

AN INTRODUCTION TO THE MATHEMATICS OF ANDERSON LOCALIZATION

GÜNTER STOLZ

ABSTRACT. We give a widely self-contained introduction to the mathematical theory of the Anderson model. After defining the Anderson model and determining its almost sure spectrum, we prove localization properties of the model. Here we discuss spectral as well as dynamical localization and provide proofs based on the fractional moments (or Aizenman-Molchanov) method.

We also discuss, in less self-contained form, the extension of the fractional moment method to the continuum Anderson model. Finally, we mention major open problems.

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1. INTRODUCTION

In 1958 the physicist P. W. Anderson introduced the model which is now named after him to explain the quantum mechanical effects of disorder, as present in materials such as alloys and amorphous media [9]. The most famous phenomena which arise in the context of this model are *Anderson localization*, i.e. the suppression of electron transport due to disorder, and the *Anderson transition* in three-dimensional disordered media which predicts the existence of a mobility edge separating energy regions of localized states from an extended states region. Anderson localization has important consequences throughout physics, in theory and experiment. Anderson's work, and that of N. F. Mott and J. H. van Vleck, won the 1977 physics Nobel prize "for their fundamental theoretical investigations of the electronic structure of magnetic and disordered systems"¹.

Mathematically rigorous studies of the Anderson Model and other models of random operators started in the 1970s, with the first proof of Anderson localization for a related one-dimensional model provided by I. Goldsheid, S. Molchanov and L. Pastur in 1977 [23], followed several years later by a proof of localization for the actual Anderson model by H. Kunz and B. Souillard [29], also initially for dimension one. Since then the study of random operators has become an important field of mathematical physics, which has led to a tremendous amount of research activity and many mathematical results.

While the Anderson transition and extended states are still an open mathematical challenge, by now a good rigorous understanding of Anderson localization has been achieved. Several powerful methods have been found to prove Anderson localization. Important differences exist between one-dimensional and multi-dimensional

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¹http://nobelprize.org/nobel_prizes/physics/laureates/1977/

models, where different physical mechanisms are responsible for localization effects. In these notes we will focus on methods which allow to prove Anderson localization in arbitrary dimension. Two such methods are available: The method of *multiscale analysis* (MSA) developed in 1983 by Fröhlich and Spencer [20], and the *fractional moments method* (FMM) introduced by Aizenman and Molchanov in 1993 [5].

MSA has produced results in situations which are out of reach for an approach through the FMM, see Section 9.1 for some related discussion. However, the FMM is mathematically more elementary, in particular for the case of the classical *discrete* Anderson model which will be our main focus here. Also, under suitable assumptions, the FMM allows to prove stronger results on dynamical localization than can be obtained by MSA. Therefore, in these lectures, after an introduction to the Anderson model and its basic spectral properties, we will discuss how to prove Anderson localization based on the FMM.

After more than 50 years of physical research and more than 30 years of mathematical work a vast literature with results on Anderson localization and, more generally, the physics of disordered quantum mechanical systems, is available. In these introductory lectures we ignore most of the literature as it can not be our goal to provide a comprehensive survey, not even of the mathematical research which has been done. Some book length presentations, or parts of such, which provide very good further reading and many more references are [18, 14, 34, 43, 27].

It is assumed below that the reader is familiar with measure and integration theory as presented in [37], with basic probabilistic concepts such as independence, and with the foundations of the theory of linear operators in Hilbert spaces, up to the spectral theorem for self-adjoint operators and consequences such as spectral types (absolutely continuous, singular continuous and pure point spectrum) and the abstract solution of the time-dependent Schrödinger equation via Stone's theorem, e.g. [45] or [35]. Otherwise, we have tried to keep these notes mostly self-contained. For much of the first seven sections we provide full proofs.

We do not aim at the most general known results, but rather want to demonstrate that simple and natural mathematical ideas can be used to rigorously establish Anderson localization. Many further developments of the ideas discussed here can be found in the literature. The references provided below can serve as a starting point for further reading. An ideal source for continued reading and learning the state of the art of much what is discussed here will be the upcoming book [8] by M. Aizenman and S. Warzel.

In Section 2 we introduce the Anderson model and, as a warm-up, prove its first important property, namely that its spectrum is almost surely deterministic. The rest of these notes exclusively deals with the phenomenon of Anderson localization. Section 3 introduces the concepts of *spectral localization* and *dynamical localization*, followed by a discussion of what is known on the physics level of rigor.

In Sections 4 and 5 we prove localization in the large disorder regime of the Anderson model. This is done via the fractional moments method, by first proving in Section 4 that fractional moments of Green's function decay exponentially, and by then showing in Section 5 that this implies dynamical as well as spectral localization. In these sections we use methods which were developed in some of the first papers on the fractional moments method, e.g. [5] and [24]. In particular, these methods work directly for the Anderson model in *infinite volume*.

Subsequently, other methods were introduced in the literature, e.g. [1] or [6], which use *finite volume* restrictions of the Anderson model. A central concept here are so-called *eigenfunction correlators*. These methods have proven to be very powerful in further-reaching work, for example in dealing with the continuum Anderson model or multi-particle Anderson models. Thus we introduce this approach in Section 6 and Appendix A and show how they yield an alternative proof of localization.

Section 7 discusses the second main regime in which multi-dimensional localization has been established rigorously, the band edge regime. Among the new ideas needed here are the phenomenon of *Lifshits tails* of the integrated density of states near spectral edges and a geometric decoupling method to control correlations in Green's function. Parts of this section have the character of an outline, referring to the literature for some of the results used.

Entirely written in form of an outline is Section 8, in which we discuss the extension of the FMM to continuum Anderson models, as accomplished in [2] and [12]. This requires considerable technical effort and we merely point out the difficulties which had to be overcome and mention some of the tools which allowed to accomplish this.

The Anderson model and, more generally, the quantum mechanics of disordered media, provides many difficult future challenges for mathematicians. We discuss some of them in our concluding Section 9.

Acknowledgement: The author's knowledge of random operators and, in particular, of the Anderson model has benefitted from many other mathematicians, through their works as well as through personal contact. We apologize for not being able to properly give credit for the origins of all of these benefits. But we need and want to make an exception for the contributions of Michael Aizenman, who was the driving force in the development of the fractional moments method and has influenced the author's way of thinking about random operators in multiple ways. Much of what we have to say here is based on ideas of Michael and his collaborators. In particular, special thanks are due to Michael and Simone Warzel for letting the author use some preliminary material from [8] in Section 6 and Appendix A below.

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2. THE ANDERSON MODEL

2.1. The Discrete Laplacian. Below we will introduce the Anderson model as a *discrete* Schrödinger operator, acting, for dimension $d \geq 1$, on the Hilbert space

$$\ell^2(\mathbb{Z}^d) = \{u : \mathbb{Z}^d \rightarrow \mathbb{C} : \sum_{n \in \mathbb{Z}^d} |u(n)|^2 < \infty\},$$

with inner product $\langle u, v \rangle = \sum_n \overline{u(n)}v(n)$.

The usual negative Laplacian $-\Delta = -\sum_j \partial^2/\partial^2 x_j$ is replaced by its discrete analogue h_0 , which acts on $u \in \ell^2(\mathbb{Z}^d)$ by

$$(1) \quad (h_0 u)(n) = - \sum_{k \in \mathbb{Z}^d, |k|=1} u(n+k), \quad n \in \mathbb{Z}^d,$$

where $|k| = |k_1| + \dots + |k_d|$ is the graph distance on \mathbb{Z}^d . More appropriately, a finite difference approximation of $-\Delta$ would be given by $h_0 + 2d$, but we neglect the mathematically trivial shift by $2d$ (which should still be kept in mind for physical interpretations). In physics, the Hamiltonian h_0 (or its negative) most frequently arises more directly, i.e. not as a discretization of a differential operator, in the context of the so-called tight-binding approximation.

Just as the continuum Laplacian, the discrete Laplacian is unitarily equivalent to a multiplication operator via Fourier transform. Here we consider the Fourier transform

$$F : L^2([0, 2\pi)^d) \rightarrow \ell^2(\mathbb{Z}^d),$$

which is the unitary operator given by

$$(Fg)(n) = (2\pi)^{-d/2} \int_{[0, 2\pi)^d} g(x) e^{-ix \cdot n} dx,$$

with inverse

$$(F^{-1}u)(x) = \text{l.i.m.} (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d, |n| \leq N} u(n) e^{ix \cdot n}.$$

Here $x \cdot n = x_1 n_1 + \dots + x_d n_d$ and l.i.m. denotes the limit $N \rightarrow \infty$ in $\ell^2(\mathbb{Z}^d)$.

A calculation shows that

$$F^{-1}h_0F = -2 \sum_{j=1}^d \cos(x_j),$$

where the right-hand side is understood as a multiplication operator on $L^2([0, 2\pi)^d)$ in the variable $x = (x_1, \dots, x_d)$. The function $g(x) = -2 \sum_j \cos(x_j)$ is real-valued and bounded. Thus h_0 is bounded and self-adjoint (which also can be checked directly from the definition (1) without use of the Fourier transform). The range of g gives the spectrum of h_0 ,

$$(2) \quad \sigma(h_0) = [-2d, 2d].$$

With a bit more effort one can show that inverse images of Lebesgue-nullsets in \mathbb{R} under the function g are Lebesgue-nullsets in \mathbb{R}^d . Thus the spectrum of h_0 is purely absolutely continuous.

Another similarity of h_0 with the continuum Laplacian is that it has plane waves as generalized eigenfunctions. To see this, let $x \in [0, 2\pi)^d$ and set

$$(3) \quad \phi_x(n) := e^{in \cdot x}.$$

While $\phi_x \notin \ell^2(\mathbb{Z}^d)$, h_0 acts on it via (1) as

$$\begin{aligned} (h_0 \phi_x)(n) &= - \sum_{|k|=1} e^{i(n+k) \cdot x} \\ &= \left(- \sum_{j=1}^d 2 \cos(x_j) \right) \phi_x(n). \end{aligned}$$

Thus ϕ_x is a bounded generalized eigenfunction of h_0 to the spectral value $-2\sum_j \cos(x_j)$.

2.2. The Anderson Model. Let $\omega = (\omega_n)_{n \in \mathbb{Z}^d}$ be a set of independent, identically distributed (i.i.d.) real-valued random variables indexed by $n \in \mathbb{Z}^d$. Recall that this means the following, where we denote probabilities by \mathbb{P} :

- The (ω_n) are identically distributed, i.e. there exists a Borel probability measure μ on \mathbb{R} such that, for all $n \in \mathbb{Z}^d$ and Borel sets $A \subset \mathbb{R}$,

$$\mathbb{P}(\omega_n \in A) = \mu(A).$$

- The (ω_n) are independent. Thus, for each finite subset $\{n_1, \dots, n_\ell\}$ of \mathbb{Z}^d and arbitrary Borel sets $A_1, \dots, A_\ell \subset \mathbb{R}$,

$$\begin{aligned} \mathbb{P}(\omega_{n_1} \in A_1, \dots, \omega_{n_\ell} \in A_\ell) &= \prod_{j=1}^{\ell} \mathbb{P}(\omega_{n_j} \in A_j) \\ &= \prod_{j=1}^{\ell} \mu(A_j). \end{aligned}$$

It is sometimes useful to think of a concrete way in which i.i.d. random variables can be realized as measurable functions on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The standard construction is the infinite product space

$$(\Omega, \mathcal{A}, \mathbb{P}) = \bigotimes_{n \in \mathbb{Z}^d} (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu),$$

with \mathcal{A} and \mathbb{P} denoting the σ -algebra and measure generated by the pre-measure induced by μ on the Borel cylinder sets in $\Omega = \mathbb{R}^{\mathbb{Z}^d}$. This is consistent with the notation $\omega = (\omega_n)_{n \in \mathbb{Z}^d}$ as the components ω_n of $\omega \in \Omega$ are now i.i.d. random variables on Ω with common distribution μ .

It is also convenient to work on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, which in the above realization is achieved by completing the product algebra \mathcal{A} under \mathbb{P} , for which the same notation will be kept.

The *Anderson Model* is a random Hamiltonian h_ω on $\ell^2(\mathbb{Z}^d)$, defined for $\omega \in \Omega$ by

$$(4) \quad (h_\omega u)(n) = (h_0 u)(n) + \omega_n u(n), \quad n \in \mathbb{Z}^d.$$

Introducing the *random potential* $V_\omega : \mathbb{Z}^d \rightarrow \mathbb{R}$ by $V_\omega(n) = \omega_n$, we may also write

$$h_\omega = h_0 + V_\omega.$$

Note here that h_ω is not a single operator, but rather an operator-valued function on a probability space. Its operator theoretic properties will generally depend on ω . Our goal will typically be that a certain property of h_ω holds *almost surely* or *with probability one*, meaning that it holds for $\omega \in \Omega_0$, a measurable subset of Ω with $\mathbb{P}(\Omega_0) = 1$. It lies within the nature of random operator theory that the most interesting properties will only hold almost surely rather than for *all* $\omega \in \Omega$.

One may think of the Anderson model h_ω as the Hamiltonian governing the quantum mechanical motion of a single electron in a discretized alloy-type random medium. In this view the random potential $V_\omega(n) = \omega_n$, $n \in \mathbb{Z}^d$, represents a solid formed by nuclei located at the sites n of the lattice \mathbb{Z}^d and carrying random electrical charges ω_n . Assuming that h_ω is self-adjoint, the dynamics of the electron

is given through the unitary group e^{-ith_ω} , defined via the spectral theorem, which provides the solution $\psi(t) = e^{-ith_\omega}\psi_0$ of the time-dependent Schrödinger equation $h_\omega\psi(t) = i\psi'(t)$, $\psi(0) = \psi_0$. The possible energies of the electron are given by the spectrum $\sigma(h_\omega)$ of the Anderson Hamiltonian h_ω .

In the above discussion we have assumed self-adjointness of the Hamiltonian, which is the first mathematical fact to be checked. This is particularly easy for discrete Schrödinger operators such as h_ω because the discrete Laplacian h_0 is bounded and self-adjoint.

Theorem 1. *For every $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, the operator h_ω is self-adjoint on*

$$D(V_\omega) = \{u \in \ell^2(\mathbb{Z}^d) : \sum_n |\omega_n u(n)|^2 < \infty\},$$

the domain of the maximal multiplication operator by the potential V_ω .

Proof. Self-adjointness of the maximal multiplication operator by a real-valued function is a standard fact. Perturbation of the self-adjoint maximal multiplication operator V_ω by the bounded self-adjoint operator h_0 preserves self-adjointness with same domain, e.g. [45]. \square

If we assume, as will be done later, that the distribution μ of the ω_n has bounded support, i.e. that

$$\text{supp } \mu := \{t \in \mathbb{R} : \mu((t - \varepsilon, t + \varepsilon)) > 0 \text{ for all } \varepsilon > 0\}$$

is bounded in \mathbb{R} , then the potential V_ω is bounded and therefore defines a bounded multiplication operator. Thus h_ω is a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^d)$ as well. On the other hand, if $\text{supp } \mu$ is unbounded, then it is not hard to see that V_ω , and thus h_ω , is almost surely unbounded.

