Probabilistic I/O Automata: Theories of Two Equivalences*†

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Abstract

Working in the context of a process-algebraic language for Probabilistic I/O Automata (PIOA), we study the notion of *PIOA behavior equivalence* by obtaining a complete axiomatization of its equational theory and comparing the results with a complete axiomatization of a more standard equivalence, *weighted bisimulation*. The axiomatization of behavior equivalence is achieved by adding to the language an operator for forming *convex combinations* of terms.

1 Introduction

In previous work [SCS03], we presented a process-algebraic language, motivated by the *probabilistic I/O automaton* model, that provides a compositional formalism for defining continuous-time Markov chains (CTMCs). The constructs in our language are similar to those in other "Markovian process algebra" languages that have been studied by a number of other researchers (see [HH02] for a survey), especially EMPA [BDG98]. In our language, we classify transitions as either *output* ("active") transitions or *input* ("passive") transitions. Output transitions, which can occur spontaneously, have associated positive *rates*. Rates

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are dimensional quantities with units of 1/time, which are regarded as the parameters of exponential probability distributions. When multiple output transitions are available for a process, the choice between them is made probabilistically by a “race policy” semantics: an exponentially distributed random future time is chosen for the occurrence of each transition (using the associated rate as the parameter of the distribution) and the transition for which the earliest time is chosen “wins the race” and becomes the next transition to occur. Input transitions, which can occur for a process only in conjunction with a similarly labeled output transition performed by its environment, have associated positive *weights*, which are dimensionless. When multiple input transitions labeled by the same action are available, the choice between them is made probabilistically on the basis of their proportionate weights.

In this paper, we consider a fragment of our language having the following syntax, where Act is a set of *actions*, variables $a, b, c, \dots$ are used to range over Act, and variables $t, u, v, \dots$ are used to range over process terms:

$$\mathrm{nil}_I \ \mid\ \langle a?_w \rangle\, t \ \mid\ \langle b!_r \rangle\, t \ \mid\ t + t' \ \mid\ t\ {}_O\|_{O'}\ t'$$

The informal meaning of the constructs is as follows:

- nil_I denotes a process that passively accepts an input from the set $I \subseteq Act$, assigning such an input a weight of 1, and then continues to behave as nil_I.

  If $I \neq \emptyset$, then term nil_I can be regarded as an abbreviation for the recursive term $\mu X.\sum_{a \in I} \langle a?_1 \rangle\, X$ in the full language of [SCS03]. The term $\mathrm{nil}_\emptyset$ is the same as the term nil of [SCS03].

- $\langle a?_w \rangle\, t$ denotes an *input-prefixed* process that can accept an input $a \in Act$ with positive *weight* w and then become the process t.

- $\langle b!_r \rangle\, t$ denotes an *output-prefixed* process that can spontaneously perform output action $b \in Act$ with positive *rate* r and then become the process denoted by t.

- $t + t'$ denotes a *choice* between alternatives offered by t and t'.

- $t\ {}_O\|_{O'}\ t'$ denotes the *parallel composition* of t and t'. Here O and O' are disjoint sets of output actions *controlled by* t and t', respectively.

The constructs we omit from consideration in the present paper are: internal prefixing, hiding, renaming, and recursion.

Synchronization of actions, which occurs between the components of a parallel composition, is restricted to the input/input and input/output cases only. Output/output synchronization is not permitted. This seems to be the simplest version of action synchronization that has an intuitively meaningful stochastic interpretation. As our goal is to understand the relationship between input and output in the simplest possible setting, we do not complicate the language with immediate actions, priorities, or other extraneous constructs.

The standard notions of process equivalence in the context of stochastic process algebra are based on variants of *probabilistic bisimulation* [LS92], which is closely related to the concept of *lumpability* in the theory of Markov chains. A typical example of such an equivalence is *Markovian bisimulation* [Hil96], in which terms regarded as equivalent are required to have the same aggregate transition rate, and which is such that equivalent terms have identical total transition probabilities to each equivalence class of terms for each action. In this paper we use *weighted bisimulation*, which uses the same fundamental idea and covers the cases of weight-labeled and rate-labeled transitions.

Behavior equivalence is an alternative to weighted bisimulation equivalence that we have studied in earlier papers. This equivalence is strictly coarser than weighted bisimulation equivalence, but still substitutive with respect to the process algebraic operations listed above. The original motivation of behavior equivalence was as a testing equivalence, and in this context a full-abstractness result was established in [WSS97]. The original definitions were reformulated in subsequent papers as our understanding of behavior equivalence improved. In [Sta03] we were able to compare weighted bisimulation equivalence and behavior equivalence by viewing them both as certain "invariant" equivalences on formal *linear combinations* of process terms, rather than as equivalences on individual terms. We showed, roughly: (1) that weighted bisimulation equivalence can be characterized as the largest invariant equivalence on combinations of terms that is in a sense generated by equations between individual terms, (2) that behavior equivalence can be characterized as the largest invariant equivalence on combinations of terms that in a sense separates terms having distinct aggregate rates, and (3) that behavior equivalence is strictly coarser than weighted bisimulation equivalence, even when restricted to individual terms.

For example, the following intuitively reasonable equation between terms in our language holds for behavior equivalence but not for weighted bisimulation equivalence:

$$\langle b!_r \rangle \; (\langle c!_{\pi s} \rangle \; t + \langle d!_{(1-\pi)s} \rangle \; u) \quad = \quad \langle b!_{\pi r} \rangle \; \langle c!_s \rangle \; t + \langle b!_{(1-\pi)r} \rangle \; \langle d!_s \rangle \; u$$

where π can be any value in the interval $(0, 1)$. Intuitively, both sides above can perform the output b with the same aggregate rate r. After doing so, the term on the left-hand side evolves to the derivative term $\langle c!_{\pi s} \rangle \; t + \langle d!_{(1-\pi)s} \rangle \; u$, which can do output c with rate πs and output d with rate $(1 - \pi)s$, for an aggregate rate of s. In contrast, there is no individual term that expresses the derivative of the right-hand side after output b has been performed. The best we can do is to think of this derivative as a *probability distribution* that assigns probability π to term $\langle c!_s \rangle \; t$ and probability $1 - \pi$ to term $\langle d!_s \rangle \; t$. Intuitively, there is no observable difference between such a probability distribution and the individual term $\langle c!_{\pi s} \rangle \; t + \langle d!_{(1-\pi)s} \rangle \; u$, which explains why the original equation is a reasonable one to expect.

Consideration of the preceding example suggests that an axiomatization of behavior equivalence might be achieved if we augment the language with an explicit notation for expressing convex combinations of terms; for example: $\langle c!_s \rangle \; t \;{}_{\pi}\!\oplus_{1-\pi} \langle d!_s \rangle \; u$. We would then be able to express the equivalence between the derivatives of the left and right-hand sides of the equation above as follows:

$$\langle c!_{\pi s} \rangle \; t + \langle d!_{(1-\pi)s} \rangle \; u \quad = \quad \langle c!_s \rangle \; t \;{}_{\pi}\!\oplus_{1-\pi} \langle d!_s \rangle \; u.$$

In fact, for the $\|$-free fragment of the language, an axiomatization of behavior equivalence can be achieved in this way and the details are the subject of the present paper. A key point, which took us a long time to discover, is that we cannot permit the the formation of combinations $t\ {}_{\pi}\!\oplus_{1-\pi}\, u$ for arbitrary terms t and u. Rather, we must require as a condition of well-formedness that terms t and u have an *identical aggregate rate*, which then becomes the aggregate rate of the combined term. Failing to impose this requirement results in the possibility of having "terms" that do not have unique aggregate rates, which produces seemingly insurmountable complications in the semantics and axiomatization. Another detail that required some care to work out properly concerns keeping track of the "types" of terms, by which we mean the sets of input and output actions in which a term is required to participate.

As a result of our investigation, we have further clarified our understanding of behavior equivalence and its relationship to weighted bisimulation equivalence. Perhaps the simplest way to summarize what we have learned is to compare the normal form used in the proofs of completeness for the axiomatization of weighted bisimulation equivalence with that used in the proof for behavior equivalence. Employing $\sum$-notation in a standard way and (for the moment) ignoring special cases that arise with empty summations, the following is a generic normal form for a term with respect to weighted bisimulation equivalence:

$$\sum_{i=1}^{m} \langle a_i ?_{w_i} \rangle\ t_i + \sum_{j=1}^{n} \langle b_j !_{r_j} \rangle\ t_j$$

In the above, the t_i and t_j are recursively required to be normal forms. Moreover, it is required that for no distinct i and i' do we have both $a_i = a_{i'}$ and t_i equivalent to $t_{i'}$ and for no distinct j and j' do we have both $b_j = b_{j'}$ and t_j equivalent to $t_{j'}$. Thus, w_i is the aggregate weight of a_i-transitions to the equivalence class of t_i, and r_j is the aggregate rate of b_j-transitions to the equivalence class of t_j.

In contrast, a generic normal form for a term with respect to behavior equivalence is the following:

$$\sum_{a \in I} \sum_{s \in R_a} \langle a ?_{w_{a,s}} \rangle\ t_{a,s} + \sum_{b \in O} \sum_{s \in R_b} \langle b !_{r_{b,s}} \rangle\ t_{b,s},$$

where each set R_a and R_b is nonempty and each term $t_{a,s}$ and $t_{b,s}$ is required recursively to be a normal form with aggregate rate s. The main point here is that, once input a has been chosen, there is a unique derivative term $t_{a,s}$ for each aggregate rate in the set R_a, and once output b has been chosen, there is a unique derivative term $t_{b,s}$ for each aggregate rate in the set R_b. Terms $t_{a,s}$ and $t_{a,s'}$ cannot be equivalent for distinct values of s because they have distinct aggregate rates. Similar considerations hold for $t_{b,s}$ and $t_{b,s'}$. A normal form for behavior equivalence is thus also a normal form for weighted bisimulation equivalence, but not conversely. So, the essential difference between weighted bisimulation equivalence and behavior equivalence is that the former will in general draw distinctions between terms based on the existence of multiple derivatives having the same aggregate rate, whereas the latter will not.

Note that, although the operator ${}_{\pi}\oplus_{1-\pi}$ does not appear in the normal form for behavior equivalence, achieving a reduction to normal form will in general require passing through terms in which explicit use is made of this operator.

The remainder of the paper is organized as follows: In Section 2, we summarize the basic definitions pertaining to our process-algebraic language and its semantics. In Section 3, we define the notion of weighted bisimulation equivalence for our language and present a sound and complete set of axioms for this equivalence. In Section 4, we define the notion of behavior equivalence, extend the language with the convex combination operator ${}_{\pi}\oplus_{1-\pi}$ discussed above, present a sound and complete set of axioms for behavior equivalence in the extended language, and sketch the main ideas of the completeness proof. Although we include the parallel composition construct in the language defined in section 2, the results of Sections 3 and 4 concern only the $\|$-free fragment. We hope to extend our results to include parallel composition in a future paper.

2 Basic Definitions

2.1 Types

As detailed in our previous paper, our PIOA language is equipped with a set of rules for inferring *typing judgements* of the form

$$t : I/J \Rightarrow O$$

where I, J, and O are sets of actions. We write $\vdash t : \phi$ to assert that a typing judgement $t : \phi$ is inferable. A term t is *well-typed* if $\vdash t : \phi$ for some ϕ. Let $\mathrm{Proc}(I/J \Rightarrow O)$ denote the set of all terms t such that $\vdash t : I/J \Rightarrow O$.

Intuitively, a typing judgement $t : I/J \Rightarrow O$ asserts that I is a set of actions for which input transitions are guaranteed to be enabled at the first step of t, that J is a set of actions for which input transitions are guaranteed to be enabled at all steps of t after the first, and O is a set of actions that includes at least all the outputs that may be produced by t (but which may be larger). The primary purpose of the typing system is to identify those terms that are *input-enabled*, in order to rule out the formation of parallel compositions involving non-input-enabled terms. Non-input-enabled terms are required in the language to permit the building up of sets of alternatives using $+$. The reason why only input-enabled terms are permitted in parallel compositions is that we do not wish to allow stochastically unclear situations in which one component in a system is attempting to perform an output with a definite rate, but is inhibited from doing so by another component that will not accept that action as an input.

Figure 1 presents the type-inference rules applicable to the language fragment we consider here. The rule given here for the abbreviation nil_I can be shown to be a derived rule of the full language. We have included an additional "weakening" rule (the last rule), which was not present in our previous paper. The purpose of the weakening rule is to ensure that if

$$\mathrm{nil}_I : I/I \Rightarrow \emptyset$$

$$\frac{t : J/J \Rightarrow O \quad a \in J}{\langle a?_w \rangle\, t : \{a\}/J \Rightarrow O} \qquad \frac{t : J/J \Rightarrow O \quad b \notin J}{\langle b!_r \rangle\, t : \emptyset/J \Rightarrow O \cup \{b\}}$$

$$\frac{t : I_t/J \Rightarrow O_t \quad u : I_u/J \Rightarrow O_u}{t + u : I_t \cup I_u/J \Rightarrow O_t \cup O_u}$$

$$\frac{t : I_t/I_t \Rightarrow O_t \quad u : I_u/I_u \Rightarrow O_u \quad I = (I_t \cup I_u) \setminus (O_t \cup O_u)}{t \;{}_{O_t}\|_{O_u}\; u : I/I \Rightarrow O_t \cup O_u}$$

$$\frac{t : I/J \Rightarrow O \quad O \subseteq O' \quad O' \cap J = \emptyset}{t : I/J \Rightarrow O'}$$

Figure 1: Type-Inference Rules

$\vdash t : I/J \Rightarrow O$ then also $\vdash t : I/J \Rightarrow O'$ for all $O' \supseteq O$ such that $O' \cap J = \emptyset$; which is a useful property that did not hold of the typing system in our previous paper.

The following metatheoretic result from our previous work will be important for our present purposes. Its truth is not affected by the introduction of the new weakening rule.

Proposition 1. *If* $\vdash t : I/J \Rightarrow O$ *for some* I, J, *and* O, *then*

1. $I \subseteq J$ *and* $J \cap O = \emptyset$.
2. *There exists* $\hat{O}$ *such that* $\vdash t : I/J \Rightarrow \hat{O}$, *and such that whenever* $\vdash t : I'/J' \Rightarrow O'$ *then* $I' = I$, $J' = J$, *and* $O' \supseteq \hat{O}$.

A technical issue with our PIOA language is that of "native" versus "non-native" actions. If $t \in \mathrm{Proc}(I/J \Rightarrow O)$, then actions $e \in J \cup O$ are called *native* to t and actions outside this set are called *non-native*. Intuitively, native actions are those in which t must participate and non-native actions are those that t ignores. This distinction is important because if t has no transition for a particular output action in which it must participate, then that action is inhibited from occurring, whereas if t ignores an action then it may occur freely. Note that whether an action is considered native or non-native depends on our having fixed a particular type $I/J \Rightarrow O$ inferable for t. All such types have the same input sets I and J, but the output sets O may differ. Thus, in the sequel it will be necessary for us to parameterize certain notions by the particular output set on which they depend.

2.2 Transition Semantics

In our previous paper, we gave structural operational semantics rules that defined the transitions that could be taken by terms in our language. Though our present purposes do not require a full presentation of the transition semantics given in our previous paper, we do need a notation for the aggregate *weight* or *rate* $\Delta_e^O(t, v)$ of e-labeled transitions from t to v.

Suppose $t \in \text{Proc}(I/J \Rightarrow O)$. Define $\Delta_e^O(t,v)$ as follows: If $e \notin J \cup O$ (non-native case), then $\Delta_e^O(t,v) = 1$ if $v = t$, and $\Delta_e^O(t,v) = 0$ otherwise. If $e \in J \cup O$ (native case), then

1. $\Delta_e^O(\text{nil}_I, v) = \begin{cases} 1, & \text{if } e \in I \text{ and } v = \text{nil}_I. \\ 0, & \text{otherwise.} \end{cases}$

2. $\Delta_e^O(\langle a?_w \rangle\ t, v) = \begin{cases} w, & \text{if } e = a \text{ and } v = t \\ 0, & \text{otherwise.} \end{cases}$

3. $\Delta_e^O(\langle b!_r \rangle\ t, v) = \begin{cases} r, & \text{if } e = b \text{ and } v = t \\ 0, & \text{otherwise.} \end{cases}$

4. $\Delta_e^O(t + u, v) = \Delta_e^O(t,v) + \Delta_e^O(u,v)$.

5. $\Delta_e^O(t\ {}_{O_t}\|_{O_u}\ u, v) = \begin{cases} \Delta_e^{O \setminus O_u}(t,t') \cdot \Delta_e^{O \setminus O_t}(u,u'), & \text{if } v = t'\ {}_{O_t}\|_{O_u}\ u' \\ 0, & \text{otherwise.} \end{cases}$

In the sequel, if $\mathcal{C}$ is a set of terms, then $\Delta_e^O(t,\mathcal{C})$ will be an abbreviation for the sum $\sum_{v \in \mathcal{C}} \Delta_e^O(t,v)$, which is always finite.

It is important for us that inferable types are preserved under transitions. Formally, we have the following result, which was stated in our previous paper and remains true in the presence of the weakening rule.

Proposition 2. *Suppose* $t \in \text{Proc}(I/J \Rightarrow O)$*. If* $\Delta_e^O(t,u) \neq 0$*, then* $u \in \text{Proc}(J/J \Rightarrow O)$*. In particular,* $\text{Proc}(J/J \Rightarrow O)$ *is closed under transitions and terms in* $\text{Proc}(I/J \Rightarrow O)$ *reach* $\text{Proc}(J/J \Rightarrow O)$ *after one transition.*

Term $t \in \text{Proc}(J/J \Rightarrow O)$ is called *input-stochastic* if $\sum_{v \in \text{Proc}(J/J \Rightarrow O)} \Delta_e^O(t,v) = 1$ for all $e \in J$.

3 Weighted Bisimulation

We first consider the equational theory of the indicated fragment under a suitable generalization of probabilistic bisimulation equivalence. The generalization we use, called *weighted bisimulation*, can be obtained by applying the standard definition of probabilistic bisimulation to total transition weights and rates, rather than transition probabilities. A technical complication is that $\text{Proc}(J/J \Rightarrow O)$ is closed under transitions, but $\text{Proc}(I/J \Rightarrow O)$ is not. Thus, we first define weighted bisimulation for $\text{Proc}(J/J \Rightarrow O)$ and then use it to define *weighted bisimulation equivalence* for general $\text{Proc}(I/J \Rightarrow O)$.

In this paper, we discuss weighted bisimulation equivalence primarily for the purposes of comparison with behavior equivalence. Modulo minor differences in the formal setup, the properties of this equivalence are standard, and have been established before by other authors (*e.g.* [HR94]). We include the full definitions and proofs below for completeness, but do not make any claims of novelty for them.

Formally, a *weighted bisimulation* on $\text{Proc}(J/J \Rightarrow O)$ is an equivalence relation $R \subseteq \text{Proc}(J/J \Rightarrow O)$ such that the following condition is satisfied:

- Whenever $t\ R\ t'$ then for all actions e and all equivalence classes $\mathcal{C}$ of R we have $\Delta_e^O(t, \mathcal{C}) = \Delta_e^O(t', \mathcal{C})$.

Clearly, the identity relation is a weighted bisimulation, and a standard argument shows that the transitive closure of the union of an arbitrary collection of weighted bisimulations is again a weighted bisimulation, hence there is a largest weighted bisimulation relation on $\text{Proc}(J/J \Rightarrow O)$.

Terms t and t' in $\text{Proc}(I/J \Rightarrow O)$ are defined to be *weighted bisimulation equivalent* if there exists a weighted bisimulation relation R on $\text{Proc}(J/J \Rightarrow O)$ such that for all actions e and all equivalence classes $\mathcal{C}$ of R we have $\Delta_e^O(t, \mathcal{C}) = \Delta_e^O(t', \mathcal{C})$. In this case we write $t \underset{O}{\sim} t'$ (there is no need to mention the input sets I and J which are uniquely determined by t and t'). The following result shows that for $\text{Proc}(J/J \Rightarrow O)$, which is closed under transitions, the definition of weighted bisimulation equivalence given above coincides with the usual formulation.

Lemma 3. *Suppose t and t' are in $\text{Proc}(I/J \Rightarrow O)$, where $I = J$. Then $t \underset{O}{\sim} t'$ if and only if there exists a weighted bisimulation relation R on $\text{Proc}(J/J \Rightarrow O)$ such that $t\ R\ t'$.*

Proof. If there exists a weighted bisimulation relation R on $\text{Proc}(J/J \Rightarrow O)$ such that $t\ R\ t'$, then clearly for all actions e and all equivalence classes $\mathcal{C}$ of R we have $\Delta_e^O(t, \mathcal{C}) = \Delta_e^O(t', \mathcal{C})$, so that $t \underset{O}{\sim} t'$.

Conversely, if $t \underset{O}{\sim} t'$, then there exists a weighted bisimulation relation R on $\text{Proc}(J/J \Rightarrow O)$ such that for all actions e and all equivalence classes $\mathcal{C}$ of R we have $\Delta_e^O(t, \mathcal{C}) = \Delta_e^O(t', \mathcal{C})$. We show that R is contained in a weighted bisimulation relation S on $\text{Proc}(J/J \Rightarrow O)$ such that $t\ S\ t'$. Let S be the least equivalence relation on $\text{Proc}(J/J \Rightarrow O)$ that contains $R \cup \{(t, t')\}$. Then $t\ S\ t'$ by construction. To show that S is a weighted bisimulation relation it suffices to show: (1) that if $u\ R\ u'$, then for all actions e and all equivalence classes $\mathcal{C}$ of S we have $\Delta_e^O(u, \mathcal{C}) = \Delta_e^O(u', \mathcal{C})$, and (2) for all actions e and all equivalence classes $\mathcal{C}$ of S we have $\Delta_e^O(t, \mathcal{C}) = \Delta_e^O(t', \mathcal{C})$, because the general case of $u\ S\ u'$ will then follow by induction on the length of an $R \cup \{(t, t')\}$-chain from u to u'. But (1) is immediate from the fact that R is a weighted bisimulation relation that refines $\mathcal{S}$. Moreover, (2) follows from the assumption defining R and the fact that R is a refinement of $\mathcal{S}$. □

For the purpose of shortening proofs in the sequel, we note here that the bisimulation condition $\Delta_e^O(t, \mathcal{C}) = \Delta_e^O(t', \mathcal{C})$ is automatically satisfied if action e is non-native, due to the fact that for such e we have $\Delta_e^O(t, \mathcal{C}) = 1$ if $t \in \mathcal{C}$ and $\Delta_e^O(t, \mathcal{C}) = 0$ otherwise. So, when proving that a particular relation is a weighted bisimulation, we need only consider native actions.

Lemma 4. *Suppose t and t' in* $\mathrm{Proc}(I/J \Rightarrow O)$ *have the property that* $\Delta_e^O(t,w) = \Delta_e^O(t',w)$ *for all actions* $e \in J \cup O$ *and all terms* $w \in \mathrm{Proc}(J/J \Rightarrow O)$. *Then t and t' are weighted bisimulation equivalent.*

Proof. Let R be the identity relation on $\mathrm{Proc}(J/J \Rightarrow O)$. Clearly R is a weighted bisimulation relation. Moreover, each equivalence class of R is a singleton set $\{w\}$. Thus for all $e \in J \cup O$ and all equivalence classes $\{w\}$ of R we have $\Delta_e^O(t,\{w\}) = \Delta_e^O(t,w) = \Delta_e^O(t',w) = \Delta_e^O(t',\{w\})$, thus establishing that t and t' are weighted bisimulation equivalent. □

3.1 Substitutivity

Lemma 5. *Weighted bisimulation equivalence is substitutive for input prefixing, output prefixing, choice, and parallel composition. That is, each of the following assertions holds for terms t and t' in* $\mathrm{Proc}(I/J \Rightarrow O)$ *whenever all the terms mentioned are well-typed and the equivalences make sense:*

1. *If* $t \underset{O}{\sim} t'$ *then* $\langle a?_p\rangle\, t \underset{O}{\sim} \langle a?_p\rangle\, t'$.
2. *If* $t \underset{O}{\sim} t'$ *then* $\langle b!_r\rangle\, t \underset{O}{\sim} \langle b!_r\rangle\, t'$.
3. *If* $t \underset{O}{\sim} t'$ *then* $t + u \underset{O}{\sim} t' + u$ *and* $u + t \underset{O}{\sim} u + t'$.
4. *If* $t \underset{O}{\sim} t'$ *then* $t \;{}_O\|_{O_u}\; u \underset{O'}{\sim} t' \;{}_O\|_{O_u}\; u$ *and* $u \;{}_{O_u}\|_O\; t \underset{O'}{\sim} u \;{}_{O_u}\|_O\; t'$.

