by induction. For  $n=1,2$ ,  $D(\delta,n) = 0$  for all  $\delta$ , and the results hold. Assume the result holds for  $n-1$ .

Case 1:  $t(n-1) = 1$ : Since by inductive hypothesis,  $D(s^*,n-1) \leq D(s,n-1)$ ,  $s(n-1) = 1$ .

$$\begin{aligned}
 D(\delta,n) &= K_\delta(1-\beta)V(0,n-1)/N_\delta + L_\delta V(1,n-1)/N_\delta \\
 &\quad - (K_\delta(1-\beta) + L_\delta)V(\delta+1,n-1)/N_\delta \\
 &= K_\delta(1-\beta)[1+\delta(1-\gamma)V(0,n-2)]/N_\delta \\
 &\quad + L_\delta[1+\delta\alpha_1(1-\beta)V(0,n-2) + \delta(1-\alpha_1)V(1,n-2)]/N_\delta \\
 &\quad - N_{\delta+1}[1+\delta K_{\delta+1}(1-\beta)V(0,n-2)/N_{\delta+1} + \delta L_{\delta+1}V(1,n-2)/N_{\delta+1}]/N_\delta
 \end{aligned}$$

$$\begin{aligned}
&= \delta(1-\beta)(\beta-\gamma)V(0,n-2)K_{\delta}/N_{\delta} \\
&\quad - \delta(\alpha_1 - \alpha_0)[V(1,n-2) - (1-\beta)V(0,n-2)]L_{\delta}/N_{\delta} . \quad (4.13)
\end{aligned}$$

By Equations (4.1) and (4.2),  $K_s/N_s > K_{s^*}/N_{s^*}$  and  $L_s/N_s < L_{s^*}/N_{s^*}$ ; by Lemma 3.2  $V(1,n-2) \geq (1-\beta)V(0,n-2)$ ; so  $D(s^*,n) \leq D(s,n)$ .

Case 2:  $t(n-1) > 1$ : Consider the cases  $s(n-1) = 1$  and  $s(n-1) > 1$  separately.

i)  $s(n-1) = 1$ .

$s^* \in [1, t(n-1)-2]$ : By Equations (4.11) and (4.13) and the fact that  $D((s+1)^*, n-1) < 0$ ,

$$\begin{aligned}
D(s^*, n) &= \delta D((s+1)^*, n-1) N_{(s+1)^*} / N_{s^*} + \delta(1-\beta)(\beta-\gamma)V(0,n-2)K_{s^*} / N_{s^*} \\
&\quad - \delta(\alpha_1 - \alpha_0)[V(1,n-2) - (1-\beta)V(0,n-2)]L_{s^*} / N_{s^*} \\
&\leq \delta(1-\beta)(\beta-\gamma)V(0,n-2)K_s / N_s \\
&\quad - \delta(\alpha_1 - \alpha_0)[V(1,n-2) - (1-\beta)V(0,n-2)]L_s / N_s \\
&= D(s, n) .
\end{aligned}$$

$s^* \in [t(n-1) - 1, \infty)$ : In this range Equation (4.13) holds and thus  $D(s^*, n) \leq D(s, n)$ .

ii)  $s(n-1) > 1$ .

$s^* \in [1, s(n-1) - 2]$ : As in Equation (4.12),

$$\begin{aligned} D(\delta, n) &= \delta D(\delta+1, n-1) N_{\delta+1} / N_{\delta} + \delta(1-\beta)(\beta-\gamma)V(0, n-2) K_{\delta} / N_{\delta} \\ &\quad - \delta G(n-1) L_{\delta} / N_{\delta} \end{aligned} \quad (4.14)$$

where  $G(n)$  is nonnegative and as defined in Lemma 4.1. Since  $D((s+1)^*, n-1) \leq D(s+1, n-1) \leq 0$  and  $N_{(s+1)^*} / N_{s^*} > N_{s+1} / N_s$ ,  $D(s^*, n) \leq D(s, n)$ .

$s^* \in [s(n-1)-1, t(n-1) - 2]$ .

$$\begin{aligned} D(s, n) &= \{K_s(1-\beta)[1+\delta(1-\gamma)V(0, n-2)] + L_s[1+\delta(1-\alpha_1\beta)V(2, n-2)] \\ &\quad - (K_s(1-\beta) + L_s)[1+\delta K_{s+1}(1-\beta)V(0, n-2)/N_{s+1} \\ &\quad + \delta L_{s+1}V(1, n-2)/N_{s+1}]\} / N_s \\ &= \delta(1-\beta)(\beta-\gamma)V(0, n-2)K_s / N_s - \delta G(n-1)L_s / N_s . \end{aligned}$$

Since  $D(s^*, n)$  is given in Equation (4.14) and  $D((s+1)^*, n-1) < 0$ ,  $D(s^*, n) \leq D(s, n)$ .

$s^* \in [t(n-1) - 1, \infty)$ : In this range  $s^* \in [s(n-1) - 1, \infty)$  so Equation (4.13) holds and  $D(s^*, n) \leq D(s, n)$ .

The assertions regarding the infinite-horizon case follow by taking limits. □

The following theorem is an immediate consequence of Lemma 4.5.

Theorem 4.9. If  $\alpha_0 > \alpha_1\beta$  and  $\gamma > \alpha_1\beta$ , then  $s(n) \leq t(n)$  for  $1 \leq n \leq \infty$ .

Thus, for the infinite-horizon problem if it is optimal to inspect a new device after it has been in service for some finite number of periods, then it is optimal to inspect an old device for every  $r$  periods and  $r$  is finite.

Table 2 summarizes the optimal policy forms for new devices.

Table 2. Forms of the optimal inspection policy for new devices based upon the relationships among  $\alpha_0$ ,  $\gamma$ , and  $\alpha_1\beta$ .

	$\alpha_0 < \alpha_1\beta$	$\alpha_0 = \alpha_1\beta$	$\alpha_0 > \alpha_1\beta$
$\gamma < \alpha_1\beta$	$\alpha_0 \leq \gamma$ never $\alpha_0 > \gamma$ periodic	periodic ( $t(n) < \infty$ )	periodic ( $t(n) < \infty$ )
$\gamma = \alpha_1\beta$	never	never	periodic ( $t(n) < \infty$ )
$\gamma > \alpha_1\beta$	never	never	periodic ( $s(n) \leq t(n)$ )

When a periodic inspection policy is followed, the survival distribution of the device can be computed for the infinite-horizon case.

Theorem 4.10. Assume the horizon is infinite and that a periodic policy with period  $s(\infty)$  is followed. Let  $Y$  be the remaining lifetime of an old device in observed state 1. Let  $\bar{F}_0(k) = \Pr\{Y > k\}$ ,

$$\bar{F}_0(k) = \delta^k L_{s(\infty)}^a N_{b+1} + \delta^k K_{s(\infty)} (1-\beta) \sum_{i=0}^{a-1} L_{s(\infty)}^i (1-\gamma)^{k - (i+1)s(\infty)}$$

where  $a = \left\lfloor \frac{k}{s(\infty)} \right\rfloor$ ,  $\frac{5/}{b} = k - a s(\infty)$ .

(Note: By convention, the summation in  $\bar{F}_0(k)$  is zero for  $a = 0$ , and  $L_{\infty}^0 = 1$ .)

Proof: If it is optimal to never inspect (i.e.,  $s(\infty) = \infty$ ),

$$\begin{aligned} \bar{F}_0(k) &= \Pr\{\text{survive } k \text{ periods}\} \\ &= \delta^k \Pr\{\text{no primary failures in } k \text{ periods}\} \\ &= \delta^k \prod_{j=1}^k P_{j, j+1}(0) \\ &= \delta^k N_{k+1}. \end{aligned}$$

If it is optimal to inspect every  $s(\infty)$  periods ( $s(\infty) < \infty$ ),

$$\begin{aligned} \bar{F}_0(k) &= \delta^k \Pr\{\text{no primary failures in } k \text{ periods}\} \\ &= \delta^k \left[ \sum_{i=0}^{a-1} \Pr\{\text{found to be OK for } i \text{ inspections} \right] \end{aligned}$$

---

$\frac{5/}{\lfloor x \rfloor} =$  greatest integer less than or equal to  $x$ .

$$\begin{aligned}
& \text{and partially failed at } i + 1^{\text{st}} \\
& \text{inspection} \} \Pr \{ \text{survived } k \text{ periods} | \\
& \text{partially failed at } i + 1^{\text{st}} \text{ inspection} \} \\
& + \Pr \{ \text{found to be OK in all } a \text{ inspections} \} \\
& \cdot \Pr \{ \text{survived } k \text{ periods} | \text{found OK in } a \\
& \text{inspections} \} ] \\
& = \delta^k \sum_{i=0}^{a-1} L_{s(\infty)}^i K_{s(\infty)} (1-\beta)(1-\gamma)^{k-(i+1)s(\infty)} + \delta^k L_{s(\infty)}^a N_{b+1}
\end{aligned}$$

where  $a = \left\lfloor \frac{k}{s(\infty)} \right\rfloor$  and  $b = k - a s(\infty)$ . □

Theorem 4.11. Assume the horizon is infinite and a periodic policy with period  $t(\infty)$  is followed. Let  $W$  be the remaining lifetime of a new device in observed state  $1^*$ . Let  $\bar{F}_N(k) = \Pr\{W > k\}$ .

$$\begin{aligned}
\bar{F}_N(k) &= \delta^k N_{(k+1)^*} && \text{for } k < t(\infty) \\
&= \delta^{t(\infty)} L_{t(\infty)^*} \bar{F}_0(k-t(\infty)) + \delta^k K_{t(\infty)^*} (1-\beta)(1-\gamma)^{k-t(\infty)} \\
&&& \text{for } k \geq t(\infty).
\end{aligned}$$

Proof: If  $k < t(\infty)$ ,

$$\begin{aligned}
\bar{F}_N(k) &= \delta^k \Pr \{ \text{a new device survives } k \text{ periods} \} \\
&= \delta^k \prod_{j=1}^k P_{j^*, (j+1)^*} (0) \\
&= \delta^k N_{(k+1)^*}.
\end{aligned}$$

If  $k \geq t(\infty)$ ,

$$\bar{F}_N(k) = \delta^k \sum_{\ell=1}^2 \Pr\{X_{t(\infty)} = \ell | X_0 = 1, a_j = 0, 0 \leq j < t(\infty)\}$$

$$\Pr\{X_k \leq 3 | X_{t(\infty)} = \ell, a_{t(\infty)} = 1\}$$

$$= \delta^k L_{t(\infty)*} \bar{F}_0(k - t(\infty)) + \delta^k K_{t(\infty)*} (1-\beta)(1-\gamma)^{k-t(\infty)}. \quad \square$$

The survival distributions can be computed for the finite-horizon case similarly.

