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Integral Input-to-output Stability for One Class of Discontinuous Dynamical Systems

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Abstract: It is well known that integral input-to-output stability(IIOS) is weaker than input-to-output stability(IOS).In this paper, IIOS problems for one class of discontinuous dynamical systems are considered. Piecewise smooth IIOS-Lyapunov functions are adopted. Furthermore, interconnections of discontinuous dynamical systems are studied. The results for a class of discontinuous dynamical systems and interconnection of discontinuous dynamical systems' IIOS character are shown, respectively.

Introduction

During the last twenty years,the notion of input-to-state stability(ISS) became a fundamental concept upon control theory. A lot of good results were obtained([1,2,4]).In 2008, W.P.M.H.Heemels and S.Weiland extended the input- to-state stability framework to continuous-time discontinuous dynamical systems adopting piecewise smooth ISS-Lyapunov functions([3]).One of the most profound contribution of this paper is the extended Filippov solution.In the sense of extended Filippov solution,authors discussed interconnected systems' ISS effectively.

Another important notion of stability is so-called input to output stability (IOS, for short) to mean that the output (as opposed to the full state) must be eventually small, no matter what the initial conditions, if future inputs are small([1,7]).It is well known that integral input-to-output stability (IIOS) is weaker than input-to-output stability(IOS).In this paper,IIOS problems for one class of discontinuous dynamical systems are considered.Piecewise smooth IIOS-Lyapunov functions are adopted.Furthermore, interconnections of discontinuous dynamical systems are studied.The results for a class of discontinuous dynamical systems and interconnections of discontinuous dynamical systems' IIOS character are shown,respectively.

Main results

Consider the discontinuous differential equation with outputs:

$$\dot{x}(t) = f(x(t), u(t)),\ y = h(x) \tag{1}$$

with $x(t) \in R^n$ and $u(t) \in R^m$, the state and control input at time $t \in R_+$, respectively. Assume the output map $h: R^n \to R^p$ is continuous and $h(0) = 0$.For each $\xi \in R^n$ and each input u,we let

$y(t,\xi,u)$ be the output function of the system i.e. $y(t,\xi,u)=h(t,\xi,u)$.The vector field f is assumed to be a piecewise continuous function from $R^n \times R^m$ to R^n in the following sense

$$f(x,u)=f_i(x,u)$$

when $(x,u)^T \in \Omega_i, i \in \overline{N}=\{1,2,\cdots,N\}$ .Here $\Omega_1,\cdots,\Omega_N$ form a partitioning of the space $R^n \times R^m$,which satisfies that

1. $\operatorname{int}\Omega_i \bigcap \operatorname{int}\Omega_j = \Phi$,when $i \neq j$.

2. $\bigcup_{i=1}^{N} \Omega_i = R^n \times R^m$.

3. $f_i : \Omega_i \to R^n$ are locally Lipschitz continuous functions on their domains $\Omega_i, i \in \overline{N}$.

Note that $\operatorname{int}\Omega$ denotes $\Omega's$ interior in this paper.

Now recall the definition of extended Filippov solutions for system (1).

Definition 1 A function $x:[a,b] \to R^n$ is an extended Filippov solution to system (1) for $u \in L_\infty([a,b] \to R^m)$,if x is locally absolutely continuous and satisfies $\dot{x}(t) \in C_f(x(t),u(t))$ for almost all $t \in [a,b]$. Here

$$C_f(x(t),u(t)) := co\{f_i(x,u) \mid i \in I(x,u)\} \text{ and } I(x,u) := \{i \in \overline{N} \mid (x,u)^T \in \Omega_i\}.$$

The extended Filippov solutions can be used to discuss the input to state stability for system (1) and interconnections of the discontinuous dynamical systems effectively(W.P.M.H.Heemels and S.Weiland 2008).In this paper, the integral input-to-output stability for system (1) is defined as follows.

Definition 2 The system (1) is said to be integral input-to-output stability(IIOS) if there exist a KL-function β and a K_∞-function α ,K-function γ such that for each intial condition $x(0)=x_0$ and each L_∞-input function u,

1.All of extended Filippov solutions x of the system (1) exist on some interval $[0,T_{x_0,u})$.

2.All of output functions satisfy that

$$\alpha(|y(t,x_0,u)|) \leq \beta(|x_0|,t)+\int_0^t \gamma(|u(s)|)ds,\ \forall t \in [0,T_{x_0,u}).$$

Similar to D.Angeli and E.D.Sontag 2000,condition 2 is equivalent to that:there exist $\beta \in KL, \gamma_1, \gamma_2 \in K$ such that

$$|y(t,x_0,u)| \leq \beta(|x_0|,t)+\gamma_2\left(\int_0^t \gamma_1(|u(s)|)ds\right),\ \forall t \in [0,T_{x_0,u}).$$

In the study of the IIOS characters for systems IIOS-Lyapunov function are used usually.We have next definition.

