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Inst. of Computer Science, Univ. of Białystok

Relations and Their Basic Properties

Edmund Woronowicz
Warsaw University
Białystok

Summary. We define here: mode Relation as a set of pairs, the domain, the codomain, and the field of relation; the empty and the identity relations, the composition of relations, the image and the inverse image of a set under a relation. Two predicates, = and ⊆, and three functions, ∪, ∩ and \ are redefined. Basic facts about the above mentioned notions are presented.

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The articles [2] and [1] provide the notation and terminology for this paper.

In this paper A, X, Y, Y_1, Y_2, a, b, c, d, x, y, z are sets.

Let I_1 be a set. We say that I_1 is relation-like if and only if:

(Def. 1) If $x \in I_1$, then there exist y, z such that $x = \langle y, z \rangle$.

Let us note that there exists a set which is relation-like and empty.

A binary relation is a relation-like set.

In the sequel P, P_1, P_2, Q, R, S denote binary relations.

We now state four propositions:

(3)[1] If $A \subseteq R$, then A is relation-like.

(4) $\{\langle x, y \rangle\}$ is relation-like.

(5) $\{\langle a, b \rangle, \langle c, d \rangle\}$ is relation-like.

(6) $[: X, Y :]$ is relation-like.

The scheme *Rel Existence* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a binary relation R such that for all x, y holds $\langle x, y \rangle \in R$ iff $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x,y]$

for all values of the parameters.

Let us consider P, R. Let us observe that $P = R$ if and only if:

(Def. 2) For all a, b holds $\langle a, b \rangle \in P$ iff $\langle a, b \rangle \in R$.

Let us consider P, R. One can verify the following observations:

* $P \cap R$ is relation-like,
* $P \cup R$ is relation-like, and

[1] The propositions (1) and (2) have been removed.

∗ $P \setminus R$ is relation-like.

Let us consider P, R. Let us observe that $P \subseteq R$ if and only if:

(Def. 3) For all a, b such that $\langle a, b\rangle \in P$ holds $\langle a, b\rangle \in R$.

The following two propositions are true:

(9)[2] $X \cap R$ is a binary relation.

(10) $R \setminus X$ is a binary relation.

Let us consider R. The functor $\operatorname{dom} R$ yields a set and is defined by:

(Def. 4) $x \in \operatorname{dom} R$ iff there exists y such that $\langle x, y\rangle \in R$.

Next we state three propositions:

(13)[3] $\operatorname{dom}(P \cup R) = \operatorname{dom} P \cup \operatorname{dom} R$.

(14) $\operatorname{dom}(P \cap R) \subseteq \operatorname{dom} P \cap \operatorname{dom} R$.

(15) $\operatorname{dom} P \setminus \operatorname{dom} R \subseteq \operatorname{dom}(P \setminus R)$.

Let us consider R. The functor $\operatorname{rng} R$ yields a set and is defined as follows:

(Def. 5) $y \in \operatorname{rng} R$ iff there exists x such that $\langle x, y\rangle \in R$.

The following propositions are true:

(18)[4] If $x \in \operatorname{dom} R$, then there exists y such that $y \in \operatorname{rng} R$.

(19) If $y \in \operatorname{rng} R$, then there exists x such that $x \in \operatorname{dom} R$.

(20) If $\langle x, y\rangle \in R$, then $x \in \operatorname{dom} R$ and $y \in \operatorname{rng} R$.

(21) $R \subseteq [\!: \operatorname{dom} R, \operatorname{rng} R :\!]$.

(22) $R \cap [\!: \operatorname{dom} R, \operatorname{rng} R :\!] = R$.

(23) If $R = \{\langle x, y\rangle\}$, then $\operatorname{dom} R = \{x\}$ and $\operatorname{rng} R = \{y\}$.

(24) If $R = \{\langle a, b\rangle, \langle x, y\rangle\}$, then $\operatorname{dom} R = \{a, x\}$ and $\operatorname{rng} R = \{b, y\}$.

(25) If $P \subseteq R$, then $\operatorname{dom} P \subseteq \operatorname{dom} R$ and $\operatorname{rng} P \subseteq \operatorname{rng} R$.

(26) $\operatorname{rng}(P \cup R) = \operatorname{rng} P \cup \operatorname{rng} R$.

(27) $\operatorname{rng}(P \cap R) \subseteq \operatorname{rng} P \cap \operatorname{rng} R$.

(28) $\operatorname{rng} P \setminus \operatorname{rng} R \subseteq \operatorname{rng}(P \setminus R)$.