## V. Computational Procedures and Results

In this chapter, some efficient methods will be introduced to compute the maximal expected remaining life of a new or an old device, and to determine the critical numbers  $s(n)$ ,  $t(n)$ ,  $z(n)$  which indicate whether or not to inspect the device. Also, the rates of convergence of the finite-horizon values to their infinite-horizon limits will be discussed. Some numerical examples will be presented for some interesting situations.

### 5.1. Computational Procedures

The finite-horizon results presented so far have been included largely to facilitate the infinite-horizon proofs. In most instances there will be no fixed time horizon and thus an infinite horizon will be appropriate.

To approximate the infinite-horizon  $V(\delta)$  by  $V(\delta, n)$  for a large value of  $n$  is quite inefficient computationally, requiring values of  $V(\cdot, k)$  for all  $k \leq n$ . Having established the form of the optimal inspection policy, the necessary computations can be performed much more efficiently.

The case where  $\alpha_0 > \alpha_1\beta$  and  $\gamma > \alpha_1\beta$  will be considered first. This is the most common case, because for small deterioration probabilities  $\alpha_0$ ,  $\alpha_1$ ,  $\beta$ ,  $\gamma$  of roughly similar orders of magnitude, the product of any two will be smaller than any one. Also, this is the case for which the optimal initiation and inspection policies are nontrivial and intuitive. Let



$$F(s) = \left[ \sum_{i=0}^{s-1} \delta^i N_{i+1} + \delta^s K_s (1-\beta) V(0) \right] / (1-\delta^s L_s) \quad (5.1)$$

Theorem 5.1. If  $\alpha_0 > \alpha_1 \beta$  and  $\gamma > \alpha_1 \beta$ , then  $\max\{F(s) : s \geq 1\} = F(s(\infty))$

=  $V(1)$  and if  $s < t \leq s(\infty)$ , then  $F(s) < F(t)$ . Moreover,

$V(s) = N_{s-1} (V(s-1) - 1) / (\delta N_s)$  for  $s < s(\infty)$  and

$V(s) = 1 + \delta K_s (1-\beta) V(0) / N_s + \delta L_s V(1) / N_s$  for  $s \geq s(\infty)$ .

Proof: By Proposition 3.3 and Theorem 4.2, for  $s < s(\infty)$

$$\begin{aligned} V(1) &= \sum_{i=0}^{s-1} \delta^i N_{i+1} + \delta^s (K_s (1-\beta) + L_s) V(s+1) \\ &= \sum_{i=0}^{s-1} \delta^i N_{i+1} + \delta^s (K_s (1-\beta) V(0) + L_s V(1) - D(s) N_s). \end{aligned}$$

By Equation (5.1),

$$F(s) = \sum_{i=0}^{s-1} \delta^i N_{i+1} + \delta^s K_s (1-\beta) V(0) + \delta^s L_s F(s).$$

Thus,

$$V(1) - F(s) = \delta^s L_s (V(1) - F(s)) - \delta^s D(s) N_s$$

i.e.,

$$V(1) - F(s) = - \delta^s D(s) N_s / (1-\delta^s L_s). \quad (5.2)$$

Since  $s < s(\infty)$ ,  $D(s) < 0$ ; also,  $1 - \delta^s L_s > 0$ . Therefore  $V(1) - F(s) > 0$ . The quantities  $N_s$  and  $\delta^s L_s$  are decreasing in  $s$ , and by Lemma 4.1,  $D(s)$  is nondecreasing in  $s$  and  $D(s) < 0$  for  $s < s(\infty)$ , so  $V(1) - F(s)$  is decreasing in  $s$  and  $F(s) < F(t)$  for  $s < t \leq s(\infty)$ .

Denoting  $s(\infty)$  by  $\hat{s}$  for the time being, by Proposition 3.3 and Theorem 4.2,

$$\begin{aligned}
V(1) &= \sum_{i=0}^{\hat{s}-2} \delta^i N_{i+1} + \delta^{\hat{s}-1} V(\hat{s}) N_{\hat{s}} \\
&= \sum_{i=0}^{\hat{s}-1} \delta^i N_{i+1} + \delta^{\hat{s}} (K_{\hat{s}} (1-\beta)V(0) + L_{\hat{s}} V(1)) \\
&= \left[ \sum_{i=0}^{\hat{s}-1} \delta^i N_{i+1} + \delta^{\hat{s}} K_{\hat{s}} (1-\beta)V(0) \right] / (1 - \delta^{\hat{s}} L_{\hat{s}}) \\
&= F(\hat{s}) .
\end{aligned}$$

For  $s \geq \hat{s}$ ,

$$\begin{aligned}
V(1) &= \sum_{i=0}^{\hat{s}-1} \delta^i N_{i+1} + \delta^{\hat{s}} (K_{\hat{s}} (1-\beta)V(0) + L_{\hat{s}} V(1)) \\
&= \sum_{i=0}^{s-1} \delta^i N_{i+1} + \sum_{i=\hat{s}}^{s-1} \delta^i [K_i (1-\beta)V(0) + L_i V(1) \\
&\quad - (N_{i+1} + \delta(K_{i+1} (1-\beta)V(0) + L_{i+1} V(1)))] \\
&\quad + \delta^s (K_s (1-\beta)V(0) + L_s V(1)) \\
&= \sum_{i=0}^{s-1} \delta^i N_{i+1} + \sum_{i=\hat{s}}^{s-1} \delta^i N_i D(i) + \delta^s (K_s (1-\beta)V(0) + L_s V(1)) .
\end{aligned}$$

By Equation (5.1),

$$\begin{aligned} V(1) - F(s) &= \sum_{i=\hat{s}}^{s-1} \delta^i D(i) N_i + \delta^s L_s (V(1) - F(s)) \\ &= \sum_{i=\hat{s}}^{s-1} \delta^i D(i) N_i / (1 - \delta^s L_s) \end{aligned}$$

Since  $D(i) \geq 0$  for  $s \geq \hat{s}$ ,  $V(1) - F(s) \geq 0$ . □

To determine the optimal ongoing inspection policy, one computes (recursively)  $F(1)$ ,  $F(2)$ , ..., stopping at the first decrease of  $F(\cdot)$ , say at  $F(u)$ . Then  $s(\infty) = u - 1$ ,  $V(1) = F(u-1)$ , and any other  $V(s)$  can be computed from  $V(1)$  via the formulas given in Theorem 5.1.

A similar procedure can be followed to efficiently determine the optimal initiation policy.

Theorem 5.2. Define  $H(s) = \sum_{i=0}^{s-1} \delta^i N_{(i+1)*} + \delta^s (K_{s*}(1-\beta)V(0) + L_{s*}V(1))$ .

If (i)  $\alpha_0 > \alpha_1 \beta$  and  $\gamma > \alpha_1 \beta$  or (ii)  $\gamma \leq \alpha_1 \beta$  and  $\gamma \leq \alpha_0$ ,

then  $\max\{H(s) : s \geq 1\} = H(t(\infty)) = V(1^*)$ . Moreover,

i) if  $r < s < t(\infty)$ , then  $H(r) < H(s)$  and

$$V(s^*) = (V((s-1)^*) - 1)N_{(s-1)*} / (\delta N_{s*});$$

ii) if  $t(\infty) \leq s < r$ , then  $H(r) \leq H(s)$  and

$$V(s^*) = 1 + \delta(K_{s*}(1-\beta)V(0) + L_{s*}V(1)) / N_{s*}.$$

Proof: Denote  $t(\infty)$  by  $t$ . For  $s < t$ , by following an argument similar to the proof of Theorem 5.2.