2.3. The spectrum of the Anderson model. Our next goal is to determine the spectrum of h_ω . It follows as a consequence of the general theory of so-called *ergodic operators* (e.g. [14]), of which the Anderson model is a special case, that $\sigma(h_\omega)$ is almost surely deterministic, i.e. there exists a closed subset Σ of \mathbb{R} such that

$$\sigma(h_\omega) = \Sigma \quad \text{almost surely.}$$

Rather than proving this within the general theory of ergodic operators we will give a direct proof of the following result, which explicitly describes the almost sure spectrum of the Anderson model:

Theorem 2. *The spectrum of the Anderson model is almost surely given by*

$$(5) \quad \sigma(h_\omega) = \sigma(h_0) + \text{supp } \mu.$$

Here the sum of two subsets A and B of \mathbb{R} is defined by $A + B := \{a + b : a \in A, b \in B\}$. In particular, this means that the almost sure spectrum of h_ω is a union of intervals, namely of translates of $[-2d, 2d]$ by the points in $\text{supp } \mu$. If $\text{supp } \mu$ doesn't have large gaps, then the almost sure spectrum of h_ω is a single interval.

Proof. We begin with the easy part of the proof, namely that

$$(6) \quad \sigma(h_\omega) \subset [-2d, 2d] + \text{supp } \mu$$

almost surely.

We first argue that $\sigma(V_\omega) = \overline{\{\omega_n : n \in \mathbb{Z}^d\}} \subset \text{supp } \mu$ almost surely. In fact, as $\mu(\text{supp } \mu) = 1$, for fixed $n \in \mathbb{Z}^d$, $\omega_n \in \text{supp } \mu$ holds almost surely, i.e. on a set

$\Omega_n \subset \Omega$ with $\mathbb{P}(\Omega_n) = 1$. The countable intersection Ω' of the Ω_n also has measure one and for $\omega \in \Omega'$ we have $\sigma(V_\omega) \subset \text{supp } \mu$ as $\text{supp } \mu$ is closed.

By a general fact from spectral theory, easily proven using a Neumann series argument, a bounded self-adjoint perturbation B does not shift the spectrum of a self-adjoint operator A by more than $\|B\|$, i.e. $\sigma(A + B) \subset \sigma(A) + [-\|B\|, \|B\|]$. Thus (6) holds for $\omega \in \Omega'$ by (2).

The proof of

$$(7) \quad [-2d, 2d] + \text{supp } \mu \subset \sigma(h_\omega)$$

with probability one is more involved and falls into a probabilistic part and a spectral theoretic part.

For the probabilistic part, start with fixed $t \in \text{supp } \mu$, $\varepsilon > 0$ and $N \in \mathbb{N}$ and let

$$\Omega_{t,N,\varepsilon} := \{\omega \in \Omega : \text{There exists a cube } \Lambda_N \subset \mathbb{Z}^d \text{ of side length } N \text{ such that } |\omega_n - t| < \varepsilon \text{ for all } n \in \Lambda_N\}.$$

As $t \in \text{supp } \mu$, we have $p := \mu((t - \varepsilon, t + \varepsilon)) > 0$. Thus, for each fixed cube Λ of side length N in \mathbb{Z}^d , the probability that $|\omega_n - t| < \varepsilon$ for all $n \in \Lambda$ is $p^{N^d} > 0$. We can cover \mathbb{Z}^d by infinitely many disjoint cubes of side length N , where these events are independent. It follows that $\mathbb{P}(\Omega_{t,N,\varepsilon}) = 1$.

Next, let $\Omega_{t,\varepsilon} := \bigcap_{N \in \mathbb{N}} \Omega_{t,N,\varepsilon}$. Thus $\mathbb{P}(\Omega_{t,\varepsilon}) = 1$ and we will prove the following below: For each $\omega \in \Omega_{t,\varepsilon}$ and $a \in [-2d, 2d]$ it holds that

$$(8) \quad [a + t - \varepsilon, a + t + \varepsilon] \cap \sigma(h_\omega) \neq \emptyset.$$

Assuming that (8) is true, we proceed as follows: Define $\Omega_t := \bigcap_{\ell \in \mathbb{N}} \Omega_{t,1/\ell}$, such that $\mathbb{P}(\Omega_t) = 1$. For $\omega \in \Omega_t$ we have by (8) that, for all $a \in [-2d, 2d]$,

$$(a + t - \frac{1}{\ell}, a + t + \frac{1}{\ell}) \cap \sigma(h_\omega) \neq \emptyset$$

for all $\ell \in \mathbb{N}$. As $\sigma(h_\omega)$ is closed, this implies that $a + t \in \sigma(h_\omega)$ for all $a \in [-2d, 2d]$, and thus $[-2d, 2d] + t \subset \sigma(h_\omega)$.

For one last argument involving countable intersections of full measure sets, let B be a countable subset of $\text{supp } \mu$ which is dense in $\text{supp } \mu$ and let $\Omega_0 := \bigcap_{t \in B} \Omega_t$. Then $\mathbb{P}(\Omega_0) = 1$ and for $\omega \in \Omega_0$ we have $[-2d, 2d] + B \subset \sigma(h_\omega)$. Using again that $\sigma(h_\omega)$ is closed completes the proof of (7).

We still need to show (8), which is the spectral-theoretic part of the proof. Let $\omega \in \Omega_{t,\varepsilon}$. Thus, by assumption, for each $N \in \mathbb{N}$ there exists a cube Λ_N of side length N such that $|\omega_n - t| < \varepsilon$ for all $n \in \Lambda_N$.

To $a \in [-2d, 2d]$ pick $x = (x_1, \dots, x_d)$ such that $a = -2 \sum_j \cos(x_j)$ and consider the corresponding generalized eigenfunction ϕ_x from (3). Then $\psi_N := \chi_{\Lambda_N} \phi_x$ has finite support and, in particular, lies in $\ell^2(\mathbb{Z}^d)$. We claim that

$$(9) \quad \limsup_{N \rightarrow \infty} \frac{\|(h_\omega - (a + t))\psi_N\|}{\|\psi_N\|} \leq \varepsilon.$$

To find a norm bound for $(h_\omega - (a + t))\psi_N = (h_0 - a)\chi_{\Lambda_N} \phi_x + (V_\omega - t)\chi_{\Lambda_N} \phi_x$, we first note that by assumption $\|(V_\omega - t)\chi_{\Lambda_N} \phi_x\| \leq \varepsilon \|\psi_N\|$. Moreover, as $(h_0 - a)\phi_x = 0$, it follows that $((h_0 - a)\chi_{\Lambda_N} \phi_x)(n)$ is non-zero only for n close to the boundary of Λ_N , where its values are bounded by a constant independent of N . Thus

$$\|(h_\omega - (a + t))\psi_N\| \leq CN^{(d-1)/2} + \varepsilon \|\psi_N\|.$$

On the other hand we have $\|\psi_N\| = N^{d/2}$. This proves (9).

We conclude by a standard argument: If $h_\omega - (a + t)$ is invertible, then, by (9),

$$\|(h_\omega - (a + t))^{-1}\| \geq \frac{1}{\varepsilon}.$$

This implies (8) by using the fact that for general self-adjoint operators A it holds that

$$\|(A - z)^{-1}\| = \frac{1}{\text{dist}(z, \sigma(A))}.$$

□

3. LOCALIZATION PROPERTIES

We will be interested in *localization properties* of the Anderson model, which can be described either by spectral properties or by dynamical properties of the Hamiltonian.

To be more precise, let $I \subset \mathbb{R}$ be an open interval. We say that h_ω exhibits *spectral localization* in I if h_ω almost surely has pure point spectrum in I , i.e. I does not contain any continuous spectrum of h_ω , and its eigenfunctions to all eigenvalues in I decay exponentially.

If I is a non-trivial interval contained in the almost sure spectrum of h_ω , which is a union of intervals, then spectral localization in I necessarily means that the spectrum consists of a dense set of eigenvalues (whose closure fills all of I). This phenomenon is very different and much more subtle than the appearance of *discrete* isolated eigenvalues, which is the classical situation encountered in atomic or molecular hamiltonians. In fact, the possibility of dense pure point spectrum historically can be considered as the biggest mathematical surprise provided by the investigation of the Anderson model.

On the other hand, we say that h_ω exhibits *dynamical localization* in I if there exist constants $C < \infty$ and $\mu > 0$ such that

$$(10) \quad \mathbb{E} \left(\sup_{t \in \mathbb{R}} | \langle e_j, e^{-it h_\omega} \chi_I(h_\omega) e_k \rangle | \right) \leq C e^{-\mu |j-k|},$$

for all $j, k \in \mathbb{Z}^d$. Here $\{e_j\}_{j \in \mathbb{Z}^d}$ is the canonical orthonormal basis in \mathbb{Z}^d , $e_j(k) = \delta_{jk}$, and $\mathbb{E}(\cdot)$ denotes the expectation with respect to the probability measure \mathbb{P} , meaning $\mathbb{E}(X) = \int_\Omega X d\mathbb{P}$ for random variables X on Ω . Both, $e^{-it h_\omega}$ as well as $\chi_I(h_\omega)$, are defined via the functional calculus for self-adjoint operators. By χ_I we denote the characteristic function of I , so that $\chi_I(h_\omega)$ is the spectral projection for h_ω onto I .

Dynamical localization in the form (10) is a strong form of asserting that solutions of the time-dependent Schrödinger equation $h_\omega \psi(t) = i \partial_t \psi(t)$ are staying localized in space, uniformly for all times, and thus shows the absence of *quantum transport*. Let us illustrate this by showing that dynamical localization implies that all moments of the position operator are bounded in time, i.e. for all $p > 0$ and all finitely supported $\psi \in \ell^2(\mathbb{Z}^d)$,

$$(11) \quad \sup_{t \in \mathbb{R}} \| |X|^p e^{-it h_\omega} \chi_I(h_\omega) \psi \| < \infty \quad \text{almost surely,}$$

where the position operator $|X|$ is defined by $(|X|\phi)(n) = |n|\phi(n)$. To see how (11) follows from (10), assume that $\psi(k) = 0$ for $|k| > R$. Then

$$\begin{aligned} \||X|^p e^{-ith_\omega} \chi_I(h_\omega) \psi\|^2 &= \sum_j |\langle e_j, |X|^p e^{-ith_\omega} \chi_I(h_\omega) \psi \rangle|^2 \\ &= \sum_j |j|^{2p} \left| \sum_{|k| \leq R} \langle e_j, e^{-ith_\omega} \chi_I(h_\omega) e_k \rangle \psi(k) \right|^2 \\ &\leq \sum_j \sum_{|k| \leq R} |j|^{2p} |\langle e_j, e^{-ith_\omega} \chi_I(h_\omega) e_k \rangle|^2 \|\psi\|^2, \end{aligned}$$

where the last step used the Cauchy-Schwarz inequality. We can drop the square from $|\langle e_j, e^{-ith_\omega} \chi_I(h_\omega) e_k \rangle|^2$ (as this number is bounded by 1) and then take expectations to get

$$\begin{aligned} &\mathbb{E} \left(\sup_t \||X|^p e^{-ith_\omega} \chi_I(h_\omega) \psi\|^2 \right) \\ &\leq \sum_j \sum_{|k| \leq R} |j|^{2p} \mathbb{E} \left(\sup_t |\langle e_j, e^{-ith_\omega} \chi_I(h_\omega) e_k \rangle| \right) \|\psi\|^2 \\ &\leq C \sum_j \sum_{|k| \leq R} |j|^{2p} e^{-\mu|j-k|} \|\psi\|^2 \\ &< \infty. \end{aligned}$$

This implies the almost sure statement in (11) (with square at the norm and therefore also without).

Dynamical localization is not only the physically more interesting statement than spectral localization (as physicists usually have little patience and limited appreciation for spectral theory), it is also the mathematically stronger property: We will show later that dynamical localization in I implies spectral localization in I .

Let us discuss situations in which localization, spectral or dynamical, is expected physically. For this it will help to introduce an additional disorder parameter $\lambda > 0$ in the Anderson model and define

$$(12) \quad h_{\omega, \lambda} = h_0 + \lambda V_\omega,$$

with $V_\omega(n) = \omega_n$ as above. Formally, this fits into the same framework as (4), using the re-scaled distribution

$$(13) \quad \mathbb{P}(\lambda \omega_n \in B) = \mu_\lambda(B) := \mu(B/\lambda)$$

of the i.i.d. random variables $\lambda \omega_n$. The distribution μ_λ is spread out over larger supports for larger λ , corresponding to a wider range of possible random charges in an alloy-type medium. Thus $\lambda \gg 1$ is the case of *large disorder* and $\lambda \ll 1$ represents *small disorder*.

Physicists know all of the following:

In dimension $d = 1$ the entire spectrum of $h_{\omega, \lambda}$ is localized for *any* value of the disorder $\lambda > 0$.

In dimension $d \geq 2$ the entire spectrum is localized at *large disorder*, i.e. for $\lambda \gg 1$.

For small disorder λ different behavior arises in dimensions $d = 2$ and $d = 3$. For $d = 2$ one still has localization of the entire spectrum, but possibly in a weaker form than for $d = 1$, e.g. a small amount (or weak type) of quantum transport might be possible. On the other hand, in dimension $d = 3$ one observes the *Anderson transition*. There are localized regions near the band edges of the almost sure spectrum, separated by *mobility edges* from a region of *extended states* in the center of the spectrum. Extended states are interpreted as the existence of quantum transport in the sense that the moments (11) should be infinite for sufficiently large p . The physical expectation for $d = 3$ is that this starts at $p = 1/2$, which corresponds to the presence of diffusive motion.

Mathematically, localization has been proven for three different regimes: (i) for all energies and arbitrary disorder in $d = 1$, (ii) in any dimension and for all energies at sufficiently large disorder, and (iii) near band edges of the spectrum in any dimension and for arbitrary disorder.

The mechanisms which cause localization in the Anderson model are fundamentally different for the one-dimensional and multi-dimensional case, which is also reflected in the mathematical methods which have been used to prove this. In $d = 1$ strong tools from the theory of one-dimensional dynamical systems are available, in particular results on the asymptotics of products of independent random variables which allow to prove positivity of Lyapunov exponents. Large parts of the books [14] and [34] are devoted to the presentation of the one-dimensional theory. A complete presentation of the Kunz-Souillard proof of localization for the one-dimensional Anderson model can be found in [18]. For a somewhat later survey of results on one-dimensional localization see [44].

As discussed in the introduction, we will focus here on methods which allow to prove multi-dimensional localization and, among the two methods which have been shown to accomplish this, focus on the fractional moments method. Using this method we will give a detailed proof of large disorder localization and also explain how it works to show band edge localization, in each case in arbitrary dimension.

We will not discuss localization proofs via multiscale analysis. Excellent introductions to this method can be found in [27] and [43], while the state of the art of what can be obtained from Fröhlich-Spencer-type multiscale analysis is presented in [21] and the review [28]. We also mention the recent powerful extension of the ideas behind multiscale analysis in [13], which allow to prove localization for continuum Anderson models (see Section 8) with discretely distributed random couplings, a result which is beyond what can be obtained by the fractional moments method.

4. LOCALIZATION AT LARGE DISORDER

Consider the Anderson model (12) at disorder $\lambda > 0$ and in any dimension $d \geq 1$.

Throughout the rest of these notes we will work with a stronger assumption on the distribution μ of the random parameters ω_n , namely that μ is absolutely continuous with density ρ ,

$$(14) \quad \mu(B) = \int_B \rho(v) dv \quad \text{for } B \subset \mathbb{R} \text{ Borel, } \rho \in L_0^\infty(\mathbb{R}),$$

i.e. ρ is bounded and has compact support. In particular, this means that the Anderson hamiltonian $h_{\omega,\lambda}$ is a bounded self-adjoint operator in $\ell^2(\mathbb{Z}^d)$.

