

Some Properties of the Intervals

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MML Identifier: MEASURE6.

WWW: <http://mizar.org/JFM/Vol6/measure6.html>

The articles [8], [7], [10], [1], [9], [11], [5], [6], [2], [3], and [4] provide the notation and terminology for this paper.

The following propositions are true:

- (1) There exists a function F from \mathbb{N} into $[\mathbb{N}, \mathbb{N}]$ such that F is one-to-one and $\text{dom } F = \mathbb{N}$ and $\text{rng } F = [\mathbb{N}, \mathbb{N}]$.
- (2) For every function F from \mathbb{N} into $\overline{\mathbb{R}}$ such that F is non-negative holds $0_{\overline{\mathbb{R}}} \leq \sum F$.
- (3) Let F be a function from \mathbb{N} into $\overline{\mathbb{R}}$ and x be an extended real number. Suppose there exists a natural number n such that $x \leq F(n)$ and F is non-negative. Then $x \leq \sum F$.
- (8)¹ For all extended real numbers x, y such that x is a real number holds $(y - x) + x = y$ and $(y + x) - x = y$.
- (10)² For all extended real numbers x, y, z such that $z \in \mathbb{R}$ and $y < x$ holds $(z + x) - (z + y) = x - y$.
- (11) For all extended real numbers x, y, z such that $z \in \mathbb{R}$ and $x \leq y$ holds $z + x \leq z + y$ and $x + z \leq y + z$ and $x - z \leq y - z$.
- (12) For all extended real numbers x, y, z such that $z \in \mathbb{R}$ and $x < y$ holds $z + x < z + y$ and $x + z < y + z$ and $x - z < y - z$.

Let x be a real number. The functor $\overline{\mathbb{R}}(x)$ yielding an extended real number is defined as follows:

(Def. 1) $\overline{\mathbb{R}}(x) = x$.

We now state a number of propositions:

- (13) For all real numbers x, y holds $x \leq y$ iff $\overline{\mathbb{R}}(x) \leq \overline{\mathbb{R}}(y)$.
- (14) For all real numbers x, y holds $x < y$ iff $\overline{\mathbb{R}}(x) < \overline{\mathbb{R}}(y)$.
- (15) For all extended real numbers x, y, z such that $x < y$ and $y < z$ holds y is a real number.
- (16) Let x, y, z be extended real numbers. Suppose x is a real number and z is a real number and $x \leq y$ and $y \leq z$. Then y is a real number.
- (17) For all extended real numbers x, y, z such that x is a real number and $x \leq y$ and $y < z$ holds y is a real number.

¹ The propositions (4)–(7) have been removed.

² The proposition (9) has been removed.

- (18) For all extended real numbers x, y, z such that $x < y$ and $y \leq z$ and z is a real number holds y is a real number.
- (19) For all extended real numbers x, y such that $0_{\overline{\mathbb{R}}} < x$ and $x < y$ holds $0_{\overline{\mathbb{R}}} < y - x$.
- (20) For all extended real numbers x, y, z such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq z$ and $z + x < y$ holds $z < y - x$.
- (21) For every extended real number x holds $x - 0_{\overline{\mathbb{R}}} = x$.
- (22) For all extended real numbers x, y, z such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq z$ and $z + x < y$ holds $z \leq y$.
- (23) For every extended real number x such that $0_{\overline{\mathbb{R}}} < x$ there exists an extended real number y such that $0_{\overline{\mathbb{R}}} < y$ and $y < x$.
- (24) Let x, z be extended real numbers. Suppose $0_{\overline{\mathbb{R}}} < x$ and $x < z$. Then there exists an extended real number y such that $0_{\overline{\mathbb{R}}} < y$ and $x + y < z$ and $y \in \mathbb{R}$.
- (25) Let x, z be extended real numbers. Suppose $0_{\overline{\mathbb{R}}} \leq x$ and $x < z$. Then there exists an extended real number y such that $0_{\overline{\mathbb{R}}} < y$ and $x + y < z$ and $y \in \mathbb{R}$.
- (26) For every extended real number x such that $0_{\overline{\mathbb{R}}} < x$ there exists an extended real number y such that $0_{\overline{\mathbb{R}}} < y$ and $y + y < x$.

Let x be an extended real number. Let us assume that $0_{\overline{\mathbb{R}}} < x$. The functor $\text{Seg}x$ yields a non empty subset of $\overline{\mathbb{R}}$ and is defined by:

(Def. 2) For every extended real number y holds $y \in \text{Seg}x$ iff $0_{\overline{\mathbb{R}}} < y$ and $y + y < x$.

Let x be an extended real number. The functor $\text{len}x$ yields an extended real number and is defined by:

(Def. 3) $\text{len}x = \sup \text{Seg}x$.