$$V(1^*) = \sum_{i=0}^{s-1} \delta^i N_{(i+1)^*} + \delta^s [K_{s^*}(1-\beta)V(0) + L_{s^*}V(1) - D(s^*)N_{s^*}]$$

and

$$V(1^*) - H(s) = -\delta^s D(s^*)N_{s^*}.$$

Since  $s < t$ ,  $D(s^*) < 0$ , and so  $V(1^*) - H(s) > 0$ . By Lemma 4.4  $D(s^*)$  is nondecreasing, and  $\delta^s N_{s^*}$  is decreasing, so  $V(1^*) - H(s)$  is decreasing in  $s$  and  $H(r) < H(s)$  for  $r < s < t$ . Also,

$$\begin{aligned} V(1^*) &= \sum_{i=1}^{t-2} \delta^i N_{(i+1)^*} + \delta^{t-1} V(t^*) N_{t^*} \\ &= \sum_{i=1}^{t-1} \delta^i N_{(i+1)^*} + \delta^t (K_{t^*}(1-\beta)V(0) + L_{t^*}V(1)) \\ &= H(t). \end{aligned}$$

For  $s > t$ ,

$$V(1^*) = \sum_{i=0}^{s-1} \delta^i N_{(i+1)^*} + \sum_{i=t}^{s-1} \delta^i D(i^*) N_{i^*} + \delta^s (K_{s^*}(1-\beta)V(0) + L_{s^*}V(1)).$$

So,

$$V(1^*) - H(s) = \sum_{i=t}^{s-1} \delta^i D(i^*) N_{i^*}.$$

Since  $D(i^*) \geq 0$ , for  $i > t$ ,  $V(1^*) - H(s) \geq 0$ . It is clear that  $V(1^*) - H(s)$  is nondecreasing in  $s$  when  $s > t$ , so  $H(r) \leq H(s)$  when  $t \leq s < r$ . □

If  $\alpha_0 \leq \gamma$  and  $\alpha_0 \leq \alpha_1 \beta$ , then it is optimal to never initiate inspection. One can compute  $V(s^*)$  as follows.

Theorem 5.3. If it is optimal to never initiate inspection (i.e., when  $\alpha_0 \leq \gamma$  and  $\alpha_0 \leq \alpha_1 \beta$ ), then

$$V(1^*) = [1 - \delta(1-\alpha_0)(1-\beta)] / [(1-\delta(1-\alpha_0))(1-\delta(1-\beta))]$$

and

$$V(s^*) = [1 + \delta V(1^*)] L_{s^*} / N_{s^*} + [1 - \delta(1-\beta)]^{-1} K_{s^*} / N_{s^*}.$$

Proof:

$$\begin{aligned} V(s^*) = & E[\text{Remaining life} \mid X_s = 1] \Pr\{X_s = 1 \mid X_0 = 1, a_k = 0, 0 \leq k < s\} \\ & + E[\text{Remaining life} \mid X_s = 2] \Pr\{X_s = 2 \mid X_0 = 1, a_k = 0, 0 \leq k < s\}. \end{aligned}$$

Since only a secondary failure can cause a device which is OK to fail within one period, and since by hypothesis inspection is never initiated,

$$E(\text{Remaining life} \mid X_s = 1) = 1 + \delta V(1^*).$$

Also, the remaining life of the device given  $X_s = 2$  has a geometric distribution with mean  $1/(1-\delta(1-\beta))$ . Thus

$$V(s^*) = (1 + \delta V(1^*))L_{s^*}/N_{s^*} + [1 - \delta(1 - \beta)]^{-1} K_{s^*}/N_{s^*} \quad (5.3)$$

Replacing  $s^*$  by  $1^*$  in Equation (5.3),

$$\begin{aligned} V(1^*) &= [1 + \delta V(1^*)](1 - \alpha_0) + [1 - \delta(1 - \beta)]^{-1} \alpha_0 \\ &= [1 - \delta(1 - \alpha_0)(1 - \beta)] / [(1 - \delta(1 - \alpha_0))(1 - \delta(1 - \beta))] \quad \square \end{aligned}$$

When  $\alpha_0 < \alpha_1 \beta$  and  $\gamma < \alpha_1 \beta$ , for an old device facing an infinite horizon which is in state  $s \leq z(\infty)$ , or when  $\gamma \leq \alpha_1 \beta \leq \alpha_0$ , for an old device in state  $s$ , it is optimal to inspect. Thus by Proposition 3.3,  $V(s) = 1 + \delta K_s(1 - \beta)V(0)/N_s + \delta L_s V(1)/N_s$ . This equation can be solved for  $V(1)$  and an explicit formula for  $V(s)$  can be obtained. This can be summarized as follows.

Theorem 5.4. (Butler, 1979) If  $\gamma \leq \alpha_1 \beta \leq \alpha_0$  then

$$V(1) = (1 + \delta \alpha_1(1 - \beta)V(0)) / (1 - \delta(1 - \alpha_1))$$

and

$$V(s) = 1 + \delta K_s(1 - \beta)V(0)/N_s + \delta L_s V(1)/N_s.$$

The cases left are those for an old device when it is optimal to never initiate inspection. The maximal expected remaining lifetime of an old device for these cases can be computed via the following theorems. (Since a new device should never reach an observed state  $s \geq 1$ , these theorems are not very useful practically. However, they are included for completeness.)

Theorem 5.5. If it is optimal to never inspect an old device (i.e., when  $\alpha_0 \leq \alpha_1 \beta \leq \gamma$ ), then

$$V(1) = 1/(1-\delta(1-\alpha_0)) + \delta(\alpha_0 - \alpha_1 \beta) / [(1-\delta(1-\beta))(1-\delta(1-\alpha_0))]$$

and

$$V(s) = (V(s-1) - 1) N_{s-1} / (\delta N_s) \quad \text{for } s \geq 2.$$

Proof: By Theorem 4.1, when  $\alpha_0 \leq \alpha_1 \beta \leq \gamma$ , it is optimal to never inspect for an old device. Thus,

$$\begin{aligned} V(1) &= E(\text{Remaining life} \mid X_{n+1} = 1) \Pr\{X_{n+1} = 1 \mid X_n = 1, a_n = 1\} \\ &\quad + E(\text{Remaining life} \mid X_{n+1} = 2) \Pr\{X_{n+1} = 2 \mid X_n = 1, a_n = 1\} \\ &= (1 + \delta V(1^*)) (1 - \alpha_1) + (1 - \delta(1 - \beta))^{-1} \alpha_1. \end{aligned}$$

By Theorem 4.4, the optimal initiation policy is to never inspect in this case. Computing  $V(1^*)$  via Theorem 5.3 and simplifying,

$$V(1) = 1 / (1 - \delta(1 - \alpha_0)) + \delta(\alpha_0 - \alpha_1 \beta) / [(1 - \delta(1 - \beta))(1 - \delta(1 - \alpha_0))].$$

By Proposition 3.3,  $V(s) = 1 + \delta N_{s+1} V(s+1) / N_s$ . So,

$$V(s) = (V(s-1) - 1) N_{s-1} / (\delta N_s).$$

□

The last case to consider is when  $\alpha_0 < \alpha_1 \beta$  and  $\gamma < \alpha_1 \beta$ .

Theorem 5.6. Suppose  $\alpha_0 < \alpha_1\beta$  and  $\gamma < \alpha_1\beta$ . Let

$$\hat{V} = 1 / (1 - \delta(1 - \alpha_0)) + \delta(\alpha_0 - \alpha_1\beta) / [(1 - \delta(1 - \beta))(1 - \delta(1 - \alpha_0))]$$

and

$$\tilde{V} = [1 + \delta\alpha_1(1 - \beta) / (1 - \delta(1 - \gamma))] / [1 - \delta(1 - \alpha_1)] .$$

- i) If  $\hat{V} > \tilde{V}$ , then the optimal ongoing policy and the optimal initiation policy are to never inspect.
- ii) If  $\hat{V} \leq \tilde{V}$ , then  $z(\infty) = \max\{s: E(s) \geq 0\}$ , where

$$E(s) = (1 - \beta)(\beta - \gamma)V(0) K_s / N_s - (\alpha_1 - \alpha_0) [\tilde{V} - (1 - \beta)V(0)] L_s / N_s .$$

Proof: By Theorem 4.3, when  $\gamma < \alpha_1\beta$  and  $\alpha_0 < \alpha_1\beta$ , it is either optimal to inspect when the device is in state 1 or else it is optimal to never inspect a previously inspected device. If it is optimal to never inspect, then by Theorem 5.5,

$$V(1) = 1 / (1 - \delta(1 - \alpha_0)) + \delta(\alpha_0 - \alpha_1\beta) / [(1 - \delta(1 - \beta))(1 - \delta(1 - \alpha_0))] = \hat{V} .$$

If it is optimal to inspect when the device is in state 1, then

$$\begin{aligned} V(1) &= 1 + \delta(K_1(1 - \beta)V(0) + L_1V(1)) / N_1 \\ &= [1 + \delta\alpha_1(1 - \beta)V(0)] / [1 - \delta(1 - \alpha_1)] = \tilde{V} . \end{aligned}$$



Since the optimal action when in state 1 is determined by  $\max\{\hat{V}, \tilde{V}\}$ , if  $\hat{V} > \tilde{V}$ , the optimal ongoing policy is to never inspect. By Theorems 4.4 and 4.8 the optimal initiation policy is also to never inspect.