Let us consider R. The functor $\operatorname{field} R$ yields a set and is defined by:

(Def. 6) $\operatorname{field} R = \operatorname{dom} R \cup \operatorname{rng} R$.

The following propositions are true:

(29) $\operatorname{dom} R \subseteq \operatorname{field} R$ and $\operatorname{rng} R \subseteq \operatorname{field} R$.

(30) If $\langle a, b\rangle \in R$, then $a \in \operatorname{field} R$ and $b \in \operatorname{field} R$.

(31) If $P \subseteq R$, then $\operatorname{field} P \subseteq \operatorname{field} R$.

[2] The propositions (7) and (8) have been removed.
[3] The propositions (11) and (12) have been removed.
[4] The propositions (16) and (17) have been removed.

(32) If $R = \{\langle x, y\rangle\}$, then $\mathrm{field}\,R = \{x, y\}$.

(33) $\mathrm{field}(P \cup R) = \mathrm{field}\,P \cup \mathrm{field}\,R$.

(34) $\mathrm{field}(P \cap R) \subseteq \mathrm{field}\,P \cap \mathrm{field}\,R$.

Let us consider R. The functor $R^{\smile}$ yields a binary relation and is defined as follows:

(Def. 7) $\langle x, y\rangle \in R^{\smile}$ iff $\langle y, x\rangle \in R$.

Let us note that the functor $R^{\smile}$ is involutive.
We now state several propositions:

(37)[5] $\mathrm{rng}\,R = \mathrm{dom}(R^{\smile})$ and $\mathrm{dom}\,R = \mathrm{rng}(R^{\smile})$.

(38) $\mathrm{field}\,R = \mathrm{field}(R^{\smile})$.

(39) $(P \cap R)^{\smile} = P^{\smile} \cap R^{\smile}$.

(40) $(P \cup R)^{\smile} = P^{\smile} \cup R^{\smile}$.

(41) $(P \setminus R)^{\smile} = P^{\smile} \setminus R^{\smile}$.

Let us consider P, R. The functor $P \cdot R$ yielding a binary relation is defined as follows:

(Def. 8) $\langle x, y\rangle \in P \cdot R$ iff there exists z such that $\langle x, z\rangle \in P$ and $\langle z, y\rangle \in R$.

One can prove the following propositions:

(44)[6] $\mathrm{dom}(P \cdot R) \subseteq \mathrm{dom}\,P$.

(45) $\mathrm{rng}(P \cdot R) \subseteq \mathrm{rng}\,R$.

(46) If $\mathrm{rng}\,R \subseteq \mathrm{dom}\,P$, then $\mathrm{dom}(R \cdot P) = \mathrm{dom}\,R$.

(47) If $\mathrm{dom}\,P \subseteq \mathrm{rng}\,R$, then $\mathrm{rng}(R \cdot P) = \mathrm{rng}\,P$.

(48) If $P \subseteq R$, then $Q \cdot P \subseteq Q \cdot R$.

(49) If $P \subseteq Q$, then $P \cdot R \subseteq Q \cdot R$.

(50) If $P \subseteq R$ and $Q \subseteq S$, then $P \cdot Q \subseteq R \cdot S$.

(51) $P \cdot (R \cup Q) = P \cdot R \cup P \cdot Q$.

(52) $P \cdot (R \cap Q) \subseteq (P \cdot R) \cap (P \cdot Q)$.

(53) $P \cdot R \setminus P \cdot Q \subseteq P \cdot (R \setminus Q)$.

(54) $(P \cdot R)^{\smile} = R^{\smile} \cdot P^{\smile}$.

(55) $(P \cdot R) \cdot Q = P \cdot (R \cdot Q)$.

Let us mention that every set which is empty is also relation-like.
Let us observe that $\emptyset$ is relation-like.
Let us observe that there exists a binary relation which is non empty.
Let f be a non empty binary relation. One can check that $\mathrm{dom}\,f$ is non empty and $\mathrm{rng}\,f$ is non empty.
Next we state three propositions:

(56) If for all x, y holds $\langle x, y\rangle \notin R$, then $R = \emptyset$.

[5] The propositions (35) and (36) have been removed.
[6] The propositions (42) and (43) have been removed.

(60)[7] $\operatorname{dom}\emptyset = \emptyset$ and $\operatorname{rng}\emptyset = \emptyset$.

(62)[8] $\emptyset \cdot R = \emptyset$ and $R \cdot \emptyset = \emptyset$.