Introduce the *Green function* as the matrix-elements of the resolvent of $h_{\omega,\lambda}$,

$$(15) \quad G_{\omega,\lambda}(x, y; z) := \langle e_x, (h_{\omega,\lambda} - z)^{-1} e_y \rangle.$$

Our first goal is to prove

Theorem 3 ([5]). *Let $0 < s < 1$. Then there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$ there are $C < \infty$ and $\mu > 0$ with*

$$(16) \quad \mathbb{E}(|G_{\omega,\lambda}(x, y; z)|^s) \leq C e^{-\mu|x-y|}$$

uniformly in $x, y \in \mathbb{Z}^d$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

It is the appearance of fractional moments of the form $\mathbb{E}(|\cdot|^s)$, $0 < s < 1$, in the above theorem which prompted the name “fractional moments method” for the circle of ideas which we want to present here. The method is also frequently called the “Aizenman-Molchanov method”, as Aizenman and Molchanov did not only realize that results such as Theorem 3 hold, but that they imply spectral and dynamical localization. These implications will be discussed in the next section.

The following proof of Theorem 3 closely follows the original ideas from [5]. We start with two lemmas, an *a-priori bound* on the fractional moments of Green’s function and a *decoupling lemma*, which contain central ideas behind the method and, in increasing degree of sophistication, have been used in all subsequent developments of the method.

Lemma 4.1 (A priori bound). *There exists a constant $C_1 = C_1(s, \rho) < \infty$ such that*

$$(17) \quad \mathbb{E}_{x,y}(|G_{\omega,\lambda}(x, y; z)|^s) \leq C_1 \lambda^{-s}$$

for all $x, y \in \mathbb{Z}^d$, $z \in \mathbb{C} \setminus \mathbb{R}$, and $\lambda > 0$.

Here

$$\mathbb{E}_{x,y}(\dots) = \int \int \dots \rho(\omega_x) d\omega_x \rho(\omega_y) d\omega_y$$

is the conditional expectation with $(\omega_u)_{u \in \mathbb{Z}^d \setminus \{x,y\}}$ fixed. After averaging over ω_x and ω_y the bound in (17) does not depend on the remaining random parameters. Thus we get also that

$$\mathbb{E}(|G_{\omega,\lambda}(x, y; z)|^s) \leq C_1 \lambda^{-s}.$$

Proof. We first prove (17) for the case $x = y$, which demonstrates the simplicity of the fundamental idea underlying the FMM. For fixed $x \in \mathbb{Z}^d$, write $\omega = (\hat{\omega}, \omega_x)$ where $\hat{\omega}$ is short for $(\omega_u)_{u \in \mathbb{Z}^d \setminus \{x\}}$. With $P_{e_x} := \langle e_x, \cdot \rangle e_x$, the orthogonal projection onto the span of e_x , we can separate the ω_x and $\hat{\omega}$ dependence of $h_{\omega,\lambda}$ as

$$h_{\omega,\lambda} = h_{\hat{\omega},\lambda} + \lambda \omega_x P_{e_x}.$$

The resolvent identity yields

$$(18) \quad (h_{\omega,\lambda} - z)^{-1} = (h_{\hat{\omega},\lambda} - z)^{-1} - \lambda \omega_x (h_{\hat{\omega},\lambda} - z)^{-1} P_{e_x} (h_{\omega,\lambda} - z)^{-1}.$$

Taking matrix-elements we conclude for the corresponding diagonal Green functions that

$$(19) \quad G_{\omega,\lambda}(x, x; z) = G_{\hat{\omega},\lambda}(x, x; z) - \lambda \omega_x G_{\hat{\omega},\lambda}(x, x; z) G_{\omega,\lambda}(x, x; z)$$

or

$$(20) \quad G_{\omega,\lambda}(x, x; z) = \frac{1}{a + \lambda \omega_x} \quad \text{with } a = \frac{1}{G_{\hat{\omega},\lambda}(x, x; z)}.$$

Note that the latter is well-defined since one can easily check the Herglotz property $\text{Im } G_{\hat{\omega}, \lambda}(x, x; z)/\text{Im } z > 0$ of the Green function.

The important fact is that a is a complex number which does not depend on ω_x . Thus, writing $\mathbb{E}_x(\dots) := \int \dots \rho(\omega_x) d\omega_x$, we find that

$$(21) \quad \mathbb{E}_x(|G_{\omega, \lambda}(x, x; z)|^s) \leq \frac{\|\rho\|_\infty}{\lambda^s} \int_{\text{supp } \rho} \frac{d\omega_x}{|\frac{a}{\lambda} + \omega_x|^s} \leq \frac{C(\rho, s)}{\lambda^s},$$

with $C(\rho, s)$ independent of λ and a , and thus independent of $\hat{\omega}$, z and x .

The proof of (17) for $x \neq y$ is based on the same idea, replacing the rank-one-perturbation arguments above with rank-two-perturbation arguments. Write $\omega = (\hat{\omega}, \omega_x, \omega_y)$, $P = P_{e_x} + P_{e_y}$ and

$$h_{\omega, \lambda} = h_{\hat{\omega}, \lambda} + \lambda \omega_x P_{e_x} + \lambda \omega_y P_{e_y}.$$

Using the resolvent identity similar to above one arrives at

$$(22) \quad P(h_{\omega, \lambda} - z)^{-1}P = \left(A + \lambda \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix} \right)^{-1},$$

where

$$A = (P(h_{\hat{\omega}, \lambda} - z)^{-1}P)^{-1},$$

both to be read as identities for 2×2 -matrices in the range of P . This is a special case of the *Krein formula* which characterizes the resolvents of finite-rank perturbations of general self adjoint operators. For the matrix A one can check that $\text{Im } A = \frac{1}{2i}(A - A^*) < 0$ if $\text{Im } z > 0$ and $\text{Im } A > 0$ if $\text{Im } z < 0$. It is also independent of ω_x and ω_y .

Using that $G_{\omega, \lambda}(x, y; z)$ is one of the matrix-elements of $P(h_{\omega, \lambda} - z)^{-1}P$, we find

$$\begin{aligned} \mathbb{E}_{x, y}(|G_{\omega, \lambda}(x, y; z)|^s) &\leq \mathbb{E}_{x, y} \left(\left\| \left(A + \lambda \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix} \right)^{-1} \right\|^s \right) \\ &= \lambda^{-s} \mathbb{E}_{x, y} \left(\left\| \left(-\frac{1}{\lambda} A - \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix} \right)^{-1} \right\|^s \right) \\ &\leq \frac{\|\rho\|_\infty^2}{\lambda^s} \int_{-r}^r \int_{-r}^r \left\| \left(-\frac{1}{\lambda} A - \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix} \right)^{-1} \right\|^s d\omega_x d\omega_y, \end{aligned}$$

where $[-r, r]$ is an interval containing $\text{supp } \rho$. In the double integral we change variables to

$$u = \frac{1}{2}(\omega_x + \omega_y), \quad v = \frac{1}{2}(\omega_x - \omega_y),$$

which gives a Jacobian factor of 2. As $(\omega_x, \omega_y) \in [-r, r]^2$ implies $(u, v) \in [-r, r]^2$ we arrive at the bound

$$\begin{aligned} &\mathbb{E}_{x, y}(|G_{\omega, \lambda}(x, y; z)|^s) \\ &\leq \frac{2\|\rho\|_\infty^2}{\lambda^s} \int_{-r}^r \int_{-r}^r \left\| \left(-\frac{1}{\lambda} A + \begin{pmatrix} -v & 0 \\ 0 & v \end{pmatrix} - uI \right)^{-1} \right\|^s du dv \\ &\leq \frac{4r\|\rho\|_\infty^2}{\lambda^s} C(r, s) = \frac{C(s, \rho)}{\lambda^s}. \end{aligned}$$

That the latter bound is uniform in x, y and z as well as in the random parameters $(\omega_u)_{u \in \mathbb{Z}^d \setminus \{x, y\}}$ follows from the fact that the matrix

$$-\frac{1}{\lambda}A + \begin{pmatrix} -v & 0 \\ 0 & v \end{pmatrix}$$

has either positive or negative imaginary part and the following general result:

For every $s \in (0, 1)$ and $r > 0$ there exists $C(r, s) < \infty$ such that

$$(23) \quad \int_{-r}^r \|(B - uI)^{-1}\|^s du \leq C(r, s)$$

for all 2×2 -matrices B such that either $\text{Im } B \geq 0$ or $\text{Im } B \leq 0$.

Let us reproduce an elementary proof of this fact, e.g. Lemma 4.1 in [25], starting with the observation that, by Schur's Theorem, B may be assumed upper triangular. We also may assume without loss that $\text{Im } B \geq 0$.

Thus

$$(24) \quad B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}$$

and

$$(25) \quad (B - uI)^{-1} = \begin{pmatrix} \frac{1}{b_{11}-u} & -\frac{b_{12}}{(b_{11}-u)(b_{22}-u)} \\ 0 & \frac{1}{b_{22}-u} \end{pmatrix}.$$

The bound (23) follows if we can establish a corresponding fractional integral bound for the absolute value of each entry of (25) separately. For the diagonal entries this is obvious.

We bound the upper right entry of (25) by

$$(26) \quad \begin{aligned} \left| \frac{b_{12}}{(b_{11}-u)(b_{22}-u)} \right| &\leq \frac{|b_{12}|}{|\text{Im}((b_{11}-u)(b_{22}-u))|} \\ &= \frac{1}{\left| u \frac{\text{Im } b_{11} + \text{Im } b_{22}}{|b_{12}|} - \frac{\text{Im}(b_{11}b_{22})}{|b_{12}|} \right|}. \end{aligned}$$

The positive matrix

$$\text{Im } B = \begin{pmatrix} \text{Im } b_{11} & \frac{1}{2i} b_{12} \\ -\frac{1}{2i} \bar{b}_{12} & \text{Im } b_{22} \end{pmatrix}$$

has positive determinant, i.e. $\det \text{Im } B = \text{Im } b_{11} \text{Im } b_{22} - |b_{12}|^2/4$. We thus get

$$\left| \frac{\text{Im } b_{11} + \text{Im } b_{22}}{b_{12}} \right|^2 \geq \frac{2\text{Im } b_{11} \text{Im } b_{22}}{|b_{12}|^2} \geq \frac{1}{2}.$$

The latter allows to conclude the required integral bound for (26). □

The other result needed for the proof of Theorem 3 is

Lemma 4.2 (Decoupling Lemma). *For a compactly supported and bounded density function ρ as above there exists a constant $C_2 < \infty$ such that*

$$(27) \quad \frac{\int \frac{1}{|v-\beta|^s} \rho(v) dv}{\int \frac{|v-\eta|^s}{|v-\beta|^s} \rho(v) dv} \leq C_2$$

uniformly in $\eta, \beta \in \mathbb{C}$.

This can be understood as a consequence of the following two facts: (i) The two integrals on the left hand side of (27) are continuous functions of η and β . As both of them neither vanish nor diverge, the same is true for the ratio of the integrals. (ii) As $|\beta|$ and $|\eta|$ become large, the left hand side of (27) has finite limits. This combines to give a uniform bound in β and η . The details are left as an exercise, or can be found in [24].

We are now prepared to complete the proof of Theorem 3:

Proof. Given the a-priori bound from Lemma 4.1 we may assume $y \neq x$. Then

$$\begin{aligned}
(28) \quad 0 &= \langle e_x, e_y \rangle \\
&= \langle e_x, (h_{\omega, \lambda} - z)^{-1} (h_{\omega, \lambda} - z) e_y \rangle \\
&= \left\langle e_x, (h_{\omega, \lambda} - z)^{-1} \left(- \sum_{u:|u-y|=1} e_u + (\lambda\omega_y - z) e_y \right) \right\rangle \\
&= - \sum_{u:|u-y|=1} G_{\omega, \lambda}(x, u; z) + (\lambda\omega_y - z) G_{\omega, \lambda}(x, y; z).
\end{aligned}$$

Note that $G_{\omega, \lambda}(x, y; z)$ is the upper left entry of the matrix on the left hand side of the Krein formula (22). Explicitly inverting the right hand side of (22) we find that

$$G_{\omega, \lambda}(x, y; z) = \frac{\alpha}{\lambda\omega_y - \beta},$$

where α and β do not depend on ω_y (and it will not matter that they depend on λ). Using Lemma 4.2, the bound $(\sum_n |a_n|)^s \leq \sum_n |a_n|^s$ and (28) we find

$$\begin{aligned}
(29) \quad \mathbb{E}(|G_{\omega, \lambda}(x, y; z)|^s) &= \frac{1}{\lambda^s} \mathbb{E} \left(\left| \frac{\alpha}{\omega_y - \frac{\beta}{\lambda}} \right|^s \right) \\
&\leq \frac{C_2}{\lambda^s} \mathbb{E} \left(|\alpha|^s \frac{|\omega_y - \frac{z}{\lambda}|^s}{|\omega_y - \frac{\beta}{\lambda}|^s} \right) \\
&= \frac{C_2}{\lambda^s} \mathbb{E} (|\lambda\omega_y - z|^s |G_{\omega, \lambda}(x, y; z)|^s) \\
&\leq \frac{C_2}{\lambda^s} \sum_{u:|u-y|=1} \mathbb{E} (|G_{\omega, \lambda}(x, u; z)|^s).
\end{aligned}$$

If none of the lattice sites u are equal to x , then the argument can be iterated. For given x and y one can iterate $|x - y|$ times, in each step picking up a factor $2dC_2/\lambda^s$ after a maximum is taken over the $2d$ terms in the sums over next neighbors. This results in a bound

$$\mathbb{E}(|G_{\omega, \lambda}(x, y; z)|^s) \leq \left(\frac{2dC_2}{\lambda^s} \right)^{|x-y|} \sup_{u \in \mathbb{Z}^d} \mathbb{E}(|G_{\omega, \lambda}(x, u; z)|^s).$$

For the last term we use the a-priori bound C_1/λ^s provided by Lemma 4.1. We get the exponential decay in (16) for $\lambda \geq \lambda_0$ if we choose λ_0 such that $2dC_2/\lambda_0^s < 1$. \square

We conclude this section by remarking that the exponential decay bound found in Theorem 3 also holds for finite volume restrictions of the Anderson Hamiltonian. More precisely, let $L \in \mathbb{N}$ and $\Lambda_L := [-L, L]^d \cap \mathbb{Z}^d$. By $h_{\omega, \lambda}^{\Lambda_L}$ and $G_{\omega, \lambda}^{\Lambda_L}$ we denote

the restriction of $h_{\omega,\lambda}$ to $\ell^2(\Lambda_L)$ as well as its Green function. By the same proof as above one finds that, for $\lambda \geq \lambda_0$,

$$(30) \quad \mathbb{E}(|G_{\omega,\lambda}^{\Lambda_L}(x,y;z)|^s) \leq C e^{-\mu|x-y|},$$

where the constants $C < \infty$ and $\mu > 0$ are now also uniform in L .