Proof.

1. Suppose $t \underset{O}{\sim} t'$, where $t, t' \in \mathrm{Proc}(I/J \Rightarrow O)$. Note that for $\langle a?_p\rangle\, t$ and $\langle a?_p\rangle\, t'$ to be well-typed, we must have $I = J$, hence in fact $t, t' \in \mathrm{Proc}(J/J \Rightarrow O)$. Let R be a weighted bisimulation on $\mathrm{Proc}(J/J \Rightarrow O)$ such that $t\, R\, t'$. If $\mathcal{C}$ is an equivalence class of R then
$$\Delta_e^O(\langle a?_p\rangle\, t, \mathcal{C}) = \begin{cases} p, & \text{if } e = a \text{ and } t \in \mathcal{C}, \\ 0, & \text{otherwise.} \end{cases}$$
Similarly,
$$\Delta_e^O(\langle a?_p\rangle\, t', \mathcal{C}) = \begin{cases} p, & \text{if } e = a \text{ and } t' \in \mathcal{C}, \\ 0, & \text{otherwise.} \end{cases}$$
However, $t \in \mathcal{C}$ if and only if $t' \in \mathcal{C}$, so these two quantities are identical.

2. Suppose $t \underset{O}{\sim} t'$, where $t, t' \in \mathrm{Proc}(I/J \Rightarrow O)$. Note that for $\langle b!_r\rangle\, t$ and $\langle b!_r\rangle\, t'$ to be well-typed, we must have $I = J$ and $b \in O$. The argument that $\langle b!_r\rangle\, t \underset{O}{\sim} \langle b!_r\rangle\, t'$ is entirely analogous to the previous case.

3. Suppose $t \underset{O}{\sim} t'$, where $t, t' \in \mathrm{Proc}(I/J \Rightarrow O)$. Then there exists a weighted bisimulation R on $\mathrm{Proc}(J/J \Rightarrow O)$ such that for all actions $e \in J \cup O$ and all equivalence classes $\mathcal{C}$ of R we have $\Delta_e^O(t, \mathcal{C}) = \Delta_e^O(t', \mathcal{C})$. Let u be such that $t + u$ and $t' + u$ are well-typed; note then that we must have $\vdash u : I'/J \Rightarrow O'$ where $O' \subseteq O$. We claim that $\Delta_e^O(t + u, \mathcal{C}) = \Delta_e^O(t' + u, \mathcal{C})$ for all actions $e \in J \cup O$ and all equivalence classes $\mathcal{C}$ of R, thus showing that $\Delta_e^O(t + u, \mathcal{C}) \underset{O}{\sim} \Delta_e^O(t' + u, \mathcal{C})$. For, by definition

$$\Delta_e^O(t + u, \mathcal{C}) = \Delta_e^O(t, \mathcal{C}) + \Delta_e^O(u, \mathcal{C}).$$

Similarly,

$$\Delta_e^O(t' + u, \mathcal{C}) = \Delta_e^O(t', \mathcal{C}) + \Delta_e^O(u, \mathcal{C}).$$

But $\Delta_e^O(t, \mathcal{C}) = \Delta_e^O(t', \mathcal{C})$ for all e and $\mathcal{C}$, so the above two quantities are identical.

A symmetric proof shows that $u + t \underset{O}{\sim} u + t'$.

4. Suppose $t \underset{O}{\sim} t'$, where $t, t' \in \mathrm{Proc}(I/J \Rightarrow O)$. Note that for $t \;{}_{O}\|_{O_u}\; u$ and $t' \;{}_{O}\|_{O_u}\; u$ to be well-typed we must have $I = J$, hence $t, t' \in \mathrm{Proc}(J/J \Rightarrow O)$. Then there exists a weighted bisimulation R on $\mathrm{Proc}(J/J \Rightarrow O)$ such that for all actions $e \in J \cup O$ and all equivalence classes $\mathcal{C}$ of R we have $\Delta_e^O(t, \mathcal{C}) = \Delta_e^O(t', \mathcal{C})$. Let u be such that $t \;{}_{O}\|_{O_u}\; u$ and $t' \;{}_{O}\|_{O_u}\; u$ are well-typed; then $\vdash u : I_u/J_u \Rightarrow O_u$ for some sets I_u, J_u, and O_u. Suppose $\vdash t \;{}_{O}\|_{O_u}\; u : I'/J' \Rightarrow O'$, and $\vdash t' \;{}_{O}\|_{O_u}\; u : I'/J' \Rightarrow O'$, then $O \cup O_u \subseteq O'$ and $I' = J' = (J \cup J_u) \setminus O'$.

Let binary relation S on $\mathrm{Proc}(J'/J' \Rightarrow O')$ be the union of the identity relation and the set

$$\bigcup_{w \in \mathrm{Proc}(J_u/J_u \Rightarrow O_u)} \{(v \;{}_{O}\|_{O_u}\; w, v' \;{}_{O}\|_{O_u}\; w) : v \; R \; v'\}.$$

Clearly S is an equivalence relation such that $(t \;{}_{O}\|_{O_u}\; u) \; S \; (t' \;{}_{O}\|_{O_u}\; u)$. We claim that in fact S is a weighted bisimulation on $\mathrm{Proc}(J'/J' \Rightarrow O')$, thereby showing that $(t \;{}_{O}\|_{O_u}\; u) \underset{O'}{\sim} (t' \;{}_{O}\|_{O_u}\; u)$. The proof that $(u \;{}_{O_u}\|_{O}\; t) \underset{O'}{\sim} (u \;{}_{O_u}\|_{O}\; t')$ is symmetric.

To show that S is a weighted bisimulation on $\mathrm{Proc}(J'/J' \Rightarrow O')$, suppose $x \; S \; x'$. Then either $x = x'$ or else $x = v \;{}_{O}\|_{O_u}\; w$ and $x' = v' \;{}_{O}\|_{O_u}\; w$ for some $v \; R \; v'$ and $w \in \mathrm{Proc}(J_u/J_u \Rightarrow O_u)$. If $x = x'$ then trivially $\Delta_e^O(x, \mathcal{C}) = \Delta_e^O(x', \mathcal{C})$ for all $e \in J' \cup O'$ and all equivalence classes $\mathcal{C}$ of S. Suppose $x = v \;{}_{O}\|_{O_u}\; w$ and $x' = v' \;{}_{O}\|_{O_u}\; w$ for some $v \; R \; v'$ and $w \in \mathrm{Proc}(J_u/J_u \Rightarrow O_u)$. Then for all y we have that

$$\Delta_e^{O'}(x, y) = \begin{cases} \Delta_e^{O' \setminus O_u}(v, v'') \cdot \Delta_e^{O' \setminus O}(w, w''), & \text{if } y = v'' \;{}_{O}\|_{O_u}\; w'' \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\Delta_e^{O'}(x', y) = \begin{cases} \Delta_e^{O' \setminus O_u}(v', v'') \cdot \Delta_e^{O' \setminus O}(w, w''), & \text{if } y = v'' \;{}_{O}\|_{O_u}\; w'' \\ 0, & \text{otherwise.} \end{cases}$$

Now, an equivalence class $\mathcal{C}$ of S is either a singleton set $\{y\}$ where y is not a parallel composition, or else it is a set of the form:

$$\{v'' \ {}_O\|_{O_u} \ w'' : v'' \ R \ v_0\}$$

for some $v_0 \in \mathrm{Proc}(J/J \Rightarrow O)$ and $w'' \in \mathrm{Proc}(J_u/J_u \Rightarrow O_u)$. In the first case, $\mathcal{C}$ does not contain any terms that are parallel compositions, hence $\Delta_e^O(x, \mathcal{C}) = 0$. In the second case, letting $\mathcal{D}$ denote the R-equivalence class of v_0, we have

$$\begin{aligned}
\Delta_e^{O'}(x, \mathcal{C}) &= \sum_{y \in \mathcal{C}} \Delta_e^{O'}(x, y) \\
&= \sum_{v'' \in \mathcal{D}} \Delta_e^{O'}(x, v'' \ {}_O\|_{O_u} \ w'') \\
&= \Delta_e^{O' \setminus O_u}(v, \mathcal{D}) \cdot \Delta_e^{O' \setminus O}(w, w'') \\
&= \Delta_e^{O' \setminus O_u}(v', \mathcal{D}) \cdot \Delta_e^{O' \setminus O}(w, w'') \\
&= \Delta_e^{O'}(x', \mathcal{C}),
\end{aligned}$$

where the replacement of $\Delta_e^O(v, \mathcal{D})$ by $\Delta_e^O(v', \mathcal{D})$ is justified by the fact that $v \ R \ v'$.

$\square$

3.2 Axioms

Axioms for behavior equivalence are shown in Table 2. Note that an equation is only regarded an axiom if all the terms involved are well-formed and the same type can be inferred for the left and right-hand sides.

$$t + \mathrm{nil}_\emptyset = t \qquad \text{(choice-unit)}$$

$$t + u = u + t \qquad \text{(choice-comm)}$$

$$(t + u) + v = t + (u + v) \qquad \text{(choice-assoc)}$$

$$\langle a?_p \rangle \ t + \langle a?_q \rangle \ t = \langle a?_{p+q} \rangle \ t \qquad \text{(input-choice)}$$

$$\langle b!_r \rangle \ t + \langle b!_s \rangle \ t = \langle b!_{r+s} \rangle \ t \qquad \text{(output-choice)}$$

$$\text{If } I \neq \emptyset, \text{ then} \qquad \sum_{a \in I} \langle a?_1 \rangle \ \mathrm{nil}_I = \mathrm{nil}_I \qquad \text{(nil-fold)}$$

Table 1: Axioms for Weighted Bisimulation Equivalence

In axiom (nil-fold), we have used the summation notation

$$\sum_{a \in I} \langle a?_1 \rangle \ \mathrm{nil}_I.$$

We regard this as an abbreviation, the general case of which takes the form:

$$\sum_{i=1}^{m} \langle a_i ?_1 \rangle \; \mathrm{nil}_J$$

where $m > 0$. This is defined by induction on m as follows:

- If $m = 1$, then $\sum_{i=1}^{m} \langle a_i ?_1 \rangle \; \mathrm{nil}_J$ abbreviates $\langle a_1 ?_1 \rangle \; \mathrm{nil}_J$.
- If $m > 1$, then $\sum_{i=1}^{m} \langle a_i ?_1 \rangle \; \mathrm{nil}_J$ abbreviates

$$\sum_{i=1}^{m-1} \langle a_i ?_1 \rangle \; \mathrm{nil}_J + \langle a_m ?_1 \rangle \; \mathrm{nil}_J .$$

Later, once we have removed the possibility of ambiguity by establishing the soundness of axioms (choice-unit), (choice-comm), and (choice-assoc), we will use summation notation in the conventional fashion without further comment.

Lemma 6. *The axioms shown in Table 1 are sound for weighted bisimulation equivalence.*

Proof. In each of the following, assume that the terms on the left and right-hand side of the equation being considered are in $\mathrm{Proc}(I/J \Rightarrow O)$. By Lemma 4, it suffices to show that if t and t' are the left and right-hand sides of the equation being considered, then for all $e \in J \cup O$ and all terms w in $\mathrm{Proc}(J/J \Rightarrow O)$ we have $\Delta_e^O(t, w) = \Delta_e^O(t', w)$.

- (choice-unit) (choice-comm) (choice-assoc)

  We calculate, for $e \in J \cup O$ and $w \in \mathrm{Proc}(J/J \Rightarrow O)$:

$$\begin{array}{rcl} \Delta_e^O(t + \mathrm{nil}_\emptyset, w) & = & \Delta_e^O(t + \mathrm{nil}_\emptyset, w) \\ & = & \Delta_e^O(t, w) + \Delta_e^O(\mathrm{nil}_\emptyset, w) \\ & = & \Delta_e^O(t, w) + 0 \\ & = & \Delta_e^O(t, w). \end{array}$$

$$\begin{array}{rcl} \Delta_e^O(t + u, w) & = & \Delta_e^O(t + u, w) \\ & = & \Delta_e^O(t, w) + \Delta_e^O(u, w) \\ & = & \Delta_e^O(u, w) + \Delta_e^O(t, w) \\ & = & \Delta_e^O(u + t, w). \end{array}$$

$$\begin{array}{rcl} \Delta_e^O((t + u) + v, w) & = & \Delta_e^O((t + u) + v, w) \\ & = & \Delta_e^O(t + u, w) + \Delta_e^O(v, w) \\ & = & \Delta_e^O(t, w) + \Delta_e^O(u, w) + \Delta_e^O(v, w) \\ & = & \Delta_e^O(t + (u + v), w). \end{array}$$

- (nil-fold)

  We calculate:

$$\begin{array}{rcl} \Delta_e^O(\sum_{a\in I}\langle a?_1\rangle \ \mathrm{nil}_I, \mathcal{C}) & = & \sum_{a\in I}\Delta_e^O(\langle a?_1\rangle \ \mathrm{nil}_I, \mathcal{C}) \\ & = & \begin{cases} 1, & \text{if } e\in I \text{ and } \mathrm{nil}_I \in \mathcal{C} \\ 0, & \text{otherwise.} \end{cases} \\ & = & \Delta_e^O(\mathrm{nil}_I, \mathcal{C}). \end{array}$$

- (input-choice)

  We calculate, for $e \in J \cup O$ and $w \in \mathrm{Proc}(J/J \Rightarrow O)$:

$$\begin{array}{rcl} \Delta_e^O(\langle a?_p\rangle \ t + \langle a?_q\rangle \ t, w) & = & \Delta_e^O(\langle a?_p\rangle \ t, w) + \Delta_e^O(\langle a?_q\rangle \ t, w) \\ & = & \begin{cases} p+q, & \text{if } e = a \text{ and } t = w \\ 0, & \text{otherwise.} \end{cases} \\ & = & \Delta_e^O(\langle a?_{p+q}\rangle \ t, w). \end{array}$$

- (output-choice)

  This case is entirely analogous to case of (input-choice).

□

3.3 Completeness

We say that two terms are *identical up to permutation of sums* if they can be proved equivalent to each other using only axioms (choice-comm) and (choice-assoc). Define a term t to be *reduced with respect to axiom* (nil-fold), if there is no term t', identical to t up to permutation of sums, such that t' contains an instance of the left-hand side of axiom (nil-fold) as a subterm.

Let the notions *input normal form*, *output normal form*, and *normal form* be defined mutually recursively as follows:

- An *input normal form* is a well-typed term u that is either nil_I for some $I \neq \emptyset$, or else has the form:

$$\sum_{i=1}^{m}\langle a_i?_{p_i}\rangle \ t_i$$

  where we require that:

  1. Each t_i is a normal form.
  2. For no distinct i, i' do we have $a_i = a_{i'}$ and t_i identical to $t_{i'}$ up to permutation of sums.

3. u is reduced with respect to (nil-fold).

- An *output normal form* is a well-typed term v that is either $\mathrm{nil}_\emptyset$ or else has the form

$$\sum_{j=1}^{n} \langle b_j!_{r_j}\rangle\ t_j$$

where we require that:

1. Each t_j is a normal form.
2. For no distinct j, j' do we have $b_j = b_{j'}$ and t_j identical to $t_{j'}$ up to permutation of sums.

An output normal form is called *nontrivial* if it is not $\mathrm{nil}_\emptyset$.

- A *normal form* is either an input normal form, an output normal form, or a sum $u+v$, where u is an input normal form and v is a nontrivial output normal form.

Lemma 7.

1. *If u and u' are input normal forms, then $u + u'$ can be proved equivalent to an input normal form.*

2. *If v and v' are output normal forms, then $v + v'$ can be proved equivalent to an output normal form, which is nontrivial if either v or v' are nontrivial.*

3. *If u is an input normal form and t' is a normal form, then $u+t'$ can be proved equivalent to a normal form. If t is a normal form and u' is an input normal form, then $t + u'$ can be proved equivalent to a normal form.*

4. *If t and t are normal forms, then $t + t'$ can be proved equivalent to a normal form.*

Proof.

1. Suppose u and u' are input normal forms. If u is nil_I, then axiom (nil-fold) can be used to prove u equivalent to $\sum_{a\in I}\langle a?_1\rangle\ \mathrm{nil}_I$, and similarly for u'. We therefore assume in what follows that u has the form $\sum_{i=1}^{m}\langle a_i?_{p_i}\rangle\ t_i$ and u' has the form $\sum_{i=1}^{m'}\langle a'_i?_{p'_i}\rangle\ t'_i$. Let $I = \{a_1, a_2, \ldots, a_m\}$ and $I' = \{a'_1, a'_2, \ldots, a'_m\}$. Axioms (choice-comm) and (choice-assoc) can be used to prove $u + u'$ equivalent to the summation

$$\sum_{i=1}^{m+m'} \langle a''_i?_{p''_i}\rangle\ t''_i$$

where a''_i, p''_i, and t''_i are a_i, p_i, and t_i, respectively, if $1 \le i \le m$, and are a'_{i-m}, p'_{i-m}, and t'_{i-m}, respectively, if $m + 1 \le i \le m + m'$.

If the summation above fails to be reduced with respect to axiom (nil-fold) then we may apply that axiom (after possibly rearranging terms using (choice-comm) and (choice-assoc)). If for some distinct i, i' we have $a_i = a_{i'}$ and t_i identical to $t_{i'}$ up to permutation of sums, then we may apply axiom (input-choice) (again after possibly rearranging terms). Either of these reductions strictly decreases the number of summands in the term. As the number of summands cannot decrease forever, the reduction eventually terminates with an input normal form.

2. Suppose v and v' are output normal forms. If both v and v' are $\mathrm{nil}_\emptyset$, then axiom (choice-unit) can be used to prove $v + v'$ equivalent to the output normal form $\mathrm{nil}_\emptyset$. If one of v or v' is $\mathrm{nil}_\emptyset$ and the other is not, then axiom (choice-unit) can be used to prove $v + v'$ equivalent to the one of v, v' that is not $\mathrm{nil}_\emptyset$. If neither v nor v' is $\mathrm{nil}_\emptyset$, then v has the form $\sum_{j=1}^{n} \langle b_j!_{r_j} \rangle\ t_j$ and v' has the form $\sum_{j=1}^{n'} \langle b'_j!_{r'_j} \rangle\ t'_j$. Axioms (choice-comm) and (choice-assoc) can now be used to prove $v + v'$ equivalent to the summation

$$\sum_{j=1}^{n+n'} \langle b''_j!_{r''_j} \rangle\ t''_j$$

where b''_j, r''_j, and t''_j are b_j, r_j, and t_j, respectively, if $1 \leq j \leq n$, and are b'_{j-n}, p'_{j-n}, and t'_{j-n}, respectively, if $n + 1 \leq j \leq n + n'$.

If for distinct j, j' we have $b_j = b_{j'}$ and t_j identical to $t_{j'}$ up to permutation of sums, then we may apply axiom (output-choice) (after possibly rearranging terms using (choice-comm) and (choice-assoc)) to reduce the number of summands. As the number of summands cannot decrease forever, the reduction eventually terminates with an output normal form.

3. Suppose u is an input normal form and t' is a normal form. If t' is in fact an input normal form u', then the proof reduces to case 1 above.

   If t' is an output normal form v', then either t' is trivial or it is not. If t' is not trivial, then $u + t'$ is already a normal form. If t' is trivial, then axiom (choice-unit) can be used to prove $u + t'$ equivalent to u, which is an input normal form, hence a normal form.

   If t' is $u' + v'$, where u' is an input normal form and v' is a nontrivial output normal form, then axioms (choice-comm) and (choice-assoc) can be used to prove $u + t'$ equivalent to $(u + u') + v'$. Then $u + u'$ can be proved equivalent to an input normal form u'' by case 1 above, hence $u + t'$ can be proved equivalent to the normal form $u'' + v'$.

   The case in which t is a normal form and u' is an input normal form is symmetric to that just considered.

4. Suppose t and t' are normal forms. If either t or t' is an input normal form, then this case reduces to case 1 above, so we suppose that neither t nor t' is an input normal form.

Suppose t is an output normal form and t' is $u' + v'$, where u' is an input normal form and v' is a nontrivial output normal form. Using axioms (choice-comm) and (choice-assoc) we can prove $t + t'$ equivalent to $u' + (t + v')$. By case 3 above, $t + v'$ can be proved equivalent to a nontrivial output normal form v'', thereby showing that $t + t'$ can be proved equivalent to the normal form $u' + v''$.

Suppose both t and t' are output normal forms. Then by case 2, $t + t'$ can be proved equivalent to an output normal form, hence normal form, t''.

Suppose t is $u+v$, where u is an input normal form and v is a nontrivial output normal form, and t' is $u' + v'$, where u' is an input normal form and v' is a nontrivial output normal form. Then $t + t'$ can be proved equivalent to $(u + u') + (v + v')$ using axioms (choice-comm) and (choice-assoc). By case 1, $u + u'$ can be proved equivalent to an input normal form u'', and by case 2, $v + v'$ can be proved equivalent to a nontrivial output normal form v''. Thus, $t + t'$ can be proved equivalent to the normal form $u'' + v''$.

□

Lemma 8. *Any* $\|$*-free term* t *in* $\mathrm{Proc}(I/J \Rightarrow O)$ *can be proved equivalent to a normal form using the axioms in Table 1.*

Proof. The proof is by structural induction on t.

- Suppose t is nil_I. If $I = \emptyset$, then t is an output normal form, hence a normal form. If $I \neq \emptyset$, then t is an input normal form, hence a normal form.

- Suppose t is $\langle a?_p \rangle\ u$. By induction, u can be proved equal to a normal form u', from which it follows by substitutivity that t can be proved equivalent to $t' = \langle a?_p \rangle\ u'$. If t' is reduced with respect to axiom (nil-fold), then it is an input normal form, hence also a normal form. If t' is not reduced with respect to axiom (nil-fold), then we can apply axiom (nil-fold) to prove t' equivalent to $\mathrm{nil}_{\{a\}}$, which is an input normal form, hence a normal form.

- Suppose t is $\langle b!_r \rangle\ u$. By induction, u can be proved equal to a normal form u', from which it follows by substitutivity that t can be proved equivalent to $t' = \langle b!_r \rangle\ u'$, which is an output normal form and a normal form.

- Suppose t is $t_1 + t_2$. By induction, t_1 can be proved equivalent to a normal form t_1' and t_2 can be proved equivalent to a normal form t_2', so that t can be proved equivalent to $t_1' + t_2'$ by substitutivity. Lemma 7 now shows that $t_1' + t_2'$ can be proved equivalent to a normal form, hence t can be proved equivalent to that same normal form.