Let X be an empty set. Observe that $\operatorname{dom} X$ is empty and $\operatorname{rng} X$ is empty. Let us consider R. Note that $X \cdot R$ is empty and $R \cdot X$ is empty.

The following four propositions are true:

(63) $R \cdot \emptyset = \emptyset \cdot R$.

(64) If $\operatorname{dom} R = \emptyset$ or $\operatorname{rng} R = \emptyset$, then $R = \emptyset$.

(65) $\operatorname{dom} R = \emptyset$ iff $\operatorname{rng} R = \emptyset$.

(66) $\emptyset^{\smile} = \emptyset$.

Let X be an empty set. One can verify that $X^{\smile}$ is empty.

The following proposition is true

(67) If $\operatorname{rng} R$ misses $\operatorname{dom} P$, then $R \cdot P = \emptyset$.

Let R be a binary relation. We say that R is non-empty if and only if:

(Def. 9) $\emptyset \notin \operatorname{rng} R$.

Let us consider X. The functor id_X yields a binary relation and is defined as follows:

(Def. 10) $\langle x, y \rangle \in \operatorname{id}_X$ iff $x \in X$ and $x = y$.

The following propositions are true:

(71)[9] $\operatorname{dom}(\operatorname{id}_X) = X$ and $\operatorname{rng}(\operatorname{id}_X) = X$.

(72) $(\operatorname{id}_X)^{\smile} = \operatorname{id}_X$.

(73) If for every x such that $x \in X$ holds $\langle x, x \rangle \in R$, then $\operatorname{id}_X \subseteq R$.

(74) $\langle x, y \rangle \in \operatorname{id}_X \cdot R$ iff $x \in X$ and $\langle x, y \rangle \in R$.

(75) $\langle x, y \rangle \in R \cdot \operatorname{id}_Y$ iff $y \in Y$ and $\langle x, y \rangle \in R$.

(76) $R \cdot \operatorname{id}_X \subseteq R$ and $\operatorname{id}_X \cdot R \subseteq R$.

(77) If $\operatorname{dom} R \subseteq X$, then $\operatorname{id}_X \cdot R = R$.

(78) $\operatorname{id}_{\operatorname{dom} R} \cdot R = R$.

(79) If $\operatorname{rng} R \subseteq Y$, then $R \cdot \operatorname{id}_Y = R$.

(80) $R \cdot \operatorname{id}_{\operatorname{rng} R} = R$.

(81) $\operatorname{id}_\emptyset = \emptyset$.

(82) If $\operatorname{dom} R = X$ and $\operatorname{rng} P_2 \subseteq X$ and $P_2 \cdot R = \operatorname{id}_{\operatorname{dom} P_1}$ and $R \cdot P_1 = \operatorname{id}_X$, then $P_1 = P_2$.

Let us consider R, X. The functor $R{\restriction}X$ yielding a binary relation is defined by:

(Def. 11) $\langle x, y \rangle \in R{\restriction}X$ iff $x \in X$ and $\langle x, y \rangle \in R$.

One can prove the following propositions:

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[7] The propositions (57)–(59) have been removed.

[8] The proposition (61) has been removed.

[9] The propositions (68)–(70) have been removed.

(86)[10] $x \in \mathrm{dom}(R \restriction X)$ iff $x \in X$ and $x \in \mathrm{dom}\, R$.

(87) $\mathrm{dom}(R \restriction X) \subseteq X$.

(88) $R \restriction X \subseteq R$.

(89) $\mathrm{dom}(R \restriction X) \subseteq \mathrm{dom}\, R$.

(90) $\mathrm{dom}(R \restriction X) = \mathrm{dom}\, R \cap X$.

(91) If $X \subseteq \mathrm{dom}\, R$, then $\mathrm{dom}(R \restriction X) = X$.

(92) $(R \restriction X) \cdot P \subseteq R \cdot P$.

(93) $P \cdot (R \restriction X) \subseteq P \cdot R$.

(94) $R \restriction X = \mathrm{id}_X \cdot R$.

(95) $R \restriction X = \emptyset$ iff $\mathrm{dom}\, R$ misses X.

(96) $R \restriction X = R \cap [: X, \mathrm{rng}\, R :]$.

(97) If $\mathrm{dom}\, R \subseteq X$, then $R \restriction X = R$.

(98) $R \restriction \mathrm{dom}\, R = R$.

(99) $\mathrm{rng}(R \restriction X) \subseteq \mathrm{rng}\, R$.

(100) $R \restriction X \restriction Y = R \restriction (X \cap Y)$.

(101) $R \restriction X \restriction X = R \restriction X$.