Definition 3 A function V is said to be an IIOS-Lyapunov function for the system (1) if 1. V is a piecewise smooth Lyapunov function, which means that

$$V(x)=V_j(x), x\in\Gamma_j, j\in\bar{M}=\{1,2,\cdots,M\}$$

where $\Gamma_1,\cdots,\Gamma_M$ form a partitioning of the space R^n and the $V_j^{'}s$ are continuously different tiable functions on some open domain containing Γ_j .

2. V is continuous.

3.There exist functions $\psi_1,\psi_2\in K_\infty$ such that

$$\psi_1(h(x))\le V(x)\le\psi_2(|x|),\forall x\in R^n .$$

4.There exist K-function σ and a positive definite continuous function $\bar{\alpha}$ such that for all $x\in R^n$ and $u\in R^m$

$$\nabla V_j(x)f_i(x,u)\le-\bar{\alpha}(V(x))+\sigma(|u|),$$

$i\in I(x,u), j\in J(x):=\{j\in\bar{M}\mid x\in\Gamma_j\}$.

Here, V is contnuous which means when $x\in\Gamma_j\bigcap\Gamma_i$, we have $V_j(x)=V_i(x)$.

Now a key theorem is shown as follows.

Theorem 1 If there exists an IIOS-Lyapunov function V for system (1),then for all $u\in L_\infty[R_+\to R^m]$ and extended Filippov solution $x:[0,T_{x_0,u})\to R^n$ it holds that

$$\frac{d}{dt}V(x(t))\le-\bar{\alpha}(V(x(t)))+\sigma(|u(t)|),$$

for almost all time $t\in[0,T_{x_0,u})$.

Proof: Similar to W.P.M.H. Heemels and S.Weiland 2008, we have V is a locally Lipschitz function.Note that $x(t)$ is locally absolutely continuous. Therefore we obtain $V(x(t))$ is locally absolutely continuous.Obviously, $V(x(t))$ is differentiable almost everywhere with respect to $t\in[0,T_{x_0,u})$. Suppose that both $\frac{d}{dt}V(x(t))$ and $\dot{x}(t)$ exist at time t. Notice that

$$\dot{x}(t)=y\in C_f(x(t),u(t))=\sum_{i\in I(x,u)}\alpha_i f_i(x(t),u(t))$$

with some $\alpha_i\ge 0$. We obtained that

$$\begin{aligned}\frac{d}{dt}V(x(t))&=\lim_{h\downarrow 0}\frac{V(x(t)+hy)-V(x(t))}{h}\\&=\lim_{h\downarrow 0}\frac{V_j(x(t)+hy)-V_j(x(t))}{h}\end{aligned}$$

for all $j \in \overline{J}(x(t),y) = \bigcap_{h_0>0} \bigcup_{0<h<h_0} J(x(t)+hy) \subseteq J(x(t))$. Therefore, we can draw a conclusion that

$$\begin{aligned}
\frac{d}{dt}V(x(t)) &\le \max_{j\in J(x(t))} \lim_{h\to 0} \frac{V(x(t)+hy)-V(x(t))}{h} \\
&= \max_{j\in J(x(t))} \nabla V_j(x(t))y \\
&= \max_{j\in J(x(t))} \sum_{i\in I(x(t),u(t))} \alpha_i \nabla V_j(x(t)) f_i(x(t),u(t)) \\
&\le \max_{i\in I(x(t),u(t)),\, j\in J(x(t))} \nabla V_j(x(t)) f_i(x(t),u(t)) \\
&\le \max_{i\in I(x(t),u(t)),\, j\in J(x(t))} -\overline{\alpha}(V(x(t))) + \max_{i\in I(x(t),u(t)),\, j\in J(x(t))} \sigma(|u(t)|) \\
&= -\overline{\alpha}(V(x(t))) + \sigma(|u(t)|)
\end{aligned}$$

Furthermore, the proof is completed.

Now we will introduce a lemma which came from D.Angeli and E.D.Sontag 2000.

Lemma 1 Given any continuous positive definite function $\rho: R_{\ge 0} \to R_{\ge 0}$,there exists a KL -function β with the following property. For any $0 < \overline{t} \le \infty$,and for any(locally) absolutely continuous function $y:[0,\overline{t}) \to R_{\ge 0}$ and any measurable, locally essentially bounded function $v:[0,\overline{t}) \to R_{\ge 0}$ if

$$\dot{y}(t) \le -\rho(y(t)) + v(t)$$

holds for almost all $t \in [0,\overline{t})$,then the following estimate holds:

$$y(t) \le \beta(y(0),t) + \int_0^t 2v(s)ds$$

for all $t \in [0,\overline{t})$.

Following from lemma 1,next theorem is shown.

Theorem 2 If there exists an IIOS-Lyapunov function, then system (1) is integral input-to-output stable.

Consider the interconnections of discontinuous dyanmical systems as follows:

$$\sum^{a}: \dot{x}_a = f^a(x_a,x_b,u_a) = f^a_{i_a}(x_a,x_b,u_a),\ \ y_a = h_a(x_a) \tag{2}$$

when $(x_a,x_b,u_a)^T \in \Omega^a_{i_a}$ for $i_a \in \overline{N}^a$.

$$\sum^{b}: \dot{x}_b = f^b(x_a,x_b,u_a) = f^b_{i_b}(x_a,x_b,u_a),\ \ y_b = h_b(x_b) \tag{3}$$

when $(x_a,x_b,u_b)^T \in \Omega^b_{i_b}$ for $i_b \in \overline{N}^b$.