Moreover, in the finite volume case the bound (30) is uniform in $z \in \mathbb{C}$, allowing for real energy. The reason for this is that the operators $h_{\omega,\lambda}^{\Lambda_L}$ are finite-dimensional and that any given real number E is almost surely not one of their eigenvalues, which implicitly follows from the above proof. In the finite volume case this also holds for the a-priori bound in Lemma 4.1. This explains why such bounds play a role in the FMM similar to the role played by Wegner estimates in localization proofs via MSA. They demonstrate that eigenvalues are sensitive to the disorder parameters. A good way to think of the main idea behind the FMM is that eigenvalues are singularities of the resolvent which move linearly under the random parameters. Thus the Green function can be made integrable by taking fractional moments.

5. FROM FRACTIONAL MOMENT BOUNDS TO LOCALIZATION

We will now discuss methods which show that exponential decay of fractional moments of Green's function as shown in Theorem 3 implies spectral as well as dynamical localization. For the sake of stating a general result of this form we will absorb the disorder parameter into the random parameters ω_x (re-scaling their distribution as in (13)). Thus we consider the Anderson Hamiltonian in its original form (4) with single-site distribution μ satisfying (14).

From now on we will generally leave the dependence of various quantities on the random variable ω implicit and write $h = h_\omega$, $G = G_\omega$, etc.

Our goal is to prove

Theorem 4. *Let $I \subset \mathbb{R}$ be an open bounded interval. If there exist $s \in (0, 1)$, $C < \infty$ and $\mu > 0$ such that*

$$(31) \quad \mathbb{E}(|G(x,y;E+i\varepsilon)|^s) \leq C e^{-\mu|x-y|}$$

uniformly in $E \in I$ and $\varepsilon > 0$, then dynamical localization in the form (10) holds on the interval I .

As a first consequence, by Theorem 3 this implies that at sufficiently large disorder λ the Anderson model is dynamically localized in the entire spectrum. In Section 7 below, we will also use the criterion provided by Theorem 4 to prove band edge localization.

The most direct way to conclude spectral localization, i.e. pure point spectrum with exponentially decaying eigenfunctions, from bounds such as (31) is by the Simon-Wolff method. It was developed in [42] to serve a similar purpose in the context of multiscale analysis, where it showed that the Green function bounds established in [20] indeed imply spectral localization. A short argument, showing that the Simon-Wolff criterion also can be combined with (31) to show spectral localization, is provided in [5].

Here we will instead discuss the proof of Theorem 4, i.e. focus on how (31) implies dynamical localization. We have two reasons for doing so: First, dynamical localization is the physically more relevant property. Second, as we will show at the end of this section, dynamical localization implies spectral localization with a straightforward argument using the RAGE theorem.

There are two substantially different arguments which prove Theorem 4. In this section we will present a modification of an argument provided by Graf in [24]. This version of the argument has recently also been used in [25] to prove dynamical localization for the so-called *unitary Anderson model*.

The second method, via the use of eigenfunction correlators, will be discussed in the next section.

Graf's argument starts with the realization that fractional moments of Green's functions of the Anderson model can be used to bound the second moment of Green's function as long as a small factor (the imaginary part of the energy) is introduced to control the singularities of Green's function at real energy.

Proposition 5.1. *For every $s \in (0, 1)$ there exists a constant $C_1 < \infty$ only depending on s and ρ such that*

$$(32) \quad |\operatorname{Im} z| \mathbb{E}_x(|G(x, y; z)|^2) \leq C_1 \mathbb{E}_x(|G(x, y; z)|^s)$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ and $x, y \in \mathbb{Z}^d$.

Here \mathbb{E}_x denotes averaging over ω_x as in the proof of Lemma 4.1. Integrating over the remaining variables, we see that (32) also holds with \mathbb{E}_x replaced by \mathbb{E} . Our proof follows the proof of Lemma 3 in [24] almost line by line.

Proof. As in the proof of Lemma 4.1 write $\omega = (\hat{\omega}, \omega_x)$. Keep $\hat{\omega}$ fixed and consider the Hamiltonian

$$h^{(\alpha)} = h_{(\hat{\omega}, \omega_x + \alpha)} = h_\omega + \alpha P_{e_x}$$

obtained by “wiggling the potential at x ”. Its Green function will be denoted by $G^{(\alpha)}$. Similar to (18) to (20) we find

$$(h_\omega - z)^{-1} = (h^{(\alpha)} - z)^{-1} + \alpha (h_\omega - z)^{-1} P_{e_x} (h^{(\alpha)} - z)^{-1},$$

and

$$(33) \quad \begin{aligned} G^{(\alpha)}(x, y; z) &= \frac{G_\omega(x, y; z)}{1 + \alpha G_\omega(x, x; z)} \\ &= \frac{1}{\alpha + G_\omega(x, x; z)^{-1}} \cdot \frac{G_\omega(x, y; z)}{G_\omega(x, x; z)}. \end{aligned}$$

For the special case $x = y$ and $\tilde{\alpha} = -\operatorname{Re} G_\omega(x, x; z)^{-1}$ we get from (33) that

$$\left| \frac{1}{\operatorname{Im} G(x, x; z)^{-1}} \right| = \left| G^{(\tilde{\alpha})}(x, x; z) \right| \leq \frac{1}{|\operatorname{Im} z|},$$

i.e. $|\operatorname{Im} G(x, x; z)^{-1}| \geq |\operatorname{Im} z|$. Inserting this into (33) gives

$$(34) \quad |\operatorname{Im} z| |G^{(\alpha)}(x, y; z)|^2 \leq \frac{|\operatorname{Im} G_\omega(x, x; z)^{-1}|}{|\alpha + G_\omega(x, x; z)|} \cdot \frac{|G_\omega(x, y; z)|^2}{|G_\omega(x, x; z)|^2}.$$

On the other hand, we can bound the same expression by

$$\begin{aligned}
(35) \quad |\operatorname{Im} z| |G^{(\alpha)}(x, y; z)|^2 &\leq |\operatorname{Im} z| \sum_{y' \in \mathbb{Z}^d} |G^{(\alpha)}(x, y'; z)|^2 \\
&= |\operatorname{Im} z| \langle e_x, (h^{(\alpha)} - \bar{z})^{-1} (h^{(\alpha)} - z)^{-1} e_x \rangle \\
&= |\operatorname{Im} z| \langle e_x, \frac{1}{z - \bar{z}} [(h^{(\alpha)} - z)^{-1} - (h^{(\alpha)} - \bar{z})^{-1}] e_x \rangle \\
&= \left| \operatorname{Im} G^{(\alpha)}(x, x; z) \right| \\
&= \frac{|\operatorname{Im} G_\omega(x, x; z)^{-1}|}{|\alpha + G_\omega(x, x; z)^{-1}|^2},
\end{aligned}$$

where the last step used (33) with $x = y$.

For $t \geq 0$ one has $\min(1, t^2) \leq t^s$. Using this to interpolate between (34) and (35) we get

$$(36) \quad |\operatorname{Im} z| |G^{(\alpha)}(x, y; z)|^2 \leq \frac{|\operatorname{Im} G_\omega(x, x; z)^{-1}|}{|\alpha + G_\omega(x, x; z)^{-1}|^2} \cdot \frac{|G_\omega(x, y; z)|^s}{|G_\omega(x, x; z)|^s}.$$

We will now use the following “re-sampling trick”, which has the effect of creating an additional random variable (here α) to average over. For a non-negative Borel function f on \mathbb{R} ,

$$\begin{aligned}
(37) \quad &\int \int f(\omega_x + \alpha) \rho(\omega_x + \alpha) d\alpha \rho(\omega_x) d\omega_x \\
&= \int \int f(\omega_x + \alpha) \rho(\omega_x + \alpha) \rho(\omega_x) d\omega_x d\alpha \\
&= \int \int f(\omega_x) \rho(\omega_x) \rho(\omega_x - \alpha) d\omega_x d\alpha \\
&= \int f(\omega_x) \rho(\omega_x) \left(\int \rho(\omega_x - \alpha) d\alpha \right) d\omega_x \\
&= \int f(\omega_x) \rho(\omega_x) d\omega_x,
\end{aligned}$$

where the integration order was interchanged in the first and third steps and translation invariance of Lebesgue measure was used in the second.

Choose $f(\omega_x) = |G_{(\hat{\omega}, \omega_x)}(x, y; z)|^2$, then (37) and (36) yield

$$\begin{aligned}
(38) \quad &|\operatorname{Im} z| \mathbb{E}_x(|G_\omega(x, y; z)|^2) \\
&= |\operatorname{Im} z| \mathbb{E}_x \left(\int |G^{(\alpha)}(x, y; z)|^2 \rho(\omega_x + \alpha) d\alpha \right) \\
&\leq \mathbb{E}_x \left(|\operatorname{Im} G_\omega(x, x; z)^{-1}| \frac{|G_\omega(x, y; z)|^s}{|G_\omega(x, x; z)|^s} \int \frac{\rho(\omega_x + \alpha)}{|\alpha + G_\omega(x, x; z)^{-1}|^2} d\alpha \right).
\end{aligned}$$

We now use Lemma 5.2 below with $w = G_\omega(x, x; z)^{-1}$ to conclude

$$|\operatorname{Im} z| \mathbb{E}_x(|G_\omega(x, x; z)|^2) \leq C \mathbb{E}_x(|G_\omega(x, y; z)|^s)$$

with a constant $C < \infty$ which only depends on $\operatorname{supp} \rho$, but not on x, y and z . \square

In the above proof we have used

Lemma 5.2. *There exists a constant $C = C(\rho) < \infty$ such that*

$$|\operatorname{Im} w| \cdot |w|^s \int \frac{\rho(\omega_x + \alpha)}{|\alpha + w|^2} d\alpha \leq C$$

uniformly in $w \in \mathbb{C}$ and $\omega_x \in \operatorname{supp} \rho$.

Proof. Using $|w|^s \leq |\alpha|^s + |\alpha + w|^s$, we need two estimates:

(i)

$$\begin{aligned} |\operatorname{Im} w| \int \frac{|\alpha|^s \rho(\omega_x + \alpha)}{|\alpha + w|^2} d\alpha &\leq \pi \|\alpha^s \rho(\omega_x + \alpha)\|_\infty \\ &\leq \pi (|\omega_x|^s \|\rho\|_\infty + \|\lambda^s \rho(\lambda)\|_\infty). \end{aligned}$$

(ii)

$$\begin{aligned} |\operatorname{Im} w| \int \frac{\rho(\omega_x + \alpha)}{|\alpha + w|^{2-s}} d\alpha &\leq \min\left(\frac{1}{|\operatorname{Im} w|^{1-s}}, C \|\rho\|_\infty |\operatorname{Im} w|^s\right) \\ &\leq C \|\rho\|_\infty^{1-s}. \end{aligned}$$

□

We now complete the proof of Theorem 4:

Proof. Consider the mixed spectral measures $\mu_{x,y}$ of h , the complex Borel measures defined by

$$(39) \quad \mu_{x,y}(B) = \langle e_x, \chi_B(h) e_y \rangle$$

for Borel sets $B \subset \mathbb{R}$. The total variation $|\mu_{x,y}|$ of $\mu_{x,y}$ is a regular bounded Borel measure which can be characterized by

$$(40) \quad \begin{aligned} |\mu_{x,y}|(B) &= \sup_{\substack{g: \mathbb{R} \rightarrow \mathbb{C} \text{ Borel} \\ |g| \leq 1}} \left| \int g(\lambda) d\mu_{x,y}(\lambda) \right| \\ &= \sup_{|g| \leq 1} |\langle e_x, g(h) \chi_B(h) e_y \rangle|, \end{aligned}$$

e.g. [37]. The particular choice $g_t(x) = e^{-itx}$ in (40) shows that

$$(41) \quad |\mu_{x,y}|(I) \geq \sup_{t \in \mathbb{R}} |\langle e_x, e^{-ith} \chi_I(h) e_y \rangle|.$$

Therefore Theorem 4 will follow from a corresponding exponential decay bound for $\mathbb{E}(|\mu_{x,y}|(I))$.

As I is an open bounded interval, it follows from Lusin's Theorem ([37]) that one can replace Borel functions in (40) by continuous functions with compact support in I ,

$$(42) \quad |\mu_{x,y}|(I) = \sup_{\substack{g \in C_c(I) \\ |g| \leq 1}} |\langle e_x, g(h) e_y \rangle|.$$

For $g \in C_c(I)$ it follows by elementary analysis (using that g is bounded and uniformly continuous) that, uniformly in $\lambda \in \mathbb{R}$,

$$g(\lambda) = \lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon}{\pi} \int \frac{g(E)}{(\lambda - E)^2 + \varepsilon^2} dE.$$

By the spectral theorem this implies

$$\langle e_x, g(h)e_y \rangle = \lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon}{\pi} \int_I g(E) \langle e_x, (h - E - i\varepsilon)^{-1} (h - E + i\varepsilon)^{-1} e_y \rangle dE.$$

This allows to estimate the expected value of (42) by

$$\begin{aligned} & \mathbb{E}(|\mu_{x,y}|(I)) \\ & \leq \mathbb{E} \left(\liminf_{\varepsilon \rightarrow 0^+} \frac{\varepsilon}{\pi} \int_I \sum_{z \in \mathbb{Z}^d} |\langle e_x, (h - E - i\varepsilon)^{-1} e_z \rangle| |\langle e_z, (h - E + i\varepsilon)^{-1} e_y \rangle| dE \right) \\ & \leq \liminf_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_I \sum_z (\mathbb{E}(\varepsilon |\langle e_x, (h - E - i\varepsilon)^{-1} e_z \rangle|^2))^{1/2} \\ & \quad \cdot (\mathbb{E}(\varepsilon |\langle e_z, (h - E + i\varepsilon)^{-1} e_y \rangle|^2))^{1/2} dE, \end{aligned}$$

where, in this order, Fatou, Fubini and Cauchy-Schwarz (on \mathbb{E}) have been used. Now Proposition 5.1 can be applied, allowing to bound further by

$$\begin{aligned} & \leq \liminf_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_I \sum_z (\mathbb{E}(|G(x, z; E + i\varepsilon)|^s))^{1/2} (\mathbb{E}(|G(z, y; E - i\varepsilon)|^s))^{1/2} dE \\ & \leq \frac{C_1 C |I|}{\pi} \sum_z e^{-\mu|x-z|/2} e^{-\mu|z-y|/2}. \end{aligned}$$

In the last step the assumption of Theorem 4 was used (which also applies to $|G(z, y; E - i\varepsilon)| = |G(y, z; E + i\varepsilon)|$). The elementary bound, based on the triangle inequality,

$$e^{-\mu|x-z|/2} e^{-\mu|z-y|/2} \leq e^{-\mu|x-z|/4} e^{-\mu|x-y|/4} e^{-\mu|z-y|/4}$$

and another use of Cauchy-Schwarz (on the z -summation) complete the proof of Theorem 4. \square

It deserves mentioning here that we have actually proven a stronger result than dynamical localization in the form (10). The above proof shows that for an open interval I on which (31) holds there are constants $C < \infty$ and $\mu > 0$ such that

$$(43) \quad \mathbb{E}(|\mu_{x,y}|(I)) = \mathbb{E} \left(\sup_{\substack{g: \mathbb{R} \rightarrow \mathbb{C} \text{ Borel} \\ |g| \leq 1}} |\langle e_x, g(h)\chi_I(h)e_y \rangle| \right) \leq C e^{-\mu|x-y|}$$

for all $x, y \in \mathbb{Z}^d$. An interesting special case is $g = 1$, where (43) establishes exponential decay of correlations in the spectral projection $\chi_I(h)$. Consequences of this for the conductivity of an electron gas in response to an electric field have been discussed in [4]. Another consequence is mentioned at the end of this section.