If  $\hat{V} \leq \tilde{V}$ , then it is optimal to inspect when in state 1, and  $\tilde{V} = V(1)$ . For  $s \leq z(\infty) - 1$ ,

$$\begin{aligned} N_s D(s) &= K_s (1-\beta)V(0) + L_s V(1) - N_{s+1} V(s+1) \\ &= K_s (1-\beta) [1 + \delta(1-\gamma)V(0)] + L_s [1 + \delta\alpha_1 (1-\beta)V(0) + \delta(1-\alpha_1)V(1)] \\ &\quad - N_{s+1} [1 + \delta(1-\beta)V(0)K_{s+1}/N_{s+1} + \delta V(1)L_{s+1}/N_{s+1}]. \end{aligned}$$

$$N_s D(s)/\delta = (1-\beta)V(0) [(1-\gamma)K_s + \alpha_1 L_s - K_{s+1}] + V(1) [(1-\alpha_1)L_s - L_{s+1}].$$

Since  $K_{s+1} = (1-\beta)K_s + \alpha_0 L_s$  and  $L_{s+1} = (1-\alpha_0)L_s$ ,

$$D(s)/\delta = (1-\beta)(\beta-\gamma)V(0) K_s / N_s - (\alpha_1 - \alpha_0) [V(1) - (1-\beta)V(0)] L_s / N_s = E(s). \quad (5.4)$$

By Lemma 3.2, Proposition 3.4, and the assumptions that  $\beta > \gamma$  and  $\alpha_1 > \alpha_0$ ,  $E(s)$  is decreasing in  $s$ . Also, by Equation (5.4)  $E(z(\infty)) \geq 0$  and  $E(z(\infty) + 1) < 0$ . Thus  $z(\infty) = \max\{s: E(s) \geq 0\}$ .  $\square$

When  $\alpha_0 < \alpha_1 \beta$  and  $\gamma < \alpha_1 \beta$ , if  $\hat{V} > \tilde{V}$ , then it is optimal to never inspect and one can compute  $V(s)$  via Theorem 5.5; if  $\hat{V} \leq \tilde{V}$ , then  $V(s) = 1 + \delta K_s (1-\beta)V(0)/N_s + \delta L_s \tilde{V}/N_s$  for  $s \leq z(\infty)$  and  $V(s) = (1 + \delta V(1^*)) K_s / N_s + \alpha_1 L_s / [(1-\delta(1-\beta))N_s]$  for  $s > z(\infty)$ .

Recall that when  $\alpha_0 > \alpha_1 \beta$  and  $\gamma > \alpha_1 \beta$ , the optimal initiation policy is to inspect if and only if  $s^* \geq t(n)$  and the optimal ongoing

policy is to inspect if and only if  $s \geq s(n)$ . For an infinite-horizon problem, one would like to know whether or not  $t(\infty)$  and  $s(\infty)$  are finite. When  $\alpha_1\beta \leq \gamma < \alpha_0$ , it can be shown that  $t(\infty) < \infty$  and  $s(\infty) < \infty$ ; however, when  $\alpha_1\beta < \alpha_0 \leq \gamma$ , either to never inspect or to periodically inspect may be optimal.

Theorem 5.7. If  $\alpha_1\beta \leq \gamma < \alpha_0$ , then  $t(\infty) < \infty$  and  $s(\infty) < \infty$ .

Proof: Assume to the contrary that  $s(\infty) = \infty$ , i.e., it is optimal to never inspect for an old device. Then by Theorem 5.5,

$$\begin{aligned} V(1) &= 1/(1-\delta(1-\alpha_0)) + \delta(\alpha_0 - \alpha_1\beta)/[(1-\delta(1-\beta))(1-\delta(1-\alpha_0))] \\ &\geq 1/(1-\delta(1-\alpha_0)) \end{aligned}$$

and

$$\begin{aligned} \lim_{s \rightarrow \infty} V(s) &= 1 + \delta \lim_{s \rightarrow \infty} (1-\beta K_s/N_s) \lim_{s \rightarrow \infty} V(s) \\ &= 1 + \delta(1-\alpha_0) \lim_{s \rightarrow \infty} V(s) \end{aligned}$$

i.e.,

$$\lim_{s \rightarrow \infty} V(s) = 1/(1-\delta(1-\alpha_0)).$$

But, by Proposition 3.3,

$$\begin{aligned} \lim_{s \rightarrow \infty} V(s) &= 1 + \delta \max\{(1-\beta)V(0) \lim_{s \rightarrow \infty} K_s/N_s + V(1) \lim_{s \rightarrow \infty} L_s/N_s, \\ &\quad \lim_{s \rightarrow \infty} [(1-\beta K_s/N_s)V(s)]\} \end{aligned}$$

$$= 1 + \delta \max \{ \alpha_0(1-\beta)V(0)/\beta + (\beta - \alpha_0)V(1)/\beta, \\ (1-\alpha_0) \lim_{s \rightarrow \infty} V(s) \},$$

and since  $\gamma < \alpha_0$ ,  $1/(1-\delta(1-\gamma)) > 1/(1-\delta(1-\alpha_0))$ . Thus,

$$\begin{aligned} & \alpha_0(1-\beta)V(0)/\beta + (\beta - \alpha_0)V(1)/\beta \\ & \geq \alpha_0(1-\beta)/[\beta(1-\delta(1-\gamma))] + (\beta - \alpha_0)/[\beta(1-\delta(1-\alpha_0))] \\ & > [\alpha_0(1-\beta) + (\beta - \alpha_0)]/[\beta(1-\delta(1-\alpha_0))] \\ & = (1-\alpha_0)/(1-\delta(1-\alpha_0)) \\ & = (1-\alpha_0) \lim_{s \rightarrow \infty} V(s). \end{aligned}$$

This is a contradiction. So  $s(\infty) < \infty$ . The inequality  $t(\infty) < \infty$  can be proved similarly. □

## 5.2. Rate of Convergence

The finite-horizon optimal lifetimes  $V(\delta, n)$  will converge to their infinite-horizon counterparts  $V(\delta)$  as  $n$  approaches infinity. Since  $V(\delta)$  has been computed,  $V(\delta, n)$  can be approximated by  $V(\delta)$  if  $n$  is sufficiently large. How good the approximation is depends on the rate of convergence and how large  $n$  is. The rate is expected to be geometric and will be discussed in the following theorems.

Theorem 5.8. If it is optimal to never inspect for the infinite-horizon problem, then

$$V(\delta) - V(\delta, n) \leq \delta^n N_{n+\delta} V(n+\delta) / N_\delta .$$

Proof: Since it is optimal to never inspect, by Proposition 3.3,

$$\begin{aligned} V(\delta) &= 1 + \delta N_{\delta+1} V(\delta+1) / N_\delta \\ &= \sum_{i=0}^{n-1} \frac{\delta^i N_{\delta+i}}{N_\delta} + \frac{\delta^n V(n+\delta) N_{n+\delta}}{N_\delta} , \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} V(\delta, n) &\geq 1 + \delta N_{\delta+1} V(\delta+1, n-1) / N_\delta \\ &\geq \sum_{i=0}^{n-1} \frac{\delta^i N_{\delta+i}}{N_\delta} , \end{aligned} \quad (5.6)$$

$$V(\delta) - V(\delta, n) \leq \delta^n V(n+\delta) N_{n+\delta} / N_\delta . \quad \square$$

Another way to look at  $N_{n+\delta} / N_\delta$  is that  $N_{n+\delta} / N_\delta = \prod_{i=\delta}^{n+\delta-1} (1 - \beta K_i / N_i)$ .

This term becomes smaller as  $n$  increases. When  $\gamma \leq \alpha_1 \beta \leq \alpha_0$ , the optimal ongoing policy is to always inspect and the rate of convergence is geometric, too.

Theorem 5.9. If  $\gamma \leq \alpha_1 \beta \leq \alpha_0$ , then

$$V(s) - V(s, n) \leq \delta^n (1-\gamma)^{n-1} [V(1) + \alpha_1 (1-\beta) V(0) L_s / ((\alpha_1 - \gamma) N_s)] .$$

Proof: If it is optimal to always inspect, then

$$V(s) = 1 + \delta K_s (1-\beta)V(0)/N_s + \delta L_s V(1)/N_s$$

and

$$V(s,n) = 1 + \delta K_s (1-\beta)V(0,n-1)/N_s + \delta L_s V(1,n-1)/N_s .$$

Thus,

$$V(s) - V(s,n) \leq \delta [K_s (1-\beta)(V(0) - V(0,n-1)) + L_s (V(1) - V(1,n-1))] / N_s .$$

Repeating this procedure for  $s = 1$  yields

$$\begin{aligned} & V(1) - V(1,n) \\ & \leq \delta \alpha_1 (1-\beta) \sum_{i=1}^n (\delta(1-\alpha_1))^{i-1} (V(0) - V(0,n-i)) + (\delta(1-\alpha_1))^n V(1) \\ & = \delta \alpha_1 (1-\beta) \sum_{i=1}^n (\delta(1-\alpha_1))^{i-1} (\delta(1-\gamma))^{n-i} / (1-\delta(1-\gamma)) + (\delta(1-\alpha_1))^n V(1) \\ & = \delta^n \alpha_1 (1-\beta) [(1-\gamma)^n - (1-\alpha_1)^n] / [(1-\delta(1-\gamma))(\alpha_1-\gamma)] + (\delta(1-\alpha_1))^n V(1) . \end{aligned}$$

Combining these results yields

$$\begin{aligned} & V(s) - V(s,n) \leq \delta K_s (1-\beta)(V(0) - V(0,n-1))/N_s + \delta L_s (V(1) - V(1,n-1))/N_s \\ & \leq \delta^n K_s (1-\beta)(1-\gamma)^{n-1} / ((1-\delta(1-\gamma))N_s) + \delta^n (1-\alpha_1)^{n-1} L_s V(1) / N_s \\ & \quad + \delta^n \alpha_1 (1-\beta) [(1-\gamma)^{n-1} - (1-\alpha_1)^{n-1}] L_s / [(1-\delta(1-\gamma))(\alpha_1-\gamma) N_s] \quad (5.7) \end{aligned}$$

$$\begin{aligned}
&\leq \delta^n K_s (1-\gamma)^{n-1} (1-\beta) V(0) / N_s + \delta^n (1-\gamma)^{n-1} L_s V(1) / N_s \\
&\quad + \delta^n \alpha_1 (1-\beta) (1-\gamma)^{n-1} L_s V(0) / [(\alpha_1 - \gamma) N_s] \\
&\leq \delta^n (1-\gamma)^{n-1} [V(1) + \alpha_1 (1-\beta) V(0) L_s / ((\alpha_1 - \gamma) N_s)] \quad \square
\end{aligned}$$

Equation (5.7) gives a tighter bound for  $V(s) - V(s, n)$  and Theorem 5.9 gives a simpler form. The next case to consider is periodic inspection (with  $s(\infty) > 1$ ).