(102) If $X \subseteq Y$, then $R \restriction X \restriction Y = R \restriction X$.

(103) If $Y \subseteq X$, then $R \restriction X \restriction Y = R \restriction Y$.

(104) If $X \subseteq Y$, then $R \restriction X \subseteq R \restriction Y$.

(105) If $P \subseteq R$, then $P \restriction X \subseteq R \restriction X$.

(106) If $P \subseteq R$ and $X \subseteq Y$, then $P \restriction X \subseteq R \restriction Y$.

(107) $R \restriction (X \cup Y) = R \restriction X \cup R \restriction Y$.

(108) $R \restriction (X \cap Y) = (R \restriction X) \cap (R \restriction Y)$.

(109) $R \restriction (X \setminus Y) = R \restriction X \setminus R \restriction Y$.

(110) $R \restriction \emptyset = \emptyset$.

(111) $\emptyset \restriction X = \emptyset$.

(112) $(P \cdot R) \restriction X = (P \restriction X) \cdot R$.

Let us consider Y, R. The functor $Y \restriction R$ yields a binary relation and is defined by:

(Def. 12) $\langle x, y \rangle \in Y \restriction R$ iff $y \in Y$ and $\langle x, y \rangle \in R$.

We now state a number of propositions:

(115)[11] $y \in \mathrm{rng}(Y \restriction R)$ iff $y \in Y$ and $y \in \mathrm{rng}\, R$.

(116) $\mathrm{rng}(Y \restriction R) \subseteq Y$.

(117) $Y \restriction R \subseteq R$.

[10] The propositions (83)–(85) have been removed.
[11] The propositions (113) and (114) have been removed.

(118) $\mathrm{rng}(Y \upharpoonright R) \subseteq \mathrm{rng}\, R$.

(119) $\mathrm{rng}(Y \upharpoonright R) = \mathrm{rng}\, R \cap Y$.

(120) If $Y \subseteq \mathrm{rng}\, R$, then $\mathrm{rng}(Y \upharpoonright R) = Y$.

(121) $(Y \upharpoonright R) \cdot P \subseteq R \cdot P$.

(122) $P \cdot (Y \upharpoonright R) \subseteq P \cdot R$.

(123) $Y \upharpoonright R = R \cdot \mathrm{id}_Y$.

(124) $Y \upharpoonright R = R \cap [:\mathrm{dom}\, R, Y:]$.

(125) If $\mathrm{rng}\, R \subseteq Y$, then $Y \upharpoonright R = R$.

(126) $\mathrm{rng}\, R \upharpoonright R = R$.

(127) $Y \upharpoonright (X \upharpoonright R) = (Y \cap X) \upharpoonright R$.

(128) $Y \upharpoonright (Y \upharpoonright R) = Y \upharpoonright R$.

(129) If $X \subseteq Y$, then $Y \upharpoonright (X \upharpoonright R) = X \upharpoonright R$.

(130) If $Y \subseteq X$, then $Y \upharpoonright (X \upharpoonright R) = Y \upharpoonright R$.

(131) If $X \subseteq Y$, then $X \upharpoonright R \subseteq Y \upharpoonright R$.

(132) If $P_1 \subseteq P_2$, then $Y \upharpoonright P_1 \subseteq Y \upharpoonright P_2$.

(133) If $P_1 \subseteq P_2$ and $Y_1 \subseteq Y_2$, then $Y_1 \upharpoonright P_1 \subseteq Y_2 \upharpoonright P_2$.

(134) $(X \cup Y) \upharpoonright R = X \upharpoonright R \cup Y \upharpoonright R$.

(135) $(X \cap Y) \upharpoonright R = (X \upharpoonright R) \cap (Y \upharpoonright R)$.

(136) $(X \setminus Y) \upharpoonright R = X \upharpoonright R \setminus Y \upharpoonright R$.

(137) $\emptyset \upharpoonright R = \emptyset$.

(138) $Y \upharpoonright \emptyset = \emptyset$.

(139) $Y \upharpoonright (P \cdot R) = P \cdot (Y \upharpoonright R)$.

(140) $(Y \upharpoonright R) \upharpoonright X = Y \upharpoonright (R \upharpoonright X)$.

Let us consider R, X. The functor $R^\circ X$ yielding a set is defined as follows:

(Def. 13) $y \in R^\circ X$ iff there exists x such that $\langle x, y \rangle \in R$ and $x \in X$.