Next we show that dynamical localization implies pure point spectrum via the RAGE-Theorem. The underlying idea is very simple: The RAGE-Theorem characterizes states in the continuous spectral subspace as scattering states (in time-mean). Dynamical localization excludes scattering states and thus continuous spectrum.

Proposition 5.3. *Suppose that dynamical localization in the form (10) holds in an open interval I . Then h_ω almost surely has pure point spectrum in I .*

Proof. For a discrete Schrödinger operator $h = h_0 + V$ in $\ell^2(\mathbb{Z}^d)$ let $P_{cont}(h)$ be the projection onto its continuous spectral subspace. Then the RAGE-Theorem, e.g. [18], says that for every $\psi \in \ell^2(\mathbb{Z}^d)$,

$$(44) \quad \|P_{cont}(h)\chi_I(h)\psi\|^2 = \lim_{R \rightarrow \infty} \lim_{T \rightarrow \infty} \int_0^T \frac{dt}{T} \|\chi_{\{|x| \geq R\}} e^{-ith} \chi_I(h)\psi\|^2.$$

If ψ has finite support, say $\text{supp } \psi \subset \{|x| \leq r\}$, then

$$\begin{aligned} \|\chi_{\{|x| \geq R\}} e^{-ith} \chi_I(h)\psi\|^2 &\leq \|\chi_{\{|x| \geq R\}} e^{-ith} \chi_I(h)\chi_{\{|x| \leq r\}}\| \|\psi\|^2 \\ &\leq \sum_{|x| \geq R, |y| \leq r} |\langle e_x, e^{-ith} \chi_I(h)e_y \rangle| \|\psi\|^2, \end{aligned}$$

where dropping a square is allowed as $\|\chi_{\{|x| \geq R\}} e^{-ith} \chi_I(h)\chi_{\{|x| \leq r\}}\| \leq 1$.

Taking expectations in (44) implies, after using Fatou and Fubini,

$$(45) \quad \begin{aligned} &\mathbb{E}(\|P_{cont}(h_\omega)\chi_I(h_\omega)\psi\|^2) \\ &\leq \lim_{R \rightarrow \infty, T \rightarrow \infty} \int_0^T \frac{dt}{T} \sum_{|x| \geq R, |y| \leq r} \mathbb{E}(|\langle e_x, e^{-ith_\omega} \chi_I(h_\omega)e_y \rangle|) \|\psi\|^2. \end{aligned}$$

By (10) we have $\mathbb{E}(|\langle e_x, e^{-ith_\omega} \chi_I(h_\omega)e_y \rangle|) \leq C e^{-\mu|x-y|}$ uniformly in t , which bounds the right hand side of (45) by

$$\leq \lim_{R \rightarrow \infty} \tilde{C} \sum_{|x| \geq R, |y| \leq r} e^{-\mu|x-y|} = 0.$$

We conclude that $P_{cont}(h_\omega)\chi_I(h_\omega)\psi = 0$ for almost every ω and every ψ of finite support. The latter are dense in $\ell^2(\mathbb{Z}^d)$ and thus $P_{cont}(h_\omega)\chi_I(h_\omega) = 0$ almost surely, meaning that the spectrum in I is pure point. \square

We note that the above proof of pure point spectrum does not imply exponential decay of corresponding eigenfunctions. It is shown in [5] how this follows directly from exponential decay of fractional moments (31), using the Simon-Wolff-method [42]. It can also be deduced from (43) by considering $g(h) = \delta_E(h)$, $E \in I$, using the result from [39] that almost surely all eigenvalues of h_ω in I are non-degenerate. For details on this see Section 2.5 of [2], where a corresponding argument for the continuum Anderson model is provided which also applies to the discrete Anderson model considered here.

6. FINITE VOLUME METHODS

The localization proof provided in Section 5 proceeds directly in *infinite volume*, i.e. does not require to consider restrictions of the hamiltonian h to finite subsets of \mathbb{Z}^d . However, it is also possible to work in finite volume, prove the relevant bounds with volume-independent constants, and then deduce localization by taking the infinite volume limit (sometimes referred to as “thermodynamical limit”). This has conceptual advantages such as having to only deal with discrete spectra, thus allowing to express functions of the hamiltonian by eigenfunction expansions and, as described at the end of Section 4, to directly study Green’s function at real energy. Moreover, finite volume methods have proven very robust under generalizations, for example in the extension to continuum Anderson models which we will discuss

in Section 8. For these reasons we will use this section to provide a different proof of dynamical localization, i.e. Theorem 4 above, using finite volume methods.

Many of the ideas involved here can already be found in the Kunz-Souillard approach to localization [29] for the one-dimensional Anderson model. They were first combined with the fractional moment method in [1] to prove dynamical localization for the multi-dimensional Anderson model. A central object are so-called *finite volume eigenfunction correlators*, arising from eigenfunction expansions. Eigenfunction correlators are also used in similar form in proofs of dynamical localization via multiscale analysis, see [43] or [28] and references therein.

The methods to be described here are not completely disjoint from the methods of Section 5. As before, we consider the mixed spectral measures $\mu_{x,y}$ of h introduced in (39) as well as their total variation $|\mu_{x,y}|$ given by (40). As will become clear in (49) below, $|\mu_{x,y}|$ can be considered as an *infinite volume eigenfunction correlator* for h . We will find bounds for it by finding bounds for finite volume eigenfunction correlators which hold uniformly in the volume.

Let $h_\omega^{\Lambda_L}$ be the restriction of h_ω to $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$ and denote its Green function by $G_\omega^{\Lambda_L}$.

Proposition 6.1. *Let $0 < s < 1$ and I an open bounded interval. Then there exists $C = C(s, \rho, d) < \infty$ such that*

$$(46) \quad \mathbb{E}(|\mu_{x,y}|(I)) \leq C \liminf_{L \rightarrow \infty} \left(\int_I \mathbb{E}(|G_\omega^{\Lambda_L}(x, y; E)|^s) dE \right)^{\frac{1}{2-s}}.$$

Results of this form were first used in implicit form in [1] and later stated more explicitly in [6]. The exact statement given here as well as its proof below and in Appendix A follow notes provided to us in private communication by M. Aizenman and S. Warzel. They used similar results also in [7].

Based on (41), we see that Proposition 6.1 may be applied to provide a proof of dynamical localization in I in situations where it can be shown that

$$(47) \quad \mathbb{E}(|G_\omega^{\Lambda_L}(x, y; E)|^s) \leq C e^{-\mu|x-y|}$$

with constants which are uniform in L and $E \in I$. In fact, dynamical localization follows under the somewhat weaker assumption that the energy average over I of the fractional moments of Green's function is exponentially decaying. However, in all our applications we have uniform pointwise bounds available. For example, as discussed at the end of Section 4, a bound of the form (47) holds on the entire spectrum for sufficiently large disorder, thus providing a second proof of dynamical localization in this regime.

Proof of Proposition 6.1. We start by reducing the claim (46) to properties of finite-volume spectral measures. We again use the characterization (42) of $|\mu_{x,y}(I)|$ for open bounded intervals I . Strong resolvent convergence of h^L to h implies for continuous g of compact support that $\langle e_x, g(h^{\Lambda_L})e_y \rangle \rightarrow \langle e_x, g(h)e_y \rangle$ and thus, by (42),

$$(48) \quad |\mu_{x,y}(I)| \leq \liminf_{L \rightarrow \infty} \sup_{|g| \leq 1} |\langle e_x, g(h^{\Lambda_L})e_y \rangle|.$$

Here the regularity assumption on g can be dropped since h^{Λ_L} has discrete spectrum.

Let h_x^L be the restriction of h^{Λ_L} to the reducing subspace \mathcal{H}_x for h^{Λ_L} generated by e_x and let P_x be the orthogonal projection onto \mathcal{H}_x . Then e_x is a cyclic vector for h_x^L and all eigenvalues E of h_x^L are simple. Thus we may label the corresponding normalized eigenvectors by ψ_E^L . We use the notation ψ_E^L also for $\psi_x^L \oplus 0$ in $\ell^2(\Lambda_L) = \mathcal{H}_x \oplus \mathcal{H}_x^\perp$.

By expanding into eigenvectors we get

$$\begin{aligned} |\langle e_x, g(h^{\Lambda_L})e_y \rangle| &= |\langle e_x, g(h_x^L)P_x e_y \rangle| \\ &= \left| \sum_{E \in I \cap \sigma(h_x^L)} g(E) \langle e_x, \psi_E^L \rangle \langle \psi_E^L, e_y \rangle \right| \\ &\leq \sum_{E \in I \cap \sigma(h_x^L)} |\psi_E^L(x)| |\psi_E^L(y)| \\ &=: Q_L(x, y; I), \end{aligned}$$

and, in particular,

$$\sup_{|g| \leq 1} |\langle e_x, g(h^{\Lambda_L})e_y \rangle| \leq Q_L(x, y; I).$$

The latter will be referred to as *eigenfunction correlators*. Using Fatou's lemma we conclude from (48) that

$$(49) \quad \mathbb{E}(|\mu_{x,y}|(I)) \leq \liminf_{L \rightarrow \infty} \mathbb{E}(Q_L(x, y; I)).$$

In order to establish a relation to the fractional moments of Green's function we will also introduce *fractional eigenfunction correlators* through

$$(50) \quad Q_L(x, y; I, r) := \sum_{E \in I \cap \sigma(h_x^L)} |\psi_E^L(x)|^{2-r} |\psi_E^L(y)|^r$$

for $0 < r \leq 2$, noting that $Q_L(x, y; I) = Q_L(x, y; I, 1)$. We claim that for $0 < s < 1$,

$$(51) \quad \mathbb{E}Q_L(x, y; I) \leq (\mathbb{E}Q_L(x, y; I, s))^{\frac{1}{2-s}}.$$

To see this, interpolate $s < 1 < 2$ via $1 = \frac{s}{p} + \frac{2}{q}$ with the conjugate exponents $p = 2 - s$ and $q = \frac{2-s}{1-s}$. Applying Hölder to expectation as well as to summation yields

$$\mathbb{E}Q_L(x, y; I, 1) \leq (\mathbb{E}Q_L(x, y; I, s))^{\frac{1}{2-s}} (\mathbb{E}Q_L(x, y; I, 2))^{\frac{1-s}{2-s}}.$$

This implies (51) after observing that $Q_L(x, y; I, 2) = \sum_{E \in I \cap \sigma(h_x^L)} |\psi_E^L(y)|^2 \leq 1$.

We will now be able to relate the fractional eigenfunction correlators to fractional moments of Green's function by showing that there exists a constant $C = C(s, \rho, d)$ such that

$$(52) \quad \mathbb{E}Q_L(x, y; I, s) \leq C \int_I \mathbb{E}(|G^{\Lambda_L}(x, y; E)|^s) dE.$$

This, combined with (49) and (51), implies (46).

In the proof of (52) we will use the fractional eigenfunction correlators $Q_{L,v}(x, y; I, s)$ which are defined as in (50), but with the summation being over the eigenvalues and eigenfunctions of $h_x^L + vP_{e_x}$. Note that, as e_x is a cyclic vector for h_x^L , $h_x^L + vP_{e_x}$ is the same as the restriction of $h^{\Lambda_L} + vP_{e_x}$ to \mathcal{H}_x and that e_x is a cyclic vector for this operator for all values of $v \in \mathbb{R}$. This makes Proposition A.2 in Appendix A applicable to our situation, which we will now use to finish the proof of Proposition 6.1 by invoking a *resampling argument*.

For this note that $\int \frac{\rho(u)}{|u-\alpha|^s} du$ is continuous and non-vanishing as a function of $\alpha \in \mathbb{R}$. Thus there exists a constant $C = C(s, \rho) < \infty$ such that

$$(53) \quad \frac{\rho(\alpha)}{\int \frac{\rho(u)}{|u-\alpha|^s} du} \leq C \quad \text{for all } \alpha \in \mathbb{R}.$$

Writing $\omega = (\hat{\omega}, \omega_x)$ and denoting the expectation over $\hat{\omega}$ by $\hat{\mathbb{E}}$, we get

$$(54) \quad \begin{aligned} \mathbb{E}(Q_L^\omega(x, y; I, s)) &= \hat{\mathbb{E}} \int_{\mathbb{R}} Q_L^{(\hat{\omega}, \omega_x)}(x, y; I, s) \rho(\omega_x) d\omega_x \\ &\leq C \hat{\mathbb{E}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} Q_L^{(\hat{\omega}, \omega_x)}(x, y; I, s) \frac{d\omega_x}{|u - \omega_x|^s} \right) \rho(u) du. \end{aligned}$$

After the change of variable $\omega_x \mapsto v := \omega_x - u$ we see that the inner integral is equal to $\int Q_L^{(\hat{\omega}, v)}(x, y; I, s) \frac{dv}{|v|^s}$. By Proposition A.2 this coincides with $\int_I |G_{(\hat{\omega}, u)}^{\Lambda_L}(x, y; E)|^s dE$. Inserting into (54) we arrive at

$$\begin{aligned} \mathbb{E}(Q_L^\omega(x, y; I, s)) &\leq C \hat{\mathbb{E}} \int_{\mathbb{R}} \int_I |G_{(\hat{\omega}, u)}^{\Lambda_L}(x, y; E)|^s dE \\ &= C \mathbb{E} \int_I |G_\omega^{\Lambda_L}(x, y; E)|^s dE. \end{aligned}$$

□

7. LIFSHTS TAILS AND BAND EDGE LOCALIZATION

7.1. Band edge localization. So far the only regime in which we have proven localization is the case of large disorder $\lambda \gg 1$ in (12). In this section we consider the Anderson model in its original form (4), i.e. at fixed disorder. Our goal is a proof of localization at energies near the bottom of the spectrum. The arguments involved can be easily modified to show the same near the upper spectral edge.

For notational convenience we will assume that the density ρ of the distribution of the ω_x satisfies $\text{supp } \rho = [0, \omega_{max}]$. We also write $E_0 = -2d$, which according to Theorem 2 becomes the bottom of the almost sure spectrum

$$\Sigma = [E_0, 2d + \omega_{max}]$$

of h_ω .

Our localization proof will again proceed via showing exponential decay of the fractional moments of Green's function:

Theorem 5. *For every $s \in (0, 1)$ there exist $\delta > 0$, $\mu > 0$ and $C < \infty$ such that*

$$(55) \quad \mathbb{E}(|G_\omega(x, y; E + i\epsilon)|^s) \leq C e^{-\mu|x-y|}$$

for all $x, y \in \mathbb{Z}^d$, $E \in [E_0, E_0 + \delta]$ and $\epsilon > 0$.

As discussed at the end of Section 4, our methods again yield a bound on the finite volume Green function as in (30), uniform in the volume and in $E \in [E_0, E_0 + \delta]$, allowing for $\epsilon = 0$. Thus we can conclude spectral and dynamical localization at the bottom of the spectrum from either of the methods discussed in Sections 5 of 6, working in infinite volume or in finite volume.