Theorem 5.10. Denote  $s(\infty)$  by  $\hat{s}$ . If  $\alpha_0 > \alpha_1 \beta$  and  $\gamma > \alpha_1 \beta$ , then

$$\begin{aligned}
V(s) - V(s, n) &\leq \delta^n N_{n+s} V(n+s) / N_s && \text{for } s \leq \hat{s} - n, \\
&\leq \delta^{\hat{s}-s} N_{\hat{s}} (V(\hat{s}) - V(\hat{s}, n - \hat{s} + s)) / N_s && \text{for } \hat{s} - n < s < \hat{s}, \\
&\leq \delta^n [K_{\hat{s}} (1-\beta) (1-\gamma)^{n-1} V(0) + L_{\hat{s}} N_n V(n)] / N_{\hat{s}} && \text{for } s = \hat{s} > 1, \hat{s} \geq n. \\
&\leq (\delta^{\hat{s}} L_{\hat{s}})^t V(\hat{s}) + \frac{\delta^n K_{\hat{s}} (1-\beta) (1-\gamma)^{n-t} \hat{s} + \hat{s} - 1 [(1-\gamma)^{t\hat{s}} - (1-\alpha_1)^t (1-\alpha_0)^{t(\hat{s}-1)}]}{N_{\hat{s}} (1-\delta(1-\gamma)) [(1-\gamma)^{\hat{s}} - (1-\alpha_1)(1-\alpha_0)^{\hat{s}-1}]} \\
&&& \text{for } s = \hat{s} > 1, \hat{s} < n, \\
&\leq \delta^n K_s (1-\beta) (1-\gamma)^{n-1} V(0) / N_s + \delta L_s (V(1) - V(1, n-1)) / N_s \\
&&& \text{for } \hat{s} < s,
\end{aligned}$$

where  $V(1) - V(1, n-1)$  can be computed from the first formulas and

$$t = \left\lfloor \frac{n}{\hat{s}} \right\rfloor.$$

Proof: If  $s+n \leq \hat{s}$ , then by Equations (5.5) and (5.6),

$$V(s) - V(s,n) \leq \delta^n N_{n+s} V(n+s)/N_s.$$

If  $\hat{s}-n < s < \hat{s}$ , then

$$V(s) = \sum_{i=0}^{\hat{s}-s-1} \delta^i N_{s+i}/N_s + \delta^{\hat{s}-s} V(\hat{s}) N_{\hat{s}}/N_s$$

and

$$V(s,n) \geq \sum_{i=0}^{\hat{s}-s-1} \delta^i N_{s+i}/N_s + \delta^{\hat{s}-s} V(\hat{s}, n-\hat{s}+s) N_{\hat{s}}/N_s$$

So,

$$V(s) - V(s,n) \leq \delta^{\hat{s}-s} N_{\hat{s}} (V(\hat{s}) - V(\hat{s}, n-\hat{s}+s))/N_s.$$

To prove the third part of the inequality, note that

$$V(\hat{s}) - V(\hat{s},n) \leq \delta^n K_{\hat{s}} (1-\beta) (1-\gamma)^{n-1} V(0)/N_{\hat{s}} + \delta L_{\hat{s}} (V(1) - V(1,n-1))/N_{\hat{s}}. \quad (5.8)$$

If  $s = \hat{s} \geq n$  (i.e.,  $1 \leq \hat{s} - n + 1$ ), then by the first formula

$$V(\hat{s}) - V(\hat{s},n) \leq \delta^n [K_{\hat{s}} (1-\beta) (1-\gamma)^{n-1} V(0) + L_{\hat{s}} N_{\hat{s}} V(n)]/N_{\hat{s}}.$$

If  $s = \hat{s} < n$  (i.e.,  $\hat{s}-n+1 < 1 < \hat{s}$ ), then by the second formula

$$V(1) - V(1,n-1) \leq \delta^{\hat{s}-1} N_{\hat{s}} (V(\hat{s}) - V(\hat{s}, n-\hat{s}))$$

substituting into Equation (5.8),

$$V(\hat{s}) - V(\hat{s}, n) \leq \delta^n K_{\hat{s}} (1-\beta)(1-\gamma)^{n-1} V(0)/N_{\hat{s}} + \delta^{\hat{s}} L_{\hat{s}} (V(\hat{s}) - V(\hat{s}, n-\hat{s})) .$$

Repeating this procedure  $t$  times

$$V(\hat{s}) - V(\hat{s}, n) \leq \delta^n K_{\hat{s}} (1-\beta)(1-\gamma)^{n-1} V(0) \sum_{k=0}^{t-1} (\delta^{\hat{s}} L_{\hat{s}} (\delta(1-\gamma))^{-\hat{s}})^k / N_{\hat{s}} \\ + (\delta^{\hat{s}} L_{\hat{s}})^t (V(\hat{s}) - V(\hat{s}, n - t\hat{s}))$$

where  $t = \left\lfloor \frac{n}{\hat{s}} \right\rfloor$ . The result regarding  $V(\hat{s}) - V(\hat{s}, n)$  can be obtained by simplifying this.

For  $s > s(\infty)$ ,

$$V(s) - V(s, n) \leq \delta K_s (1-\beta)(V(0) - V(0, n-1))/N_s + \delta L_s (V(1) - V(1, n-1))/N_s$$

Since  $V(0) - V(0, n-1) = (\delta(1-\gamma))^{n-1} V(0)$ , the proof is completed.  $\square$

When the optimal ongoing policy is all-or-none, i.e., inspect if and only if  $s \leq z(n)$ , the bound for  $V(s) - V(s, n)$  can be computed as follows. If  $s \leq z(\infty)$ , apply Theorem 5.9, otherwise, use Theorem 5.8.

When the optimal initiation policy is to periodically inspect, the convergence rate is also geometric and can be computed by the following theorem.

Theorem 5.11. Denote  $t(\infty)$  by  $t$ . Then

$$V(s^*) - V(s^*, n) \leq \delta^n N_{(n+s)^*} V((n+s)^*) / N_{s^*} \quad \text{for} \quad s \leq t-n .$$



$$\begin{aligned}
V(s^*) - V(s^*,n) &\leq \delta^n K_{t^*} (1-\beta) (1-\gamma)^{n-t+s-1} V(0) / N_{s^*} \\
&\quad + \delta^{t-s+1} L_{s^*} (V(1) - V(1, n-t+s-1)) / N_{s^*}
\end{aligned}$$

for  $t-n < s < t$ .

$$\begin{aligned}
V(s^*) - V(s^*,n) &\leq \delta^n K_{s^*} (1-\beta) (1-\gamma)^{n-1} V(0) / N_{s^*} \\
&\quad + \delta L_{s^*} (V(1) - V(1, n-1)) / N_{s^*}
\end{aligned}$$

for  $t \leq s$ .

Proof: For  $s \leq t-n$ , the inequality follows from Equations (5.5) and (5.6).

For  $t-n < s < t$ ,

$$V(s^*) = \sum_{i=0}^{t-s-1} \delta^i N_{(i+s)^*} / N_{s^*} + \delta^{t-s} V(t^*) N_{t^*} / N_{s^*}$$

$$V(s^*,n) \geq \sum_{i=0}^{t-s-1} \delta^i N_{(i+s)^*} / N_{s^*} + \delta^{t-s} V(t^*, n-t+s) N_{t^*} / N_{s^*},$$

So

$$\begin{aligned}
V(s^*) - V(s^*,n) &\leq \delta^{t-s} N_{t^*} (V(t^*) - V(t^*, n-t+s)) / N_{s^*} \\
&\leq \delta^{t-s} (K_{t^*} (1-\beta) (V(0) - V(0, n-t+s-1)) \\
&\quad + \delta L_{t^*} (V(1) - V(1, n-t+s-1))) / N_{s^*}
\end{aligned}$$

$$\begin{aligned}
&= \delta^{t-s+1} (K_{t^*} (1-\beta) (\delta(1-\gamma))^{n-t+s-1} / (1-\delta(1-\gamma))) \\
&\quad + L_{t^*} (V(1) - V(1, n-t+s-1)) / N_{s^*}
\end{aligned}$$

For  $t < s$ , the result follows from Proposition 3.3. □

For all cases, when  $n$  approaches infinity, the bounds will approach zero, and  $V(\delta) - V(\delta, n) \leq \delta^n C$  (for some constant  $C$ ). Actually, these bounds will converge faster than this.

### 5.3. Computational Results

Tables 3-6 present sample calculations for four different sets of parameter values. Each set corresponds to one of the four regions found to determine the form of the optimal inspection policies for an old device.