Next we state several propositions:

- (27) For every extended real number x such that $0_{\overline{\mathbb{R}}} < x$ holds $0_{\overline{\mathbb{R}}} < \text{len}x$.
- (28) For every extended real number x such that $0_{\overline{\mathbb{R}}} < x$ holds $\text{len}x \leq x$.
- (29) For every extended real number x such that $0_{\overline{\mathbb{R}}} < x$ and $x < +\infty$ holds $\text{len}x$ is a real number.
- (30) For every extended real number x such that $0_{\overline{\mathbb{R}}} < x$ holds $\text{len}x + \text{len}x \leq x$.
- (31) Let e_1 be an extended real number. Suppose $0_{\overline{\mathbb{R}}} < e_1$. Then there exists a function F from \mathbb{N} into $\overline{\mathbb{R}}$ such that for every natural number n holds $0_{\overline{\mathbb{R}}} < F(n)$ and $\sum F < e_1$.
- (32) Let e_1 be an extended real number and X be a non empty subset of $\overline{\mathbb{R}}$. Suppose $0_{\overline{\mathbb{R}}} < e_1$ and $\inf X$ is a real number. Then there exists an extended real number x such that $x \in X$ and $x < \inf X + e_1$.
- (33) Let e_1 be an extended real number and X be a non empty subset of $\overline{\mathbb{R}}$. Suppose $0_{\overline{\mathbb{R}}} < e_1$ and $\sup X$ is a real number. Then there exists an extended real number x such that $x \in X$ and $\sup X - e_1 < x$.
- (34) Let F be a function from \mathbb{N} into $\overline{\mathbb{R}}$. Suppose F is non-negative and $\sum F < +\infty$. Let n be a natural number. Then $F(n) \in \mathbb{R}$.

$-\infty$ is an extended real number. Then $+\infty$ is an extended real number.

One can prove the following propositions:

- (35) \mathbb{R} is an interval and $\mathbb{R} =]-\infty, +\infty[$ and $\mathbb{R} = [-\infty, +\infty]$ and $\mathbb{R} = [-\infty, +\infty[$ and $\mathbb{R} =]-\infty, +\infty]$.

- (36) For all extended real numbers a, b such that $b = -\infty$ holds $]a, b[= \emptyset$ and $[a, b] = \emptyset$ and $[a, b[= \emptyset$ and $]a, b] = \emptyset$.
- (37) For all extended real numbers a, b such that $a = +\infty$ holds $]a, b[= \emptyset$ and $[a, b] = \emptyset$ and $[a, b[= \emptyset$ and $]a, b] = \emptyset$.
- (38) Let A be an interval, a, b be extended real numbers, and c, d, e be real numbers. If $A =]a, b[$ and $c \in A$ and $d \in A$ and $c \leq e$ and $e \leq d$, then $e \in A$.
- (39) Let A be an interval, a, b be extended real numbers, and c, d, e be real numbers. If $A = [a, b]$ and $c \in A$ and $d \in A$ and $c \leq e$ and $e \leq d$, then $e \in A$.
- (40) Let A be an interval, a, b be extended real numbers, and c, d, e be real numbers. If $A =]a, b]$ and $c \in A$ and $d \in A$ and $c \leq e$ and $e \leq d$, then $e \in A$.
- (41) Let A be an interval, a, b be extended real numbers, and c, d, e be real numbers. If $A = [a, b[$ and $c \in A$ and $d \in A$ and $c \leq e$ and $e \leq d$, then $e \in A$.
- (42) Let A be a non empty subset of $\overline{\mathbb{R}}$ and m, M be extended real numbers. Suppose $m = \inf A$ and $M = \sup A$. Suppose that
- (i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \leq e$ and $e \leq d$ holds $e \in A$,
 - (ii) $m \notin A$, and
 - (iii) $M \notin A$.
- Then $A =]m, M[$.
- (43) Let A be a non empty subset of $\overline{\mathbb{R}}$ and m, M be extended real numbers. Suppose $m = \inf A$ and $M = \sup A$. Suppose that
- (i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \leq e$ and $e \leq d$ holds $e \in A$,
 - (ii) $m \in A$,
 - (iii) $M \in A$, and
 - (iv) $A \subseteq \mathbb{R}$.
- Then $A = [m, M]$.
- (44) Let A be a non empty subset of $\overline{\mathbb{R}}$ and m, M be extended real numbers. Suppose $m = \inf A$ and $M = \sup A$. Suppose that
- (i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \leq e$ and $e \leq d$ holds $e \in A$,
 - (ii) $m \in A$,
 - (iii) $M \notin A$, and
 - (iv) $A \subseteq \mathbb{R}$.
- Then $A = [m, M[$.
- (45) Let A be a non empty subset of $\overline{\mathbb{R}}$ and m, M be extended real numbers. Suppose $m = \inf A$ and $M = \sup A$. Suppose that
- (i) for all real numbers c, d such that $c \in A$ and $d \in A$ and for every real number e such that $c \leq e$ and $e \leq d$ holds $e \in A$,
 - (ii) $m \notin A$,
 - (iii) $M \in A$, and
 - (iv) $A \subseteq \mathbb{R}$.
- Then $A =]m, M]$.