□

Lemma 9. *Suppose u and u' are input normal forms in* $\mathrm{Proc}(I/J \Rightarrow O)$ *and v and v' are output normal forms in* $\mathrm{Proc}(\emptyset/J \Rightarrow O)$. *Then $u + v$ and $u' + v'$ in* $\mathrm{Proc}(I/J \Rightarrow O)$ *are weighted bisimulation equivalent if and only if u and u' are weighted bisimulation equivalent in* $\mathrm{Proc}(I/J \Rightarrow O)$ *and v and v' are weighted bisimulation equivalent in* $\mathrm{Proc}(\emptyset/J \Rightarrow O)$.

Proof. If u and u' are weighted bisimulation equivalent in $\mathrm{Proc}(I/J \Rightarrow O)$ and v and v' are weighted bisimulation equivalent in $\mathrm{Proc}(\emptyset/J \Rightarrow O)$, then $u + v$ and $u' + v'$ are weighted bisimulation equivalent in $\mathrm{Proc}(I/J \Rightarrow O)$ by substitutivity.

Conversely, suppose $u+v$ and $u'+v'$ are weighted bisimulation equivalent in $\mathrm{Proc}(I/J \Rightarrow O)$. Let R be a weighted bisimulation relation on $\mathrm{Proc}(J/J \Rightarrow O)$ such that for all $e \in J \cup O$ and all equivalence classes $\mathcal{C}$ of R we have $\Delta_e^O(u+v, \mathcal{C}) = \Delta_e^O(u'+v', \mathcal{C})$. We claim that for all $e \in J \cup O$ and all equivalence classes $\mathcal{C}$ of R we have $\Delta_e^O(u, \mathcal{C}) = \Delta_e^O(u', \mathcal{C})$ and $\Delta_e^O(v, \mathcal{C}) = \Delta_e^O(v', \mathcal{C})$, so that u and u' are weighted bisimulation equivalent in $\mathrm{Proc}(I/J \Rightarrow O)$ and v and v' are weighted bisimulation equivalent in $\mathrm{Proc}(\emptyset/J \Rightarrow O)$.

For, observe that

$$\Delta_e^O(u+v, \mathcal{C}) = \begin{cases} \Delta_e^O(u, \mathcal{C}), & \text{if } e \in J \\ \Delta_e^O(v, \mathcal{C}), & \text{if } e \in O \end{cases}$$

and similarly

$$\Delta_e^O(u'+v', \mathcal{C}) = \begin{cases} \Delta_e^O(u', \mathcal{C}), & \text{if } e \in J \\ \Delta_e^O(v', \mathcal{C}), & \text{if } e \in O \end{cases}$$

Now, $\Delta_e^O(u+v, \mathcal{C}) = \Delta_e^O(u'+v', \mathcal{C})$ by hypothesis. If $e \in J$, then $\Delta_e^O(u, \mathcal{C}) = \Delta_e^O(u+v, \mathcal{C}) = \Delta_e^O(u'+v', \mathcal{C}) = \Delta_e^O(u', \mathcal{C})$ by the observation above. If $e \in O$, then $\Delta_e^O(u, \mathcal{C}) = 0 = \Delta_e^O(u', \mathcal{C})$. Hence $\Delta_e^O(u, \mathcal{C}) = \Delta_e^O(u', \mathcal{C})$ for all $e \in J \cup O$. Similarly, if $e \in O$, then $\Delta_e^O(v, \mathcal{C}) = \Delta_e^O(u+v, \mathcal{C}) = \Delta_e^O(u'+v', \mathcal{C}) = \Delta_e^O(v', \mathcal{C})$ by the observation above, and if $e \in J$, then $\Delta_e^O(v, \mathcal{C}) = 0 = \Delta_e^O(v', \mathcal{C})$. Hence $\Delta_e^O(v, \mathcal{C}) = \Delta_e^O(v', \mathcal{C})$ for all $e \in J \cup O$, completing the proof. □

Lemma 10. *For all I, J, and O:*

1. *If output normal forms t and t' in* $\mathrm{Proc}(I/J \Rightarrow O)$ *are weighted bisimulation equivalent, then they are identical up to permutation of sums.*

2. *If input normal forms t and t' in* $\mathrm{Proc}(I/J \Rightarrow O)$ *are weighted bisimulation equivalent, then they are identical up to permutation of sums.*

3. *If normal forms t and t' in* $\mathrm{Proc}(I/J \Rightarrow O)$ *are weighted bisimulation equivalent, then they are identical up to permutation of sums.*

Proof. We prove all three conclusions simultaneously by induction on the sum of the depths of nesting of input and output prefix operators in terms t and t'. Suppose we have established the result for all pairs of terms u and u' in $\mathrm{Proc}(I'/J' \Rightarrow O')$ whose prefix depths sum to strictly less than some $d \geq 0$, and suppose t and t' are terms in $\mathrm{Proc}(I/J \Rightarrow O)$ whose prefix depths sum to d.

1. Suppose output normal forms t and t' in $\mathrm{Proc}(I/J \Rightarrow O)$ are weighted bisimulation equivalent. Then t has the form $\sum_{j=1}^{n} \langle b_j !_{r_j} \rangle\ t_j$ and t' has the form $\sum_{j=1}^{n'} \langle b'_j !_{r'_j} \rangle\ t'_j$. We claim that there can be no distinct k, l with $1 \leq k, l \leq n$ such that $b_k = b_l$ and such that t_k and t_l are weighted bisimulation equivalent. For if there were such k, l, then by induction hypothesis t_k and t_l would be identical up to permutation of sums. But then since also $b_k = b_l$ we would have a contradiction with the assumption that t is an output normal form. Similar reasoning applies to t'.

   Now, since t and t' are weighted bisimulation equivalent, for each j with $1 \leq j \leq n$ there must exist j' with $1 \leq j' \leq n'$ such that $b_j = b'_{j'}$ and such that t_j and $t'_{j'}$ are weighted bisimulation equivalent. Conversely, for each j' with $1 \leq j' \leq n'$ there must exist j with $1 \leq j \leq n$ such that the same relationships hold. Moreover, by the claim of the previous paragraph, j' is uniquely determined by j and vice versa and we must therefore have $r_j = \Delta^O_{b_j}(t, t_j) = \Delta^O_{b'_{j'}}(t, t'_{j'}) = r'_{j'}$ for corresponding j and j'. It follows that $n = n'$ and there is a bijection $\phi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n'\}$ such that $b'_{\phi(j)} = b_j$, $r'_{\phi(j)} = r_j$, and $t'_{\phi(j)}$ and t_j are weighted bisimulation equivalent for $1 \leq j \leq n$.

   Since the sum of the prefix depths of t_j and $t'_{\phi(j)}$ is strictly less than the sum of the prefix depths of t and t', by induction we can conclude that t_j and $t'_{\phi(j)}$ are identical up to permutation of sums. Thus, for $1 \leq j \leq n$ we have $b'_{\phi(j)} = b_j$, $r'_{\phi(j)} = r_j$, and $t'_{\phi(j)}$ and t_j are identical up to permutation of sums. But this implies that t and t' are identical up to permutation of sums, as was to be proved.

2. Suppose input normal forms t and t' in $\mathrm{Proc}(I/J \Rightarrow O)$ are weighted bisimulation equivalent. If both t and t' are nil_J, then they are identical and there is nothing to prove, so suppose at least one of t and t' is not nil_J. We consider the case in which t is not nil_J; the case in which t' is not nil_J is symmetric. If t is not nil_J, then t has the form
$$\sum_{i=1}^{m} \langle a_i ?_{p_i} \rangle\ t_i.$$
   We claim that in this case t' also must not be nil_J. Suppose the contrary, then the following must hold by the assumption that t and t' are weighted bisimulation equivalent:

   (a) $I = \{a_1, a_2, \ldots, a_m\} = J$.

   (b) Each t_i is weighted bisimulation equivalent to nil_J.

   (c) $\sum_{\{i \in I : a_i = a\}} p_i = 1$ for all $a \in I$.

   Because the sum of the prefix depths of t_i and nil_J is strictly less than d, by induction it follows that each t_i is identical up to permutation of sums to nil_J, hence each t_i *is* nil_J. Because of this and the fact that t is an input normal form, for each $a \in I$ there can be at most one i such that $a_i = a$. Thus, all the p_i are equal to 1. But we have thus shown that t has the form

$$\sum_{a \in J} \langle a?_1 \rangle \ \mathrm{nil}_J.$$

This is a contradiction with the fact that t is reduced with respect to axiom (nil-fold), so we conclude that t' cannot be nil_J.

It remains to consider the case in which both t and t' are not nil_I. Then t has the form

$$\sum_{i=1}^{m} \langle a_i ?_{p_i} \rangle \ t_i.$$

and t' has the form

$$\sum_{i=1}^{m'} \langle a'_i ?_{p'_i} \rangle \ t'_i.$$

We may now argue as in case (1) above, using the fact that there can be no distinct k, l such that $a_k = a_l$ and such that t_k is identical to t_l up to permutation of sums, and the symmetric fact for t', to conclude that t and t' are weighted bisimulation equivalent.

3. Suppose normal forms t and t' in $\mathrm{Proc}(I/J \Rightarrow O)$ are weighted bisimulation equivalent.

   Suppose one of t or t' is an output normal form. We claim the other must be as well. For, if t is an output normal form, then $I = \emptyset$, hence in view of the assumption that t and t' are weighted bisimulation equivalent, t' cannot be nil_J, nor can t' have any input-prefixed summands. The only remaining possibility is that t' is an output normal form. Symmetric reasoning applies in case t' is an output normal form. Thus, if one of t or t' is an output normal form, the other is as well, and the proof then reduces to that in case (1) above.

   Suppose one of t or t' is an input normal form. We claim the other must be as well. For, if t is an input normal form, then in view of the assumption that t and t' are weighted bisimulation equivalent, t' cannot have any output-prefixed summands. The only remaining possibility is that t' is an input normal form. Thus, if one of t or t' is an input normal form, the other is as well, and the proof then reduces to that in case (2) above.

   Finally, suppose that t is $u + v$ where u is an input normal form and v is a nontrivial output normal form and that t' is $u' + v'$ where u' is an input normal form and v' is a nontrivial output normal form. By Lemma 9, u and u' are weighted bisimulation equivalent in $\mathrm{Proc}(I/J \Rightarrow O)$ and v and v' are weighted bisimulation equivalent in $\mathrm{Proc}(\emptyset/J \Rightarrow O)$. By case (1) already established, v and v' are identical up to permutation of sums. By case (2) already established, u and u' are identical up to permutation of sums. It follows that t and t' are identical up to permutation of sums.

□

Theorem 1. *The axioms in Table 1 are sound and complete for weighted bisimulation equivalence of $\|$-free terms.*

Proof. Soundness was shown in Lemma 6.

Suppose terms t and u are weighted bisimulation equivalent. Then t can be proved equivalent to a normal form t', and u can be proved equivalent to a normal form u'. By soundness, t' and u' are weighted bisimulation equivalent. Since t' and u' are weighted bisimulation equivalent normal forms, they are identical up to permutation of sums. Thus t and u can be proved equivalent to terms identical up to permutation of sums, hence they can be proved equivalent to each other. □

4 Behavior Equivalence

We now consider the theory of behavior equivalence. To define behavior equivalence, we need some auxiliary concepts. First is the notion of the *aggregate rate* $\mathrm{rt}(t)$ of a term $t \in \mathrm{Proc}(I/J \Rightarrow O)$. This is defined by: $\mathrm{rt}(t) = \sum_{e \in O} \sum_{t'} \Delta_e^O(t, t')$. The following result gives a syntax-directed characterization of $\mathrm{rt}(t)$.

Lemma 11. *Suppose* $t \in \mathrm{Proc}(I/J \Rightarrow O)$. *Then*

- *If t has the form* nil_I *or* $\langle a?_w \rangle\ u$, *then* $\mathrm{rt}(t) = 0$.
- *If t has the form* $\langle b!_r \rangle\ u$, *then* $\mathrm{rt}(t) = r$.
- *If t has the form* $u + v$, *then* $\mathrm{rt}(t) = \mathrm{rt}(u) + \mathrm{rt}(v)$.
- *If t has the form* $u\ {}_{O_u}\|_{O_v}\ v$, *where u and v are input-stochastic (*cf. *Section 2.2), then* $\mathrm{rt}(t) = \mathrm{rt}(u) + \mathrm{rt}(v)$.

Proof. Structural induction on t, using the definitions of $\mathrm{rt}(t)$ and Δ_e^O. □

Next, we define a *rated action* to be a pair $\langle e, r \rangle \in \mathit{Act} \times [0, \infty)$. A *rated trace* is a finite sequence of rated actions:

$$\langle e_1, r_1 \rangle \langle e_2, r_2 \rangle \ldots \langle e_n, r_n \rangle.$$

We use ϵ to denote the empty rated trace.

An *observable* is a mapping from rated traces to real numbers. We use Obs to denote the set of all observables. The *derivative* of an observable Φ by a rated action $\langle e, r \rangle$ is the observable $\langle e, r \rangle^{-1}\Phi$ defined by

$$(\langle e, r \rangle^{-1}\Phi)(\alpha) = \Phi(\langle e, r \rangle\ \alpha)$$

for all all rated traces α.

To each term t in $\mathrm{Proc}(I/J \Rightarrow O)$ we associate a *behavior map* $\mathcal{B}_t^O : \mathrm{Obs} \to \mathrm{Obs}$ defined by induction on the length of a rated trace as follows:

1. $\mathcal{B}_t^O[\Phi](\epsilon) = \Phi(\epsilon)$.

2. $\mathcal{B}_t^O[\Phi](\langle e, r\rangle\alpha) = \sum_u \Delta_e^O(t,u) \cdot \mathcal{B}_u^O[\langle e, r + \mathrm{rt}(t)\rangle^{-1}\Phi](\alpha)$.

Terms t and t' in $\mathrm{Proc}(I/J \Rightarrow O)$ are defined to be *behavior equivalent*, and we write $t \underset{O}{\equiv} u$, if $\mathcal{B}_t^O = \mathcal{B}_u^O$.

4.1 Properties of Behavior Maps

The following result helps to clarify the nature of the somewhat obscure-looking definition of $\mathcal{B}_t^O$. The inductive definition of $\mathcal{B}_t^O$ basically serves to describe the computation of the coefficients c_k and "rate variants" α_k of the rated trace α in the indicated linear combination.

Lemma 12. *Let sets of actions I, J, and O be given. Then for all rated traces α, for all terms t with inferable type $I/J \Rightarrow O$, and for all observables Φ, $\mathcal{B}_t^O[\Phi](\alpha)$ can be expressed as a linear combination of the values of Φ at a finite number of arguments. That is:*

$$\mathcal{B}_t^O[\Phi](\alpha) = \sum_{k \in K} c_k \cdot \Phi(\alpha_k).$$

for some finite set K, values $\{c_k : k \in K\} \subseteq (0, \infty)$ and rated traces $\{\alpha_k : k \in K\}$.

Proof. We prove, by induction on the length of α, that for all α, for all t, and forall Φ, $\mathcal{B}_t^O[\Phi](\alpha)$ can be expressed as a finite linear combination of values of Φ. In the basis case, $\alpha = \epsilon$, we have

$$\mathcal{B}_t^O[\Phi](\alpha) = \Phi(\epsilon),$$

so given any t we may take $K = \{*\}$ (a one-point set), $c_* = 1$, and $\alpha_* = \epsilon$ to obtain the result.

Now suppose we have shown the result for α and consider a rated trace of the form $\langle e, r\rangle\alpha$. Then we have

$$\begin{aligned}
\mathcal{B}_t^O[\Phi](\langle e, r\rangle\alpha) &= \sum_u \Delta_e^O(t,u) \cdot \mathcal{B}_u^O[\langle e, r + \mathrm{rt}(t)\rangle^{-1}\Phi](\alpha) \\
&= \sum_u \Delta_e^O(t,u) \cdot \sum_{k \in K'} c_k \cdot (\langle e, r + \mathrm{rt}(t)\rangle^{-1}\Phi)(\alpha_k) \\
&= \sum_u \sum_{k \in K'} \Delta_e^O(t,u)\; c_k \cdot \Phi(\langle e, r + \mathrm{rt}(t)\rangle\alpha_k),
\end{aligned}$$

where we have applied the induction hypothesis in the second line. Since there are only finitely many u for which $\Delta_e^O(t,u) \neq 0$, we may take

- K to be the set of all pairs (u, k) such that $k \in K'$ and $\Delta_e^O(t,u) \neq 0$,
- $c_{(u,k)} = \Delta_e^O(t,u)\; c_k$, and

- $\alpha_{(u,k)} = \langle e, r + \mathrm{rt}(t)\rangle\alpha_k$.

to obtain the desired result:

$$\mathcal{B}_t^O[\Phi](\alpha) = \sum_{(u,k)\in K} c_{(u,k)} \cdot \Phi(\alpha_{(u,k)}).$$

□

Lemma 13. *The mapping that takes Φ to $\langle e, r\rangle^{-1}\Phi$ is a linear operator on the set of observables Obs, considered as a vector space under pointwise addition and scalar multiplication.*

Proof. We calculate, using the definitions:

$$\begin{aligned}\langle e, r\rangle^{-1}(c\cdot\Phi + d\cdot\Psi)(\alpha) &= (c\cdot\Phi + d\cdot\Psi)(\langle e, r\rangle\alpha)\\ &= c\cdot\Phi(\langle e, r\rangle\alpha) + d\cdot\Psi(\langle e, r\rangle\alpha)\\ &= c\cdot\langle e, r\rangle^{-1}\Phi(\alpha) + d\cdot\langle e, r\rangle^{-1}\Psi(\alpha).\end{aligned}$$

□

Lemma 14. *Behavior maps $\mathcal{B}_t^O$ are linear operators on the set of observables Obs, considered as a vector space under pointwise addition and scalar multiplication.*

Proof. We prove, by induction on the length of a rated trace, that for all rated traces α, all t, all observables Φ and Ψ, and all real numbers c and d we have

$$\mathcal{B}_t^O[c\cdot\Phi + d\cdot\Psi](\alpha) = c\cdot\mathcal{B}_t^O[\Phi](\alpha) + d\cdot\mathcal{B}_t^O[\Psi](\alpha).$$

In the basis case, $\alpha = \epsilon$ we have:

$$\begin{aligned}\mathcal{B}_t^O[c\cdot\Phi + d\cdot\Psi](\epsilon) &= (c\cdot\Phi + d\cdot\Psi)(\epsilon)\\ &= c\cdot\Phi(\epsilon) + d\cdot\Psi(\epsilon)\\ &= c\cdot\mathcal{B}_t^O[\Phi](\epsilon) + d\cdot\mathcal{B}_t^O[\Psi](\epsilon).\end{aligned}$$

For the induction step, suppose the result has been shown for α and consider the rated trace $\langle e, r\rangle\alpha$. We calculate, using the definitions, the induction hypothesis, and Lemma 13:

$$\begin{aligned}\mathcal{B}_t^O[c\cdot\Phi + d\cdot\Psi](\langle e, r\rangle\alpha) &= \sum_u \Delta_e^O(t, u)\cdot\mathcal{B}_u^O[\langle e, r + \mathrm{rt}(t)\rangle^{-1}(c\cdot\Phi + d\cdot\Psi)](\alpha)\\ &= \sum_u \Delta_e^O(t, u)\cdot(c\cdot\mathcal{B}_u^O[\langle e, r + \mathrm{rt}(t)\rangle^{-1}\Phi](\alpha)\\ &\qquad + d\cdot\mathcal{B}_u^O[\langle e, r + \mathrm{rt}(t)\rangle^{-1}\Psi](\alpha))\\ &= \left(c\cdot\sum_u \Delta_e^O(t, u)\cdot\mathcal{B}_u^O[\langle e, r + \mathrm{rt}(t)\rangle^{-1}\Phi](\alpha)\right)\\ &\qquad + \left(d\cdot\sum_u \Delta_e^O(t, u)\cdot\mathcal{B}_u^O[\langle e, r + \mathrm{rt}(t)\rangle^{-1}\Psi](\alpha)\right)\\ &= c\cdot\mathcal{B}_t^O[\Phi](\langle e, r\rangle\alpha) + d\cdot\mathcal{B}_t^O[\Psi](\langle e, r\rangle\alpha)\end{aligned}$$

completing the induction step and the proof. □

Lemma 15. *For all $t \in \mathrm{Proc}(I/J \Rightarrow O)$, all observables Φ, all rated actions $\langle e, r\rangle$ and all rated traces α:*

$$\langle e, r\rangle^{-1}\mathcal{B}_t^O[\Phi](\alpha) \;=\; \sum_u \Delta_e^O(t,u)\cdot \mathcal{B}_u^O[\langle e, r+\mathrm{rt}(t)\rangle^{-1}\Phi](\alpha).$$

Proof. Calculation using the definitions shows:

$$\begin{aligned}\langle e, r\rangle^{-1}\mathcal{B}_t^O[\Phi](\alpha) &= \mathcal{B}_t^O[\Phi](\langle e, r\rangle\alpha)\\ &= \sum_u \Delta_e^O(t,u)\cdot \mathcal{B}_u^O[\langle e, r+\mathrm{rt}(t)\rangle^{-1}\Phi](\alpha).\end{aligned}$$

□

The following theorem gives a syntax-directed characterization of $\mathcal{B}_t^O$ that is useful in proofs. Case (1) is concerned with non-native actions. It reflects the fact that, although a term t does not participate in non-native actions, the occurrence of such an action implies that t has "lost the race" with its environment for control of that action, and consequently the rate of t must be taken into account in determining the value of the behavior map on an observable.

Lemma 16. *For all terms $t \in \mathrm{Proc}(I/J \Rightarrow O)$, all observables Φ, all rated traces α and rated actions $\langle e, r\rangle$:*

1. $\mathcal{B}_t^O[\Phi](\langle e, r\rangle\alpha) = \mathcal{B}_t^O[\langle e, r+\mathrm{rt}(t)\rangle^{-1}\Phi](\alpha)$, if $e \notin J \cup O$.
2. $\mathcal{B}_{\mathrm{nil}_J}^O[\Phi](\langle e, r\rangle\alpha) = \begin{cases} \mathcal{B}_{\mathrm{nil}_J}^O[\langle e, r\rangle^{-1}\Phi](\alpha), & \text{if } e \in J,\\ 0, & \text{if } e \in O.\end{cases}$
3. $\mathcal{B}_{\langle a?_p\rangle\, t}^O[\Phi](\langle e, r\rangle\alpha) = \begin{cases} p\cdot\mathcal{B}_t^O[\langle e, r\rangle^{-1}\Phi](\alpha), & \text{if } e = a\\ 0, & \text{if } e \in (J\cup O)\setminus\{a\}.\end{cases}$
4. $\mathcal{B}_{\langle b!_s\rangle\, t}^O[\Phi](\langle e, r\rangle\alpha) = \begin{cases} s\cdot\mathcal{B}_t^O[\langle e, r+s\rangle^{-1}\Phi](\alpha), & \text{if } e = b\\ 0, & \text{if } e \in (J\cup O)\setminus\{b\}.\end{cases}$
5. $\mathcal{B}_{t+u}^O[\Phi](\langle e, r\rangle\alpha) = \mathcal{B}_t^O[\Phi](\langle e, r+\mathrm{rt}(u)\rangle\alpha) + \mathcal{B}_u^O[\Phi](\langle e, r+\mathrm{rt}(t)\rangle\alpha)$.
6. $\mathcal{B}_{t\ {}_{O_t}\|_{O_u}\ u}^O = \mathcal{B}_t^{O\setminus O_u}\circ\mathcal{B}_u^{O\setminus O_t} = \mathcal{B}_u^{O\setminus O_t}\circ\mathcal{B}_t^{O\setminus O_u}$, *assuming t and u are input-stochastic.*

Proof. (1)-(4) are straightforward consequences of the definitions.