The most typical case would be when  $\alpha_0 > \alpha_1 \beta$  and  $\gamma > \alpha_1 \beta$ . Here a periodic policy is optimal. Sample numerical values of  $V(\delta, n)$  are given in Table 3, and  $D(s)$  and  $D(s^*)$  are plotted in Figure 1.

When  $\alpha_0 < \alpha_1 \beta$  and  $\gamma < \alpha_1 \beta$  an all-or-none policy is optimal. In this case the maximal expected remaining lifetime is increasing in  $s$  for an old device. Table 4 gives sample numerical values of  $V(\delta, n)$ , and Figure 2 shows  $D(s)$  and  $D(s^*)$ .

When  $\alpha_0 \geq \alpha_1 \beta \geq \gamma$ , it is optimal to always inspect an old device and to periodically inspect a new device; when  $\alpha_0 \leq \alpha_1 \beta \leq \gamma$ , it is optimal to never inspect either a new or an old device. Sample values of  $V(\delta, n)$  for these two ranges are shown in Table 5 and Table 6.

TABLE 3: SAMPLE RUN FOR  $\alpha_0 > \alpha_1\beta$  AND  $\gamma > \alpha_1\beta$

( $\alpha_0 = .10$   $\alpha_1 = .20$   $\beta = .30$   $\gamma = .15$   $\delta = .90$ )

FOR AN OLD DEVICE ( OPTIMAL POLICY: PERIODIC )

s \ n	<u>V(s,n)</u>										s(n)
	0	1	2	3	4	5	10	20	30	60	
2	1.76500	1.84600	1.83681	1.83018	1.82532	1.82169	1.81318	1.81025	1.81002	1.81000	1
3	2.35023	2.55394	2.53281	2.51932	2.50942	2.50204	2.48472	2.47877	2.47829	2.47825	2
4	2.79792	3.14275	3.10819	3.08705	3.07231	3.06133	3.03554	3.02668	3.02597	3.02591	3
5	3.14041	3.62953	3.58327	3.55289	3.53367	3.51937	3.48577	3.47422	3.47330	3.47322	3
10	3.96321	5.03271	4.94022	4.87648	4.83397	4.80811	4.74737	4.72649	4.72483	4.72468	4
20	4.23527	5.71344	5.59221	5.50765	5.44898	5.40948	5.33199	5.30534	5.30322	5.30304	5
30	4.25394	5.79289	5.66767	5.58017	5.51922	5.47783	5.39732	5.36988	5.36769	5.36750	6
60	4.25532	5.80273	5.67699	5.58908	5.52779	5.48613	5.40519	5.37764	5.37545	5.37525	6
inf	4.25532	5.80273	5.67699	5.58908	5.52780	5.48613	5.40519	5.37764	5.37545	5.37526	6

FOR A NEW DEVICE ( OPTIMAL POLICY: PERIODIC )

s \ n	<u>V(s*,n)</u>										t(n)
	0	1	2	3	4	5	10	20	30	60	
2	1.76500	1.87300	1.85546	1.84348	1.83501	1.82887	1.81500	1.81039	1.81003	1.81000	1
3	2.35023	2.61982	2.57703	2.54780	2.52914	2.51665	2.48842	2.47905	2.47831	2.47825	4
4	2.79792	3.24975	3.17955	3.13328	3.10167	3.08307	3.04104	3.02711	3.02601	3.02591	4
5	3.14041	3.77574	3.68041	3.61620	3.57438	3.54769	3.49294	3.47477	3.47334	3.47322	5
10	3.96321	5.31545	5.13003	5.00683	4.92269	4.86495	4.76033	4.72750	4.72491	4.72468	6
20	4.23527	6.07870	5.84053	5.68004	5.56910	5.49170	5.34852	5.30663	5.30333	5.30304	7
30	4.25394	6.16946	5.92405	5.75838	5.64364	5.56339	5.41434	5.37120	5.36780	5.36750	7
60	4.25532	6.18064	5.93430	5.76799	5.65275	5.57210	5.42228	5.37897	5.37555	5.37525	7
inf	4.25532	6.18065	5.93431	5.76799	5.65275	5.57211	5.42228	5.37897	5.37556	5.37526	7

TABLE 4: SAMPLE RUN FOR  $\gamma < \alpha_0 < \alpha_1 \beta$

( $\alpha_0 = .05$   $\alpha_1 = .20$   $\beta = .40$   $\gamma = .03$   $\delta = .90$ )

FOR AN OLD DEVICE ( OPTIMAL POLICY: ALL-OR-NONE )

		<u>V(s,n)</u>										
<u>s</u>	<u>n</u>	0	1	2	3	4	5	10	20	30	60	z(n)
2		1.87300	1.82800	1.83739	1.84364	1.84773	1.85037	1.85453	1.85500	1.85500	1.85500	inf
3		2.63513	2.52136	2.54385	2.55883	2.56862	2.57494	2.58490	2.58601	2.58602	2.58603	0
4		3.30047	3.10631	3.14274	3.16700	3.18286	3.19309	3.20922	3.21103	3.21105	3.21105	0
5		3.88131	3.60219	3.65202	3.68520	3.70688	3.72088	3.74295	3.74542	3.74545	3.74545	0
10		5.84942	5.17500	5.27301	5.33828	5.38093	5.40846	5.45186	5.45673	5.45678	5.45678	0
20		7.35344	6.24245	6.34878	6.43657	6.49394	6.53098	6.58936	6.59591	6.59598	6.59598	1
30		7.74016	6.51412	6.57331	6.66582	6.72628	6.76531	6.82683	6.83373	6.83380	6.83380	1
60		7.87174	6.60694	6.65117	6.72576	6.78703	6.82658	6.88891	6.89591	6.89598	6.89598	2
inf		7.87402	6.60855	6.65279	6.72631	6.78758	6.82714	6.88948	6.89648	6.89655	6.89655	2

FOR A NEW DEVICE ( OPTIMAL POLICY: NEVER INSPECT )

		<u>V(s*,n)</u>										
<u>s</u>	<u>n</u>	0	1	2	3	4	5	10	20	30	60	t(n)
2		1.87300	1.88200	1.87153	1.86524	1.86139	1.85901	1.85540	1.85500	1.85500	1.85500	1
3		2.63513	2.65069	2.62562	2.61056	2.60133	2.59562	2.58698	2.58603	2.58603	2.58603	inf
4		3.30047	3.31579	3.27518	3.25078	3.23585	3.22660	3.21260	3.21107	3.21105	3.21105	inf
5		3.88131	3.88871	3.83316	3.79979	3.77937	3.76671	3.74756	3.74547	3.74545	3.74545	inf
10		5.84942	5.73856	5.62930	5.56367	5.52349	5.49860	5.46094	5.45682	5.45678	5.45678	inf
20		7.35344	6.97501	6.82804	6.73976	6.68571	6.65223	6.60157	6.59603	6.59598	6.59598	inf
30		7.74016	7.23322	7.07834	6.98532	6.92836	6.89308	6.83969	6.83386	6.83380	6.83380	inf
60		7.87174	7.30073	7.14379	7.04952	6.99181	6.95605	6.90195	6.89604	6.89598	6.89598	inf
inf		7.87402	7.30135	7.14439	7.05011	6.99239	6.95663	6.90252	6.89661	6.89655	6.89655	inf

TABLE 5: SAMPLE RUN FOR  $\gamma < \alpha_1 \beta < \alpha_0$

( $\alpha_0 = .10$     $\alpha_1 = .20$     $\beta = .40$     $\gamma = .05$     $\delta = .90$ )

FOR AN OLD DEVICE ( OPTIMAL POLICY: ALWAYS INSPECT )

s \ n	<u>V(s,n)</u>										s(n)
	0	1	2	3	4	5	10	20	30	60	
2	1.85500	1.82800	1.82174	1.81771	1.81509	1.81338	1.81044	1.81001	1.81000	1.81000	1
3	2.58603	2.51650	2.50531	2.49811	2.49343	2.49036	2.48511	2.48434	2.48433	2.48433	1
4	3.21105	3.09117	3.07607	3.06636	3.06004	3.05589	3.04881	3.04777	3.04775	3.04775	1
5	3.74545	3.57244	3.55421	3.54249	3.53486	3.52986	3.52131	3.52005	3.52003	3.52003	1
10	5.45678	5.05297	5.02605	5.00875	4.99749	4.99009	4.97747	4.97562	4.97558	4.97558	1
20	6.59598	5.99007	5.95842	5.93807	5.92483	5.91614	5.90130	5.89912	5.89908	5.89908	1
30	6.83380	6.18129	6.14875	6.12783	6.11422	6.10528	6.09003	6.08778	6.08774	6.08774	1
60	6.89598	6.23107	6.19830	6.17724	6.16353	6.15453	6.13917	6.13690	6.13687	6.13686	1
inf	6.89655	6.23153	6.19876	6.17769	6.16398	6.15498	6.13962	6.13736	6.13732	6.13732	1