Compared to the case of large disorder, we face essentially two new difficulties, which are illustrated by the shortcomings of Lemmas 4.1 and 4.2. The a-priori bound from Lemma 4.1 is still valid and will be used. But, as the disorder λ is fixed,

we can not hope that the a-priori bound also provides a “smallness mechanism”, which can be used to iteratively prove exponential decay. We will again proceed by iteration, but a different mechanism will be needed to get it started. Also, it will turn out that we need a different decoupling argument. Lemma 4.2, which was used in this context in the proof of Theorem 3, is too case-specific and will not work for the geometric situations which we will encounter here.

7.2. Lifshits tails. Physically, the new smallness mechanism is provided by the fact that the bottom of the spectrum $E_0 = \inf \Sigma$ is a *fluctuation boundary*. This means that finite volume restrictions of h_ω rarely have eigenvalues close to 0. To make this precise, as before let $h_\omega^{\Lambda_L}$ be the restriction of h_ω to $\ell^2(\Lambda_L)$, $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$.

Lemma 7.1. *For every $\beta \in (0, 1)$ there are $\eta > 0$ and $C < \infty$ such that*

$$(56) \quad \mathbb{P}(\inf \sigma(h_\omega^{\Lambda_L}) \leq E_0 + L^{-\beta}) \leq CL^d e^{-\eta L^{\beta d/2}}$$

for all $L \in \mathbb{N}$.

To illustrate why this means that small eigenvalues are rare, let us assume that the $|\Lambda_L| = (2L + 1)^d$ eigenvalues of h_ω^L are uniformly distributed in Σ . Then the smallest eigenvalue should be no larger than C/L^d . But by (56) this is extremely rare for large L . In fact, the methods used to prove (56) can also be used to prove that the integrated density of states $N(E)$ of h_ω satisfies Lifshits-tail asymptotics near the bottom of the spectrum:

$$(57) \quad N(E) \leq C e^{-\eta |E - E_0|^{-d/2}},$$

which is much “thinner” near E_0 than the corresponding IDS $N_0(E) = C|E - E_0|^{d/2}$ of the Laplacian h_0 .

For detailed proofs of Lemma 7.1 as well as (57) we refer to [27] or [43], with the latter working in the setting of the continuum Anderson model (but applicable to the discrete model as well). Here we only briefly outline the reasons behind Lemma 7.1. By the variational principle

$$(58) \quad \begin{aligned} \inf \sigma(h_\omega^{\Lambda_L} - E_0) &= \inf_{\|\varphi\|=1} \langle (h_\omega^{\Lambda_L} - E_0)\varphi, \varphi \rangle \\ &= \inf_{\|\varphi\|=1} \left(\langle (h_0^{\Lambda_L} - E_0)\varphi, \varphi \rangle + \sum_{i \in \Lambda_L} \omega_i |\varphi(i)|^2 \right). \end{aligned}$$

Note that both terms on the right hand side of (58), the kinetic and potential energy, are non-negative. In order to find a low lying eigenvalue, they both need to be small. By reasons of the uncertainty principle, small kinetic energy requires that φ is approximately constant, $\varphi(i) \sim C = |\Lambda_L|^{-1/2}$, to be normalized. For such φ the potential energy is approximately $\sum_{i \in \Lambda_L} \omega_i / |\Lambda_L|$, which by the central limit theorem with large probability is close to the expected value $\mathbb{E}(\omega_0) > 0$. The event $\sum_i \omega_i / |\Lambda_L| < L^{-\beta} < \mathbb{E}(\omega_0)$ is a large deviation and has probability exponentially small in $|\Lambda_L|$.

The weakest part of the above heuristics is the reference to the uncertainty principle. Slightly different ways to make this rigorous are provided in [27] and [43], both requiring arguments which control the separation of the second lowest eigenvalue from the lowest eigenvalue. The proof provided in [27] (going back to work from the 1980s) uses Temple’s inequality in this context, while [43] uses an argument based on analytic perturbation theory.

In the context of proving Theorem 5, Lemma 7.1 provides a first step, a so-called *initial length estimate*:

Lemma 7.2 (Initial Length Estimate). *For every $s \in (0, 1)$ there exist $C < \infty$ and $\eta > 0$ such that*

$$(59) \quad \mathbb{E}(|G^{\Lambda_L}(x, y; E + i\epsilon)|^s) \leq CL^d e^{-\eta L^{d/(d+2)}}$$

for all $L \in \mathbb{N}$, $x, y \in \Lambda_L$ with $|x - y| \geq L/2$, $E \in [E_0, E_0 + \frac{1}{2}L^{-2/(d+2)}]$ and $\epsilon > 0$.

Proof. Let $\beta \in (0, 1)$ and, motivated by Lemma 7.1, define the “good” and “bad” sets as $\Omega_G := \{\omega : \inf \sigma(h_\omega^{\Lambda_L} - E_0) \leq L^{-\beta}\}$ and $\Omega_B := \Omega_G^c$. Then

$$(60) \quad \mathbb{E}(|G^{\Lambda_L}(x, y; E + i\epsilon)|^s) = \mathbb{E}(|G^{\Lambda_L}(x, y; E + i\epsilon)|^s \chi_{\Omega_G}) + \mathbb{E}(|G^{\Lambda_L}(x, y; E + i\epsilon)|^s \chi_{\Omega_B}).$$

Pick $p > 1$ sufficiently small such that $sp < 1$ and let q be conjugate to p , $\frac{1}{p} + \frac{1}{q} = 1$. Hölder applied to the second term on the right hand side of (60) gives

$$(61) \quad \begin{aligned} \mathbb{E}(|G^{\Lambda_L}(x, y; E + i\epsilon)|^s \chi_{\Omega_B}) &\leq (\mathbb{E}(|G^{\Lambda_L}(x, y; E + i\epsilon)|^{sp}))^{1/p} \mathbb{P}(\Omega_B)^{1/q} \\ &\leq CL^{d/q} e^{-\frac{\eta}{q} L^{\beta d/2}}, \end{aligned}$$

where we have used the a-priori bound from Lemma 4.1 as well as the probability bound from Lemma 7.1. The first term on the right hand side of (60) concerns the event where E has distance at least $\frac{1}{2}L^{-\beta}$ from the bottom of the spectrum, which allows to conclude exponential decay of $|G^{\Lambda_L}(x, y; E + i\epsilon)|$ in $|x - y|$ from a Combes-Thomas estimate (see e.g. [27]), giving

$$(62) \quad \begin{aligned} \mathbb{E}(|G^{\Lambda_L}(x, y; E + i\epsilon)|^s \chi_{\Omega_G}) &\leq CL^{\beta s} e^{-s\eta|x-y|/L^\beta} \\ &\leq CL^{\beta s} e^{-\eta_1 L^{1-\beta}} \end{aligned}$$

for constants $\eta > 0$, $\eta_1 > 0$ and $C < \infty$.

The choice $\beta = 2/(2 + d)$ leads to equal exponents in (61) and (62), which combine to give (59). \square

7.3. Geometric decoupling. We will eventually fix $L = L_0$, choosing L_0 such that the right hand side of (59) is sufficiently small (how small still to be determined). After making this choice we will pick $\delta = \frac{1}{2}L_0^{-2/(d+2)}$, thus determining the interval $[E_0, E_0 + \delta]$ in which Theorem 5 establishes localization. In order to derive the exponential decay bound (55) from this we have to develop a decoupling method which will allow to proceed iteratively, splitting the path from x to y into segments of length L_0 .

The description of this so-called *geometric decoupling method* will fill the remainder of this section. Our argument will closely follow a construction introduced in [6].

In addition to $h_\omega^{\Lambda_L}$, consider $h_\omega^{\Lambda_L^c}$, the restriction of h_ω to $\ell^2(\Lambda_L^c)$, where $\Lambda_L^c = \mathbb{Z}^d \setminus \Lambda_L$. Let

$$h_\omega^{(L)} = h_\omega^{\Lambda_L} \oplus h_\omega^{\Lambda_L^c}.$$

This means that

$$(63) \quad h_\omega = h_\omega^{(L)} + T^{(L)},$$

where $T^{(L)}$ is the operator containing the “hopping terms” introduced by the discrete Laplacian between sites of Λ_L and Λ_L^c . We write $G_\omega^{(L)}(z) = (h_\omega^{(L)} - z)^{-1}$.

More precisely, the matrix-elements of $T^{(L)}$ are

$$(64) \quad T^{(L)}(u, u') = \begin{cases} -1, & \text{if } (u, u') \in \Gamma_L, \\ 0, & \text{else,} \end{cases}$$

where Γ_L is the boundary of Λ_L defined as the set of pairs (u, u') with $|u - u'| = 1$ and $u \in \Lambda_L$, $u' \in \Lambda_L^c$, or vice versa.

We now perform a *double decoupling* of the resolvent $G_\omega = G_\omega(z)$ by using the resolvent equation for (63) twice, first at L and then at $L + 1$:

$$(65) \quad \begin{aligned} G_\omega &= G_\omega^{(L)} - G_\omega^{(L)} T^{(L)} G_\omega \\ &= G_\omega^{(L)} - G_\omega^{(L)} T^{(L)} G_\omega^{(L+1)} + G_\omega^{(L)} T^{(L)} G_\omega T^{(L+1)} G_\omega^{(L+1)}. \end{aligned}$$

Equations of this form are often referred to as *geometric resolvent identities*.

By translation invariance it suffices to prove (55) for $x = 0$. If $|y| \geq L + 2$, then the first two terms on the right hand side of (65) do not contribute to the matrix-element $G_\omega(0, y; z)$ and thus

$$\begin{aligned} G_\omega(0, y; z) &= \langle e_0, G_\omega^{(L)} T^{(L)} G_\omega T^{(L+1)} G_\omega^{(L+1)} e_y \rangle \\ &= \sum_{(u, u') \in \Gamma_L} \sum_{(v, v') \in \Gamma_{L+1}} G_\omega^{(L)}(0, u; z) G_\omega(u', v; z) G_\omega^{(L+1)}(v', y; z). \end{aligned}$$

For $s \in (0, 1)$ we get

$$(66) \quad \begin{aligned} \mathbb{E}(|G_\omega(0, y; z)|^s) &\leq \sum_{\substack{(u, u') \in \Gamma_L \\ (v, v') \in \Gamma_{L+1}}} \mathbb{E} \left(|G_\omega^{\Lambda_L}(0, u; z) G_\omega(u', v; z) G_\omega^{\Lambda_{L+1}^c}(v', y; z)|^s \right). \end{aligned}$$

Here we have replaced $G_\omega^{(L)}$ by $G_\omega^{\Lambda_L}$ as 0 and u are both in Λ_L . Similarly, $G_\omega^{(L+1)}$ was replaced by $G_\omega^{\Lambda_{L+1}^c}$. For fixed (u, u') and (v, v') consider the corresponding term on the right hand side of (66) and note that the first and last of the three factors are independent of $\omega_{u'}$ and ω_v . Thus, in taking the expectation we may integrate over $\omega_{u'}$ and ω_v first and use Lemma 4.1 to conclude

$$(67) \quad \mathbb{E} \left(|G_\omega^{\Lambda_L}(0, u; z) G_\omega(u', v; z) G_\omega^{\Lambda_{L+1}^c}(v', y; z)|^s \right)$$

$$(68) \quad \leq C \mathbb{E} \left(|G_\omega^{\Lambda_L}(0, u; z)|^s |G_\omega^{\Lambda_{L+1}^c}(v', y; z)|^s \right)$$

$$(69) \quad = C \mathbb{E}(|G_\omega^{\Lambda_L}(0, u; z)|^s) \mathbb{E}(|G_\omega^{\Lambda_{L+1}^c}(v', y; z)|^s).$$

In the last step we have used that the remaining two factors in (68) are stochastically independent. Now let $z = E + i\epsilon$ with $E \in [E_0, E_0 + \frac{1}{2}L^{-2/(2+d)}]$. Then we may estimate the first factor in (69) by the bound obtained in Lemma 7.2 and, after inserting into (66), find

$$(70) \quad \mathbb{E}(|G_\omega(0, y; z)|^s) \leq CL^{2d-1} e^{-\eta L^{d/(d+2)}} \sum_{\|v'\|_\infty = L+2} \mathbb{E}(|G_\omega^{\Lambda_{L+1}^c}(v', y; z)|^s).$$

We want to use (70) as the first step in an iteration. The second step would consist in finding a bound for $\mathbb{E}(|G_\omega^{\Lambda_{L+1}^c}(v', y; z)|^s)$ similar to the bound for $\mathbb{E}(|G_\omega(0, y; z)|^s)$ given by (70), with v' serving as the new origin. A problem arises from the fact that the underlying domain is not any longer \mathbb{Z}^d , but Λ_{L+1}^c . Iterating

this would result in more and more complex geometries and we would be faced with the problem to check if all the constants involved in the estimates leading to (70) can be chosen uniform in those geometries.

An elegant way around this is the following result of [6], see Lemma 2.3 there, which allows to bound the depleted Green function $G_\omega^{(L+1)}$ in terms of the full Green function G_ω :

Lemma 7.3. *There exists a constant $C = C(s, \rho) < \infty$ such that*

$$\mathbb{E}(|G_\omega^{(L+1)}(v', y; z)|^s) \leq \mathbb{E}(|G_\omega(v', y; z)|^s) + C \sum_{\|u'\|_\infty = L+2} \mathbb{E}(|G_\omega(u', y; z)|^s).$$

The proof of this starts from the geometric resolvent identity $G_\omega^{(L+1)} = G_\omega + G_\omega^{(L+1)}T^{(L+1)}G_\omega$. In the resulting Green function expansion over $(u, u') \in \Gamma_{L+1}$ crucial use is made of the bound

$$\mathbb{E}(|G_\omega^{(L+1)}(v', u; z)|^s |G_\omega(u', y; z)|^s) \leq C \mathbb{E}(|G_\omega(u', y; z)|^s).$$

The proof of this uses another special case of the Krein formula similar to (22) (but this time tracking the dependence on all four variables $\omega_{v'}$, ω_u , $\omega_{u'}$ and ω_y) and a decoupling argument similar to Lemma 4.2 (but for functions of two variables which are linear separately in each variable). For the details we refer to [6].

Lemma 7.3 combines with (70) to yield

$$(71) \quad \mathbb{E}(|G_\omega(0, y; z)|^s) \leq CL^{4d} e^{-\eta L^{d/(d+2)}} \sup_{\|u_1\|_\infty \leq L+2} \mathbb{E}(|G_\omega(u_1, y; z)|^s)$$

for some constant $C < \infty$. With that constant we fix $L = L_0$ such that $\rho := CL_0^{4d} e^{-\eta L_0^{d/(d+2)}} < 1$. We also choose $\delta := \frac{1}{2}L_0^{-2/(2+d)}$ now. For $E \in [E_0, E_0 + \delta]$ we can use (71) to start an iteration,

$$\mathbb{E}(|G_\omega(u_1, y; z)|^s) \leq \rho \sup_{\|u_2\|_\infty \leq 2(L_0+2)} \mathbb{E}(|G_\omega(u_2, y; z)|^s),$$

and so forth. This iteration can be carried out approximately $|y|/L_0$ times before the chains u_1, u_2, \dots may reach y . After this number of steps we use the a-priori bound from Lemma 4.1 to bound the last fractional moment in the chain. We have proven Theorem 5 with exponential decay rate $\mu = |\log \rho|/L_0$.

8. THE CONTINUUM ANDERSON MODEL

It took somewhat more than a decade to find a generalization of the fractional moment method to continuum Anderson models. Our goal in this section is to explain why this took so long and how it was eventually done. Here our presentation will be less self-contained than in previous sections. We will outline the new ideas which were needed and refer to the literature for details.