To prove (5), we calculate:

$$\begin{aligned}
\mathcal{B}^O_{t+u}[\Phi](\langle e,r\rangle\alpha) &= \sum_v \Delta^O_e(t+u,v)\cdot\mathcal{B}^O_v[\langle e, r+\mathrm{rt}(t+u)\rangle^{-1}\Phi](\alpha)\\
&= \sum_v(\Delta^O_e(t,v)+\Delta^O_e(u,v))\cdot\mathcal{B}^O_v[\langle e, r+\mathrm{rt}(t)+\mathrm{rt}(u)\rangle^{-1}\Phi](\alpha)\\
&= \sum_v \Delta^O_e(t,v)\cdot\mathcal{B}^O_v[\langle e,(r+\mathrm{rt}(u))+\mathrm{rt}(t)\rangle^{-1}\Phi](\alpha)\\
&\qquad +\sum_v \Delta^O_e(u,v)\cdot\mathcal{B}^O_v[\langle e,(r+\mathrm{rt}(t))+\mathrm{rt}(u)\rangle^{-1}\Phi](\alpha)\\
&= \mathcal{B}^O_t[\Phi](\langle e, r+\mathrm{rt}(u)\rangle\alpha)+\mathcal{B}^O_u[\Phi](\langle e, r+\mathrm{rt}(t)\rangle\alpha).
\end{aligned}$$

To establish (6), we prove, by induction on the length of a rated trace α, that for all suitably well-typed, input-stochastic t and u, all observables Φ and all rated traces α, that

$$\mathcal{B}^O_{t\ {}_{O_t}\|_{O_u}\ u}[\Phi](\langle e,r\rangle\alpha) \quad=\quad \mathcal{B}^{O\setminus O_u}_t[\mathcal{B}^{O\setminus O_t}_u[\Phi]](\alpha).$$

For the basis case, $\alpha=\epsilon$, we have

$$\begin{aligned}
\mathcal{B}^O_{t\ {}_{O_t}\|_{O_u}\ u}[\Phi](\epsilon) &= \Phi(\epsilon)\\
&= \mathcal{B}^{O\setminus O_t}_u[\Phi](\epsilon)\\
&= \mathcal{B}^{O\setminus O_u}_t[\mathcal{B}^{O\setminus O_t}_u[\Phi]](\epsilon).
\end{aligned}$$

For the induction step, suppose the result has been established for α and consider a rated trace $\langle e,r\rangle\alpha$. We calculate, using the definitions, the induction hypothesis, Lemma 14 and Lemma 15:

$$\begin{aligned}
\mathcal{B}^O_{t\ {}_{O_t}\|_{O_u}\ u}[\Phi](\langle e,r\rangle\alpha) &= \sum_{t'}\sum_{u'}\Delta^{O\setminus O_u}_e(t,t')\cdot\Delta^{O\setminus O_t}_e(u,u')\\
&\qquad\cdot\mathcal{B}^O_{t'\ {}_{O_t}\|_{O_u}\ u'}[\langle e, r+\mathrm{rt}(t)+\mathrm{rt}(u)\rangle^{-1}\Phi](\alpha)\\
&= \sum_{t'}\sum_{u'}\Delta^{O\setminus O_u}_e(t,t')\cdot\Delta^{O\setminus O_t}_e(u,u')\\
&\qquad\cdot\mathcal{B}^{O\setminus O_u}_{t'}[\mathcal{B}^{O\setminus O_t}_{u'}[\langle e, r+\mathrm{rt}(t)+\mathrm{rt}(u)\rangle^{-1}\Phi]](\alpha)\\
&= \sum_{t'}\Delta^{O\setminus O_u}_e(t,t')\cdot\mathcal{B}^{O\setminus O_u}_{t'}\\
&\qquad[\sum_{u'}\Delta^{O\setminus O_t}_e(u,u')\cdot\mathcal{B}^{O\setminus O_t}_{u'}[\langle e, r+\mathrm{rt}(t)+\mathrm{rt}(u)\rangle^{-1}\Phi]](\alpha)\\
&= \sum_{t'}\Delta^{O\setminus O_u}_e(t,t')\cdot\mathcal{B}^{O\setminus O_u}_{t'}[\langle e, r+\mathrm{rt}(t)\rangle^{-1}\mathcal{B}^{O\setminus O_t}_u[\Phi]](\alpha)\\
&= \mathcal{B}^{O\setminus O_u}_t[\mathcal{B}^{O\setminus O_t}_u[\Phi]](\langle e,r\rangle\alpha)
\end{aligned}$$

completing the induction step and the proof. Note that $t\ {}_{O_t}\|_{O_u}\ u=\mathrm{rt}(t)+\mathrm{rt}(u)$, which follows from the assumption of input-stochasticity, has been used in the first line. □

As the terms nil_I play an important role, the following characterization of their behaviors is useful.

Lemma 17. *Suppose* $\alpha = \langle e_1, r_1\rangle\langle e_2, r_2\rangle \ldots \langle e_k, r_k\rangle$. *Then*

$$\mathcal{B}^O_{\mathrm{nil}_I}[\Phi](\alpha) = \begin{cases} \Phi(\alpha), & \text{if } \{e_1, e_2, \ldots, e_k\} \cap O = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Straightforward calculation from the definitions. □

The following is a special case of a more general result, proved in Section 4.5, on the information about a term that can be extracted from its behavior map.

Lemma 18. *If* $t \underset{O}{\equiv} u$ *then* $\mathrm{rt}(t) = \mathrm{rt}(u)$.

Proof. Suppose $t, u \in \mathrm{Proc}(I/J \Rightarrow O)$ are such that $t \underset{O}{\equiv} u$. Let $*$ be an arbitrarily chosen (non-native) action in $Act \setminus (J \cup O)$. Let Φ be the observable defined as follows:

$$\Phi(\alpha) = \begin{cases} s, & \text{if } \alpha = \langle *, s\rangle, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \mathcal{B}^O_t[\Phi](\langle *, 0\rangle) &= \mathcal{B}^O_{t'}[\langle *, \mathrm{rt}(t)\rangle^{-1}\Phi](\epsilon) \\ &= \Phi(\langle *, \mathrm{rt}(t)\rangle) \\ &= \mathrm{rt}(t). \end{aligned}$$

Similarly, $\mathcal{B}^O_u[\Phi](\langle *, 0\rangle) = \mathrm{rt}(u)$. Since $t \underset{O}{\equiv} u$ by hypothesis, it follows that $\mathrm{rt}(t) = \mathrm{rt}(u)$. □

Further discussion of behavior maps and their properties can be found in our previous papers [WSS97, Sta03, SCS03].

4.2 Combinations

As indicated in the introduction, in order to axiomatize behavior equivalence, we extend our language by adding a construct for forming (convex) *combinations* of terms. Specifically, we add an additional binary operator ${}_{\pi}\!\oplus_{1-\pi}$, where the parameter π is a real number in the open interval $(0, 1)$. The following typing rule applies to this new operator:

$$\frac{t : I/J \Rightarrow O \quad u : I/J \Rightarrow O \quad \mathrm{rt}(t) = \mathrm{rt}(u)}{t \;{}_{\pi}\!\oplus_{1-\pi}\, u : I/J \Rightarrow O}$$

This rule requires that, for $t \;{}_{\pi}\!\oplus_{1-\pi}\, u$ to be well-typed, terms t and u must have the same aggregate rate as well as a common type. In this case we extend the notion of aggregate rate by defining $\mathrm{rt}(t \;{}_{\pi}\!\oplus_{1-\pi}\, u)$ to be the common value $\mathrm{rt}(t) = \mathrm{rt}(u)$.

We formally extend the transition semantics Δ_e^O given in Section 2.2 to encompass terms containing $t\ {}_{\pi}\!\oplus_{1-\pi}\, u$ by adding to the defining clauses given there the additional clause:

$$\Delta_e^O(t\ {}_{\pi}\!\oplus_{1-\pi}\, u, v) = \pi \cdot \Delta_e^O(t, v) + (1-\pi) \cdot \Delta_e^O(u, v).$$

Note that in making the extension we are implicitly re-interpreting the original clauses from Section 2.2 by allowing for the possibility of terms containing ${}_{\pi}\!\oplus_{1-\pi}$. For example, we now have

$$\Delta_a^O(\langle a?_w\rangle\ (t\ {}_{\pi}\!\oplus_{1-\pi}\, u),\ t\ {}_{\pi}\!\oplus_{1-\pi}\, u) \;=\; w.$$

We similarly extend the definition of $\mathcal{B}_t^O$ given earlier in this section to allow for the possibility of terms containing ${}_{\pi}\!\oplus_{1-\pi}$. These particular definitions are intuitively motivated by our desire for $t\ {}_{\pi}\!\oplus_{1-\pi}\, u$ to represent a probabilistic choice between (or superposition of) t and u; as considered, for example, in [And99]. Formally, we obtain the following result:

Lemma 19. *For all terms t, u in $\mathrm{Proc}(I/J \Rightarrow O)$ such that $\mathrm{rt}(t) = \mathrm{rt}(u)$, for all observables Φ and all rated traces α we have:*

$$\mathcal{B}^O_{t\ {}_{\pi}\oplus_{1-\pi}\, u}[\Phi](\alpha) \;=\; \pi \cdot \mathcal{B}_t^O[\Phi](\alpha) + (1-\pi) \cdot \mathcal{B}_u^O[\Phi](\alpha).$$

Proof. We proceed by induction on the length of a rated trace α. If $\alpha = \epsilon$, then

$$\begin{aligned}
\mathcal{B}^O_{t\ {}_{\pi}\oplus_{1-\pi}\, u}[\Phi](\alpha) &= \Phi(\epsilon) \\
&= \pi \cdot \Phi(\epsilon) + (1-\pi) \cdot \Phi(\epsilon) \\
&= \pi \cdot \mathcal{B}_t^O[\Phi](\alpha) + (1-\pi) \cdot \mathcal{B}_u^O[\Phi](\alpha).
\end{aligned}$$

If $\alpha = \langle e, r\rangle\alpha'$, then we have

$$\begin{aligned}
\mathcal{B}^O_{t\ {}_{\pi}\oplus_{1-\pi}\, u}[\Phi](\alpha) &= \sum_v \Delta_e^O(t\ {}_{\pi}\!\oplus_{1-\pi}\, u, v) \cdot \mathcal{B}_v^O[\langle e, r + \mathrm{rt}(t\ {}_{\pi}\!\oplus_{1-\pi}\, u)\rangle^{-1}\Phi](\alpha') \\
&= \sum_v (\pi \cdot \Delta_e^O(t, v) + (1-\pi) \cdot \Delta_e^O(u, v)) \\
&\qquad\qquad \cdot \mathcal{B}_v^O[\langle e, r + \mathrm{rt}(t\ {}_{\pi}\!\oplus_{1-\pi}\, u)\rangle^{-1}\Phi](\alpha') \\
&= \pi \cdot \sum_v \Delta_e^O(t, v) \cdot \mathcal{B}_v^O[\langle e, r + \mathrm{rt}(t)\rangle^{-1}\Phi](\alpha') \\
&\qquad + (1-\pi) \cdot \sum_v \Delta_e^O(u, v) \cdot \mathcal{B}_v^O[\langle e, r + \mathrm{rt}(u)\rangle^{-1}\Phi](\alpha') \\
&= \pi \cdot \mathcal{B}_t^O[\Phi](\alpha) + (1-\pi) \cdot \mathcal{B}_u^O[\Phi](\alpha)
\end{aligned}$$

completing the induction step and the proof. □

It will also be convenient to make use of an additional summation notation tailored to combinations. Specifically, for $n > 0$, for $0 < \pi_i < 1$ such that $\sum_{i=1}^n \pi_i = 1$, and for $\{t_1, t_2, \ldots, t_n\}$ sharing a common inferable type $I/J \Rightarrow O$ and total rate r, we define $\sum_{i=1}^n \pi_i \cdot t_i$ as follows:

1. For $n = 1$,

$$\sum_{i=1}^{n} \pi_i \cdot t_i = t_1$$

2. For $n > 1$,

$$\sum_{i=1}^{n} \pi_i \cdot t_i = \left(\sum_{i=1}^{n} \frac{\pi_i}{1-\pi_n} \cdot t_i \right) {}_{1-\pi_n}\!\oplus_{\pi_n} t_n$$

We will establish the soundness of laws for ${}_{\pi}\!\oplus_{1-\pi}$ that make it possible to manipulate such summations without ambiguity.

Lemma 20. *Behavior equivalence is substitutive for input prefixing, output prefixing, choice, combination of arbitrary terms, and also for parallel composition of input-stochastic terms. That is, each of the following assertions holds for terms t and t' in $\mathrm{Proc}(I/J \Rightarrow O)$ whenever all the terms mentioned are well-typed and the equivalences make sense:*

1. *If $t \underset{O}{\equiv} t'$ then $\langle a?_w \rangle\, t \underset{O}{\equiv} \langle a?_w \rangle\, t'$.*
2. *If $t \underset{O}{\equiv} t'$ then $\langle b!_r \rangle\, t \underset{O}{\equiv} \langle b!_r \rangle\, t'$.*
3. *If $t \underset{O}{\equiv} t'$ then $t + u \underset{O}{\equiv} t' + u$ and $u + t \underset{O}{\equiv} u + t'$.*
4. *If $t \underset{O}{\equiv} t'$ then $t \;{}_{O}\|_{O_u}\; u \underset{O'}{\equiv} t' \;{}_{O}\|_{O_u}\; u$ and $u \;{}_{O_u}\|_{O}\; t \underset{O'}{\equiv} u \;{}_{O_u}\|_{O}\; t'$, assuming t, t', and u are input-stochastic.*
5. *If $t \underset{O}{\equiv} t'$ then $t \;{}_{\pi}\!\oplus_{1-\pi}\, u \underset{O}{\equiv} t' \;{}_{\pi}\!\oplus_{1-\pi}\, u$ and $u \;{}_{\pi}\!\oplus_{1-\pi}\, t \underset{O}{\equiv} u \;{}_{\pi}\!\oplus_{1-\pi}\, t'$.*

Proof.

1. If $t \underset{O}{\equiv} t'$, then

$$\begin{aligned}
\mathcal{B}^{O}_{\langle a?_w \rangle\, t}[\Phi](\langle e,r\rangle\alpha) &= \sum_{v} \Delta^{O}_{e}(\langle a?_w \rangle\, t, v) \cdot \mathcal{B}^{O}_{t}[\langle e, r+0\rangle^{-1}\Phi](\alpha) \\
&= \begin{cases} w \cdot \mathcal{B}^{O}_{t}[\langle a,r\rangle^{-1}\Phi](\alpha), & \text{if } e = a \\ 0, & \text{otherwise.} \end{cases} \\
&= \begin{cases} w \cdot \mathcal{B}^{O}_{t'}[\langle a,r\rangle^{-1}\Phi](\alpha), & \text{if } e = a \\ 0, & \text{otherwise.} \end{cases} \\
&= \sum_{v} \Delta^{O}_{e}(\langle a?_w \rangle\, t', v) \cdot \mathcal{B}^{O}_{t'}[\langle e, r+0\rangle^{-1}\Phi](\alpha) \\
&= \mathcal{B}^{O}_{\langle a?_w \rangle\, t'}[\Phi](\langle e,r\rangle\alpha).
\end{aligned}$$

2. If $t \underset{O}{\equiv} t'$, then

$$\begin{array}{rcl}
\mathcal{B}^O_{\langle b!_r\rangle\ t}[\Phi](\langle e,s\rangle\alpha) & = & \sum_v \Delta^O_e(\langle b!_r\rangle\ t, v)\cdot \mathcal{B}^O_t[\langle e, r+s\rangle^{-1}\Phi](\alpha) \\
& = & \left\{\begin{array}{ll} r\cdot \mathcal{B}^O_t[\langle b, r+s\rangle^{-1}\Phi](\alpha), & \text{if } e=b \\ 0, & \text{otherwise.}\end{array}\right. \\
& = & \left\{\begin{array}{ll} r\cdot \mathcal{B}^O_{t'}[\langle b, r+s\rangle^{-1}\Phi](\alpha), & \text{if } e=b \\ 0, & \text{otherwise.}\end{array}\right. \\
& = & \sum_v \Delta^O_e(\langle b!_r\rangle\ t', v)\cdot \mathcal{B}^O_{t'}[\langle e, r+s\rangle^{-1}\Phi](\alpha) \\
& = & \mathcal{B}^O_{\langle b!_r\rangle\ t'}[\Phi](\langle e,s\rangle\alpha).
\end{array}$$

3. If $t \underset{O}{\equiv} t'$, then

$$\begin{array}{rcl}
\mathcal{B}^O_{t+u}[\Phi](\langle e,r\rangle\alpha) & = & \mathcal{B}^O_t[\Phi](\langle e, r+\mathrm{rt}(u)\rangle\alpha) + \mathcal{B}^O_u[\Phi](\langle e, r+\mathrm{rt}(t)\rangle\alpha) \\
& = & \mathcal{B}^O_{t'}[\Phi](\langle e, r+\mathrm{rt}(u)\rangle\alpha) + \mathcal{B}^O_u[\Phi](\langle e, r+\mathrm{rt}(t')\rangle\alpha) \\
& = & \mathcal{B}^O_{t'+u}[\Phi](\langle e,r\rangle\alpha),
\end{array}$$

where we have used Lemma 18 and the assumption that $t \underset{O}{\equiv} t'$ to conclude that $\mathrm{rt}(t) = \mathrm{rt}(t')$.

4. If $t \underset{O}{\equiv} t'$, then

$$\begin{array}{rcl}
\mathcal{B}^{O'}_{t\ {}_O\|_{O_u}\ u} & = & \mathcal{B}^{O'\setminus O_u}_t \circ \mathcal{B}^{O'\setminus O_t}_u \\
& = & \mathcal{B}^{O'\setminus O_u}_{t'} \circ \mathcal{B}^{O'\setminus O_t}_u \\
& = & \mathcal{B}^{O'}_{t\ {}_O\|_{O_u}\ u}.
\end{array}$$

5. If $t \underset{O}{\equiv} t'$, then

$$\begin{array}{rcl}
\mathcal{B}^O_{t\ {}_\pi\oplus_{1-\pi}\ u}[\Phi](\langle e,r\rangle\alpha) & = & \pi\cdot \mathcal{B}^O_t[\Phi](\langle e,r\rangle\alpha) + (1-\pi)\cdot \mathcal{B}^O_u[\Phi](\langle e,r\rangle\alpha) \\
& = & \pi\cdot \mathcal{B}^O_{t'}[\Phi](\langle e,r\rangle\alpha) + (1-\pi)\cdot \mathcal{B}^O_u[\Phi](\langle e,r\rangle\alpha) \\
& = & \mathcal{B}^O_{t'\ {}_\pi\oplus_{1-\pi}\ u}[\Phi](\langle e,r\rangle\alpha).
\end{array}$$

□

4.3 Axioms

Axioms for behavior equivalence are shown in Table 2. Note that an equation is only regarded an axiom if all the terms involved are well-formed and the same type can be inferred for the left and right-hand sides. Particular care must be taken when using axioms (input-distr) and (output-distr), to see that these equations are never applied in such a way as to create combinations whose operands have different rates.

In contrast to the axioms for weighted bisimulation equivalence, the axioms for behavior equivalence expose some distinction between input and output. For example, comparison of axiom (input-comb) and (output-comb) reveals that in (output-comb) the two output actions are permitted to be distinct. This is not permitted in (input-comb), because in that case the right-hand side would never be well-typed. Also, the axiom (input-extract) exhibits a special property of input that is not shared by output. Intuitively, since $\mathrm{rt}(\langle a?_p\rangle\ t) = 0$, its presence in $\langle a?_p\rangle\ t + u$ does not impact the behavior of u. Also, the presence or absence of $\mathrm{rt}(\langle a?_p\rangle\ t)$ does not affect the aggregate rate of the first argument to ${}_\pi\!\oplus_{1-\pi}$, so the combination is well-formed (*i.e.* the arguments have same aggregate rates) in either case. These statements would not hold if $\langle a?_p\rangle\ t$ were replaced by a term with nonzero aggregate rate. The content of axiom (interchange) is that the two types of sums commute freely with each other, subject only to the conditions on rates imposed by well-typedness.

Note that Table 2 includes all the axioms for weighted bisimulation equivalence, except for the axioms (input-choice) and (output-choice). However, these axioms are derivable, as we now show, so that all equations provable for weighted bisimulation equivalence are also provable for behavior equivalence.

Lemma 21. *Equations* (input-choice) *and* (output-choice) *are derivable from the axioms in Table 2.*

Proof. (input-choice)

$$\begin{aligned}
\langle a?_p\rangle\ t + \langle a?_q\rangle\ t &= \langle a?_{p+q}\rangle\ t\ {}_{\frac{p}{p+q}}\!\oplus_{\frac{q}{p+q}}\ \langle a?_{p+q}\rangle\ t && \text{by (input-comb)}\\
&= \langle a?_{p+q}\rangle\ (t\ {}_{\frac{p}{p+q}}\!\oplus_{\frac{q}{p+q}}\ t) && \text{by (input-distr)}\\
&= \langle a?_{p+q}\rangle\ t && \text{by (comb-idemp)}
\end{aligned}$$

(output-choice)

$$\begin{aligned}
\langle b!_r\rangle\ t + \langle b!_s\rangle\ t &= \langle b!_{r+s}\rangle\ t\ {}_{\frac{r}{r+s}}\!\oplus_{\frac{s}{r+s}}\ \langle b!_{r+s}\rangle\ t && \text{by (output-comb)}\\
&= \langle b!_{r+s}\rangle\ (t\ {}_{\frac{r}{r+s}}\!\oplus_{\frac{s}{r+s}}\ t) && \text{by (output-distr)}\\
&= \langle b!_{r+s}\rangle\ t && \text{by (comb-idemp)}
\end{aligned}$$

□

The theory generated by the axioms in Table 2 is not a conservative extension of that generated by the axioms in Table 1. A typical example of an equation that is provable from the axioms in Table 2, but which is not sound for weighted bisimulation equivalence is the following:

$$\langle b!_r\rangle\ (\langle c!_{s\pi}\rangle\ t + \langle d!_{s(1-\pi)}\rangle\ u) = \langle b!_{r\pi}\rangle\ \langle c!_s\rangle\ t + \langle b!_{r(1-\pi)}\rangle\ \langle d!_s\rangle\ u$$

where $0 < \pi < 1$. As we shall see, the axioms in Table 2 permit us to prove each term t equivalent to a normal form in which the same rate, $\mathrm{rt}(t)$, is displayed on all output-prefixed terms that are summed together.

$$t + \mathrm{nil}_\emptyset = t \qquad \text{(choice-unit)}$$

$$t + u = u + t \qquad \text{(choice-comm)}$$

$$(t + u) + v = t + (u + v) \qquad \text{(choice-assoc)}$$

$$\sum_{a \in I} \langle a?_1 \rangle \ \mathrm{nil}_I = \mathrm{nil}_I \qquad \text{(nil-fold)}$$

$$\langle a?_p \rangle \ t + \langle a?_q \rangle \ u = \langle a?_{p+q} \rangle \ t \ {}_{\frac{p}{p+q}}\!\oplus_{\frac{q}{p+q}} \langle a?_{p+q} \rangle \ u \qquad \text{(input-comb)}$$

$$\langle b!_r \rangle \ t + \langle c!_s \rangle \ u = \langle b!_{r+s} \rangle \ t \ {}_{\frac{r}{r+s}}\!\oplus_{\frac{s}{r+s}} \langle c!_{r+s} \rangle \ u \qquad \text{(output-comb)}$$

$$t = t \ {}_\pi\!\oplus_{1-\pi} t \qquad \text{(comb-idemp)}$$

$$t \ {}_\pi\!\oplus_{1-\pi} u = u \ {}_{1-\pi}\!\oplus_\pi t \qquad \text{(comb-comm)}$$

$$(t \ {}_\pi\!\oplus_{1-\pi} u) \ {}_\rho\!\oplus_{1-\rho} v = t \ {}_\sigma\!\oplus_{1-\sigma} (u \ {}_\tau\!\oplus_{1-\tau} v), \qquad \text{(comb-assoc)}$$
$$\text{whenever } \pi\rho = \sigma \text{ and } (1-\rho) = (1-\sigma)(1-\tau).$$

$$\langle a?_p \rangle \ t \ {}_\pi\!\oplus_{1-\pi} \langle a?_p \rangle \ u = \langle a?_p \rangle \ (t \ {}_\pi\!\oplus_{1-\pi} u) \qquad \text{(input-distr)}$$

$$\langle b!_r \rangle \ t \ {}_\pi\!\oplus_{1-\pi} \langle b!_r \rangle \ u = \langle b!_r \rangle \ (t \ {}_\pi\!\oplus_{1-\pi} u) \qquad \text{(output-distr)}$$

$$(t \ {}_\pi\!\oplus_{1-\pi} w) + (u \ {}_\pi\!\oplus_{1-\pi} v) = (t + u) \ {}_\pi\!\oplus_{1-\pi} (w + v) \qquad \text{(interchange)}$$

$$\langle a?_{\pi p} \rangle \ t + (u \ {}_\pi\!\oplus_{1-\pi} v) = (\langle a?_p \rangle \ t + u) \ {}_\pi\!\oplus_{1-\pi} v \qquad \text{(input-extract)}$$

Table 2: Axioms for Behavior Equivalence

Lemma 22. *The axioms in Table 2 are sound for behavior equivalence.*

Proof. For each equation $L = R$, where $\vdash L : I/J \Rightarrow O$ and $\vdash R : I/J \Rightarrow O$, we show, by induction on the length of a rated trace α, that for all rated traces alpha and all observables Φ we have $\mathcal{B}_L^O[\Phi](\alpha) = \mathcal{B}_R^O[\Phi](\alpha)$. In the basis case $\alpha = \epsilon$ the reasoning is always the same: $\mathcal{B}_L^O[\Phi](\epsilon) = \mathcal{B}_R^O[\Phi](\epsilon)$, which is immediate because by definition each side is equal to $\Phi(\epsilon)$. Hence, in the remainder of the proof we assume that $\alpha = \langle e, r\rangle\beta$. It is easy to check for each equation $L = R$ that $\mathrm{rt}(L) = \mathrm{rt}(R)$. In case $e \notin J \cup O$ (*i.e.* e is non-native), we have

$$\begin{array}{rcl} \mathcal{B}_L^O[\Phi](\langle e, r\rangle\beta) & = & \mathcal{B}_L^O[\langle e, r + \mathrm{rt}(L)\rangle^{-1}\Phi](\beta) \\ & = & \mathcal{B}_R^O[\langle e, r + \mathrm{rt}(R)\rangle^{-1}\Phi](\beta) \\ & = & \mathcal{B}_R^O[\Phi](\langle e, r\rangle\beta) \end{array}$$

where we have used the induction hypothesis in the second step.