FOR A NEW DEVICE ( OPTIMAL POLICY: PERIODIC )

s \ n	<u>V(s*,n)</u>										t(n)
	0	1	2	3	4	5	10	20	30	60	
2	1.85500	1.86400	1.84375	1.83160	1.82403	1.81919	1.81118	1.81002	1.81000	1.81000	1
3	2.58603	2.59300	2.54541	2.52294	2.50940	2.50075	2.48643	2.48436	2.48433	2.48433	3
4	3.21105	3.19924	3.12916	3.09985	3.08158	3.06992	3.05059	3.04780	3.04775	3.04775	2
5	3.74545	3.70360	3.61829	3.58292	3.56087	3.54679	3.52345	3.52009	3.52003	3.52003	2
10	5.45678	5.24408	5.12068	5.06845	5.03588	5.01510	4.98064	4.97567	4.97559	4.97558	2
20	6.59598	6.20821	6.06968	6.00826	5.96998	5.94554	5.90502	5.89918	5.89908	5.89908	2
30	6.83380	6.40390	6.26314	6.20000	6.16064	6.13551	6.09385	6.08785	6.08774	6.08774	2
60	6.89598	6.45480	6.31350	6.24991	6.21027	6.18497	6.14302	6.13697	6.13687	6.13686	2
inf	6.89655	6.45526	6.31396	6.25037	6.21073	6.18542	6.14347	6.13742	6.13732	6.13732	2

TABLE 6: SAMPLE RUN FOR  $\alpha_0 < \alpha_1 \beta < \gamma$

( $\alpha_0 = .05$   $\alpha_1 = .20$   $\beta = .40$   $\gamma = .10$   $\delta = .90$ )

FOR AN OLD DEVICE ( OPTIMAL POLICY: NEVER INSPECT )

		$V(s,n)$										
$s \backslash n$	0	1	2	3	4	5	10	20	30	60	$s(n)$	
2	1.81000	1.82800	1.83739	1.84364	1.84773	1.85037	1.85453	1.85500	1.85500	1.85500	1	
3	2.46610	2.52136	2.54385	2.55883	2.56862	2.57494	2.58490	2.58601	2.58602	2.58603	inf	
4	2.99754	3.10631	3.14274	3.16700	3.18286	3.19309	3.20922	3.21103	3.21105	3.21105	inf	
5	3.42801	3.60219	3.65202	3.68520	3.70688	3.72088	3.74295	3.74542	3.74545	3.74545	inf	
10	4.62328	5.17500	5.27301	5.33828	5.38093	5.40846	5.45186	5.45673	5.45678	5.45678	inf	
20	5.18536	6.21694	6.34878	6.43657	6.49394	6.53098	6.58936	6.59591	6.59598	6.59598	inf	
30	5.25370	6.43438	6.57331	6.66582	6.72628	6.76531	6.82683	6.83373	6.83380	6.83380	inf	
60	5.26314	6.49123	6.63201	6.72576	6.78703	6.82658	6.88891	6.89591	6.89598	6.89598	inf	
inf	5.26316	6.49175	6.63255	6.72631	6.78758	6.82714	6.88948	6.89648	6.89655	6.89655	inf	

FOR A NEW DEVICE ( OPTIMAL POLICY: NEVER INSPECT )

		$V(s^*,n)$										
$s \backslash n$	0	1	2	3	4	5	10	20	30	60	$t(n)$	
2	1.81000	1.88200	1.87153	1.86524	1.86139	1.85901	1.85540	1.85500	1.85500	1.85500	1	
3	2.46610	2.65069	2.62562	2.61056	2.60133	2.59562	2.58698	2.58603	2.58603	2.58603	inf	
4	2.99754	3.31579	3.27518	3.25078	3.23585	3.22660	3.21260	3.21107	3.21105	3.21105	inf	
5	3.42801	3.88871	3.83316	3.79979	3.77937	3.76671	3.74756	3.74547	3.74545	3.74545	inf	
10	4.62328	5.73856	5.62930	5.56367	5.52349	5.49860	5.46094	5.45682	5.45678	5.45678	inf	
20	5.18536	6.97501	6.82804	6.73976	6.68571	6.65223	6.60157	6.59603	6.59598	6.59598	inf	
30	5.25370	7.23322	7.07834	6.98532	6.92836	6.89308	6.83969	6.83386	6.83380	6.83380	inf	
60	5.26314	7.30073	7.14379	7.04952	6.99181	6.95605	6.90195	6.89604	6.89598	6.89598	inf	
inf	5.26316	7.30135	7.14439	7.05011	6.99239	6.95663	6.90252	6.89661	6.89655	6.89655	inf	

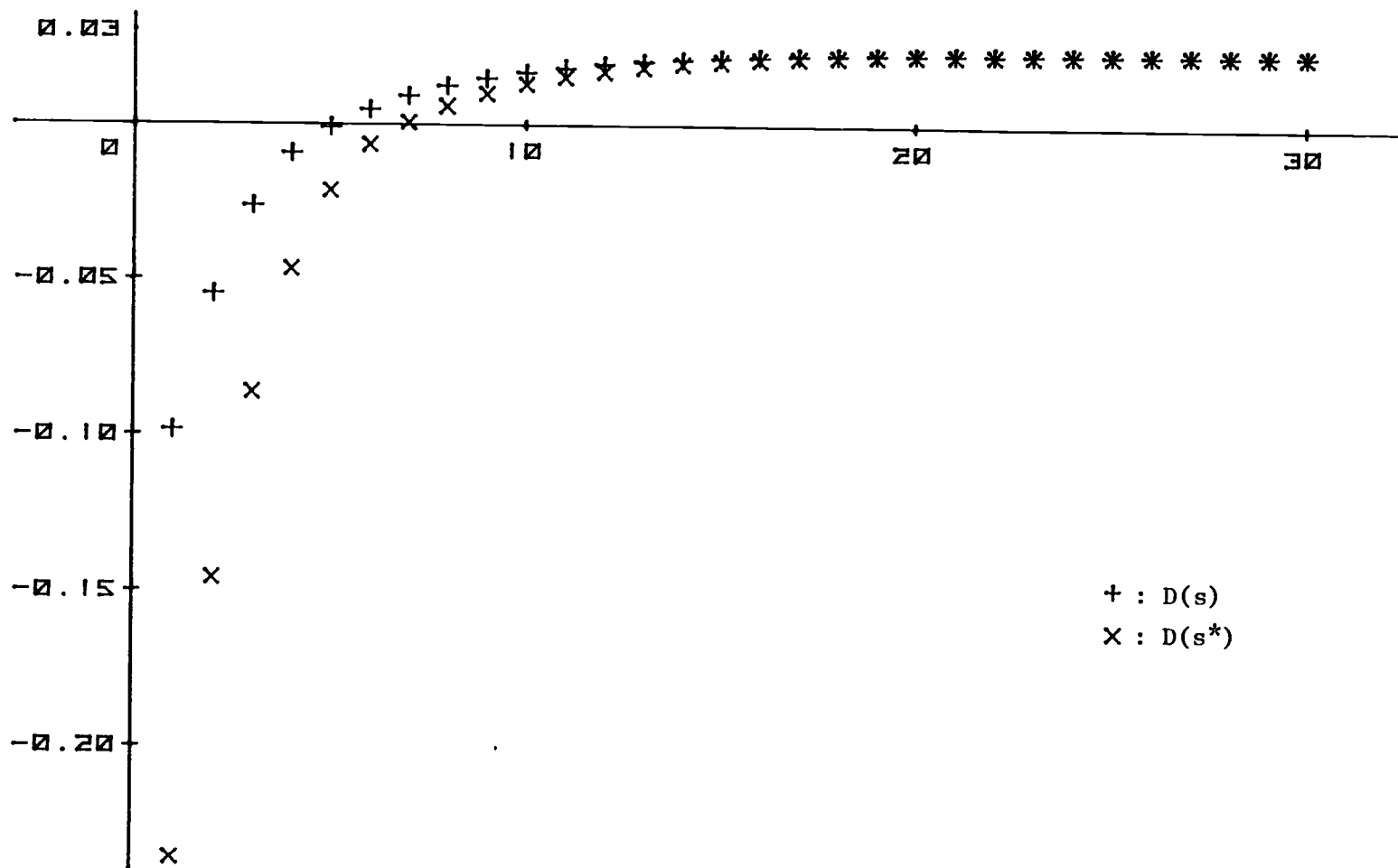


Figure 1. Plots of  $D(s)$  and  $D(s^*)$ . ( $\alpha_0 = .10$ ,  $\alpha_1 = .20$ ,  $\beta = .30$ ,  $\gamma = .15$ ,  $\delta = .90$ )