The main difficulty is that the rather elementary arguments from rank-one and rank-two perturbation theory, which worked so well for the discrete Anderson model, fall far short of applying in the continuum. In the latter, each single site potential is a perturbation of infinite rank, which at best has certain compactness properties relative to the Laplacian. To make the central ideas behind the fractional moment method work in this setting required a much deeper understanding of some of the operator-theoretic aspects involved. Here we will follow the works [2] and [12], where these questions were settled. Earlier work in [26] extended certain

aspects of the fractional moment method to continuum models, but still relied on finite-rank perturbation arguments by, for example, considering continuum models with random point interactions.

For our presentation here we choose to work with the deterministic background operator

$$(72) \quad H_0 = -\Delta + V_0$$

in $L^2(\mathbb{R}^d)$, where V_0 is a real-valued, \mathbb{Z}^d -periodic potential in $L^\infty(\mathbb{R}^d)$. Let $E_0 := \inf \sigma(H_0)$ denote its spectral minimum.

A continuum Anderson-type model is then given by

$$(73) \quad H_\omega = H_0 - \sum_{n \in \mathbb{Z}^d} \omega_n U_n,$$

where $\omega = (\omega_n)_{n \in \mathbb{Z}^d}$ is an array of i.i.d. random variables with bounded density ρ such that $\text{supp } \rho = [0, \omega_{max}]$.

The single-site potentials $U_n(x) = U(x-n)$ are translates of a non-negative bump function U characterized by the existence of $0 < r_1 \leq r_2 < \infty$ and $0 < c_1 \leq c_2 < \infty$ such that

$$(74) \quad c_1 \chi_{\{|x| \leq r_1\}} \leq U \leq c_2 \chi_{\{|x| \leq r_2\}}.$$

The spectrum of H_ω is almost surely deterministic,

$$\sigma(H_\omega) = \Sigma \quad \text{a.s.},$$

and

$$E_1 := \inf \Sigma = \inf \sigma(H_0 - \omega_{max} \sum_n U_n)$$

is characterized by choosing all couplings maximal and thus, due to our sign-convention, the potential minimal. It can be shown under the assumption (74) that the spectral minimum is strictly decreased by the random potential: $E_1 < E_0$.

We will use the notation $\chi_n = \chi_{\Lambda_1(n)}$, where $\Lambda_1(n)$ refers to the unit cube in \mathbb{R}^d centered at $n \in \mathbb{Z}^d$.

The following theorem is a special case of a result in [12]. Similar results were first obtained in [2], where a ‘‘covering condition’’ of the form

$$(75) \quad U \geq c\chi_0, \quad c > 0,$$

was required for the single-site potential.

Theorem 6. *Let $d \leq 3$ and $0 < s < \frac{1}{3}$. Then there exist $\delta > 0$, $\mu > 0$ and $C < \infty$ such that*

$$(76) \quad \mathbb{E}(\|\chi_k(H_\omega - E - i\epsilon)^{-1}\chi_\ell\|^s) \leq Ce^{-\mu|k-\ell|}$$

for all $E \in [E_1, E_1 + \delta]$, $\epsilon > 0$ and $k, \ell \in \mathbb{Z}^d$.

In Theorem 6 we use the norm of the localized resolvent $\chi_k(H_\omega - E - i\epsilon)^{-1}\chi_\ell$ (sometimes called a ‘‘smeared Green function’’) as a continuum analogue of the discrete Green function $G(x, y; E + i\epsilon)$. This has also been found to be the correct object to consider in continuum extensions of multiscale analysis.

Without going into the details here (which for the continuum case can be done similar to what was described in Section 6, see [2]), we state that exponential decay of fractional moments of the smeared Green function, as established in (76), implies spectral and dynamical localization:

Corollary 8.1. *Under the assumptions of Theorem 6 the following holds:*

(a) *For almost every ω , H_ω has pure point spectrum in $[E_1, E_1 + \delta]$ with exponentially decaying eigenfunctions.*

(b) *There are constants $\mu > 0$ and $C < \infty$ such that*

$$(77) \quad \mathbb{E} \left(\sup_{|g| \leq 1} \|\chi_k g(H_\omega) \chi_{[E_1, E_1 + \delta]}(H_\omega) \chi_\ell\| \right) \leq C e^{-\mu|k-\ell|}$$

for all $k, \ell \in \mathbb{Z}^d$, with the supremum taken over Borel functions $g : \mathbb{R} \rightarrow \mathbb{C}$.

The overall approach to proving Theorem 6 is similar to the proof of Theorem 5 in the previous section. The main steps are:

(i) **A priori-bound:** It can be shown that to every $E_2 \in (E_1, E_0)$ and $0 < s < 1$ there exists $C < \infty$ such that

$$(78) \quad \mathbb{E}(\|\chi_k(H_\omega - E - i\epsilon)^{-1}\chi_\ell\|^s) \leq C$$

uniformly in $E \in [E_1, E_2]$, $\epsilon > 0$ and $k, \ell \in \mathbb{Z}^d$.

Note here that, as opposed to the discrete case Lemma 4.1, the a-priori bound is only shown for energies below the spectrum of the unperturbed operator H_0 . This is a consequence of not requiring the covering condition (75) for the single-site potential. If a covering condition holds, then it was shown in [2] that the bound (78) holds at all energies, with a constant C on the right which grows polynomially in E .

(ii) **Lifshits tails:** The bottom E_1 of the almost sure spectrum is again a fluctuation boundary and close analogues to Lemmas 7.1 and 7.2 as well as Lifshits tail asymptotics (57) of the IDS hold in the continuum, see e.g. [43] and [2]. As in the discrete case, this provides the start of an iterative procedure for the proof of exponential decay in (76).

(iii) **Geometric decoupling:** The geometric decoupling procedure described at the end of Section 7 can be carried out similarly in the continuum. Additional technical difficulties arise mostly due to the fact that the required geometric resolvent identities (compare (65)) are less straightforward in the continuum. One consequence of this is the restriction of Theorem 6 to $s < 1/3$, which is due to the need of an additional three-factor Hölder bound used in the decoupling procedure. Also, elementary decoupling bounds such as Lemma 4.2 have to be replaced by a more systematic construction involving *resampling* of the random variables ω_n near the surfaces at which the decoupling is carried out. For details in the setting of Theorem 6 see [12].

The only one of the above three points which we want to address in some more detail is the a-priori bound (78), as the existence of such a bound can be seen as the crucial test for the possibility of using the fractional moment method in the continuum.

For simplicity, we only consider the “diagonal” case $k = \ell = 0$ here and will assume the covering condition (75). We will discuss reasons why we could hope that

$$(79) \quad \sup_{\epsilon > 0} \mathbb{E}(\|U(H_\omega - E - i\epsilon)^{-1}U\|^s) < \infty$$

for energies near $\inf \Sigma$. Under the covering condition, this implies the same result with U replaced by χ_0 .

When trying to implement ideas similar to the ones used in the proof of Lemma 4.1, we are faced with having to find an analogue to the Krein formula. It turns out that this is done by the identities known from Birman-Schwinger theory. Write

$$\omega = (\hat{\omega}, \omega_0), \quad H_\omega = H_{\hat{\omega}} - \omega_0 U.$$

Then, at least formally, it is easy to derive by the resolvent identity that

$$(80) \quad U^{1/2}(H_\omega - z)^{-1}U^{1/2} = (A_{BS} - \omega_0 I)^{-1}$$

in $L^2(\text{supp } U)$, with the Birman-Schwinger operator

$$(81) \quad A_{BS} = \left(U^{1/2}(H_{\hat{\omega}} - z)^{-1}U^{1/2} \right)^{-1}.$$

It can be justified that the inverses in (81) and (80) exist and that A_{BS} is maximally dissipative. Here an operator A is called maximally dissipative if it is dissipative, i.e. $\text{Im}\langle \phi, A\phi \rangle \geq 0$ for all ϕ in its domain, and it has no proper dissipative extension. This can also be characterized by the fact that $\{e^{itA}\}_{t \geq 0}$ is a contraction semigroup.

The identity (80) looks promising since the right hand side separates the dependence on ω_0 from the dependence on $\hat{\omega}$. Indeed, if the bound (23) could be generalized from dissipative 2×2 -matrices to general maximally dissipative operators B , then it would immediately give us (79). While (23) extends to dissipative $N \times N$ -matrices, the bound $C(r, s)$ on the right will become N -dependent and diverge for $N \rightarrow \infty$, as is seen by choosing B to be a diagonal matrix with entries $1, \dots, N$. Thus it is not possible to directly extend (23) to the Hilbert space setting.

However, the extension to the Hilbert space setting becomes possible if additional Hilbert-Schmidt multipliers are introduced. This is most naturally stated in terms of a closely related weak- L^1 -bound:

Theorem 7. *Let \mathcal{H}_0 and \mathcal{H}_1 be separable Hilbert spaces, let A be maximally dissipative in \mathcal{H}_0 , and let $M : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ be a Hilbert-Schmidt operator. Then*

(a) *the boundary value*

$$M(A - v + i0)^{-1}M^* := \lim_{\epsilon \rightarrow 0} M(A - v + i\epsilon)^{-1}M^*$$

exists in Hilbert-Schmidt norm for almost every $v \in \mathbb{R}$,

(b) *there exists a constant $C < \infty$ (independent of A and M) such that*

$$(82) \quad |\{v \in \mathbb{R} : \|M(A - v + i0)^{-1}M^*\|_{HS} > t\}| \leq \frac{C\|M\|_{HS}^2}{t}$$

for all $t > 0$.

In (82) $|\cdot|$ denotes Lebesgue measure and $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm.

Part (a) is well known in mathematical physics and has been frequently used in scattering theory. Less well known is part (b), which describes the value-distribution of the boundary values guaranteed to exist by (a). This was proven in [33] (see also an appendix in [2] for a reproduction of the proof), based on the weak- L^1 -property of the Hilbert transform of Hilbert space-valued functions, the latter being a quite classical result in harmonic analysis.

The weak- L^1 -bound (82) can be turned into the fractional moment bound

$$(83) \quad \int \|M(A - v + i0)^{-1}M^*\|_{HS}^s \rho(v) dv \leq C(s, \rho) \|M\|_{HS}^{2s},$$

where the constant $C(s, \rho)$ can be chosen uniform for all Hilbert-Schmidt operators M and maximally dissipative A . This is done by the standard layer-cake integration argument: If $F(v) := \|M(A - v + i0)^{-1}M^*\|_{HS}$, then

$$\begin{aligned} \int |F(v)|^s \rho(v) dv &\leq \|\rho\|_\infty \int_{\text{supp } \rho} |F(v)|^s dv \\ &= \|\rho\|_\infty \int_0^\infty |\{v \in \text{supp } \rho : |F(v)|^s > t\}| dt. \end{aligned}$$

By (82) the integrand is bounded by $\min\{\tilde{C}, C\|M\|_{HS}^2/t^{1/s}\}$, where $\tilde{C} = |\text{supp } \rho|$. Splitting the integral at the t -value where $\tilde{C} = C\|M\|_{HS}^2/t^{1/s}$ leads to (83).

When trying to use (83) for a proof of (79) we see from (80) that

$$(84) \quad U(H_\omega - z)^{-1}U = U^{1/2}(A_{BS} - \omega_0 I)^{-1}U^{1/2}.$$

This leaves us with one more problem to deal with: The multiplication operator $U^{1/2}$ in $L^2(\mathbb{R}^d)$ is *not* Hilbert-Schmidt. In fact, multiplication operators with non-vanishing functions in the continuum are *never* compact.

The key to solving this last problem is that $U^{1/2}$ is *relatively* Hilbert-Schmidt with respect to $-\Delta$ (meaning that $U^{1/2}(-\Delta + 1)^{-1}$ is Hilbert-Schmidt), at least for $d \leq 3$, see e.g. [38]. Arguments as typical in relative perturbation theory allow to split the left hand side of (84) into a sum of terms, some of which trivially satisfy a fractional moment bound, while others include additional multipliers which lead to the Hilbert-Schmidt property required in (83). These arguments only work at energies below the spectrum of the unperturbed operator H_0 , which is the reason for the corresponding assumption which we made when stating (78).

For further details on these relative perturbation arguments as well as on the ‘‘off-diagonal’’ case $k \neq \ell$ in (78) we refer to [12] and conclude our sketch of the proof of Theorem 6 here.

9. OPEN PROBLEMS AND DIRECTIONS FOR FUTURE WORK

To conclude this introduction into the theory of Anderson localization, we mention some open problems and discuss some wide open issues which mathematicians need to understand better in the future. Here we will not restrict ourselves to further developments of the fractional moment method, but will address broader aspects of the quantum mechanical description of disordered media. We will be relatively brief here and note that a more complete and more detailed recent discussion of open problems in this field can be found in [10]. In particular, we do not attempt here to give complete references to related works.

9.1. Singular distributions. Consider the discrete and continuous Anderson models h_ω and H_ω , but allow for singular distributions of the random coupling parameters ω_i , $i \in \mathbb{Z}^d$. The most extreme case would be the case of independent Bernoulli variables, i.e. $\mathbb{P}(\omega_i = a) = p$, $\mathbb{P}(\omega_i = b) = 1 - p$. This models the physically interesting case of a two-component alloy. Both, the fractional moments method and the Fröhlich-Spencer multiscale analysis, fail to provide localization proofs in this situation. The reason for this is that both methods to a large extent use *local*

averaging arguments in the random parameters, as demonstrated very clearly by the proof of Lemma 4.1 above. While it is possible to deal with Hölder-continuous distributions, the Bernoulli case is out of reach for the traditional approaches.

However, Bourgain and Kenig [13] have shown

Theorem 8. *Consider the continuum Anderson model H_ω defined by (72) and (73) with $V_0 = 0$ and independent Bernoulli random variables (ω_i) . Then H_ω is spectrally localized near $E_1 = \inf \Sigma$.*

Their proof is based on a substantial enhancement of the multiscale analysis approach and, in particular, a deeper understanding of the underlying averaging mechanisms (such as the role of the so-called *Wegner estimates*). It has also been shown in [3] how the argument provided in [13] can be used to handle (ω_i) with arbitrary non-trivial distribution.

However, somewhat surprisingly, the same question remains open for the *discrete* Anderson model (4) with Bernoulli distributed random couplings. The technical reason for this is that [13] uses subtle unique continuation properties of the eigenfunctions of Schrödinger operators which are not available for lattice models.

More generally, one can easily imagine various other models of random operators where the random parameters naturally have discrete distribution and where the available mathematical methods fail to prove localization. One such model would be discrete Laplacians on random subgraphs of the edges of \mathbb{Z}^d . An open question is to decide if in the supercritical percolation regime, where the graph has a unique infinite component, the Laplacian has localized spectrum. For a recent survey on these models see [32]

9.2. Extended states. Every list of open problems in random operator theory needs to mention the somewhat embarrassing fact that mathematicians are still far from understanding the physically conjectured extended states regime in the three-dimensional Anderson model.

A proof of the existence of continuous (or absolutely continuous) spectrum or of diffusive solutions to the time-dependent Schrödinger equation for this model would be an important break-through. Here we would like to mention another way to characterize the Anderson transition from localized to extended states, namely the *level statistics conjecture*. In fact, this is how physicists can numerically distinguish the two regimes, which provides the most important evidence for the correctness of the physical heuristics explaining the transition.