It thus remains in each case for us to show that $\mathcal{B}_L^O[\Phi](\langle e, r\rangle\beta) = \mathcal{B}_R^O[\Phi](\langle e, r\rangle\beta)$ for all observables Φ, all rated actions $\langle e, r\rangle$ with $e \in J \cup O$, and all rated traces β. We consider each axiom in turn:

- Axiom (choice-unit). Recalling that $\mathrm{rt}(\mathrm{nil}_\emptyset) = 0$, we have

$$\begin{array}{rcl} \mathcal{B}_{t+\mathrm{nil}_\emptyset}^O[\Phi](\langle e, r\rangle\beta) & = & \mathcal{B}_t^O[\Phi](\langle e, r + \mathrm{rt}(\mathrm{nil}_\emptyset)\rangle\beta) + \mathcal{B}_{\mathrm{nil}_\emptyset}^O[\Phi](\langle e, r + \mathrm{rt}(t)\rangle\beta) \\ & = & \mathcal{B}_t^O[\Phi](\langle e, r\rangle\beta) + \mathcal{B}_{\mathrm{nil}_\emptyset}^O[\Phi](\langle e, r + \mathrm{rt}(t)\rangle\beta) \end{array}$$

  for all observables Φ, all rated actions $\langle e, r\rangle$ and all rated traces β. Now,

$$\mathcal{B}_{\mathrm{nil}_\emptyset}^O[\Phi](\langle e, r + \mathrm{rt}(t)\rangle\beta) = \begin{cases} 0, & \text{if } e \in O, \\ \mathcal{B}_{\mathrm{nil}_\emptyset}^O[\langle e, r + \mathrm{rt}(t)\rangle^{-1}\Phi](\beta), & \text{otherwise.} \end{cases}$$

  If $e \in O$, then $\mathcal{B}_{t+\mathrm{nil}_\emptyset}^O[\Phi](\langle e, r\rangle\beta) = \mathcal{B}_t^O[\Phi](\langle e, r\rangle\beta)$ follows immediately. If $e \notin O$, then in fact $e \notin J \cup O$ (*i.e.* e is non-native) because we can only have $\vdash \mathrm{nil}_\emptyset : I/J \Rightarrow O$ if $I = J = \emptyset$. The case of non-native e has already been treated above.

- Axiom (choice-comm).

$$\begin{array}{rcl} \mathcal{B}_{t+u}^O[\Phi](\langle e, r\rangle\beta) & = & \mathcal{B}_t^O[\Phi](\langle e, r + \mathrm{rt}(u)\rangle\beta) + \mathcal{B}_u^O[\Phi](\langle e, r + \mathrm{rt}(t)\rangle\beta) \\ & = & \mathcal{B}_u^O[\Phi](\langle e, r + \mathrm{rt}(t)\rangle\beta) + \mathcal{B}_t^O[\Phi](\langle e, r + \mathrm{rt}(u)\rangle\beta) \\ & = & \mathcal{B}_{u+t}^O[\Phi](\langle e, r\rangle\beta). \end{array}$$

- Axiom (choice-assoc).

$$\begin{array}{rcl} \mathcal{B}_{t+(u+v)}^O[\Phi](\langle e, r\rangle\beta) & = & \mathcal{B}_t^O[\Phi](\langle e, r + \mathrm{rt}(u+v)\rangle\beta) + \mathcal{B}_{u+v}^O[\Phi](\langle e, r + \mathrm{rt}(t)\rangle\beta) \\ & = & \mathcal{B}_t^O[\Phi](\langle e, r + \mathrm{rt}(u) + \mathrm{rt}(v)\rangle\beta) \\ & & \quad +\mathcal{B}_u^O[\Phi](\langle e, r + \mathrm{rt}(t) + \mathrm{rt}(v)\rangle\beta) \\ & & \quad +\mathcal{B}_v^O[\Phi](\langle e, r + \mathrm{rt}(t) + \mathrm{rt}(u)\rangle\beta) \\ & = & \mathcal{B}_{t+u}^O[\Phi](\langle e, r + \mathrm{rt}(v)\rangle\beta) + \mathcal{B}_v^O[\Phi](\langle e, r + \mathrm{rt}(t+u)\rangle\beta) \\ & = & \mathcal{B}_{(t+u)+v}^O[\Phi](\langle e, r\rangle\beta). \end{array}$$

- Axiom (nil-fold). Note that in this case well-typedness implies that $I = J$. Since $\mathrm{rt}(\langle a?_1\rangle\ \mathrm{nil}_I) = 0 = \mathrm{rt}(\mathrm{nil}_I)$ we have

$$\begin{aligned}
\mathcal{B}^O_{\sum_{a\in I}\langle a?_1\rangle\ \mathrm{nil}_I}[\Phi](\langle e,r\rangle\beta) &= \sum_{a\in I}\mathcal{B}^O_{\langle a?_1\rangle\ \mathrm{nil}_I}[\Phi](\langle e,r+0\rangle\beta) \\
&= \begin{cases} \mathcal{B}^O_{\mathrm{nil}_I}[\langle e,r\rangle^{-1}\Phi](\beta), & \text{if } e\in I, \\ 0, & \text{if } e\in O \end{cases} \\
&= \mathcal{B}^O_{\mathrm{nil}_I}[\Phi](\langle e,r\rangle\beta).
\end{aligned}$$

- Axiom (input-distr).

$$\begin{aligned}
&\mathcal{B}^O_{\langle a?_p\rangle\ (t\ {}_\pi\oplus_{1-\pi}\ u)}[\Phi](\langle e,r\rangle\beta) \\
&\quad= \begin{cases} p\cdot\mathcal{B}^O_{t\ {}_\pi\oplus_{1-\pi}\ u}[\langle e,r\rangle^{-1}\Phi](\beta), & \text{if } e=a \\ 0, & \text{otherwise} \end{cases} \\
&\quad= \begin{cases} p\pi\cdot\mathcal{B}^O_t[\langle e,r\rangle^{-1}\Phi](\beta) + p(1-\pi)\cdot\mathcal{B}^O_u[\langle e,r\rangle^{-1}\Phi](\beta) & \text{if } e=a \\ 0, & \text{otherwise} \end{cases} \\
&\quad= \pi\cdot\mathcal{B}^O_{\langle a?_p\rangle\ t}[\Phi](\langle e,r\rangle\beta) + (1-\pi)\cdot\mathcal{B}^O_{\langle a?_p\rangle\ u}[\Phi](\langle e,r\rangle\beta) \\
&\quad= \mathcal{B}^O_{\langle a?_p\rangle\ t\ {}_\pi\oplus_{1-\pi}\ \langle a?_p\rangle\ u}[\Phi](\langle e,r\rangle\beta).
\end{aligned}$$

- Axiom (output-distr).

$$\begin{aligned}
&\mathcal{B}^O_{\langle b!_s\rangle\ (t\ {}_\pi\oplus_{1-\pi}\ u)}[\Phi](\langle e,r\rangle\beta) \\
&\quad= \begin{cases} s\cdot\mathcal{B}^O_{t\ {}_\pi\oplus_{1-\pi}\ u}[\langle e,r+s\rangle^{-1}\Phi](\beta), & \text{if } e=a \\ 0, & \text{otherwise} \end{cases} \\
&\quad= \begin{cases} s\cdot\pi\cdot\mathcal{B}^O_t[\langle e,r+s\rangle^{-1}\Phi](\beta) & \\ \qquad +\ s\cdot(1-\pi)\cdot\mathcal{B}^O_u[\langle e,r+s\rangle^{-1}\Phi](\beta) & \text{if } e=a \\ 0, & \text{otherwise} \end{cases} \\
&\quad= \pi\cdot\mathcal{B}^O_{\langle b!_s\rangle\ t}[\Phi](\langle e,r\rangle\beta) + (1-\pi)\cdot\mathcal{B}^O_{\langle b!_s\rangle\ u}[\Phi](\langle e,r\rangle\beta) \\
&\quad= \mathcal{B}^O_{\langle b!_s\rangle\ t\ {}_\pi\oplus_{1-\pi}\ \langle b!_s\rangle\ u}[\Phi](\langle e,r\rangle\beta).
\end{aligned}$$

- Axiom (comb-idemp).

$$\begin{aligned}
\mathcal{B}^O_{t\ {}_\pi\oplus_{1-\pi}\ t}[\Phi](\langle e,r\rangle\beta) &= \pi\cdot\mathcal{B}^O_t[\Phi](\langle e,r\rangle\beta) + (1-\pi)\cdot\mathcal{B}^O_t[\Phi](\langle e,r\rangle\beta) \\
&= \mathcal{B}^O_t[\Phi](\langle e,r\rangle\beta).
\end{aligned}$$

- Axiom (comb-comm).

$$\begin{aligned}
\mathcal{B}^O_{t\ {}_\pi\oplus_{1-\pi}\ u}[\Phi](\langle e,r\rangle\beta) &= \pi\cdot\mathcal{B}^O_t[\Phi](\langle e,r\rangle\beta) + (1-\pi)\cdot\mathcal{B}^O_u[\Phi](\langle e,r\rangle\beta) \\
&= (1-\pi)\cdot\mathcal{B}^O_u[\Phi](\langle e,r\rangle\beta) + \pi\cdot\mathcal{B}^O_t[\Phi](\langle e,r\rangle\beta) \\
&= \mathcal{B}^O_{u\ {}_{1-\pi}\oplus_\pi\ t}[\Phi](\langle e,r\rangle\beta).
\end{aligned}$$

- Axiom (comb-assoc). Suppose $\pi\rho = \sigma$ and $(1-\rho) = (1-\sigma)(1-\tau)$. Then also $(1-\pi)\rho = (1-\sigma)\tau$ and we have

$$\begin{aligned}
&\mathcal{B}^O_{(t\ {}_\pi\!\oplus_{1-\pi}\ u)\ {}_\rho\!\oplus_{1-\rho}\ v}[\Phi](\langle e,r\rangle\beta)\\
&= \rho\cdot\mathcal{B}^O_{t\ {}_\pi\!\oplus_{1-\pi}\ u}[\Phi](\langle e,r\rangle\beta) + (1-\rho)\cdot\mathcal{B}^O_v[\Phi](\langle e,r\rangle\beta)\\
&= \pi\rho\cdot\mathcal{B}^O_t[\Phi](\langle e,r\rangle\beta) + (1-\pi)\rho\cdot\mathcal{B}^O_u[\Phi](\langle e,r\rangle\beta) + (1-\pi)(1-\rho)\cdot\mathcal{B}^O_v[\Phi](\langle e,r\rangle\beta)\\
&= \pi\rho\cdot\mathcal{B}^O_t[\Phi](\langle e,r\rangle\beta) + (1-\sigma)\cdot(\tau\cdot\mathcal{B}^O_u[\Phi](\langle e,r\rangle\beta) + (1-\tau)\cdot\mathcal{B}^O_v[\Phi](\langle e,r\rangle\beta))\\
&= \sigma\cdot\mathcal{B}^O_t[\Phi](\langle e,r\rangle\beta) + (1-\sigma)\cdot\mathcal{B}^O_{u\ {}_\tau\!\oplus_{1-\tau}\ v}[\Phi](\langle e,r\rangle\beta)\\
&= \mathcal{B}^O_{t\ {}_\sigma\!\oplus_{1-\sigma}\ (u\ {}_\tau\!\oplus_{1-\tau}\ v)}[\Phi](\langle e,r\rangle\beta).
\end{aligned}$$

- Axiom (input-comb).

$$\begin{aligned}
&\mathcal{B}^O_{\langle a?_p\rangle\ t+\langle a?_q\rangle\ u}[\Phi](\langle e,r\rangle\beta)\\
&= \mathcal{B}^O_{\langle a?_p\rangle\ t}[\Phi](\langle e,r+0\rangle\beta) + \mathcal{B}^O_{\langle a?_q\rangle\ u}[\Phi](\langle e,r+0\rangle\beta)\\
&= \begin{cases} p\cdot\mathcal{B}^O_t[\langle a,r\rangle^{-1}\Phi](\beta) + q\cdot\mathcal{B}^O_u[\langle a,r\rangle^{-1}\Phi](\beta) & \text{if } e=a\\ 0, & \text{otherwise}\end{cases}\\
&= \begin{cases} (p+q)\cdot\frac{p}{p+q}\cdot\mathcal{B}^O_t[\langle a,r\rangle^{-1}\Phi](\beta) & \\ \qquad + (p+q)\cdot\frac{q}{p+q}\cdot\mathcal{B}^O_u[\langle a,r\rangle^{-1}\Phi](\beta) & \text{if } e=a\\ 0, & \text{otherwise}\end{cases}\\
&= \mathcal{B}^O_{\langle a?_{p+q}\rangle\ t\ {}_{\frac{p}{p+q}}\!\oplus_{\frac{q}{p+q}}\ \langle a?_{p+q}\rangle\ u}[\Phi](\langle e,r\rangle\beta).
\end{aligned}$$

- Axiom (output-comb).

$$\begin{aligned}
&\mathcal{B}^O_{\langle b!_r\rangle\ t+\langle c!_{r'}\rangle\ u}[\Phi](\langle e,s\rangle\beta)\\
&= \mathcal{B}^O_{\langle b!_r\rangle\ t}[\Phi](\langle e,s+r'\rangle\beta) + \mathcal{B}^O_{\langle c!_{r'}\rangle\ u}[\Phi](\langle e,s+r\rangle\beta)\\
&= \begin{cases} r\cdot\mathcal{B}^O_t[\langle b,s+r+r'\rangle^{-1}\Phi](\beta) & \text{if } b=e\neq c\\ r'\cdot\mathcal{B}^O_u[\langle c,s+r+r'\rangle^{-1}\Phi](\beta) & \text{if } b\neq e=c\\ r\cdot\mathcal{B}^O_t[\langle b,s+r+r'\rangle^{-1}\Phi](\beta) & \\ \qquad + r'\cdot\mathcal{B}^O_u[\langle c,s+r+r'\rangle^{-1}\Phi](\beta) & \text{if } e=b=c\\ 0, & \text{otherwise}\end{cases}\\
&= \begin{cases} (r+r')\cdot\frac{r}{r+r'}\cdot\mathcal{B}^O_t[\langle b,s+r+r'\rangle^{-1}\Phi](\beta) & \text{if } b=e\neq c\\ (r+r')\cdot\frac{r'}{r+r'}\cdot\mathcal{B}^O_u[\langle c,s+r+r'\rangle^{-1}\Phi](\beta) & \text{if } b\neq e=c\\ (r+r')\cdot\frac{r}{r+r'}\cdot\mathcal{B}^O_t[\langle b,s+r+r'\rangle^{-1}\Phi](\beta) & \\ \qquad + (r+r')\cdot\frac{r'}{r+r'}\cdot\mathcal{B}^O_u[\langle c,s+r+r'\rangle^{-1}\Phi](\beta) & \text{if } e=b=c\\ 0, & \text{otherwise}\end{cases}\\
&= \mathcal{B}^O_{\langle b!_{r+r'}\rangle\ t\ {}_{\frac{r}{r+r'}}\!\oplus_{\frac{r'}{r+r'}}\ \langle c!_{r+r'}\rangle\ u}[\Phi](\langle e,s\rangle\beta).
\end{aligned}$$

- Axiom (interchange).

$$\begin{aligned}
&\mathcal{B}^O_{(t\ {}_\pi\!\oplus_{1-\pi}\, w)+(u\ {}_\pi\!\oplus_{1-\pi}\, v)}[\Phi](\langle e,r\rangle\beta)\\
&\quad= \mathcal{B}^O_{t\ {}_\pi\!\oplus_{1-\pi}\, w}[\Phi](\langle e, r+\mathrm{rt}(u\ {}_\pi\!\oplus_{1-\pi}\, v)\rangle\beta)\\
&\qquad\quad +\mathcal{B}^O_{u\ {}_\pi\!\oplus_{1-\pi}\, v}[\Phi](\langle e, r+\mathrm{rt}(t\ {}_\pi\!\oplus_{1-\pi}\, w)\rangle\beta)\\
&\quad= \pi\cdot\mathcal{B}^O_t[\Phi](\langle e,r+\mathrm{rt}(u)\rangle\beta) + (1-\pi)\cdot\mathcal{B}^O_w[\Phi](\langle e,r+\mathrm{rt}(v)\rangle\beta)\\
&\qquad\quad +\pi\cdot\mathcal{B}^O_u[\Phi](\langle e,r+\mathrm{rt}(t)\rangle\beta) + (1-\pi)\cdot\mathcal{B}^O_v[\Phi](\langle e,r+\mathrm{rt}(w)\rangle\beta)\\
&\quad= \pi\cdot(\mathcal{B}^O_t[\Phi](\langle e,r+\mathrm{rt}(u)\rangle\beta) + \mathcal{B}^O_u[\Phi](\langle e,r+\mathrm{rt}(t)\rangle\beta))\\
&\qquad\quad +(1-\pi)\cdot(\mathcal{B}^O_w[\Phi](\langle e,r+\mathrm{rt}(v)\rangle\beta) + \mathcal{B}^O_v[\Phi](\langle e,r+\mathrm{rt}(w)\rangle\beta))\\
&\quad= \pi\cdot\mathcal{B}^O_{t+u}[\Phi](\langle e,r\rangle\beta) + (1-\pi)\cdot\mathcal{B}^O_{w+v}[\Phi](\langle e,r\rangle\beta)\\
&\quad= \mathcal{B}^O_{(t+u)\ {}_\pi\!\oplus_{1-\pi}\,(w+v)}[\Phi](\langle e,r\rangle\beta).
\end{aligned}$$

Note that the above reasoning depends crucially on the assumptions $\mathrm{rt}(t\ {}_\pi\!\oplus_{1-\pi}\, w) = \mathrm{rt}(t) = \mathrm{rt}(w)$ and $\mathrm{rt}(u\ {}_\pi\!\oplus_{1-\pi}\, v) = \mathrm{rt}(u) = \mathrm{rt}(v)$, which are prerequisite for the well-typedness of $t\ {}_\pi\!\oplus_{1-\pi}\, w$ and $u\ {}_\pi\!\oplus_{1-\pi}\, v$.

- Axiom (input-extract).

$$\begin{aligned}
&\mathcal{B}^O_{\langle a?_{\pi p}\rangle\ t+(u\ {}_\pi\!\oplus_{1-\pi}\, v)}[\Phi](\langle e,s\rangle\beta)\\
&\quad= \mathcal{B}^O_{\langle a?_{\pi p}\rangle\ t}[\Phi](\langle e,s+\mathrm{rt}(u\ {}_\pi\!\oplus_{1-\pi}\, v)\rangle\beta) + \mathcal{B}^O_{u\ {}_\pi\!\oplus_{1-\pi}\, v}[\Phi](\langle e,s\rangle\beta)\\
&\quad= \pi\cdot\mathcal{B}^O_{\langle a?_p\rangle\ t}[\Phi](\langle e,s+\mathrm{rt}(u)\rangle\beta) + \pi\cdot\mathcal{B}^O_u[\Phi](\langle e,s\rangle\beta) + (1-\pi)\cdot\mathcal{B}^O_v[\Phi](\langle e,s\rangle\beta)\\
&\quad= \pi\cdot\mathcal{B}^O_{\langle a?_p\rangle\ t+u}[\Phi](\langle e,s\rangle\beta) + (1-\pi)\cdot\mathcal{B}^O_v[\Phi](\langle e,s\rangle\beta)\\
&\quad= \mathcal{B}^O_{(\langle a?_p\rangle\ t+u)\ {}_\pi\!\oplus_{1-\pi}\, v}[\Phi](\langle e,s\rangle\beta).
\end{aligned}$$

Note that the first and third steps crucially depend on the fact that the input-prefixed terms $\langle a?_{\pi p}\rangle\ t$ and $\langle a?_p\rangle\ t$ have aggregate rate 0, and that $\mathrm{rt}(\langle a?_{\pi p}\rangle\ t + (u\ {}_\pi\!\oplus_{1-\pi}\, v)) = \mathrm{rt}(u\ {}_\pi\!\oplus_{1-\pi}\, v) = \mathrm{rt}(u) = \mathrm{rt}(v) = \mathrm{rt}((\langle a?_p\rangle\ t+u)\ {}_\pi\!\oplus_{1-\pi}\, v) = s$.

□

The next few results obtain generalizations of axioms (output-comb), (interchange), and (input-extract) to n-ary summations. These generalizations will be needed for the completeness proof in the next section.