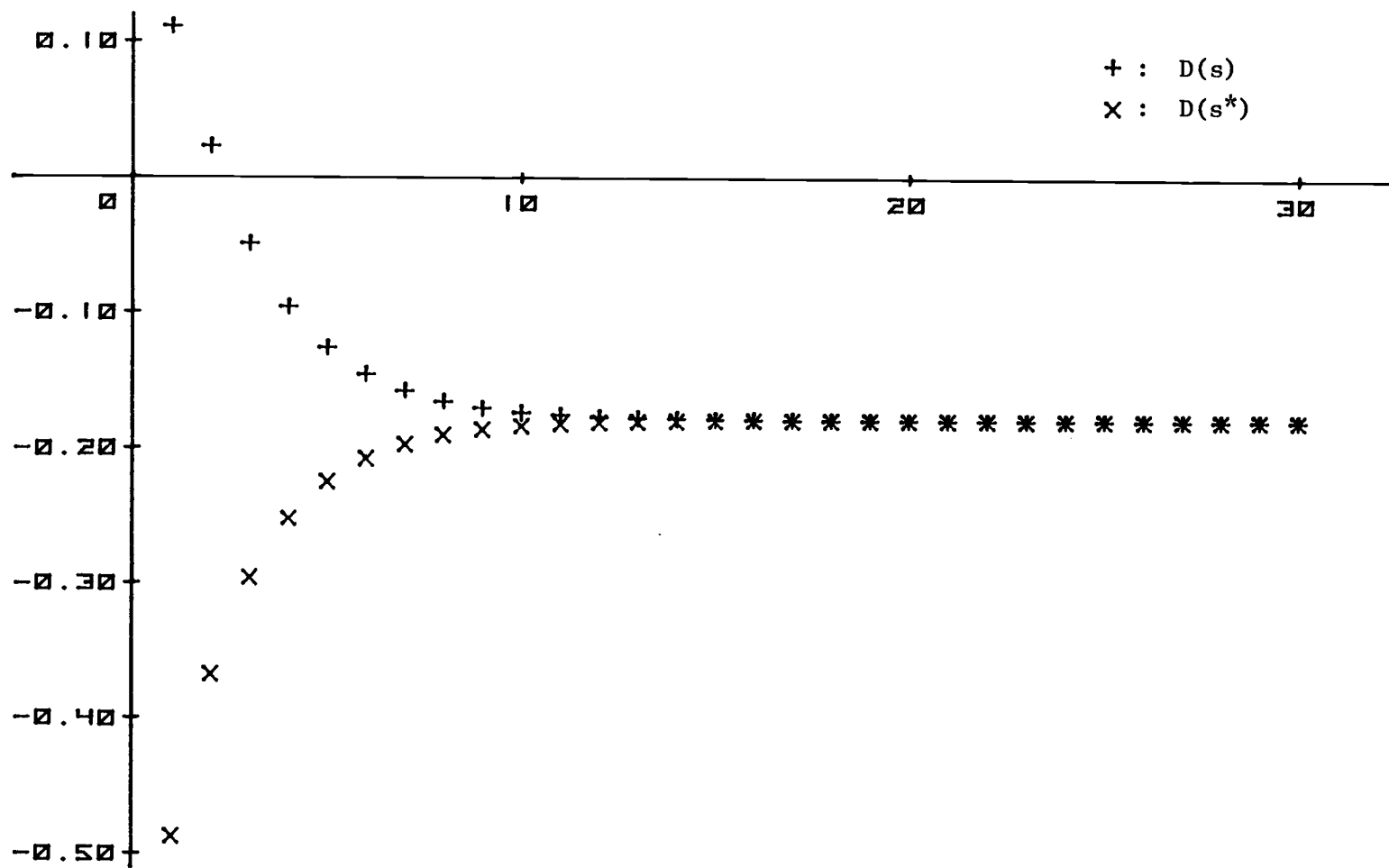


Figure 2: Plots of  $D(s)$  and  $D(s^*)$ . ( $\alpha_0 = .05$ ,  $\alpha_1 = .20$ ,  $\beta = .40$ ,  $\gamma = .03$ ,  $\delta = .90$ )



## VI. Summary and Extensions

In this chapter, what has been done for this simple inspection model and what interesting work remains to be done will be briefly discussed.

### 6.1. Summary and Conclusions

A partially observable Markov decision process model of hazardous inspections has been developed and transformed to a completely observable process. The transformed states summarize all the information which can be used to control the inspection process.

For an old device (i.e., one which has been inspected), Butler (1979) has analyzed the optimal inspection policies completely. In this thesis, the optimal inspection initiation policies for a new device have been analyzed and shown to have the form of periodic inspection (including to never inspect as a special case) regardless of the values of the transition parameters.

It has been shown that a new device is better than an old device of the same "age" (time since true state last known), under the criterion of maximizing the expected remaining lifetime. It has also been shown that if it is optimal to inspect a new device which was put into service  $s$  periods ago, then it is optimal to inspect an old device which was inspected and found to be OK  $s$  periods ago. This means that for the infinite-horizon case, if it is optimal to inspect a new device which is in state  $s^*$ , then it is optimal to inspect it every  $s$  periods with  $s \leq s^*$  thereafter, i.e., no policies which only inspect a new device and never an old device will be optimal.

Efficient methods to compute the infinite-horizon maximal expected remaining lifetime of either a new or an old device have been introduced. These methods either compute  $V(\delta)$  directly (if optimal to never inspect or to always inspect), or first compute an increasing sequence and then  $V(\delta)$  immediately. The methods are far more efficient than simply approximating the infinite-horizon  $V(\delta)$  by  $V(\delta, n)$  for a large value of  $n$ , because the latter requires values of  $V(m, k)$  for all  $m, k$  such that  $m+k \leq n+1$  and  $m \leq n$ .

The survival distribution of a device can be computed assuming a periodic-inspection policy is followed. So, once the form of the optimal inspection policies and  $s(n)$ ,  $t(n)$ , and  $z(n)$  are determined, the distribution function of the life of a device can be derived. For the infinite horizon, the survival distributions have been derived in Chapter IV.

## 6.2. Extensions and Future Directions

Research on hazardous inspections is very limited now. (There are only two published papers, Wattanapanom and Shaw (1979) and Butler (1979).) There is much interesting work which can be done, even on this simple model.

So far, only a "single-cycle" model has been considered. The device cannot enter a better state from an inferior state, and when it is partially failed, the decision process is ended because either action has the same effect. So another action, for example repair or replacement, seems a very desirable addition to the model. Such a model has been constructed as follows.

The true and observed state spaces remain the same. One action, repair, is added to the action space. Because this is not a single-cycle system anymore, the criterion of maximizing the expected remaining lifetime is not adequate. Instead, minimizing the expected discounted cost might be used. Secondary failures are not considered. The possible actions for a device which is in observed state  $s \geq 1$  are 0 (no action), 1 (inspection), and 2 (repair); for a device in observed state 0, the possible actions are 0 and 2; for a failed device, only repair is permitted. The costs are defined as follows.

		Actions		
		0	1	2
States	$s \geq 1$	$C_{01}$	$C_{11}$	$C_{21}$
	$s = 0$	$C_{02}$	—	$C_{22}$
	$s = -1$	—	—	$C_{23}$

It is assumed that  $C_{ij} \leq C_{ik}$  for  $i=0, 2$ , and  $j < k$ , and  $C_{ik} \leq C_{jk}$  for  $k=1, 2$ , and  $i < j$ .

The repair action is instantaneous and sends the device to the OK state. Thus the transition matrix for the true states when action 2 is taken is defined as  $Q_{i1}(2) = 1$  and  $Q_{ij}(2) = 0$  for  $i=1,2,3,4$ , and  $j \neq 1$ . For the observed states,  $P_{i1^*}(2) = 1$  and  $P_{ij}(2) = 0$  for  $i \in S$  and  $j \neq 1^*$ . The recursive formulas for  $V(s,n)$  are

$$\begin{aligned}
V(\delta, n) = \min\{ & C_{01} + \delta(K_\delta(1-\beta) + L_\delta)V(\delta+1, n-1)/N_\delta + \delta\beta K_\delta V(-1, n-1)/N_\delta, \\
& C_{11} + \delta(K_\delta(1-\beta)V(0, n-1) + L_\delta V(1, n-1))/N_\delta + \delta\beta K_\delta V(-1, n-1)/N_\delta, \\
& C_{21} + \delta V(1^*, n-1)\} \quad \text{for } \delta \geq 1,
\end{aligned}$$

$$V(0, n) = \min\{C_{02} + \delta(1-\gamma)V(0, n-1) + \delta\gamma V(-1, n-1), C_{22} + \delta V(1^*, n-1)\},$$

$$V(-1, n) = C_{23} + \delta V(1^*, n-1)$$

The variables  $K_\delta$ ,  $L_\delta$ , and  $N_\delta$  are the same as in the original model. The cost function  $V(-1, n)$  can be shown to be at least as big as  $V(\delta, n)$  for all  $\delta \in S$ . So, as minimizing the discounted costs is the criterion, the objective is still (roughly) to avoid entering the failed state.

This model has been investigated by computer runs for many sets of parameters. In most of the cases,  $V(\delta, n)$  is either increasing or decreasing in  $\delta$ , but counterexamples have been found and this very nice monotone property of the original process does not hold in general.

Intuitively, the optimal policies should take the form of periodic inspection or repair, but this has not been proved. A guess is that the state space can be divided into at most five regions of actions.

In the original model, no hazard is incurred by inspection when the device is partially failed. The inspection can be made hazardous by substituting two parameters for the single parameter  $\beta$  as follows.

	Q(0)					Q(1)			
	(OK)	(UPF)	(DPF)	(F)		(OK)	(UPF)	(DPF)	(F)
(OK)	$1-\alpha_0$	$\alpha_0$	0	0	(OK)	$1-\alpha_1$	$\alpha_1$	0	0
(UPF)	0	$1-\beta_0$	0	$\beta_0$	(UPF)	0	0	$1-\beta_1$	$\beta_1$
(DPF)	0	0	$1-\gamma$	$\gamma$	(DPF)	0	0	$1-\gamma$	$\gamma$
(F)	0	0	0	1	(F)	0	0	0	1

It is assumed that  $\beta_1 > \beta_0 > \gamma$ . This is an interesting model but no strong results have been obtained so far.

Some other interesting work which remains to be done on hazardous-inspection models are: i) latency between time of partial failure and its detectability, ii) long-run inspection hazards (either increasing or cumulative), iii) imperfect information revealed by inspection, iv) duration of inspection is not negligible, v) more device states, vi) multiple grades of inspection.

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