(46) Let A be a subset of \mathbb{R} . Then A is an interval if and only if for all real numbers a, b such that $a \in A$ and $b \in A$ and for every real number c such that $a \leq c$ and $c \leq b$ holds $c \in A$.

(47) For all intervals A, B such that A meets B holds $A \cup B$ is an interval.

Let A be an interval. Let us assume that $A \neq \emptyset$. The functor $\inf A$ yields an extended real number and is defined as follows:

(Def. 4) There exists an extended real number b such that $\inf A \leq b$ but $A =]\inf A, b[$ or $A =]\inf A, b]$ or $A = [\inf A, b[$ or $A = [\inf A, b]$.

Let A be an interval. Let us assume that $A \neq \emptyset$. The functor $\sup A$ yields an extended real number and is defined as follows:

(Def. 5) There exists an extended real number a such that $a \leq \sup A$ but $A =]a, \sup A[$ or $A =]a, \sup A]$ or $A = [a, \sup A[$ or $A = [a, \sup A]$.

One can prove the following propositions:

(48) For every interval A such that A is open interval and $A \neq \emptyset$ holds $\inf A \leq \sup A$ and $A =]\inf A, \sup A[$.

(49) For every interval A such that A is closed interval and $A \neq \emptyset$ holds $\inf A \leq \sup A$ and $A = [\inf A, \sup A]$.

(50) For every interval A such that A is right open interval and $A \neq \emptyset$ holds $\inf A \leq \sup A$ and $A = [\inf A, \sup A[$.

(51) For every interval A such that A is left open interval and $A \neq \emptyset$ holds $\inf A \leq \sup A$ and $A =]\inf A, \sup A]$.

(52) For every interval A such that $A \neq \emptyset$ holds $\inf A \leq \sup A$ but $A =]\inf A, \sup A[$ or $A =]\inf A, \sup A]$ or $A = [\inf A, \sup A[$ or $A = [\inf A, \sup A]$.

(54)³ For every interval A and for every real number a such that $a \in A$ holds $\inf A \leq \overline{\mathbb{R}}(a)$ and $\overline{\mathbb{R}}(a) \leq \sup A$.

(55) For all intervals A, B and for all real numbers a, b such that $a \in A$ and $b \in B$ holds if $\sup A \leq \inf B$, then $a \leq b$.

(56) For every interval A and for every extended real number a such that $a \in A$ holds $\inf A \leq a$ and $a \leq \sup A$.

(57) For every interval A such that $A \neq \emptyset$ and for every extended real number a such that $\inf A < a$ and $a < \sup A$ holds $a \in A$.

(58) For all intervals A, B such that $\sup A = \inf B$ but $\sup A \in A$ or $\inf B \in B$ holds $A \cup B$ is an interval.

Let A be a subset of \mathbb{R} and let x be a real number. The functor $x + A$ yielding a subset of \mathbb{R} is defined as follows:

(Def. 6) For every real number y holds $y \in x + A$ iff there exists a real number z such that $z \in A$ and $y = x + z$.

We now state several propositions:

(59) For every subset A of \mathbb{R} and for every real number x holds $-x + (x + A) = A$.

(60) For every real number x and for every subset A of \mathbb{R} such that $A = \mathbb{R}$ holds $x + A = A$.

(61) For every real number x holds $x + \emptyset = \emptyset$.

³ The proposition (53) has been removed.

- (62) For every interval A and for every real number x holds A is open interval iff $x + A$ is open interval.
- (63) For every interval A and for every real number x holds A is closed interval iff $x + A$ is closed interval.
- (64) Let A be an interval and x be a real number. Then A is right open interval if and only if $x + A$ is right open interval.
- (65) Let A be an interval and x be a real number. Then A is left open interval if and only if $x + A$ is left open interval.
- (66) For every interval A and for every real number x holds $x + A$ is an interval.

Let A be an interval and let x be a real number. One can check that $x + A$ is interval.

We now state the proposition

- (67) For every interval A and for every real number x holds $\text{vol}(A) = \text{vol}(x + A)$.

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Received February 5, 1994

Published January 2, 2004
