

Sparse Recovery with Coherent Tight Frames via Analysis Dantzig Selector and Analysis LASSO *

Junhong Lin and Song Li [†]
 Department of Mathematics, Zhejiang University
 Hangzhou, 310027, P. R. China

Abstract

This article considers recovery of signals that are sparse or approximately sparse in terms of a (possibly) highly overcomplete and coherent tight frame from undersampled data corrupted with additive noise. We show that the properly constrained l_1 -analysis optimization problem, called analysis Dantzig selector, stably recovers a signal which is nearly sparse in terms of a tight frame provided that the measurement matrix satisfies a restricted isometry property adapted to the tight frame. As a special case, we consider the Gaussian noise. Further, under a sparsity scenario, with high probability, the recovery error from noisy data is within a log-like factor of the minimax risk over the class of vectors which are at most s sparse in terms of the tight frame. Similar results for the analysis LASSO are shown.

The above two algorithms provide guarantees only for noise that is bounded or bounded with high probability (for example, Gaussian noise). However, when the underlying measurements are corrupted by sparse noise, these algorithms perform suboptimally. We demonstrate robust methods for reconstructing signals that are nearly sparse in terms of a tight frame in the presence of bounded noise combined with sparse noise. The analysis in this paper is based on the restricted isometry property adapted to a tight frame, which is a natural extension to the standard restricted isometry property.

Keywords. l_1 -analysis, Restricted isometry property, Sparse recovery, Dantzig selector, LASSO, Gaussian noise, Sparse noise.

1 Introduction

1.1 Standard compressed sensing

Compressed sensing predicts that sparse signals can be reconstructed from what was previously believed to be incomplete information. The seminal papers [11, 12, 19] have triggered a large research activity in mathematics, engineering and computer science with a lot of potential applications.

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[†]Corresponding author: Song Li.

E-mail address: jhlin5@hotmail.com (J. Lin), songli@zju.edu.cn (S. Li).

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Formally, in compressed sensing, one considers the following model:

$$y = Af + z, \tag{1.1}$$

where A is a known $m \times n$ measurement matrix (with $m \ll n$) and $z \in \mathbb{R}^m$ is a vector of measurement errors. The goal is to reconstruct the unknown signal f based on y and A . The key idea is that the sparsity helps in isolating the original signal under suitable conditions on A .

The approach for solving this problem, that probably comes first to mind, is to search for the sparsest vector in the feasible set of possible solutions, which leads to an l_0 -minimization problem. However, solving the l_0 -minimization directly is NP-hard in general and thus is computationally infeasible [43, 44]. It is then natural to consider the method of l_1 -minimization which can be viewed as a convex relaxation of the l_0 -minimization. Three most renowned recovery algorithms based on convex relaxation proposed in the literature are: the Basis Pursuit (BP) [7], the Dantzig selector (DS) [15], and the LASSO estimator [53] (or Basis Pursuit Denoising [7]):

$$\text{(BP)} : \min_{\tilde{f} \in \mathbb{R}^n} \|\tilde{f}\|_1 \quad \text{subject to} \quad \|A\tilde{f} - y\|_2 \leq \varepsilon,$$

$$\text{(DS)} : \min_{\tilde{f} \in \mathbb{R}^n} \|\tilde{f}\|_1 \quad \text{subject to} \quad \|A^*(A\tilde{f} - y)\|_\infty \leq \lambda_n \sigma,$$

$$\text{(LASSO)} : \min_{\tilde{f} \in \mathbb{R}^n} \frac{1}{2} \|(A\tilde{f} - y)\|_2^2 + \mu_n \sigma \|\tilde{f}\|_1,$$

here $\|\cdot\|_2$ denotes the standard Euclidean norm, $\|\cdot\|_1$ is the l_1 -norm, λ_n (or μ_n) is a turning parameter, and ε (or σ) is a measure of the noise level. All these three optimization programs can be implemented efficiently using convex programming or even linear programming.