Lemma 23. *All equations of the following form (for $m \geq 1$) are provable:*

$$\langle b!_r\rangle\ t + \left(\sum_{i=1}^{m}\sigma_i\cdot\langle c_i!_s\rangle\ u_i\right) \quad = \quad \langle b!_{r+s}\rangle\ t\ {}_{\frac{r}{r+s}}\!\oplus_{\frac{s}{r+s}}\ \sum_{i=1}^{m}\sigma_i\cdot\langle b_i!_{r+s}\rangle\ t_i$$

Proof. We proceed by induction on m. If $m = 1$, then the stated equation is an instance of axiom (output-comb). Suppose $m > 2$. Using axioms (comb-comm), (comb-assoc), (comb-idemp), and (interchange) we may prove

$$\begin{aligned}
&\langle b!_r\rangle\ t + (\textstyle\sum_{i=1}^m \sigma_i \cdot \langle c_i!_s\rangle\ u_i) \\
&\quad= \langle b!_r\rangle\ t + \left(\langle c_1!_s\rangle\ u_1\ {}_{\sigma_1}{\oplus}_{1-\sigma_1}\ \textstyle\sum_{i=2}^m \frac{\sigma_i}{1-\sigma_1} \cdot \langle c_i!_s\rangle\ u_i\right) \\
&\quad= (\langle b!_r\rangle\ t\ {}_{\sigma_1}{\oplus}_{1-\sigma_1}\ \langle b!_r\rangle\ t) + \left(\langle c_1!_r\rangle\ u_1\ {}_{\sigma_1}{\oplus}_{1-\sigma_1}\ \textstyle\sum_{i=2}^m \frac{\sigma_i}{1-\sigma_1} \cdot \langle c_i!_s\rangle\ u_i\right) \\
&\quad= (\langle b!_r\rangle\ t + \langle c_1!_s\rangle\ u_1)\ {}_{\sigma_1}{\oplus}_{1-\sigma_1}\ \left(\langle b!_r\rangle\ t + \textstyle\sum_{i=2}^m \frac{\sigma_i}{1-\sigma_1} \cdot \langle c_i!_s\rangle\ u_i\right)
\end{aligned}$$

By axiom (output-comb) we may prove

$$\langle b!_r\rangle\ t + \langle c_1!_s\rangle\ u_1 \quad = \quad \langle b!_{r+s}\rangle\ t\ {}_{\frac{r}{r+s}}{\oplus}_{\frac{s}{r+s}}\ \langle c_1!_{r+s}\rangle\ t_1$$

By induction hypothesis, we may prove

$$\langle b!_r\rangle\ t + \sum_{i=2}^{m} \frac{\sigma_i}{1-\sigma_1} \cdot \langle c_i!_s\rangle\ u_i \quad = \quad \langle b!_{r+s}\rangle\ t\ {}_{\frac{r}{r+s}}{\oplus}_{\frac{s}{r+s}}\ \sum_{i=2}^{m} \frac{\sigma_i}{1-\sigma_1} \cdot \langle c_i!_{r+s}\rangle\ u_i$$

Substituting into

$$(\langle b!_r\rangle\ t + \langle c_1!_s\rangle\ u_1)\ {}_{\sigma_1}{\oplus}_{1-\sigma_1}\ \left(\langle b!_r\rangle\ t + \sum_{i=2}^{m} \frac{\sigma_i}{1-\sigma_1} \cdot \langle c_i!_s\rangle\ u_i\right)$$

gives

$$\left(\langle b!_{r+s}\rangle\ t\ {}_{\frac{r}{r+s}}{\oplus}_{\frac{s}{r+s}}\ \langle c_1!_{r+s}\rangle\ u_1\right)\ {}_{\sigma_1}{\oplus}_{1-\sigma_1}\ \left(\langle b!_{r+s}\rangle\ t\ {}_{\frac{r}{r+s}}{\oplus}_{\frac{s}{r+s}}\ \sum_{i=2}^{m} \frac{\sigma_i}{1-\sigma_1} \cdot \langle c_i!_{r+s}\rangle\ u_i\right)$$

and then using axioms (comb-comm) and (comb-assoc) to rearrange terms yields and axiom (comb-idemp) to combine the two occurrences of $\langle b!_{r+s}\rangle\ t$ yields the result. □

Lemma 24. *All equations of the following form are provable (for $m \geq 1$):*

$$\sum_{i=1}^{m} \langle b_i!_{r\sigma_i}\rangle\ t_i \quad = \quad \sum_{i=1}^{m} \sigma_i \cdot \langle b_i!_r\rangle\ t_i.$$

Proof. If $m = 1$, then there is nothing to prove. If $m = 2$, then the stated equation is an instance of axiom (input-comb). If $m > 2$, then we may prove

$$\sum_{i=1}^{m} \langle b_i!_{r\sigma_i}\rangle\ t_i \quad = \quad \langle b_1!_{r\sigma_1}\rangle\ t_1 + \sum_{i=2}^{m} \langle b_i!_{r\sigma_i}\rangle\ t_i$$

By induction,

$$\begin{aligned}
\langle b_1!_{r\sigma_1}\rangle\ t_1 + \sum_{i=2}^{m} \langle b_i!_{r\sigma_i}\rangle\ t_i &= \langle b_1!_{r\sigma_1}\rangle\ t_1 + \sum_{i=2}^{m} \langle b_i!_{r(1-\sigma_1)\frac{\sigma_i}{1-\sigma_1}}\rangle\ t_i \\
&= \langle b_1!_{r\sigma_1}\rangle\ t_1 + \sum_{i=2}^{m} \frac{\sigma_i}{1-\sigma_1} \cdot \langle b_i!_{r(1-\sigma_1)}\rangle\ t_i
\end{aligned}$$

By Lemma 23, this last expression is provably equivalent to

$$\langle b_1!_r\rangle\ t_1\ {}_{\sigma_1}\!\oplus_{1-\sigma_1} \sum_{i=2}^{m} \frac{\sigma_i}{1-\sigma_1} \cdot \langle b_i!_r\rangle\ t_i$$

But it follows from axioms (comb-comm) and (comb-assoc) that this is provably equivalent to

$$\sum_{i=1}^{m} \sigma_i \cdot \langle b_i!_r\rangle\ t_i$$

completing the induction step and the proof. □

Lemma 25. *Suppose $t_1, t_2, \ldots, t_n$ and $t'_1, t'_2, \ldots, t'_n$ ($n \geq 1$) are well-typed terms such that t_i and t'_i have a common inferable type and $\mathrm{rt}(t_i) = \mathrm{rt}(t'_i)$ for $1 \leq i \leq n$. Then the equation*

$$\sum_{i=1}^{n} t_i\ {}_{\pi}\!\oplus_{1-\pi} \sum_{i=1}^{n} t'_i = \sum_{i=1}^{n} (t_i\ {}_{\pi}\!\oplus_{1-\pi}\ t'_i)$$

is provable.

Proof. The proof is by induction on n. If $n = 1$ there is nothing to prove. Suppose $n > 1$. Then

$$\sum_{i=1}^{n} t_i\ {}_{\pi}\!\oplus_{1-\pi} \sum_{i=1}^{n} t'_i = (t_1 + \sum_{i=2}^{n} t_i)\ {}_{\pi}\!\oplus_{1-\pi}\ (t'_1 + \sum_{i=2}^{n} t'_i)$$

Since by hypothesis t_1 and t'_1 have a common inferable type and rate and the same for $\sum_{i=2}^{n} t_i$ and $\sum_{i=2}^{n} t'_i$, we may apply axiom (interchange) to prove the last expression above equivalent to

$$(t_1\ {}_{\pi}\!\oplus_{1-\pi}\ t'_1) + \left(\sum_{i=2}^{n} t_i\ {}_{\pi}\!\oplus_{1-\pi} \sum_{i=2}^{n} t'_i \right)$$

By induction, this is provably equivalent to

$$(t_1\ {}_{\pi}\!\oplus_{1-\pi}\ t'_1) + \sum_{i=2}^{n} (t_i\ {}_{\pi}\!\oplus_{1-\pi}\ t'_i)$$

which using axioms (choice-comm) and (choice-assoc) is provably equivalent to

$$\sum_{i=1}^{n}(t_i \ {}_{\pi}\!\oplus_{1-\pi} \, t'_i)$$

completing the induction step and the proof. □

Lemma 26. *All equations of the following form, where $m \geq 1$ and $n \geq 1$, are derivable from the axioms in Table 2:*

$$\sum_{i=1}^{m}\langle a?_{p_i}\rangle \ t_i \ {}_{\pi}\!\oplus_{1-\pi} \sum_{j=1}^{n}\langle a?_{q_j}\rangle \ u_j \quad = \quad \sum_{i=1}^{m}\langle a?_{\pi p_i}\rangle \ t_i + \sum_{j=1}^{n}\langle a?_{(1-\pi)q_j}\rangle \ u_j$$

Proof. We proceed by induction on the pair (m, n). If (m, n) is $(1, 1)$, then the stated equation is an instance of axiom (input-comb).

If $m > 1$, then

$$\sum_{i=1}^{m}\langle a?_{p_i}\rangle \ t_i \ {}_{\pi}\!\oplus_{1-\pi} \sum_{j=1}^{n}\langle a?_{q_j}\rangle \ u_j \quad = \quad \left(\langle a?_{p_1}\rangle \ t_1 + \sum_{i=2}^{m}\langle a?_{p_i}\rangle \ t_i\right) {}_{\pi}\!\oplus_{1-\pi} \sum_{j=1}^{n}\langle a?_{q_j}\rangle \ u_j$$

Using axiom (input-extract), the right-hand side is provably equivalent to

$$\langle a?_{\pi p_1}\rangle \ t_1 + \left(\sum_{i=2}^{m}\langle a?_{p_i}\rangle \ t_i \ {}_{\pi}\!\oplus_{1-\pi} \sum_{j=1}^{n}\langle a?_{q_j}\rangle \ u_j\right)$$

By induction, this last expression is provably equivalent to

$$\langle a?_{\pi p_1}\rangle \ t_1 + \left(\sum_{i=2}^{m}\langle a?_{\pi p_i}\rangle \ t_i + \sum_{j=1}^{n}\langle a?_{(1-\pi)q_j}\rangle \ u_j\right)$$

Axioms (choice-comm) and (choice-assoc) can now be used to show this provably equivalent to

$$\sum_{i=1}^{m}\langle a?_{\pi p_i}\rangle \ t_i + \sum_{j=1}^{n}\langle a?_{(1-\pi)q_j}\rangle \ u_j$$

completing the induction step and the proof. □

4.4 Normal Forms

Let the notions *input normal form*, *output normal form*, and *normal form* be defined mutually recursively as follows:

- An *input normal form* is a well-typed term u that is either nil_I for some $I \neq \emptyset$, or else has the form:

$$\sum_{a \in I} \sum_{s \in R_a} \langle a?_{p_{a,s}} \rangle \; t_{a,s},$$

  where we require that:

  1. $I \neq \emptyset$.
  2. Each $t_{a,s}$ is a normal form, with $\mathrm{rt}(t_{a,s}) = s$.
  3. For each $a \in I$ the set R_a is a nonempty finite subset of $(0, \infty)$.
  4. u is not an instance (up to permutation of sums) of the left-hand side of axiom (nil-fold).

- An *output normal form* is a well-typed term u that is either $\mathrm{nil}_\emptyset$ or else has the form:

$$\sum_{b \in O} \sum_{s \in R_b} \langle b!_{\sigma_{b,s} \cdot r} \rangle \; t_{b,s},$$

  where we require that:

  1. $O \neq \emptyset$.
  2. Each $t_{b,s}$ is a normal form, with $\mathrm{rt}(t_{b,s}) = s$.
  3. For each $b \in O$ the set R_b is a nonempty finite subset of $(0, \infty)$.
  4. Each $\sigma_{b,s}$ satisfies $0 < \sigma_{b,s} \leq 1$ and $\sum_{b \in O} \sum_{s \in R_b} \sigma_{b,s} = 1$.

  An output normal form is called *nontrivial* if it is not $\mathrm{nil}_\emptyset$.

- A *normal form* is either an input normal form, an output normal form, or a sum $u + v$, where u is an input normal form and v is a nontrivial output normal form.

Lemma 27. *Suppose t_1 and t_2 are terms having a common inferable type $I/J \Rightarrow O$ and such that $\mathrm{rt}(t_1) = \mathrm{rt}(t_2) = r$.*

1. *If t_1 and t_2 are input normal forms then there exists an input normal form t' such that the equation $t_1 \;{}_\pi{\oplus}_{1-\pi}\; t_2 = t'$ is provable.*

2. *If t_1 and t_2 are output normal forms then there exists an output normal form t' such that the equation $t_1 \;{}_\pi{\oplus}_{1-\pi}\; t_2 = t'$ is provable.*

3. *If t_1 and t_2 are normal forms then there exists a normal form t' such that the equation $t_1 \;{}_\pi{\oplus}_{1-\pi}\; t_2 = t'$ is provable.*

Proof. We prove all three claims simultaneously by induction on the sum of the maximum prefix depths of t_1 and t_2. Suppose we have established (1)-(3) for all pairs of terms whose whose maximum prefix depths sum to strictly less than some $d \geq 0$ and suppose the sum of the maximum prefix depths of t_1 and t_2 is exactly d.

1. Suppose t_1 and t_2 are input normal forms. If both t_1 and t_2 are nil_I, then we may prove $t_1 \;{}_{\pi}{\oplus}_{1-\pi}\; t_2 = \mathrm{nil}_I$ using axiom (comb-idemp). For the remainder of this case, we assume that t_1 and t_2 are not both nil_I.

   Now, t_1 is either nil_I or else has the form

$$\sum_{a\in I} \sum_{r\in R_{a,1}} \langle a_{a,r,1}?_{p_{a,r,1}}\rangle \; t_{a,r,1}$$

   Similarly, t_2 is either nil_{I_2} or else has the form

$$\sum_{a\in I} \sum_{r\in R_{a,2}} \langle a_{a,r,2}?_{p_{a,r,2}}\rangle \; t_{a,r,2}$$

   If t_1 is nil_I then we may use axiom (nil-fold) to prove it equivalent to a summation in the form above. The same reasoning applies to t_2. We assume that this has been done, so that neither t_1 nor t_2 is nil_I. Note that since t_1 and t_2 are not both nil_I, axiom (nil-fold) is applied to at most one of them. Because of this, for any pair i, i', the sum of the maximum prefix depths of $t_{i,1}$ and $t_{i',2}$ is strictly less than d. This observation will enable the application of the induction hypothesis in the argument below.

   We may now apply Lemma 25 to prove $t_1 \;{}_{\pi}{\oplus}_{1-\pi}\; t_2$ equivalent to

$$\sum_{a\in I} \left(\sum_{r\in R_{a,1}} \langle a?_{p_{a,r,1}}\rangle \; t_{a,r,1} \;{}_{\pi}{\oplus}_{1-\pi} \sum_{r\in R_{a,2}} \langle a?_{p_{a,r,2}}\rangle \; t_{a,r,2} \right)$$

   Using Lemma 26, this may be proved equivalent to

$$\sum_{a\in I} \left(\sum_{r\in R_{a,1}} \langle a?_{\pi p_{a,r,1}}\rangle \; t_{a,r,1} + \sum_{r\in R_{a,2}} \langle a?_{(1-\pi) p_{a,r,2}}\rangle \; t_{a,r,2} \right) \qquad (*)$$

   Using axioms (choice-comm) and (choice-assoc), this can be proved equivalent to

$$\sum_{a\in I} \sum_{r\in R_{a,1}\cup R_{a,2}} u_{a,r}$$

   where

$$u_{a,r} \;=\; \begin{cases} \langle a?_{\pi p_{a,r,1}}\rangle \; t_{a,r,1}, & \text{if } r \in R_{a,1} \setminus R_{a,2} \\ \langle a?_{(1-\pi) p_{a,r,2}}\rangle \; t_{a,r,2}, & \text{if } r \in R_{a,2} \setminus R_{a,1} \\ \langle a?_{\pi p_{a,r,1}}\rangle \; t_{a,r,1} + \langle a?_{(1-\pi) p_{a,r,2}}\rangle \; t_{a,r,2}, & \text{if } r \in R_{a,1} \cap R_{a,2}. \end{cases}$$

   If $r \in R_{a,1} \cap R_{a,2}$, then because $\mathrm{rt}(t_{a,r,1}) = r = \mathrm{rt}(t_{a,r,2})$, by axioms (input-comb) and (input-distr) the term

$$\langle a?_{\pi p_{a,r,1}}\rangle \; t_{a,r,1} + \langle a?_{(1-\pi) p_{a,r,2}}\rangle \; t_{a,r,2}$$

is provably equivalent to

$$\langle a?_q\rangle\ (t_{a,r,1}\ {}_{\sigma_{a,r,1}}\!\oplus_{\sigma_{a,r,2}} t_{a,r,2})$$

where

$$\begin{aligned} q &= \pi p_{a,r,1} + (1-\pi)p_{a,r,2} \\ \sigma_{a,r,1} &= \pi p_{a,r,1}/q \\ \sigma_{a,r,2} &= (1-\pi)p_{a,r,2}/q. \end{aligned}$$

Now, as observed above, the sum of the maximum prefix depths of $t_{i(a,r,1),1}$ and $t_{i(a,r,2),2}$ is strictly less than d. We may therefore apply the induction hypothesis to conclude that

$$t_{a,r,1}\ {}_{\sigma_{a,r,1}}\!\oplus_{\sigma_{a,r,2}} t_{a,r,2}$$

provably equivent to a normal form $t_{a,r}$. Thus $t_1\ {}_{\pi}\!\oplus_{1-\pi} t_2$ is provably equivalent to

$$\sum_{a\in I}\ \sum_{r\in R_1\cup R_2} \langle a?_{q_{a,r}}\rangle\ u'_{a,r}$$

where

$$\begin{aligned} q_{a,r} &= \begin{cases} \pi p_{a,r,1}, & \text{if } r\in R_1\setminus R_2 \\ (1-\pi)p_{a,r,2}, & \text{if } r\in R_2\setminus R_1 \\ q, & \text{if } r\in R_1\cap R_2. \end{cases} \\ u'_{a,r} &= \begin{cases} t_{a,r,1}, & \text{if } r\in R_1\setminus R_2 \\ t_{a,r,2}, & \text{if } r\in R_2\setminus R_1 \\ t_{a,r}, & \text{if } r\in R_1\cap R_2. \end{cases} \end{aligned}$$

If this last summation is reduced with respect to axiom (nil-fold), then it is already an input normal form. If it is not so reduced, then we may apply axiom (nil-fold) to prove it equivalent to nil_I, which is an input normal form.

2. Suppose t_1 and t_2 are output normal forms. If both t_1 and t_2 are trivial, then $t_1\ {}_{\pi}\!\oplus_{1-\pi} t_2$ can be proved equivalent to the output normal form $\mathrm{nil}_\emptyset$ using axiom (comb-idemp). Note that it is impossible for one of t_1 or t_2 to be trivial and the other nontrivial, because then we would have $\mathrm{rt}(t_1)\neq \mathrm{rt}(t_2)$ So, we assume in what follows that both t_1 and t_2 are nontrivial.

   Now, t_1 has the form

$$\sum_{b\in O_1}\ \sum_{s\in R_{b,1}} \langle b!_{\sigma_{b,s,1} r_1}\rangle\ t_{b,s,1}$$

   where $r_1 = \mathrm{rt}(t_1)$, and t_2 has the form

$$\sum_{b\in O_2}\ \sum_{s\in R_{b,2}} \langle b!_{\sigma_{b,s,2} r_2}\rangle\ t_{b,s,2}.$$

where $r_2 = \mathrm{rt}(t_2)$. By Lemma 24, t_1 may be proved equivalent to t_1' which has the form

$$\sum_{b \in O_1, s \in R_{b,1}} \sigma_{b,s,1} \cdot \langle b!_{r_1} \rangle \; t_{b,s,1}.$$

Similarly, t_2 may be proved equivalent to t_2' which has the form

$$\sum_{b \in O_2, s \in R_{b,2}} \sigma_{b,s,2} \cdot \langle b!_{r_2} \rangle \; t_{b,s,2}.$$

Hence $t_1 \; {}_{\pi}\!\oplus_{1-\pi} \; t_2$ may be proved equivalent to $t_1' \; {}_{\pi}\!\oplus_{1-\pi} \; t_2'$.

Since $\mathrm{rt}(t_1) = \mathrm{rt}(t_2) = r$ by hypothesis, using axioms (comb-comm) and (comb-assoc) the term $t_1' \; {}_{\pi}\!\oplus_{1-\pi} \; t_2'$ can be proved equivalent to

$$\sum_{b \in O_1 \cup O_2, s \in R_b} \sigma_{b,s} \cdot t_{b,s}$$

where

$$R_b = \begin{cases} R_{b,1}, & \text{if } b \in O_1 \setminus O_2 \\ R_{b,2}, & \text{if } b \in O_2 \setminus O_1 \\ R_{b,1} \cup R_{b,2}, & \text{if } b \in O_1 \cap O_2. \end{cases}$$

$$\sigma_{b,s} = \begin{cases} \pi\sigma_{b,s,1}, & \text{if } b \in O_1 \setminus O_2 \\ (1-\pi)\sigma_{b,s,2}, & \text{if } b \in O_2 \setminus O_1 \\ \pi\sigma_{b,s,1}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \setminus R_{b,2} \\ (1-\pi)\sigma_{b,s,2}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,2} \setminus R_{b,1} \\ \pi\sigma_{b,s,1} + (1-\pi)\sigma_{b,s,2}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \cap R_{b,2} \end{cases}$$

$$t_{b,s} = \begin{cases} \langle b!_r \rangle \; t_{b,s,1}, & \text{if } b \in O_1 \setminus O_2 \\ \langle b!_r \rangle \; t_{b,s,2}, & \text{if } b \in O_2 \setminus O_1 \\ \langle b!_r \rangle \; t_{b,s,1}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \setminus R_{b,2} \\ \langle b!_r \rangle \; t_{b,s,2}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,2} \setminus R_{b,1} \\ \langle b!_r \rangle \; t_{b,s,1} \; {}_{\pi_b}\!\oplus_{1-\pi_b} \langle b!_r \rangle \; t_{b,s,2}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \cap R_{b,2} \end{cases}$$

and where $\pi_b = \pi\sigma_{b,s,1}/\sigma_b$ for $b \in O_1 \cap O_2$.

Using axiom (output-distr) each term $\langle b!_r \rangle \; t_{b,s,1} \; {}_{\pi_b}\!\oplus_{1-\pi_b} \langle b!_r \rangle \; t_{b,s,2}$ can be proved equivalent to $\langle b!_r \rangle \; (t_{b,s,1} \; {}_{\pi_b}\!\oplus_{1-\pi_b} \; t_{b,s,2})$. Since the sum of the maximum prefix depths of $t_{b,s,1}$ and $t_{b,s,2}$ is strictly less than d, by induction each term $t_{b,s,1} \; {}_{\pi_b}\!\oplus_{1-\pi_b} \; t_{b,s,2}$ can be proved equivalent to a normal form $u_{b,s}$. Thus $t_1' \; {}_{\pi}\!\oplus_{1-\pi} \; t_2'$ can be proved equivalent to

$$\sum_{b \in O_1 \cup O_2, s \in R_b} \sigma_{b,s} \cdot t'_{b,s}$$

where $\sigma_{b,s}$ is as above and

$$t'_{b,s} \quad = \quad \begin{cases} \langle b!_r \rangle \; t_{b,s,1}, & \text{if } b \in O_1 \setminus O_2 \\ \langle b!_r \rangle \; t_{b,s,2}, & \text{if } b \in O_2 \setminus O_1 \\ \langle b!_r \rangle \; t_{b,s,1}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \setminus R_{b,2} \\ \langle b!_r \rangle \; t_{b,s,2}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,2} \setminus R_{b,1} \\ \langle b!_r \rangle \; u_{b,s}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \cap R_{b,2} \end{cases}$$

Lemma 24 may now be applied again to show

$$\sum_{b \in O_1 \cup O_2, s \in R_b} \sigma_{b,s} \cdot t'_{b,s} \quad = \quad \sum_{b \in O_1 \cup O_2} \sum_{s \in R_b} t''_{b,s}$$

where

$$t''_{b,s} \quad = \quad \begin{cases} \langle b!_{\sigma_{b,s} r} \rangle \; t_{b,s,1}, & \text{if } b \in O_1 \setminus O_2 \\ \langle b!_{\sigma_{b,s} r} \rangle \; t_{b,s,2}, & \text{if } b \in O_2 \setminus O_1 \\ \langle b!_{\sigma_{b,s} r} \rangle \; t_{b,s,1}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \setminus R_{b,2} \\ \langle b!_{\sigma_{b,s} r} \rangle \; t_{b,s,2}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,2} \setminus R_{b,1} \\ \langle b!_{\sigma_{b,s} r} \rangle \; u_{b,s}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \cap R_{b,2} \end{cases}$$

The right-hand side of the last equation is an output normal form.