Here, the means of everything are omitted. Without loss of generality, we can combine system (2) with system (3) to a system as follows:

$$\dot{x} = f(x,u) = f_{(i_a,i_b)}(x,u),\ \ y = h(x) = (h_a(x_a),h_b(x_b))^T \tag{4}$$

when $(x,u)^T \in \Omega_{(i_a,i_b)}$,where

$$\Omega_{(i_a,i_b)} := \{(x,u)^T \mid (x_a,x_b,u_a)^T \in \Omega^a_{i_a}, and\ (x_a,x_b,u_b)^T \in \Omega^b_{i_b}\}.$$

Consider system (2) and (3), we have next theorem.

Theorem 3 Suppose that there exist IIOS Lyapunov functions V_a and V_b of the form definition 3 for system (2) and (3),respectively,that satisfy:

1.There exist functions $\psi_1^a,\psi_2^a,\psi_1^b,\psi_2^b \in K_\infty$ such that

$$\psi_1^a(h_a(x_a)) \le V^a(x_a) \le \psi_2^a(|x_a|),$$

$$\psi_1^b(h_b(x_b)) \le V^b(x_b) \le \psi_2^b(|x_b|),$$

2.There exist functions α^a,α^b positive definite and continuous, $\sigma^a,\sigma^b \in K$ such that

$$\nabla V^a_{j_a}(x_a) f^a_{j_a}(x_a,x_b,u_a) \le -\alpha^a(V^a(x_a)) + \sigma^a(|u_a|) + k_1\alpha^b(V^b(x_b)),$$

$$\nabla V^b_{j_b}(x_b) f^b_{j_b}(x_a,x_b,u_b) \le -\alpha^b(V^b(x_b)) + \sigma^a(|u_b|) + k_2\alpha^a(V^a(x_a))$$

with positive real number k_1,k_2 satisfying

$$k_1k_2 \le 1.$$

Then the interconnected system (2) and (3) is IIOS with output $y=(h_a(x_a),h_b(x_b))^T$, input $u=(u_a,u_b)^T$.

Proof: Notice $k_1k_2 \le 1$,we obtain that there exist positive real number λ_1,λ_2 such that

$$\frac{\lambda_1}{k_2} > \lambda_2 > k_1\lambda_1.$$

That means

$$\lambda_1k_1 < \lambda_2, \lambda_2k_2 < \lambda_1.$$

Choose Lyapunov function as follows:

$$V(x) = \lambda_1V^a(x_a) + \lambda_2V^b(x_b).$$

Choose K_∞ functions as follows:

$$\psi_1(x) = \min\{\lambda_1\psi_1^a(|x|)/\sqrt{2}, \lambda_2\psi_1^b(|x|)/\sqrt{2}\}$$
$$\psi_2(x) = \lambda_1\psi_2^b(|x|) + \lambda_2\psi_2^b(|x|)$$

We have

$$\psi_1(h(x)) \le V(x) \le \psi_2(|x|),$$

Then we only need to examine condition 4 of definition 3.Notice that

$$\nabla V_{(j_a,i_b)}(x) f_{(i_a,i_b)}(x,u) = \lambda_1 \nabla V_{j_a}^a(x_a) f_{j_a}^a(x_a,x_b,u_a) + \lambda_2 \nabla V_{j_b}^b(x_b) f_{j_b}^b(x_a,x_b,u_b) .$$

After tedious calculating, we have

$$\nabla V_{(j_a,i_b)}(x) f_{(i_a,i_b)}(x,u) \le -(\lambda_1 - \lambda_2 k_2)\alpha^a(V^a(x_a)) - (\lambda_2 - \lambda_1 k_1)\alpha^b(V^b(x_b))$$

$$+\lambda_1 \sigma^a(|u_a|) + \lambda_2 \sigma^b(|u_b|)$$

Let

$$\alpha(V(x)) = (\lambda_1 - \lambda_2 k_2)\alpha^a(V^a(x_a)) + (\lambda_2 - \lambda_1 k_1)\alpha^b(V^b(x_b))$$
$$\sigma(|u|) = \lambda_1 \sigma^a(|u|) + \lambda_2 \sigma^b(|u|)$$

we obtain

$$\nabla V_{(j_a,i_b)}(x) f_{(i_a,i_b)}(x,u) \le -\alpha(V(x)) + \sigma(|u|)$$

It is easy to deduce that α is positive definite and continuous, $\sigma \in K$.

Hence, V is an IIOS Lyapunov function for system (4).Furthermore, the proof is completed.

Conclusions

In this paper, IIOS problems for one class of discontinuous dynamical systems are considered. Piecewise smooth IIOS-Lyapunov functions are adopted. Furthermore, interconnections of disco ntinuous dynamical systems are studied. The results for a class of discontinuous dynamical systems and interconnections of discontinuous dynamical systems' IIOS character are shown,respectively.

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