It is now well known that the BP recovers all (approximately) s sparse vectors with small or zero errors provided that the measurement matrix A satisfies a restricted isometry property (RIP) condition $\delta_{cs} \leq \delta$ for some constants $c, \delta > 0$ and that the error bound $\|z\|_2$ is small [13, 12, 6, 16, 29, 41]. Similar results were obtained for the DS and the LASSO provided that A satisfies a RIP condition $\delta_{cs} \leq \delta$ for some constants $c, \delta > 0$ and that the error bound $\|A^*z\|_\infty$ is small [15, 5, 16]. Recall that for an $m \times n$ matrix A and $s \leq n$, the RIP constant δ_s [11, 14, 20] is defined as the smallest number δ such that for all s sparse vectors $\tilde{x} \in \mathbb{R}^n$,

$$(1 - \delta)\|\tilde{x}\|_2^2 \leq \|A\tilde{x}\|_2^2 \leq (1 + \delta)\|\tilde{x}\|_2^2.$$

So far, all good constructions of matrices with the RIP use randomness. It is well known [14, 3, 42, 50] that many types of random measurement matrices such as Gaussian matrices or Sub-Gaussian matrices have the RIP constant $\delta_s \leq \delta$ with overwhelming probability provided that $m \geq C\delta^{-2}s \log(n/s)$. Up to the constant, the lower bounds for Gelfand widths of l_1 -balls [31, 30] show that this dependence on n and s is optimal. The fast multiply partial random Fourier matrix has the RIP constant $\delta_s \leq \delta$ with very high probability provided that $m \geq C\delta^{-2}s(\log n)^4$ [14, 50, 34].

In many common settings it is natural to assume that the noise vector $z \sim N(0, \sigma^2 I)$, i.e., z is i.i.d. Gaussian noise, which is of particular interest in signal processing and in statistics. The case

of Gaussian noise was first considered in [33], which examined the performance of l_0 -minimization with noisy measurements. Since the Gaussian noise is essentially bounded (e.g. [15, 17]), all stably recovery results mentioned above for bounded error related to the BP, the DS and the LASSO can be extended directly to the Gaussian noise case. While the BP and the DS (or the Lasso) provide very similar guarantees, there are certain circumstances where the DS is preferable since the DS yields a bound that is adaptive to the unknown level of sparsity of the object we try to recover and thus providing a stronger guarantee when s is small [15]. Besides, Candès and Tao [15] established an oracle inequality for the DS. Bickel et al. [5] showed that the DS and the LASSO have analogous properties, which lead to analogous error bounds.

The above mentioned recovery algorithms provide guarantees only for noise that is bounded or bounded with high probability. However, these algorithms perform suboptimally when the measurement noise is also sparse [37]. This can occur in practice due to shot noise, malfunctioning hardware, transmission errors, or narrowband interference. Several recovery techniques have been developed for sparse noise [37, 52, 36]. We refer the readers to [37, 52, 36] and the reference therein for more details on sparse noise.

There are many other algorithmic approaches to compressed sensing based on pursuit algorithms in the literature, including Orthogonal Matching Pursuit (OMP) [48, 23], Stagewise OMP [24], Regularized OMP [47], Compressive Sampling Matching Pursuit [46], Iterative Hard Thresholding [2], Subspace Pursuit [22] and many other variants. Refer to [55] for an overview of these pursuit methods.

1.2 l_1 -synthesis

For signals which are sparse in the standard coordinate basis or sparse in terms of some other orthonormal basis, the techniques above hold. However, in practical examples, there are numerous signals of interest which are not sparse in an orthonormal basis. Often, sparsity is expressed not in terms of an orthogonal basis but in terms of an overcomplete dictionary, which means that our signal $f \in \mathbb{R}^n$ is now expressed as $f = Dx$ where $D \in \mathbb{R}^{n \times d}$ ($d \geq n$) is a redundant dictionary and x is (approximately) sparse, see e.g. [7, 4, 8] and the reference therein. Examples include signal modeling in array signal processing (oversampled array steering matrix), reflected radar and sonar signals (Gabor frames), and images with curves (Curvelet frames), etc.