3. Suppose t_1 and t_2 are normal forms. If t_1 and t_2 are both input normal forms, then this case reduces to case (1) already established. If t_1 and t_2 are both output normal forms, then this case reduces to case (2) already established.

   We assume in the remainder of the proof that t_1 is either an output normal form or else it has the form $u_1 + v_1$, where u_1 is an input normal form and v_1 is a (possibly trivial) output normal form. This is without loss of generality, since if if t_1 is an input normal form, then we may replace it by the provably equivalent $t_1 + \text{nil}_\emptyset$. Similarly, we assume that t_2 is either an output normal form or else has the form $u_2 + v_2$, where u_2 is an input normal form and v_2 is a (possibly trivial) output normal form.

   We next claim that it is impossible for t_1 to be an output normal form and t_2 to be $u_2 + v_2$, where u_2 is an input normal form and v_2 is an output normal form. For, in this case t_1 would have inferable type $\emptyset/J \Rightarrow O$ and t_2 would have inferable type $I/J \Rightarrow O$ for some $I \neq \emptyset$, contradicting the assumption that t_1 and t_2 have a common inferable type. For the same reason, it is impossible for t_2 to be an output normal form and t_1 to be $u_1 + v_1$, where u_1 is an input normal form and v_1 is an output normal form.

   It remains to consider the case in which $t_1 = u_1 + v_1$ and $t_2 = u_2 + v_2$, where u_1 and u_2 are input normal forms and v_1 and v_2 are output normal forms. In this case, using axiom (interchange) we may prove $t_1 \; {}_{\pi}\!\oplus_{1-\pi} \, t_2$ equivalent to the term

$$(u_1 \; {}_{\pi}\!\oplus_{1-\pi} \, u_2) + (v_1 \; {}_{\pi}\!\oplus_{1-\pi} \, v_2).$$

Using case (1) already established, we can prove $u_1 \,{}_{\pi}{\oplus}_{1-\pi}\, u_2$ equivalent to an input normal form u'. Using case (2) already established, we can prove $v_1 \,{}_{\pi}{\oplus}_{1-\pi}\, v_2$ equivalent to an output normal form v'. Thus, we can prove $t_1 \,{}_{\pi}{\oplus}_{1-\pi}\, t_2$ equivalent to $u' + v'$. If v' is nontrivial, then this is already a normal form. If v' is trivial, then we may use axiom (choice-unit) to prove $u' + v'$ equivalent to u', which is a normal form.

□

Lemma 28.

1. *If t_1 and t_2 are input normal forms then there exists an input normal form t' such that the equation $t_1 + t_2 = t'$ is provable.*
2. *If t_1 and t_2 are output normal forms then there exists an output normal form t' such that the equation $t_1 + t_2 = t'$ is provable.*
3. *If t_1 and t_2 are normal forms then there exists a normal form t' such that the equation $t_1 + t_2 = t'$ is provable.*

Proof.

1. Suppose t_1 and t_2 are input normal forms, where t_1 has inferable type $I_1/J \Rightarrow O_1$ and t_2 has inferable type $I_2/J \Rightarrow O_2$. Then t_1 is either nil_{I_1} or else has the form

$$\sum_{a \in I_1} \sum_{r \in R_{a,1}} \langle a?_{p_{a,r,1}} \rangle\, t_{a,r,1}$$

Similarly, t_2 is either nil_{I_2} or else has the form

$$\sum_{a \in I_2} \sum_{r \in R_{a,2}} \langle a?_{p_{a,r,2}} \rangle\, t_{a,r,2}$$

If t_1 is nil_{I_1} then we may use axiom (nil-fold) to prove it equivalent to a summation as above. The same reasoning applies to t_2. Thus, for the remainder of the proof we assume that t_1 is not nil_{I_1} and t_2 is not nil_{I_2}.

For $a \in I_1 \cup I_2$ define

$$R_a \;=\; \begin{cases} R_{a,1}, & \text{if } a \in I_1 \setminus I_2 \\ R_{a,2}, & \text{if } a \in I_2 \setminus I_1 \\ R_{a,1} \cup R_{a,2}, & \text{if } a \in I_1 \cup I_2 \end{cases}$$

Using axioms (choice-comm) and (choice-assoc) we may prove $t_1 + t_2$ equivalent to a term of the form

$$\sum_{a \in I_1 \cup I_2} \sum_{r \in R_a} t_{a,r}$$

where

$$t_{a,r} = \begin{cases} \langle a?_{p_{a,r,1}} \rangle \; t_{a,r,1}, & \text{if } a \in I_1 \setminus I_2 \\ \langle a?_{p_{a,r,2}} \rangle \; t_{a,r,2}, & \text{if } a \in I_2 \setminus I_1 \\ \langle a?_{p_{a,r,1}} \rangle \; t_{a,r,1}, & \text{if } a \in I_1 \cap I_2, r \in R_{a,1} \setminus R_{a,2} \\ \langle a?_{p_{a,r,2}} \rangle \; t_{a,r,2}, & \text{if } a \in I_1 \cap I_2, r \in R_{a,2} \setminus R_{a,1} \\ \langle a?_{p_{a,r,1}} \rangle \; t_{a,r,1} + \langle a?_{p_{a,r,2}} \rangle \; t_{a,r,2}, & \text{if } a \in I_1 \cap I_2, r \in R_{a,1} \cap R_{a,2} \end{cases}$$

Using axiom (input-comb) we may prove each term

$$\langle a?_{p_{a,r,1}} \rangle \; t_{a,r,1} + \langle a?_{p_{a,r,2}} \rangle \; t_{a,r,2}$$

equivalent to the term

$$\langle a?_{q_{a,r}} \rangle \; (t_{a,r,1} \;{}_{\pi_{a,r}}\!\oplus_{1-\pi_{a,r}} t_{a,r,2})$$

where

$$\begin{aligned} q_{a,r} &= p_{a,r,1} + p_{a,r,2} \\ \pi_{a,r} &= p_{a,r,1}/q_{a,r}. \end{aligned}$$

We may now apply Lemma 27 to conclude that the term $t_{a,r,1} \;{}_{\pi_{a,r}}\!\oplus_{1-\pi_{a,r}} t_{a,r,2}$ is provably equivalent to a normal form $u_{a,r}$. Thus we may prove $t_1 + t_2$ equivalent to

$$\sum_{a \in I_1 \cup I_2} \sum_{r \in R_a} t'_{a,r}$$

where

$$t'_{a,r} = \begin{cases} \langle a?_{p_{a,r,1}} \rangle \; t_{a,r,1}, & \text{if } a \in I_1 \setminus I_2 \\ \langle a?_{p_{a,r,2}} \rangle \; t_{a,r,2}, & \text{if } a \in I_2 \setminus I_1 \\ \langle a?_{p_{a,r,1}} \rangle \; t_{a,r,1}, & \text{if } a \in I_1 \cap I_2, r \in R_{1,a} \setminus R_{2,a} \\ \langle a?_{p_{a,r,2}} \rangle \; t_{a,r,2}, & \text{if } a \in I_2 \cap I_1, r \in R_{2,a} \setminus R_{1,a} \\ \langle a?_{q_{a,r}} \rangle \; u_{a,r}, & \text{if } a \in I_1 \cap I_2, r \in R_{1,a} \cap R_{2,a} \end{cases}$$

If the last summation above is reduced with respect to axiom (nil-fold), then it is already an input normal form. Otherwise, axiom (nil-fold) can be applied to prove it equal to $\text{nil}_{I_1 \cup I_2}$.

2. Suppose t_1 and t_2 are output normal forms. If t_1 is trivial, then $t_1 + t_2$ can be proved equivalent to t_2 using axiom (choice-unit). Similarly, if t_2 is trivial, then $t_1 + t_2$ can be proved equivalent to t_1, so we suppose in the remainder of the proof that neither t_1 nor t_2 is trivial.

   Now, t_1 has the form

$$\sum_{b \in O_1} \sum_{s \in R_{b,1}} \langle b!_{\sigma_{b,s,1} r_1} \rangle \; t_{b,s,1}$$

and t_2 has the form

$$\sum_{b \in O_2} \sum_{s \in R_{b,2}} \langle b!_{\sigma_{b,s,2} r_2} \rangle \; t_{b,s,2}.$$

so that $t_1 + t_2$ is provably equivalent to

$$\sum_{b \in O_1} \sum_{s \in R_{b,1}} \langle b!_{\sigma_{b,s,1} r_1} \rangle \; t_{b,s,1} + \sum_{b \in O_2} \sum_{s \in R_{b,2}} \langle b!_{\sigma_{b,s,2} r_2} \rangle \; t_{b,s,2}.$$

Let $\pi = \frac{r_1}{r_1 + r_2}$, then applying Lemma (24) shows that this last term is provably equivalent to

$$\sum_{b \in O_1 \cup O_2, s \in R_b} \sigma_{b,s} \cdot \langle b!_{r_1 + r_2} \rangle \; t_{b,s}$$

where

$$R_b = \begin{cases} R_{b,1}, & \text{if } b \in O_1 \setminus O_2 \\ R_{b,2}, & \text{if } b \in O_2 \setminus O_1 \\ R_{b,1} \cup R_{b,2}, & \text{if } b \in O_1 \cap O_2 \end{cases}$$

$$\sigma_{b,s} = \begin{cases} \sigma_{b,s,1}\pi, & \text{if } b \in O_1 \setminus O_2 \\ \sigma_{b,s,2}\pi, & \text{if } b \in O_2 \setminus O_1 \\ \sigma_{b,s,1}\pi, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \setminus R_{b,2} \\ \sigma_{b,s,2}\pi, & \text{if } b \in O_1 \cap O_2, s \in R_{b,2} \setminus R_{b,1} \\ \sigma_{b,s,1}\pi + \sigma_{b,s,2}(1-\pi), & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \cap R_{b,2} \end{cases}$$

$$t_{b,s} = \begin{cases} \langle b!_{r_1+r_2} \rangle \; t_{b,s,1}, & \text{if } b \in O_1 \setminus O_2 \\ \langle b!_{r_1+r_2} \rangle \; t_{b,s,2}, & \text{if } b \in O_2 \setminus O_1 \\ \langle b!_{r_1+r_2} \rangle \; t_{b,s,1}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \setminus R_{b,2} \\ \langle b!_{r_1+r_2} \rangle \; t_{b,s,2}, & \text{if } b \in O_2 \cap O_1, s \in R_{b,2} \setminus R_{b,1} \\ \langle b!_{r_1+r_2} \rangle \; t_{b,s,1} \;{}_{\rho_{b,s}}\!\oplus_{1-\rho_{b,s}} \langle b!_{r_1+r_2} \rangle \; t_{b,s,2}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \cap R_{b,2} \end{cases}$$

and $\rho_{b,s} = \frac{\sigma_{b,s,1}\pi}{\sigma_{b,s}}$.

By axiom (output-comb), we can prove:

$$\langle b!_{r_1+r_2} \rangle \; t_{b,s,1} \;{}_{\rho_{b,s}}\!\oplus_{1-\rho_{b,s}} \langle b!_{r_1+r_2} \rangle \; t_{b,s,2} \;=\; \langle b!_{r_1+r_2} \rangle \; (t_{b,s,1} \;{}_{\rho_{b,s}}\!\oplus_{1-\rho_{b,s}} t_{b,s,2}).$$

By Lemma 27, the term $t_{b,s,1} \;{}_{\rho_{b,s}}\!\oplus_{1-\rho_{b,s}} t_{b,s,2}$ is provably equivalent to a normal form $u_{b,s}$. Thus we may prove $t_1 + t_2$ equivalent to the term

$$\sum_{b \in O_1 \cup O_2, s \in R_b} \sigma_{b,s} \cdot \langle b!_{r_1 + r_2} \rangle \; t'_{b,s}$$

where

$$t_{b,s} = \begin{cases} \langle b!_{r_1+r_2} \rangle \; t_{b,s,1}, & \text{if } b \in O_1 \setminus O_2 \\ \langle b!_{r_1+r_2} \rangle \; t_{b,s,2}, & \text{if } b \in O_2 \setminus O_1 \\ \langle b!_{r_1+r_2} \rangle \; t_{b,s,1}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \setminus R_{b,2} \\ \langle b!_{r_1+r_2} \rangle \; t_{b,s,2}, & \text{if } b \in O_2 \cap O_1, s \in R_{b,2} \setminus R_{b,1} \\ \langle b!_{r_1+r_2} \rangle \; u_{b,s}, & \text{if } b \in O_1 \cap O_2, s \in R_{b,1} \cap R_{b,2} \end{cases}$$

Another application of Lemma (24) serves to prove this equal to an output normal form.

3. Suppose t_1 and t_2 are normal forms. If t_1 is an input normal form, then we may use axiom (choice-unit) to prove t_1 equivalent to $t_1 + \mathrm{nil}_\emptyset$. Similarly, if t_2 is an input normal form, then we may prove t_2 equivalent to $t_2 + \mathrm{nil}_\emptyset$. We therefore assume in the following that neither t_1 nor t_2 is an input normal form.

   If both t_1 and t_2 are output normal forms, then the proof reduces to case (2) already established.

   If $t_1 = u_1 + v_1$ and $t_2 = u_2 + v_2$, where u_1 and u_2 are input normal forms and v_1 and v_2 are output normal forms, then we may use axioms (choice-comm) and (choice-assoc) to prove $t_1 + t_2$ equivalent to $(u_1 + u_2) + (v_1 + v_2)$. By case (1) already established, $u_1 + u_2$ is provably equivalent to an output normal form u', and by case (2) already established, $v_1 + v_2$ is provably equivalent to an output normal form v', so $(u_1 + u_2) + (v_1 + v_2)$ is provably equivalent to $u' + v'$. If v' is nontrivial, then $u' + v'$ is a normal form. If v' is trivial then we may use axiom (choice-unit) to prove $u' + v'$ equivalent to u', which is a normal form.

   The remaining cases, in which one of t_1 or t_2 is an output normal form and the other is a sum of an input normal form and an output normal form, are similar to that considered in the previous paragraph. We omit the details.

□

Lemma 29. *Any $\|$-free term t in* $\mathrm{Proc}(I/J \Rightarrow O)$ *can be proved equivalent to a normal form using the axioms in Table 2.*

Proof. The proof is by structural induction on t. Suppose we have already established the result for all proper subterms of a well-typed term t such that $\vdash t : I/J \Rightarrow O$, and consider the possible syntactic forms taken by t:

- Suppose t is nil_I. If $I = \emptyset$, then t is an output normal form and if $I \neq \emptyset$, then t is an input normal form, so there is nothing to prove.

- Suppose t is $\langle a?_p \rangle\ u$. By induction, u can be proved equivalent to a normal form u'. It follows by substitutivity that t can be proved equivalent to $\langle a?_p \rangle\ u'$. If u' does not have the form nil_I, then $\langle a?_p \rangle\ u'$ is already an input normal form. Otherwise, $\langle a?_p \rangle\ u'$ can fail to be an input normal form only if it is an instance of the left-hand side of axiom (nil-fold); that is, only if p is 1 and u' is $\mathrm{nil}_{\{a\}}$. In this case, axiom (nil-fold) can be used to prove u equivalent to $\mathrm{nil}_{\{a\}}$, which is an input normal form.

- Suppose t is $\langle b!_r \rangle\ u$. By induction, u can be proved equal to a normal form u'. It follows by substitutivity and an application of axiom (choice-unit) that t can be proved equal to $\langle b!_r \rangle\ u'$, which is an output normal form.

- Suppose t is $u_1 \ {}_{\pi}\!\oplus_{1-\pi} u_2$. By induction, u_1 can be proved equal to a normal form u_1' and u_2 can be proved equal to a normal form u_2'. By substitutivity, t can be proved equal to $u_1' \ {}_{\pi}\!\oplus_{1-\pi} u_2'$. Application of Lemma 27 shows that there exists a normal form t' such that $u_1' \ {}_{\pi}\!\oplus_{1-\pi} u_2'$ is provably equal to t'.

- Suppose t is $u_1 + u_2$. By induction, u_1 can be proved equal to a normal form u_1' and u_2 can be proved equal to a normal form u_2'. By substitutivity t can be proved equal to $u_1' + u_2'$. Application of Lemma 28 shows that there exists a normal form t' such that $u_1' + u_2'$ is provably equal to t'.

□

4.5 Completeness

Key to the completeness proof is Lemma 30 below, which shows how certain information about the structure of t can be extracted from its behavior. In case t is a normal form, this information is essentially the entire structure of t, except for the ordering of terms in sums. Suppose a type $I/J \Rightarrow O$ has been fixed and let $*$ be an arbitrarily chosen (non-native) action in $Act \setminus (J \cup O)$. Given $e \in Act$ and $r \geq 0$, let $\Xi_{e,r}$ be the observable defined as follows:

$$\Xi_{e,r}(\alpha) = \begin{cases} s, & \text{if } \alpha = \langle *, s \rangle, \\ 1, & \text{if } \alpha = \langle e, s \rangle \langle *, r \rangle, \\ 0, & \text{otherwise.} \end{cases}$$

We call such an observable a *probe*.

Lemma 30. *Suppose t is a well-typed term with $\vdash t : I/J \Rightarrow O$. Then the probe $\Xi_{e,r}$ has the following properties:*

1. $\mathcal{B}_t^O[\Xi_{*,0}](\langle *, 0 \rangle) = \mathrm{rt}(t)$.
2. *For $e \in J \cup O$,* $\mathcal{B}_t^O[\Xi_{e,r}](\langle e, 0 \rangle \langle *, 0 \rangle) \ = \ \sum_{\{u : \mathrm{rt}(u) = r\}} \Delta_e^O(t, u)$.

Proof. To prove (1), we calculate, using the definition of $\mathcal{B}_t^O$ and the fact that $*$ is a non-native action:

$$\begin{aligned} \mathcal{B}_t^O[\Xi_{*,0}](\langle *, 0 \rangle) &= \mathcal{B}_t^O[\langle *, \mathrm{rt}(t) \rangle^{-1} \Xi_{*,0}](\epsilon) \\ &= (\langle *, \mathrm{rt}(t) \rangle^{-1} \Xi_{*,0})(\epsilon) \\ &= \Xi_{*,0}(\langle *, \mathrm{rt}(t) \rangle) \\ &= \mathrm{rt}(t). \end{aligned}$$

To prove (2), suppose $e \in J \cup O$. We calculate, again using the definition of $\mathcal{B}_t^O$ and the fact that $*$ is a non-native action:

$$\begin{aligned}
\mathcal{B}_t^O[\Xi_{e,r}](\langle e,0\rangle\langle *,0\rangle) &= \sum_u \Delta_e^O(t,u)\cdot \mathcal{B}_u^O[\langle e,\mathrm{rt}(t)\rangle^{-1}\Xi_{e,r}](\langle *,0\rangle) \\
&= \sum_u \Delta_e^O(t,u)\cdot \mathcal{B}_u^O[\langle *,\mathrm{rt}(u)\rangle^{-1}\langle e,\mathrm{rt}(t)\rangle^{-1}\Xi_{e,r}](\epsilon) \\
&= \sum_u \Delta_e^O(t,u)\cdot \Xi_{e,r}(\langle e,\mathrm{rt}(t)\rangle\langle *,\mathrm{rt}(u)\rangle) \\
&= \sum_{\{u:\mathrm{rt}(u)=r\}} \Delta_e^O(t,u).
\end{aligned}$$

□

Lemma 31. *Suppose term $t \in \mathrm{Proc}(I/J \Rightarrow O)$ is a normal form that is not nil_J. Then $\mathcal{B}_t^O \neq \mathcal{B}_{\mathrm{nil}_J}^O$.*

Proof. We proceed by induction on the maximum prefix depth of t.

Suppose we have established the result for all t of maximum prefix depth strictly less than k for some $k \geq 0$. Suppose t has maximum prefix depth k. If $k = 0$, then the result holds vacuously, since the only terms with prefix depth 0 are terms of the form nil_J, so suppose $k > 0$.

We first consider the case that $\mathrm{rt}(t) > 0$. By Lemma 30 we have

$$\mathcal{B}_t^O[\Xi_{*,0}](\langle *,0\rangle) \;=\; \mathrm{rt}(t) \;\neq\; 0 \;=\; \mathrm{rt}(\mathrm{nil}_J) \;=\; \mathcal{B}_{\mathrm{nil}_J}^O[\Xi_{*,0}](\langle *,0\rangle).$$

Hence in case $\mathrm{rt}(t) > 0$ we have $\mathcal{B}_t^O \neq \mathcal{B}_{\mathrm{nil}_J}^O$.

For the remainder of the proof we suppose that $\mathrm{rt}(t) = 0$. Then t must be an input normal form that is not nil_J, hence it has the form

$$\sum_{a\in I}\sum_{r\in R_a} \langle a?_{p_{a,r}}\rangle\; t_{a,r}.$$

Suppose first that $I \subset J$, so that there exists some $a \in J \setminus I$. In this case, we have

$$\mathcal{B}_t^0[\mathbf{1}](\langle a,0\rangle) \;=\; 0 \;\neq\; 1 \;=\; \mathcal{B}_{\mathrm{nil}_J}^O[\mathbf{1}](\langle a,0\rangle),$$

thus establishing $\mathcal{B}_t^O \neq \mathcal{B}_{\mathrm{nil}_J}^O$.

Suppose now that $I = J$. We next consider the case in which $R_a \neq \{0\}$ for some $a \in I$. Then there exists $a \in I$ for which there is some $r > 0$ in R_a. In this case, application of Lemma 30 shows that

$$\mathcal{B}_t^O[\Xi_{a,r}](\langle a,0\rangle\langle *,0\rangle) \;=\; \sum_{\{u:\mathrm{rt}(u)=r\}} \Delta_a^O(t,u).$$

But because t is an input normal form, the term $t_{a,r}$ is the unique term u such that $\Delta_a^O(t,u) > 0$ and $\mathrm{rt}(u) = r$. Then we have

$$\mathcal{B}_t^O[\Xi_{a,r}](\langle a,0\rangle\langle *,0\rangle) \;=\; \Delta_a^O(t,t_{a,r}) \;=\; p_{a,r} \;\neq\; 0 \;=\; \mathcal{B}_{\mathrm{nil}_J}^O[\Xi_{a,r}](\langle a,0\rangle\langle *,0\rangle),$$

showing that $\mathcal{B}^O_t \neq \mathcal{B}^O_{\mathrm{nil}_J}$ in this case as well.

Suppose now that $R_a = \{0\}$ for all $a \in I$. We next consider the case that $p_{a,0} \neq 1$ for some $a \in I$. In this case, application of Lemma 30 shows that

$$\mathcal{B}^O_t[\Xi_{a,0}](\langle a,0\rangle\langle *,0\rangle) \;=\; p_{a,0} \;\neq\; 1 \;=\; \mathcal{B}^O_{\mathrm{nil}_J}[\Xi_{a,0}](\langle *,0\rangle\langle a_i,0\rangle)$$

once again showing that $\mathcal{B}^O_t \neq \mathcal{B}^O_{\mathrm{nil}_J}$.