According to the level statistics conjecture it is possible to distinguish the localized and delocalized regimes by considering the statistical distribution of the eigenvalues (viewed as point processes) of finite volume restrictions of the Anderson model. Localized states should be characterized by Poisson statistics of the eigenvalues, while in spectral regions with extended states the finite volume eigenvalues should show GOE statistics. The latter is a special kind of level repulsion observed for Gaussian orthogonal ensembles in random matrix theory.

In the spectral regimes where mathematicians can establish localization, it has also been verified that the finite volume eigenvalues are Poisson distributed, see [31], [30], [17] and [22]. However, regarding GOE statistics in the Anderson model, as little is known as for other possible characterizations of extended states.

As discussed in the lectures by L. Erdos at this School [19], GOE statistics is a rather universal phenomenon observed in large classes of random matrices, e.g.

so-called *Wigner random matrices*. The most apparent difference between Wigner matrices and the Anderson model is that for the latter randomness is restricted to the diagonal matrix-elements while in Wigner matrices all entries are random. Understanding the transition between Anderson models and random matrices, for example by considering *random band matrices* with an increasing amount of off-diagonal random entries, could provide important insights into the localization-delocalization transition in the Anderson model.

9.3. Electron-electron interactions and many-body systems. The Anderson models discussed above are *one-electron* models, which ignore interactions between electrons (as well as interactions between nuclei, which are considered as affixed to the lattice sites). Quite recently, Anderson-type models for a fixed number N of interacting electrons in a random background have been shown to have localization properties. Chulaevsky and Suhov [15, 16] have done this by an extension of multiscale analysis, while [7] provides similar results based on the fractional moments approach.

Let us give one example of a result which can be obtained by both approaches, where we do not try to state the most general result. An N -particle discrete Anderson-type model can be defined as

$$(h_\omega^{(N)}\phi)(x) = \sum_{y:|y-x|=1} \phi(y) + (U(x) + \lambda \sum_{j=1}^N \omega_{x_j})\phi(x),$$

where $\phi \in \ell^2(\mathbb{Z}^{Nd})$, $x = (x_1, \dots, x_N) \in \mathbb{Z}^{Nd}$ and $y \in \mathbb{Z}^{Nd}$. As above, the $(\omega_x)_{x \in \mathbb{Z}^d}$ are i.i.d. random variables with bounded, compactly supported density. Assume, for simplicity, that $U(x)$ is a two-particle interaction term of finite range,

$$U(x) = \sum_{1 \leq j < k \leq N} \Phi(x_j - x_k), \quad \text{supp } \Phi \text{ finite.}$$

Then localization holds for large disorder:

Theorem 9. *If λ is sufficiently large, then $h_\omega^{(N)}$ is spectrally and dynamically localized at all energies.*

Note that a version of dynamical localization suitable for N -particle systems has to be used here, see [7].

While such results provide important mathematical progress, condensed matter physicists will object to the above model because it keeps the number of electrons fixed. The physically correct system to look at would be a model with positive *electron density*. For example, one could consider the restriction of $h_\omega^{(N)}$ to $\ell^2(\Lambda_L^N)$, where $\Lambda_L^N = \Lambda_L \times \dots \times \Lambda_L$, and study its properties if N goes to infinity together with the volume, i.e. $N \sim L^d \rightarrow \infty$. This is mathematically wide open. It is not even clear how the concepts of localization and extended states should be defined in this setting, with spectral theoretic terms most likely not being the correct language any more. Instead dynamical descriptions will have to be used. One particularly interesting question, which is still challenging even to physicists, is if electron interaction enhances or reduces localization effects. It has been argued in the physics literature that such effects could be crucial to understand localization for two-dimensional disordered systems, the critical case.

APPENDIX A. BASIC RANK-ONE PERTURBATION THEORY

Here we first provide some facts from rank-one perturbation theory. They can be considered as finite-dimensional, elementary versions of some of the much more profound insights behind Simon-Wolff theory, e.g. [42] or Chapter 11 to 13 of [40]. Then we establish a general relation between fractional moments of Green's function and eigenfunction correlators, which was used in Section 6 above.

Let \mathcal{H} be a finite-dimensional Hilbert space, $\dim \mathcal{H} = N$, h_0 a self-adjoint operator in \mathcal{H} , and φ a normalized cyclic vector for h_0 . By $P = \langle \varphi, \cdot \rangle \varphi$ we denote the orthogonal projection onto $\text{span}\{\varphi\}$. Consider the family of rank-one perturbations

$$h_v := h_0 + vP, \quad v \in \mathbb{R},$$

of h_0 . Then φ is a cyclic vector for all h_v . All h_v have simple eigenvalues, which we label as

$$E_1(v) < E_2(v) < \dots < E_N(v).$$

Also consider the self-adjoint operator

$$h_\infty := Ph_0P \quad \text{on } D(h_\infty) = \{\varphi\}^\perp.$$

Application of the Gram-Schmidt procedure to $\varphi, A\varphi, A^2\varphi, \dots$ shows that $A\varphi - \langle \varphi, A\varphi \rangle \varphi$ is a cyclic vector for h_∞ . In particular, h_∞ has simple eigenvalues which we label as

$$E_1(\infty) < E_2(\infty) < \dots < E_{N-1}(\infty).$$

In the following we will also use the notation $E_0(\infty) := -\infty$, $E_N(\infty) := \infty$.

Lemma A.1. (i) For every $k \in \{1, \dots, N\}$, $E_k(v)$ is analytic and strictly increasing as a function of $v \in \mathbb{R}$.

(ii) For every $v \in \mathbb{R}$, the eigenvalues of h_v and h_∞ are intertwined as

$$(85) \quad E_1(v) < E_1(\infty) < E_2(v) < E_2(\infty) < \dots < E_{N-1}(v) < E_{N-1}(\infty) < E_N(v).$$

(iii) For every $k \in \{1, \dots, N\}$,

$$\lim_{v \rightarrow -\infty} E_k(v) = E_{k-1}(\infty), \quad \lim_{v \rightarrow \infty} E_k(v) = E_k(\infty).$$

Proof. (i) Let $\psi_k(v)$ denote normalized eigenvectors of h_v to $E_k(v)$, $k = 1, \dots, N$. That φ is a cyclic vector for h_v means that $\langle \psi_k(v), \varphi \rangle \neq 0$ for all $k \in \{1, \dots, N\}$. By analytic perturbation theory, the functions $E_k(v)$ are analytic with

$$(86) \quad E'_k(v) = \langle \psi_k(v), P\psi_k(v) \rangle = |\langle \psi_k(v), \varphi \rangle|^2 \neq 0.$$

(ii) We will prove this by the variational characterization of eigenvalues of h_v and h_∞ , e.g. Theorem XIII.2 in [36], which says that for $k \in \{1, \dots, N\}$,

$$(87) \quad E_k(v) = \sup_{\substack{V \subset \mathcal{H} \\ \dim V = k-1}} \inf_{\substack{f \in V^\perp \\ \|f\| = 1}} \langle f, h_v f \rangle$$

and for $k \in \{1, \dots, N-1\}$,

$$(88) \quad E_k(\infty) = \sup_{\substack{\tilde{V} \in \{\varphi\}^\perp \\ \dim \tilde{V} = k-1}} \inf_{\substack{g \in \tilde{V}^\perp \\ \|g\| = 1}} \langle g, h_\infty g \rangle.$$

Note that in the infimum in (87) the orthogonal complement V^\perp is taken with respect to \mathcal{H} , while \tilde{V}^\perp in (88) is taken with respect to $\{\varphi\}^\perp$. By definition of h_∞ and h_v we also have that

$$(89) \quad \langle f, h_\infty f \rangle = \langle f, h_v f \rangle \quad \text{for all } f \in \{\varphi\}^\perp.$$

We first show that

$$E_k(v) \leq E_k(\infty) \quad \text{for all } k \in \{1, \dots, N\}.$$

This is trivial for $k = N$. For $k \leq N-1$, let V be a subspace of \mathcal{H} with $\dim V = k-1$. Then $V^\perp \cap \{\varphi\}^\perp$ is a subspace of $\{\varphi\}^\perp$ of dimension at least $N-k$. Thus it is the orthogonal complement of a subspace W of $\{\varphi\}^\perp$ of dimension at most $k-1$. Therefore, by (88) and (89),

$$\begin{aligned} E_k(\infty) &\geq \inf_{\substack{f \in W^\perp \\ \|f\|=1}} \langle f, h_\infty f \rangle = \inf_{\substack{f \in V^\perp \cap \{\varphi\}^\perp \\ \|f\|=1}} \langle f, h_v f \rangle \\ &\geq \inf_{\substack{f \in V^\perp \\ \|f\|=1}} \langle f, h_v f \rangle. \end{aligned}$$

As this holds for every subspace V of \mathcal{H} with $\dim V = k-1$, (87) implies $E_k(\infty) \geq E_k(v)$.

Next we will show that

$$E_k(\infty) \leq E_{k+1}(v) \quad \text{for all } k \in \{0, \dots, N-1\},$$

which is trivial for $k = 0$. Let $k \geq 1$ and $\tilde{V} \subset \{\varphi\}^\perp$ with $\dim \tilde{V} = k-1$. Then $V := \text{span}\{\varphi\} \oplus \tilde{V} \subset \mathcal{H}$ with $\dim V = k$ and $V^\perp = \{0\} \oplus \tilde{V}^\perp$. Thus, by (87) and (89),

$$E_{k+1}(v) \geq \inf_{\substack{f \in V^\perp \\ \|f\|=1}} \langle f, h_v f \rangle = \inf_{\substack{g \in \tilde{V}^\perp \\ \|g\|=1}} \langle g, h_\infty g \rangle.$$

As $\tilde{V} \subset \{\varphi\}^\perp$ with $\dim \tilde{V} = k-1$ was arbitrary, (88) implies $E_{k+1}(v) \geq E_k(\infty)$.

Strictness of all inequalities in (85) now is a consequence of (i).

(iii) Here we use the following general fact, which can be proven using Schur complementation (see e.g. [11] for a description of this method): For the self-adjoint 2×2 -block matrix

$$\begin{pmatrix} A & B \\ B^* & D \end{pmatrix},$$

let $E \notin \sigma(D)$, then

$$\lim_{|v| \rightarrow \infty} \begin{pmatrix} A + vI - EI & B \\ B^* & D - EI \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & (D - EI)^{-1} \end{pmatrix}.$$

Applying this to the 2×2 -block representation of h_0 in $\text{span}\{\varphi\} \oplus \{\varphi\}^\perp$ shows that $(h_v - EI)^{-1} \rightarrow 0 \oplus (h_\infty - EI)^{-1}$ as $|v| \rightarrow \infty$ for every $E \notin \sigma(h_\infty)$.

Using that for self-adjoint operators A ,

$$\|(A - EI)^{-1}\| = \frac{1}{\text{dist}(E, \sigma(A))},$$

we conclude that for every $E \in \sigma(h_\infty)$ there exists a function $E(v)$ such that $E(v) \in \sigma(h_v)$ for all v and $\lim_{v \rightarrow \infty} E(v) = E$. If $E = E_k(\infty)$ for $k = 1, \dots, N-1$, it follows from the results of (i) and (ii) that $E(v) = E_k(v)$ for v sufficiently large, i.e. $\lim_{v \rightarrow \infty} E_k(v) = E_k(\infty)$. Similarly, it follows that $\lim_{v \rightarrow -\infty} E_{k+1}(v) = E_k(\infty)$. $E_1(v) \rightarrow -\infty$ as $v \rightarrow -\infty$ and $E_N(v) \rightarrow \infty$ as $v \rightarrow \infty$ follows easily by minimizing/maximizing the quadratic form of h_v . □

To given $\chi \in \mathcal{H}$, $s \in (0, 1)$ and open interval $I \subset \mathbb{R}$ we consider the fractional eigenfunction correlators

$$(90) \quad Q_v(\varphi, \chi; I, s) := \sum_{k: E_k(v) \in I} |\langle \psi_k(v), \varphi \rangle|^{2-s} |\langle \psi_k(v), \chi \rangle|^s.$$

Proposition A.2. *The fractional eigenfunction correlators satisfy the identity*

$$(91) \quad \int_{\mathbb{R}} \frac{Q_v(\varphi, \chi; I, s)}{|v|^s} dv = \int_I |\langle \varphi, (h_0 - E)^{-1} \chi \rangle|^s dE.$$

Proof. As φ is cyclic for h_v and thus $\langle \psi_k(v), \varphi \rangle \neq 0$ for all k , we can rewrite (90) as

$$(92) \quad Q_v(\varphi, \chi; I, s) = \sum_{k: E_k(v) \in I} E'_k(v) \left| \frac{\langle \psi_k(v), \chi \rangle}{\langle \psi_k(v), \varphi \rangle} \right|^s,$$

where we have also used (86). Observe that

$$(h_0 - E_k(v))\psi_k(v) = (h_v - E_k(v) - vP)\psi_k(v) = -v\langle \varphi, \psi_k(v) \rangle \varphi.$$

Thus, using that $E_k(v) \notin \sigma(h_0)$ for all $v \neq 0$,

$$\begin{aligned} \langle \psi_k(v), \chi \rangle &= \langle (h_0 - E_k(v))\psi_k(v), (h_0 - E_k(v))^{-1} \chi \rangle \\ &= -v \langle \psi_k(v), \varphi \rangle \langle \varphi, (h_0 - E_k(v))^{-1} \chi \rangle. \end{aligned}$$

This allows to further rewrite (92) as

$$(93) \quad \frac{Q_v(\varphi, \chi; I, s)}{|v|^s} = \sum_{k: E_k(v) \in I} E'_k(v) |\langle \varphi, (h_0 - E_k(v))^{-1} \chi \rangle|^s.$$

This will allow to prove (91) by integration. Here we may assume that $I \subset (E_{k-1}(\infty), E_k(\infty))$ for a fixed $k \in \{1, \dots, N\}$ (from which the general case follows easily). In this case (93) says that

$$\frac{Q_v(\varphi, \chi; I, s)}{|v|^s} = \begin{cases} E'_k(v) |\langle \varphi, (h_0 - E_k(v))^{-1} \chi \rangle|^s, & \text{if } E_k(v) \in I, \\ 0, & \text{else.} \end{cases}$$

Integration yields

$$\begin{aligned} \int_{\mathbb{R}} \frac{Q_v(\varphi, \chi; I, s)}{|v|^s} dv &= \int_{v: E_k(v) \in I} E'_k(v) |\langle \varphi, (h_0 - E_k(v))^{-1} \chi \rangle|^s dv \\ &= \int_I |\langle \varphi, (h_0 - E)^{-1} \chi \rangle|^s dE, \end{aligned}$$

which used the substitution $v \mapsto E = E_k(v)$. \square

We conclude by noting that it is possible to prove a result corresponding to Proposition A.2 without assuming that the Hilbert space is finite-dimensional, as long as the spectral measure for h_0 corresponding to χ is purely singular and boundary values $\langle \varphi, (h_0 - E - i0)^{-1} \chi \rangle$ of Green's function are used. A corresponding streamlining of Aizenman's original arguments in [1] has been provided by Simon in [41] for the unitary models considered there and will be presented for the self-adjoint setting in [8].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, AL 35294, USA

E-mail address: `stolz@math.uab.edu`