The l_1 -synthesis (e.g. [7, 49, 25]) consists in finding the sparsest possible coefficient \hat{x} by solving an l_1 -minimization problem (BP or LASSO) with the decoding matrix AD instead of A , and then reconstruct the signal by a synthesis operation, i.e., $\hat{f} = D\hat{x}$. Empirical studies show that l_1 -synthesis often provides good recovery [7, 25]. Little is known about the theoretical performance of this method. In [49] recovery results were obtained where essentially require the frame D to have columns that are extremely uncorrelated such that AD satisfies the RIP condition imposed by the standard compressed sensing assumptions. However, if D is a coherent frame, AD does not generally satisfy the standard RIP [49, 8]. Also, the mutual incoherence property (MIP) [21] may not apply, as it is very hard for AD to satisfy the MIP as well when D is highly correlated.

1.3 l_1 -analysis

An alternative to l_1 -synthesis is l_1 -analysis, which finds the estimator \hat{f} directly by solving an l_1 -minimization problem. There are two most renowned analysis recovery algorithms proposed in the literature: the analysis Basis Pursuit (ABP) [8] and the analysis LASSO (ALASSO) [25, 54]¹:

$$\text{(ABP)} : \quad \hat{f} = \arg \min_{\tilde{f} \in \mathbb{R}^n} \|D^* \tilde{f}\|_1 \quad \text{subject to} \quad \|A\tilde{f} - y\|_2 \leq \varepsilon, \quad (1.2)$$

$$\text{(ALASSO)} : \quad \hat{f}^{AL} = \arg \min_{\tilde{f} \in \mathbb{R}^n} \frac{1}{2} \|(A\tilde{f} - y)\|_2^2 + \mu \|D^* \tilde{f}\|_1. \quad (1.3)$$

Here μ is a tuning parameter, and ε is a measure of the noise level.

Several works exist in the literature that are related to the analysis model (e.g. [25, 51, 8, 1, 39, 45]). It has been shown that l_1 -analysis and l_1 -synthesis approaches are exactly equivalent when D is orthogonal otherwise there is a remarkable difference between the two despite their apparent similarity [25], for example truly redundant dictionaries. Empirical evidence of the effectiveness of the analysis approach can be found in [25] for signal denoising and in [51] for signal and image restoration. Numerical algorithms have been proposed to solve the ALASSO, e.g. [32, 9, 40].

More recently, Candès et al. [8] showed that the ABP recovers a signal \hat{f} with an error bound

$$\|\hat{f} - f\|_2 \leq C_0 \frac{\|D^* f - (D^* f)_{[s]}\|_1}{\sqrt{s}} + C_1 \varepsilon, \quad (1.4)$$

provided that A satisfies a restricted isometry property adapted to D (D -RIP) condition with $\delta_{2s} < 0.08$, where D is a tight frame for \mathbb{R}^n . Later, the D -RIP condition is improved to $\delta_{2s} < 0.493$ [38]. Note that we denote $x_{[s]}$ to be the vector consisting of the s largest coefficients of $x \in \mathbb{R}^d$ in magnitude, i.e. $x_{[s]}$ is the best s sparse approximation to the vector x . Following [8], Liu et al. [39] provided a theoretical study on the error when the ABP is used in the context of compressed sensing with general frames. Aldroubi et al. [1] showed that the ABP is robust to measurement noise, and stable with respect to perturbations of the measurement matrix A and the general frames D . Foucart [28] studied the ABP algorithm under the setting that the measurement matrices are Weibull random matrices. Recall that the D -RIP of a measurement matrix A , which first appeared in [8] and is a natural extension to the standard RIP, is defined as follows:

Definition 1.1 (D -RIP). *Let D be an $n \times d$ matrix. A measurement matrix A is said to obey the restricted isometry property adapted to D (abbreviated as D -RIP) of order s with constant δ if*

$$(1 - \delta) \|Dv\|_2^2 \leq \|ADv\|_2^2 \leq (1 + \delta) \|Dv\|_2^2 \quad (1.5)$$

holds for all s sparse vectors $v \in \mathbb{R}^d$. The D -RIP constant δ_s is defined as the smallest number δ such that (1.5) holds for all s sparse vectors $v \in \mathbb{R}^d$.