Suppose now that $p_{a,0} = 1$ for all $a \in I$. We claim that for some $a \in I$ the term $t_{a,0}$ is different from nil_J. For if all the terms $t_{a,0}$ were nil_J then since $I = J$ and $p_{a,0} = 1$ for all $a \in I$ it would follow that t is an instance of the left-hand side of axiom (nil-fold), in contradiction to the assumption that t is an input normal form.

Let $a \in I$ be chosen such that $t_{a,0}$ is different from nil_J. Since the prefix depth of $t_{a,0}$ is strictly less than that of t, the induction hypothesis may be applied to $t_{a,0}$ to show that $\mathcal{B}^O_{t_{a,0}} \neq \mathrm{nil}_J$. In particular, there exists Ψ and β such that

$$\mathcal{B}^O_{t_{a,0}}[\Psi](\beta) \quad\neq\quad \mathcal{B}^O_{\mathrm{nil}_J}[\Psi](\beta).$$

Let $*$ be an arbitrarily chosen action in $Act \setminus (J \cup O)$. Define observable Φ as follows:

$$\Phi(\alpha) = \begin{cases} \Psi(\gamma), & \text{if } \alpha = \langle a,0\rangle\langle *,0\rangle\gamma, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\langle *,0\rangle^{-1}\langle a,0\rangle^{-1}\Phi)(\gamma) = \Phi(\langle a,0\rangle\langle *,0\rangle\gamma) = \Psi(\gamma)$ for all γ, hence

$$\langle *,0\rangle^{-1}\langle a,0\rangle^{-1}\Phi = \Psi.$$

It follows that

$$\begin{aligned}
\mathcal{B}^O_t[\Phi](\langle a,0\rangle\langle *,0\rangle\beta) &= \sum_u \Delta^O_a(t,u)\cdot\mathcal{B}^O_u[\langle a,0\rangle^{-1}\Phi](\langle *,0\rangle\beta) \\
&= \sum_u \Delta^O_a(t,u)\cdot\mathcal{B}^O_u[\langle *,\mathrm{rt}(u)\rangle^{-1}\langle a,0\rangle^{-1}\Phi](\beta) \\
&= \Delta^O_a(t,u_{a,0})\cdot\mathcal{B}^O_{u_{a,0}}[\langle *,0\rangle^{-1}\langle a,0\rangle^{-1}\Phi](\beta) \\
&= p_{a,0}\cdot\mathcal{B}^O_{u_{a,0}}[\Psi](\beta) \\
&= \mathcal{B}^O_{u_{a,0}}[\Psi](\beta) \\
&\neq \mathcal{B}^O_{\mathrm{nil}_J}[\Psi](\beta) \\
&= \mathcal{B}^O_{\mathrm{nil}_J}[\Phi](\langle a,0\rangle\langle *,0\rangle\beta).
\end{aligned}$$

Thus once again we have $\mathcal{B}^O_t \neq \mathcal{B}^O_{\mathrm{nil}_J}$, completing the induction step and the proof. □

Lemma 32. *Suppose t and t' are normal forms such that $\vdash t : I/J \Rightarrow O$ and $\vdash t' : I/J \Rightarrow O$ If $\mathcal{B}^O_t = \mathcal{B}^O_{t'}$, then t and t' are identical up to permutation of sums.*

Proof. We prove, by induction on the sum of the maximum prefix depths of t and t', that if t and t' are not identical up to permutation of sums, then $\mathcal{B}^O_t \neq \mathcal{B}^O_{t'}$

If both t and t' have depth 0, then both must be nil_J, and so the result holds vacuously.

Suppose one of t, t' has prefix depth 0 and the other has nonzero prefix depth. Then one of t, t' takes the form nil_J and the other does not. In this case, $\mathcal{B}^O_t \neq \mathcal{B}^O_{t'}$ follows from Lemma 31.

Suppose neither t nor t' has zero prefix depth. Suppose next that $\mathrm{rt}(t) = 0$ but $\mathrm{rt}(t') > 0$. Then by Lemma 30 we have

$$\mathcal{B}^O_t[\Xi_{*,0}](\langle *, 0\rangle) = \mathrm{rt}(t) = 0 \neq \mathrm{rt}(t') = \mathcal{B}^O_t[\Xi_{*,0}](\langle *, 0\rangle).$$

Thus, $\mathcal{B}^O_t \neq \mathcal{B}^O_{t'}$ in this case. A symmetric argument applies in case $\mathrm{rt}(t) > 0$ but $\mathrm{rt}(t') = 0$.

There are now two cases remaining: (1) $\mathrm{rt}(t) = 0 = \mathrm{rt}(t')$ and (2) $\mathrm{rt}(t) > 0$ and $\mathrm{rt}(t) > 0$.

Case (1): $\mathrm{rt}(t) = 0 = \mathrm{rt}(t')$.

Suppose $\mathrm{rt}(t) = 0 = \mathrm{rt}(t')$. Then both t and t' are input normal forms distinct from nil_J. We therefore have

$$t \;=\; \sum_{a\in I}\sum_{r\in R_a}\langle a?_{p_{a,r}}\rangle\; t_{a,r} \qquad\qquad t' \;=\; \sum_{a\in I'}\sum_{r\in R'_a}\langle a?_{p'_{a,r}}\rangle\; t'_{a,r}.$$

Since t and t' have the same inferable type $I/J \Rightarrow O$ it must be the case that $I = I'$.

We first claim that $R_a = R'_a$ for all $a \in I$. Suppose otherwise, that $R_a \neq R'_a$ for some $a \in I$. Then one of the sets $R_a \setminus R'_a$ or $R'_a \setminus R_a$ is nonempty. Suppose the former, the proof for the latter is symmetric. Choose $r \in R_a \setminus R'_a$. Then by Lemma 30 we have

$$\mathcal{B}^O_t[\Xi_{a,r}](\langle *, 0\rangle\langle a, 0\rangle) \;=\; p_{a,r} \;\neq\; 0 \;=\; \mathcal{B}^O_{t'}[\Xi_{a,r}](\langle *, 0\rangle\langle a, 0\rangle),$$

contradicting the assumption that $\mathcal{B}^O_t = \mathcal{B}^O_{t'}$.

We next claim that for all $a \in I$, and all $r \in R_a$, that $p_{a,r} = p'_{a,r}$ and $t_{a,r}$ and $t'_{a,r}$ are identical up to permutation of sums. Suppose for some $a \in I$ and $r \in R_a$ we have $p_{a,r} \neq p'_{a,r}$. Then by Lemma 30 we have

$$\mathcal{B}^O_t[\Xi_{a,r}](\langle *, 0\rangle\langle a, 0\rangle) \;=\; p_{a,r} \;\neq\; p'_{a,r} \;=\; \mathcal{B}^O_{t'}[\Xi_{a,r}](\langle *, 0\rangle\langle a, 0\rangle),$$

contradicting the assumption that $\mathcal{B}^O_t = \mathcal{B}^O_{t'}$. Suppose $p_{a,r} = p'_{a,r}$ but that $t_{a,r}$ and $t'_{a,r}$ are not identical up to permutation of sums. In this case, since the sum of the maximum prefix depths of $t_{a,r}$ and $t'_{a,r}$ is strictly less than that of t, we may apply the induction hypothesis to infer that $\mathcal{B}^O_{t_{a,r}} \neq \mathcal{B}^O_{t'_{a,r}}$. That is, there exist Ψ and β such that $\mathcal{B}^O_{t_{a,r}}[\Psi](\beta) \neq \mathcal{B}^O_{u'_{a,r}}[\Psi](\beta)$. Let observable Φ be defined as follows:

$$\Phi(\gamma) = \begin{cases} \Psi(\delta), & \text{if } \gamma = \langle a, 0\rangle\langle *, r\rangle\delta, \\ 0, & \text{otherwise.} \end{cases}$$

so that $\langle *, r\rangle^{-1}\langle a, 0\rangle^{-1}\Phi = \Psi$. It follows (using the assumption that $\mathrm{rt}(t) = 0$) that for all γ we have

$$\begin{aligned}\mathcal{B}^O_t[\Phi](\langle a,0\rangle\langle *,0\rangle\gamma) &= \sum_u \Delta^O_a(t,u)\cdot \mathcal{B}^O_u[\langle a,0\rangle^{-1}\Phi](\langle *,0\rangle\gamma)\\ &= \sum_u \Delta^O_a(t,u)\cdot \mathcal{B}^O_u[\langle *,\mathrm{rt}(u)\rangle^{-1}\langle a,0\rangle^{-1}\Phi](\gamma).\end{aligned}$$

Now, by Lemma 12, for all γ the quantity $\mathcal{B}^O_u[\langle *,\mathrm{rt}(u)\rangle^{-1}\langle a,0\rangle^{-1}\Phi](\gamma)$ can be expressed as a finite linear combination

$$\sum_{k\in K} c_k \cdot (\langle *,\mathrm{rt}(u)\rangle^{-1}\langle a,0\rangle^{-1}\Phi)(\gamma_i).$$

However,

$$(\langle *,\mathrm{rt}(u)\rangle^{-1}\langle a,0\rangle^{-1}\Phi)(\gamma_i) = \Phi(\langle a,0\rangle\langle *,\mathrm{rt}(u)\rangle\gamma_i),$$

which by definition of Φ equals 0 unless $\mathrm{rt}(u) = r$. Thus

$$\mathcal{B}^O_u[\langle *,\mathrm{rt}(u)\rangle^{-1}\langle a,0\rangle^{-1}\Phi](\gamma) = 0$$

unless $\mathrm{rt}(u) = r$. It follows from this observation (in the particular case that $\gamma = \beta$) that:

$$\begin{aligned}\mathcal{B}^O_t[\Phi](\langle a,0\rangle\langle *,0\rangle\beta) &= \sum_u \Delta^O_a(t,u)\cdot \mathcal{B}^O_u[\langle *,\mathrm{rt}(u)\rangle^{-1}\langle a,0\rangle^{-1}\Phi](\beta)\\ &= p_{a,r}\cdot \mathcal{B}^O_{t_{a,r}}[\langle *,r\rangle^{-1}\langle a,0\rangle^{-1}\Phi](\beta)\\ &= p_{a,r}\cdot \mathcal{B}^O_{t_{a,r}}[\Psi](\beta)\\ &\neq p'_{a,r}\cdot \mathcal{B}^O_{t'_{a,r}}[\Psi](\beta)\\ &= \mathcal{B}^O_{t'}[\Phi](\langle a,0\rangle\langle *,0\rangle\beta).\end{aligned}$$

We have thus exhibited observable Φ and rated trace $\alpha = \langle a,0\rangle\langle *,0\rangle\beta$ such that $\mathcal{B}^O_t[\Phi](\alpha) \neq \mathcal{B}^O_{t'}[\Phi](\alpha)$, contradicting the fact that $\mathcal{B}^O_t \neq \mathcal{B}^O_{t'}$. We conclude that our assumption that $t_{a,r}$ and $t'_{a,r}$ are not identical up to permutation of sums was impossible, hence $t_{a,r}$ and $t'_{a,r}$ must in fact be identical up to permutation of sums.

We have therefore shown, in case $\mathrm{rt}(t) = 0 = \mathrm{rt}(t')$, that $I = I'$, $R_a = R'_a$ for all $a \in I$, $p_{a,r} = p'_{a,r}$ for all $a \in I$ and all $r \in R_a$, and $t_{a,r}$ and $t'_{a,r}$ are identical up to permutation of sums for all $a \in I$ and all $r \in R_a$. But this implies that t and t' are identical up to permutation of sums. This completes the induction step in case $\mathrm{rt}(t) = 0 = \mathrm{rt}(t')$.

Case (2): $\mathrm{rt}(t) > 0$ and $\mathrm{rt}(t') > 0$.

Suppose $\mathrm{rt}(t) > 0$ and $\mathrm{rt}(t') > 0$. In this case, t is either an output normal form or a sum $u + v$, where u is an input normal form and v is a nontrivial output normal form. Similarly, t' is either an output normal form or a sum $u' + v'$, where u' is an input normal form and v' is a nontrivial output normal form. Because t and t' are assumed to have the same inferable type, it is impossible for t to be an output normal form and t' to be $u'+v'$ and vice versa. Thus, either t and t' are both output normal forms or else t is $u+v$ and t' is $u'+v'$.

We first consider the case in which both t and t' are output normal forms. In this case:

$$t \;=\; \sum_{b\in O}\sum_{s\in R_b}\langle b!_{\sigma_{b,s}\mathrm{rt}(t)}\rangle\; v_{b,s} \qquad\qquad t' \;=\; \sum_{b\in O}\sum_{s\in R'_b}\langle b!_{\sigma'_{b,s}\mathrm{rt}(t')}\rangle\; v'_{b,s}$$

We first consider the case in which $\mathrm{rt}(t)\neq\mathrm{rt}(t')$. Then by Lemma 30 we have

$$\mathcal{B}^O_t[\Xi_{*,0}](\langle *,0\rangle) \;=\; \mathrm{rt}(t) \;\neq\; \mathrm{rt}(t') \;=\; \mathcal{B}^O_{t'}[\Xi_{*,0}](\langle *,0\rangle)$$

thus showing that $\mathcal{B}^O_t\neq\mathcal{B}^O_{t'}$ and completing the induction step in this case.

For the remainder of the proof, we suppose that $\mathrm{rt}(t)=\mathrm{rt}(t')$. We claim that $R_b=R'_b$ for all $b\in O$, that $\sigma_{b,s}=\sigma'_{b,s}$ for all $b\in O$ and all $s\in R_b$, and that $v_{b,s}$ is identical up to permutation of sums to $v'_{b,s}$ for all $b\in O$ and all $s\in R_b$.

Suppose first that $R_b\neq R'_b$ for some $b\in O$. Then one of the sets $R_b\setminus R'_b$ and $R'_b\setminus R_b$ is nonempty. Suppose the former, the proof for the latter is symmetric. Choose $s\in R_b\setminus R'_b$. Then by Lemma 30,

$$\mathcal{B}^O_t[\Xi_{b,s}](\langle *,0\rangle\langle b,0\rangle) \;=\; \sigma_{b,s} \;\neq\; 0 \;=\; \mathcal{B}^O_{t'}[\Xi_{b,s}](\langle *,0\rangle\langle b,0\rangle)$$

a contradiction with the assumption that $\mathcal{B}^O_t=\mathcal{B}^O_{t'}$.

Suppose now that for some $b\in O$ and $s\in R_b$ we have $\sigma_{b,s}\neq\sigma'_{b,s}$. Then by Lemma 30,

$$\mathcal{B}^O_t[\Xi_{b,s}](\langle *,0\rangle\langle b,0\rangle) \;=\; \sigma_{b,s} \;\neq\; \sigma'_{b,s} \;=\; \mathcal{B}^O_{t'}[\Xi_{b,s}](\langle *,0\rangle\langle b,0\rangle)$$

so that $\mathcal{B}^O_t\neq\mathcal{B}^O_{t'}$, again contradicting the assumption that $\mathcal{B}^O_t=\mathcal{B}^O_{t'}$.

Finally, suppose that for some $b\in O$ and $s\in R_b$ the normal form $v_{b,s}$ is not identical up to permutation of sums to $v'_{b,s}$. In this case, since the sum of the maximum prefix depths of $v_{b,s}$ and $v'_{b,s}$ is strictly less than that of t and t', we may apply the induction hypothesis to conclude that $\mathcal{B}^O_{t_{b,s}}\neq\mathcal{B}^O_{t'_{b,s}}$. That is, there exist Ψ and β such that $\mathcal{B}^O_{t_{b,s}}[\Psi](\beta)\neq\mathcal{B}^O_{u'_{b,s}}[\Psi](\beta)$. Let observable Φ be defined as follows:

$$\Phi(\gamma)=\begin{cases}\Psi(\delta), & \text{if } \gamma=\langle b,\mathrm{rt}(t)\rangle\langle *,s\rangle\delta,\\ 0, & \text{otherwise.}\end{cases}$$

so that $\langle *,s\rangle^{-1}\langle b,\mathrm{rt}(t)\rangle^{-1}\Phi=\Psi$. It follows that for all γ we have

$$\begin{aligned}\mathcal{B}^O_t[\Phi](\langle b,0\rangle\langle *,0\rangle\gamma) \;&=\; \sum_u \Delta^O_b(t,u)\cdot\mathcal{B}^O_u[\langle b,\mathrm{rt}(t)\rangle^{-1}\Phi](\langle *,0\rangle\gamma)\\ &=\; \sum_u \Delta^O_b(t,u)\cdot\mathcal{B}^O_u[\langle *,\mathrm{rt}(u)\rangle^{-1}\langle b,\mathrm{rt}(t)\rangle^{-1}\Phi](\gamma).\end{aligned}$$

Now, by Lemma 12, for all γ the quantity $\mathcal{B}^O_u[\langle *,\mathrm{rt}(u)\rangle^{-1}\langle b,\mathrm{rt}(t)\rangle^{-1}\Phi](\gamma)$ can be expressed as a finite linear combination

$$\sum_{k\in K} c_k\cdot(\langle *,\mathrm{rt}(u)\rangle^{-1}\langle b,\mathrm{rt}(t)\rangle^{-1}\Phi)(\gamma_i).$$

However,

$$(\langle *, \mathrm{rt}(u)\rangle^{-1}\langle b, \mathrm{rt}(t)\rangle^{-1}\Phi)(\gamma_i) = \Phi(\langle b, \mathrm{rt}(t)\rangle\langle *, \mathrm{rt}(u)\rangle\gamma_i),$$

which by definition of Φ equals 0 unless $\mathrm{rt}(u) = s$. Thus

$$\mathcal{B}_u^O[\langle *, \mathrm{rt}(u)\rangle^{-1}\langle b, 0\rangle^{-1}\Phi](\gamma) = 0$$

unless $\mathrm{rt}(u) = s$. It follows from this observation (in the particular case that $\gamma = \beta$) that:

$$\begin{aligned}
\mathcal{B}_t^O[\Phi](\langle b, 0\rangle\langle *, 0\rangle\beta) &= \sum_u \Delta_b^O(t, u) \cdot \mathcal{B}_u^O[\langle *, \mathrm{rt}(u)\rangle^{-1}\langle b, \mathrm{rt}(t)\rangle^{-1}\Phi](\beta) \\
&= \mathrm{rt}(t) \cdot \mathcal{B}_{v_{b,s}}^O[\langle *, s\rangle^{-1}\langle b, \mathrm{rt}(t)\rangle^{-1}\Phi](\beta) \\
&= \mathrm{rt}(t) \cdot \mathcal{B}_{v_{b,s}}^O[\Psi](\beta) \\
&\neq \mathrm{rt}(t') \cdot \mathcal{B}_{v'_{b,s}}^O[\Psi](\beta) \\
&= \mathcal{B}_{t'}^O[\Phi](\langle b, 0\rangle\langle *, 0\rangle\beta).
\end{aligned}$$

We have thus exhibited observable Φ and rated trace $\alpha = \langle b, 0\rangle\langle *, 0\rangle\beta$ such that $\mathcal{B}_t^O[\Phi](\alpha) \neq \mathcal{B}_{t'}^O[\Phi](\alpha)$, thereby establishing that $\mathcal{B}_t^O \neq \mathcal{B}_{t'}^O$. As this is a contradiction with our hypothesis that $\mathcal{B}_t^O = \mathcal{B}_{t'}^O$ we conclude that our assumption that for some $b \in O$ and $s \in R_b$ the normal form $v_{b,s}$ is not identical up to permutation of sums to $v'_{b,s}$ is incorrect, and hence in fact $v_{b,s}$ and $v'_{b,s}$ are identical up to permutation of sums for all $b \in O$ and $s \in R_b$. This completes the induction step in case both t and t' are output normal forms.

It remains to consider the case in which t is $u + v$ and t' is $u' + v'$. In this case, the same argument as that just given above applies to show that v and v' are identical up to permutation of sums. The same argument as that used to prove Lemma 31 shows that it is impossible for one of u and u' to be nil_J and the other not. If both u and u' are nil_J, then they are identical up to permutation of sums. If both u and u' are input normal forms other than nil_J, then the same argument as that given for Case (1) above shows that u and u' are identical up to permutation of sums. Thus, in all cases, u is identical up to permutation of sums to u' and v is identical up to permutation of sums to v', hence t is identical up to permutation of sums to t'. □

Theorem 2. *The axioms in Table 2 are sound and complete for behavior equivalence of* $\|$*-free terms.*

Proof. Soundness was shown in Lemma 22.

Suppose terms t and u are behavior equivalent. Then t can be proved equivalent to a normal form t', and u can be proved equivalent to a normal form u'. By soundness, t' and u' are behavior equivalent. Since t' and u' are behavior equivalent normal forms, by Lemma 32 they are identical up to permutation of sums. Since t and u can be proved equivalent to terms identical up to permutation of sums, they can be proved equivalent to each other. □

5 Conclusion

By comparing complete axiomatizations (and especially the normal forms arising in the completeness proofs), we have improved our understanding of the relationship between two notions of equivalence for processes with Markovian behavior. In contrast to the axiomatization of weighted bisimulation equivalence, the axiomatization of behavior equivalence exhibits differences in the role of input actions and output actions.

If we restrict to the output-only fragment of the language, then a complete axiomatization of behavior equivalence is given by axioms (choice-unit), (choice-comm), (choice-assoc), (comb-idemp), (comb-comm), (comb-assoc), (output-comb), (output-distr), and (interchange). This axiomatization may be compared to the axiomatization given in [Ber05] for the "Markovian trace equivalence" notion originally defined in [BC00]. In fact, each of the axioms for Markovian trace equivalence is sound for behavior equivalence, so (applying Bernardo's completeness result) Markovian trace equivalent processes are also behavior equivalent.

Conversely, for $x \in Act^*$ (*i.e.* a trace) and $T \in [0, \infty)$ let observables $\Phi_{x,T}$ be defined by induction on x as follows:

$$\begin{array}{rcl} \Phi_{\epsilon,T} & = & \mathbf{1} \\ \Phi_{ax,T}(\alpha) & = & \begin{cases} \frac{1}{r} \cdot \Phi_{x,t-\frac{1}{r}}(\alpha'), & \text{if } \alpha = \langle a, r\rangle\alpha' \text{ and } \frac{1}{r} \leq T, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

It can be shown that "the probability of process t performing an execution compatible with x in average time $\leq T$" is given by $\sum_\alpha \mathcal{B}_t^O[\Phi_{x,T}](\alpha)$, where α ranges over all rated traces that contain only actions in O. Thus, behavior equivalent output-only processes are also Markovian trace equivalent. So, one part of what we have achieved is to show that the introduction of the operator ${}_\pi\oplus_{1-\pi}$ permits a finite axiomatization of Markovian trace equivalence, as opposed to the infinite axiom scheme given in [Ber05].

We have not yet succeeded in extending our results to include parallel composition. For weighted bisimulation equivalence there is an evident "expansion theorem" that permits parallel composition to be eliminated in favor of choice. For behavior equivalence, one might attempt a similar expansion for the parallel composition of two normal forms. One difficulty in doing this arises from the fact that behavior equivalence fails to be substitutive for parallel composition unless we restrict to input-stochastic terms. Thus we cannot employ various useful manipulations that move individual input-prefixed terms into and out of the scope of a parallel operator, as these do not preserve input-stochasticity, in general. Another subtlety is the following: if $t \in \mathrm{Proc}(I/J \Rightarrow O)$ and $J' \cap (J \cup O) = \emptyset$, then there is no way to eliminate parallel composition from a term of the form $t\ {}_O\|_{O'}\ \mathrm{nil}_{J'}$. Such a term amounts to a kind of "input expansion" of t which, in the absence of a recursion operator, cannot be otherwise expressed. So in the absence of recursion (or alternatively, an explicit input expansion operator) there can be no expansion theorem that completely eliminates parallel composition. To attempt an axiomatization of recursion would first require an extension of the completeness results of the present paper to open terms. We leave these explorations as subjects for future research.

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