One can prove the following propositions:

(143)[12] $y \in R^\circ X$ iff there exists x such that $x \in \mathrm{dom}\, R$ and $\langle x, y \rangle \in R$ and $x \in X$.

(144) $R^\circ X \subseteq \mathrm{rng}\, R$.

(145) $R^\circ X = R^\circ(\mathrm{dom}\, R \cap X)$.

(146) $R^\circ \mathrm{dom}\, R = \mathrm{rng}\, R$.

(147) $R^\circ X \subseteq R^\circ \mathrm{dom}\, R$.

(148) $\mathrm{rng}(R \upharpoonright X) = R^\circ X$.

(149) $R^\circ \emptyset = \emptyset$.

[12] The propositions (141) and (142) have been removed.

(150) $\emptyset^{\circ}X = \emptyset$.

(151) $R^{\circ}X = \emptyset$ iff $\mathrm{dom}\,R$ misses X.

(152) If $X \neq \emptyset$ and $X \subseteq \mathrm{dom}\,R$, then $R^{\circ}X \neq \emptyset$.

(153) $R^{\circ}(X \cup Y) = R^{\circ}X \cup R^{\circ}Y$.

(154) $R^{\circ}(X \cap Y) \subseteq R^{\circ}X \cap R^{\circ}Y$.

(155) $R^{\circ}X \setminus R^{\circ}Y \subseteq R^{\circ}(X \setminus Y)$.

(156) If $X \subseteq Y$, then $R^{\circ}X \subseteq R^{\circ}Y$.

(157) If $P \subseteq R$, then $P^{\circ}X \subseteq R^{\circ}X$.

(158) If $P \subseteq R$ and $X \subseteq Y$, then $P^{\circ}X \subseteq R^{\circ}Y$.

(159) $(P \cdot R)^{\circ}X = R^{\circ}P^{\circ}X$.

(160) $\mathrm{rng}(P \cdot R) = R^{\circ}\,\mathrm{rng}\,P$.

(161) $(R{\restriction}X)^{\circ}Y \subseteq R^{\circ}Y$.

(163)[13] $\mathrm{dom}\,R \cap X \subseteq (R^{\smile})^{\circ}R^{\circ}X$.

Let us consider R, Y. The functor $R^{-1}(Y)$ yielding a set is defined as follows:

(Def. 14) $x \in R^{-1}(Y)$ iff there exists y such that $\langle x, y\rangle \in R$ and $y \in Y$.

Next we state a number of propositions:

(166)[14] $x \in R^{-1}(Y)$ iff there exists y such that $y \in \mathrm{rng}\,R$ and $\langle x, y\rangle \in R$ and $y \in Y$.

(167) $R^{-1}(Y) \subseteq \mathrm{dom}\,R$.

(168) $R^{-1}(Y) = R^{-1}(\mathrm{rng}\,R \cap Y)$.

(169) $R^{-1}(\mathrm{rng}\,R) = \mathrm{dom}\,R$.

(170) $R^{-1}(Y) \subseteq R^{-1}(\mathrm{rng}\,R)$.

(171) $R^{-1}(\emptyset) = \emptyset$.

(172) $\emptyset^{-1}(Y) = \emptyset$.

(173) $R^{-1}(Y) = \emptyset$ iff $\mathrm{rng}\,R$ misses Y.

(174) If $Y \neq \emptyset$ and $Y \subseteq \mathrm{rng}\,R$, then $R^{-1}(Y) \neq \emptyset$.

(175) $R^{-1}(X \cup Y) = R^{-1}(X) \cup R^{-1}(Y)$.

(176) $R^{-1}(X \cap Y) \subseteq R^{-1}(X) \cap R^{-1}(Y)$.

(177) $R^{-1}(X) \setminus R^{-1}(Y) \subseteq R^{-1}(X \setminus Y)$.

(178) If $X \subseteq Y$, then $R^{-1}(X) \subseteq R^{-1}(Y)$.

(179) If $P \subseteq R$, then $P^{-1}(Y) \subseteq R^{-1}(Y)$.

(180) If $P \subseteq R$ and $X \subseteq Y$, then $P^{-1}(X) \subseteq R^{-1}(Y)$.

(181) $(P \cdot R)^{-1}(Y) = P^{-1}(R^{-1}(Y))$.

(182) $\mathrm{dom}(P \cdot R) = P^{-1}(\mathrm{dom}\,R)$.

(183) $\mathrm{rng}\,R \cap Y \subseteq (R^{\smile})^{-1}(R^{-1}(Y))$.

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[13] The proposition (162) has been removed.

[14] The propositions (164) and (165) have been removed.

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