¹Note that we use the name ABP and ALASSO as the counterparts of BP and LASSO respectively. If D is specially the concatenation of a discrete derivative and a weighted identity, then it is the Fused LASSO introduced in [54].

As noted in [8], using a standard covering argument as in [3] (also [49]), one can prove that, for any $m \times n$ matrix A obeying for any fixed $\nu \in \mathbb{R}^n$,

$$\mathbb{P} \left(\left| \|A\nu\|_2^2 - \|\nu\|_2^2 \right| \geq \delta \|\nu\|_2^2 \right) \leq ce^{-\gamma m \delta^2}, \quad \delta \in (0, 1) \quad (1.6)$$

(γ, c are positive numerical constants) will satisfy the D -RIP $\delta_s \leq \delta$ with overwhelming probability provided that $m \geq C\delta^{-2}s \log(d/s)$. Many types of random matrices satisfy (1.6). It is now well known that matrices with Gaussian, Sub-Gaussian, or Bernoulli entries satisfy (1.6) (e.g. [3]). It has also been shown [42] that if the rows of A are independent (scaled) copies of an isotropic ψ_2 vector, then A also satisfies (1.6). Recall that an isotropic ψ_2 vector a is one that satisfies for all v ,

$$\mathbb{E}|\langle a, v \rangle| = \|v\|_2^2 \quad \text{and} \quad \inf\{t : \mathbb{E} \exp(\langle a, v \rangle^2/t^2) \leq 2\} \leq \alpha \|v\|_2,$$

for some constant α [42]. Very recently, Ward and Kramer [35] showed that randomizing the column signs of any matrix that satisfies the standard RIP results in a matrix which satisfies the Johnson-Lindenstrauss lemma. Therefore, nearly all random matrix constructions which satisfy the standard RIP compressed sensing requirements will also satisfy the D -RIP. Consequently, partial random Fourier matrices (or partial circulant matrices) with randomized column signs will satisfy the D -RIP since these matrices are known to satisfy the RIP.

1.4 Motivation and contributions

In this paper, following [8], we consider recovery of signals which are (approximately) sparse in terms of a tight frame from undersampled data. Formally, let D be an $n \times d$ ($n \leq d$) matrix whose d columns D_1, \dots, D_d form a tight frame for \mathbb{R}^n , i.e.

$$f = \sum_k \langle f, D_k \rangle D_k \quad \text{for all } f \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product. Our objective in this paper is to reconstruct the unknown signal $f \in \mathbb{R}^n$, where D^*f is sparse or approximately sparse, from a collection of m linear measurements corrupted with additive noise (1.1). Motivated by the DS, we propose a reconstruction by the following algorithms:

$$\text{(ADS): } \hat{f}^{ADS} = \arg \min_{f \in \mathbb{R}^n} \|D^*f\|_1 \quad \text{subject to} \quad \|D^*A^*(Af - y)\|_\infty \leq \lambda. \quad (1.7)$$

We call this convex program the analysis Dantzig selector (ADS). It can be implemented efficiently using convex programming. For the rest of this paper, D is an $n \times d$ tight frame and δ_s denotes the D -RIP constant with order s of the measurement matrix A without special mentioning.

We first show that, the ADS recovers a signal with an error bound

$$\|\hat{f}^{ADS} - f\|_2 \leq \min_{1 \leq k \leq s} \left[C_0 \sqrt{k} \lambda + C_1 \frac{\|D^*f - (D^*f)_{[k]}\|_1}{\sqrt{k}} \right] \quad (1.8)$$

provided that A satisfies the D -RIP with $\delta_{3s} < 1/2$ and that $\|D^*A^*z\|_\infty \leq \lambda$, where C_0 and C_1 are small positive constants depending only on the D -RIP constant δ_{3s} . As a special case, we consider the Gaussian noise $z \sim N(0, \sigma^2 I)$. Under a sparsity scenario in the case of Gaussian noise, comparing the error bound derived by the ABP in the literature, e.g [8, 39], the ADS yields a bound that is adaptive to the unknown level of sparsity (with respect to D) of the object we try to recover and thus providing a stronger guarantee when s is small. Moreover, we derive a minimax over the class of vectors which are at most s sparse in terms of D , which tells us that such error bound (1.8) under a sparsity scenario is in general unimprovable if one ignores the log-like factor.

To the best of our knowledge, there are fewer results on the performance of the ALASSO in the literature related to compressed sensing. Our second contribution of this paper is that as that for the ADS, we derive similar results for the ALASSO .

The ADS, the ALASSO and the ABP provide guarantees only for noise that is bounded or bounded with high probability (for example, Gaussian noise). However, when the underlying measurements are corrupted by sparse noise [37], such algorithms fail to recover a close approximation of the signal. Our third contribution of this paper is that we propose robust methods for reconstructing signals which are nearly sparse in terms of a tight frame in the presence of bounded noise combined with sparse (with respect to a tight frame) noise. Namely, we want to reconstruct the unknown signal $f \in \mathbb{R}^n$, where D^*f is sparse or approximately sparse, from a collection of m linear measurements

$$y = Af + z + e, \quad (1.9)$$

where z is suitably bounded, e is s' sparse in terms of Ω and $\Omega \in \mathbb{R}^{m \times M}$ ($M \geq m$) is a tight frame for \mathbb{R}^m . Let $\Phi = [A, I]$ and $u = [f^*, e^*]^*$. Denote

$$W = \begin{bmatrix} D & 0 \\ 0 & \Omega \end{bmatrix}.$$

Then one has $y = \Phi u + z$ and that $W \in \mathbb{R}^{(n+m) \times (d+M)}$ is a tight frame for \mathbb{R}^{n+m} . We propose the following three approaches: the separation ABP (SABP), the separation ADS (SADS) and the separation ALASSO (SALASSO):

$$\text{(SABP): } \hat{u}^{SABP} = \arg \min_{\tilde{u} \in \mathbb{R}^{n+m}} \|W^* \tilde{u}\|_1 \quad \text{subject to} \quad \|\Phi \tilde{u} - y\|_2 \leq \varepsilon, \quad (1.10)$$

$$\text{(SADS): } \hat{u}^{SADS} = \arg \min_{\tilde{u} \in \mathbb{R}^{n+m}} \|W^* \tilde{u}\|_1 \quad \text{subject to} \quad \|W^* \Phi^* (\Phi \tilde{u} - y)\|_\infty \leq \lambda, \quad (1.11)$$

$$\text{(SALASSO): } \hat{u}^{SAL} = \arg \min_{\tilde{u} \in \mathbb{R}^{n+m}} \frac{1}{2} \|(\Phi \tilde{u} - y)\|_2^2 + \mu \|W^* \tilde{u}\|_1. \quad (1.12)$$

We will provide results on the performance of these approaches in the case when the measurement matrix A is a Gaussian matrix or Sub-Gaussian matrix. Our analysis is based on the W -RIP.

We shall restrict this work to the setting of real valued signals $f \in \mathbb{R}^n$. For perspective, it is known that compressed sensing results ([6]) such as for the BP are also valid for complex valued signals $f \in \mathbb{C}^d$, e.g., [27]. Note also that we have restricted to the tight frame case and that a signal being sparse in a non-tight frame is also interesting.

1.5 Notation

The following notation is used throughout this paper. The set of indices of the nonzero entries of a vector \tilde{x} is called the support of \tilde{x} and denoted as $\text{supp}(\tilde{x})$. Denote $\|x\|_0 = |\text{supp}(x)|$. For $n \in \mathbb{N}$, denote $[n]$ to mean $\{1, 2, \dots, n\}$. Given an index set $T \subset [n]$ and a matrix $A \in \mathbb{R}^{m \times n}$, T^c is the complement of T in $[n]$, A_T is the submatrix of A formed from the columns of A indexed by T , or the $m \times n$ matrix obtained by setting the columns of A indexed by T^c to zero. Write A^* to mean the conjugate transpose of a matrix A , A_T^* to mean $(A_T)^*$, $\lambda_{\min}(A^*A)$ and $\lambda_{\max}(A^*A)$ to mean the smallest and largest eigenvalues of A^*A , $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ to mean the smallest and largest singular values of A . $\|A\|$ is the operator norm of A . $\|A\|_{p,q}$ denotes the norm of A from l_p to l_q . For $j \in [n]$, A_j is the j th columns of A . \tilde{x}_T is the vector equal to \tilde{x} on T and zero elsewhere or a vector of \tilde{x} restricted to T . $C > 0$ (or c, C_0, C_1) denotes a universal constant that might be different in each occurrence.

1.6 Organization

This paper is organized as follows. In Section 2, we present stably recovery results for the ADS. Similar results for the ALASSO are given in Section 3. The performance of the SABP, the SADS and the SALASSO are presented in Section 4. Section 5 contains the proofs of the main results.

2 The analysis Dantzig selector

In this section, we consider model (1.1), where z is suitably bounded. Specially, z can be Gaussian noise. We will present the recovery result of the ADS, which only requires that A satisfies the D -RIP.

Theorem 2.1. *Let D be an arbitrary $n \times d$ tight frame and let A be an $m \times n$ measurement matrix satisfying the D -RIP with $\delta_{3s} < \frac{1}{2}$. Assume that λ obeys $\|D^*A^*z\|_{\infty} \leq \lambda$. Then the solution \hat{f}^{ADS} to the ADS (1.7) obeys*

$$\|\hat{f}^{ADS} - f\|_2 \leq \min_{1 \leq k \leq s} \left[C_0 \sqrt{k} \lambda + C_1 \frac{\|D^*f - (D^*f)_{[k]}\|_1}{\sqrt{k}} \right],$$

where C_0 and C_1 are small constants depending only on the D -RIP constant δ_{3s} .

The Gaussian noise is essentially bounded.

Lemma 2.2. *Let D be an arbitrary $n \times d$ tight frame and let A be an $m \times n$ matrix satisfying the D -RIP with constant $\delta_1 \in (0, 1)$. Then for arbitrary fixed constant $\alpha > 0$, the Gaussian error $z \sim N(0, \sigma^2 I_m)$ satisfies*

$$\mathbb{P} \left(\|D^*A^*z\|_{\infty} \leq \sigma \sqrt{2(1+\alpha)(1+\delta_1) \log d} \right) \geq 1 - \frac{1}{d^{\alpha} \sqrt{(1+\alpha)\pi \log d}}.$$

Combining Lemma 2.2 ($\alpha = 1$) with Theorem 2.1 and noting that $\delta_1 \leq \delta_{3s}$, we have the following result.

Theorem 2.3. *Let D be an arbitrary $n \times d$ tight frame and let A be an $m \times n$ measurement matrix satisfying the D -RIP with $\delta_{3s} < \frac{1}{2}$. Assume that $z \sim N(0, \sigma^2 I_m)$ and that \hat{f}^{ADS} is the solution of the ADS (1.7) with $\lambda = 2\sigma\sqrt{2\log d}$. Then we have*

$$\|\hat{f}^{ADS} - f\|_2 \leq \min_{1 \leq k \leq s} \left[C_0 \sigma \sqrt{k \log d} + C_1 \frac{\|D^* f - (D^* f)_{[k]}\|_1}{\sqrt{k}} \right]$$

with probability at least $1 - 1/(d\sqrt{2\pi \log d})$, where C_0 and C_1 are small constants depending only on δ_{3s} .

Remark 2.4. (a) *In the exactly s sparse case ($\|D^* f\|_0 \leq s$), the above theorem implies*

$$\|\hat{f}^{ADS} - f\|_2^2 \leq C_0 \cdot \log d \cdot \sigma^2. \quad (2.1)$$

Specially, when $D = I$, that is for the standard compressed sensing, we derive similar result as in [15, Theorem 1.1] (see also [5, 17]). Now it was shown in [15] that the standard DS achieves a loss within a logarithmic factor of the ideal mean squared error. The log-like factor is the price we pay for adaptivity, that is, for not knowing ahead of time where the nonzero coefficients actually are. In this sense, ignoring the log-like factor, the error bound (2.1) is in general unimprovable.

(b) *The Gaussian error satisfies*

$$\mathbb{P}(\|z\|_2 \leq \sigma \sqrt{m + 2\sqrt{m \log m}}) \geq 1 - \frac{1}{m}, \quad (2.2)$$

see [17, Lemma 1]. Combing this with (1.4), one would show that the solution \hat{f} to the ABP (1.2) with $\varepsilon = \sigma \sqrt{m + 2\sqrt{m \log m}}$ satisfies

$$\|\hat{f} - f\|_2 \leq C_2 \frac{\|D^* f - (D^* f)_{[s]}\|_1}{\sqrt{s}} + C_3 \sigma \sqrt{m + 2\sqrt{m \log m}} \quad (2.3)$$

with high probability provided that A satisfies the D -RIP with $\delta_{2s} < 0.493$, where C_2 and C_3 are small constants depending on δ_{2s} . Specially, if $\|D^* f\|_0 \leq s$, then

$$\|\hat{f} - f\|_2 \leq C_1 \sigma \sqrt{m + 2\sqrt{m \log m}}. \quad (2.4)$$

Ignoring the D -RIP condition, the precise constants and the probabilities with which the stated bounds hold, we observe that in the case when $m = O(s \log d)$, (2.4) and (2.1) appear to be essentially the same. However, there is a subtle difference. Specially, if m and n are fixed and we consider the effect of varying s , we can see that the ADS yields a bound that is adaptive to this change, providing a stronger guarantee when s is small, whereas the bound in (2.4) does not improve as s is reduced. What is missing in [8, 39] is achieved here is the adaptivity to the unknown level of sparsity (with respect to D) of the object we try to recover.

(c) Assume that the signal's transform coefficients in terms of D decays like a power-law, i.e., the j th largest entry of the vector $|D^*f|$ obeys

$$|D^*f|_j \leq R \cdot j^{-1/p} \quad (2.5)$$

for some positive numbers R and $p \leq 1$. Such a model is appropriate for the wavelet frame coefficients of a piecewise smooth signal, for example. Then with high probability, we have

$$\|\hat{f}^{ADS} - f\|_2^2 \leq \min_{1 \leq k \leq s} C_0 \cdot \left(\sigma^2 k \log d + R^2 k^{-2/p+1} \right).$$

(In this case, one can also compare this bound with the error estimates yielded by the ABP by applying (2.5) to (2.3).) In the case of $D = I$, that is for the standard compressed sensing, we derive similar result as in [15, Theorem 1.3].

(d) We have not tried to optimize the D -RIP condition. We expect that with a more complicated proof as in [6] or [29, 16, 26, 41], one can still improve this condition.

The error bound (2.1) is within a log-like factor of the minimax risk over the class of vectors which are at most s sparse in terms of D :

Theorem 2.5. *Let D be an arbitrary $n \times d$ tight frame. Assume that the measurement matrix A satisfies the D -RIP of order s and that $z \sim N(0, \sigma^2 I_m)$. Suppose that there exists a subset $T_0 \in [d]$ such that $|T_0| = s$ and $\Sigma_{T_0} \subset \{D^* \tilde{f} : \tilde{f} \in \mathbb{R}^n\}$, where $\Sigma_{T_0} = \{x \in \mathbb{R}^d : \text{supp}(x) \subset T_0\}$. Then*

$$\inf_{\hat{f}} \sup_{\|D^*f\|_0 \leq s} \mathbb{E} \|\hat{f} - f\|_2^2 \geq \frac{1}{1 + \delta_s} s \cdot \sigma^2,$$

where the infimum is over all measurable functions $\hat{f}(y)$ of y .

Remark 2.6. *When D is an identity matrix or an orthonormal basis, the condition $\Sigma_{T_0} \subset \{D^* \tilde{f} : \tilde{f} \in \mathbb{R}^n\}$ is satisfied.*

The exacting reading may argue that while this lower bound is in expectation, the upper bound holds with high probability. Thus, we provide the following complementary theorem.

Theorem 2.7. *Under the assumptions of Theorem 2.5, any estimator $\hat{f}(y)$ obeys*

$$\sup_{\|D^*f\|_0 \leq s} \mathbb{P} \left(\|\hat{f} - f\|_2^2 \geq \frac{1}{2(1 + \delta_s)} s \cdot \sigma^2 \right) \geq 1 - e^{-\frac{s}{16}}.$$

3 The analysis LASSO

In this section, we will present the performance of the ALASSO from the noisy measurements (1.1), where z is suitably bounded. Specially, z can be Gaussian noise. Note that our results are similar as that for the ADS.

Theorem 3.1. *Let D be an arbitrary $n \times d$ tight frame and let A be an $m \times n$ measurement matrix satisfying the D -RIP with $\delta_{3s} < \frac{1}{4}$. Assume that μ obeys $\|D^*A^*z\|_\infty \leq \mu/2$. Then the solution \hat{f}^{AL} to the ALASSO (1.3) obeys*

$$\|\hat{f}^{AL} - f\|_2 \leq \min_{1 \leq k \leq s} \left[C_0 \sqrt{k} \mu + C_1 \frac{\|D^*f - (D^*f)_{[k]}\|_1}{\sqrt{k}} \right],$$

where C_1 is small constant depending only on δ_{3s} and C_0 is depending on δ_{3s} and $\|D^*D\|_{1,1}$.

Combining Lemma 2.2 with Theorem 3.1, we have the following result.

Theorem 3.2. *Let D be an arbitrary $n \times d$ tight frame and let A be an $m \times n$ measurement matrix satisfying the D -RIP with $\delta_{3s} < \frac{1}{4}$. Assume that $z \sim N(0, \sigma^2 I_m)$ and that \hat{f}^{AL} is the solution of the ALASSO with $\mu = 4\sigma\sqrt{2\log d}$. Then we have*

$$\|\hat{f}^{AL} - f\|_2 \leq \min_{1 \leq k \leq s} \left[C_0 \sigma \sqrt{k \log d} + C_1 \frac{\|D^*f - (D^*f)_{[k]}\|_1}{\sqrt{k}} \right]$$

with probability exceeding $1 - 1/(d\sqrt{2\pi \log d})$, where C_1 is small constant depending only on δ_{3s} and C_0 is depending on δ_{3s} and $\|D^*D\|_{1,1}$.

Remark 3.3. (a) *From the proof, one can see that $C_0 = 2\sqrt{2}(1 + 2\|D^*D\|_{1,1})/(1 - 4\delta_{3s})$. When D is an identity matrix or an orthonormal basis, $\|D^*D\|_{1,1} = 1$. For general tight frame D , we hope that with some more delicate proof, the depending on $\|D^*D\|_{1,1}$ can be deleted.*

(b) *In the exactly s sparse case ($\|D^*f\|_0 \leq s$), the above theorem implies*

$$\|\hat{f}^{AL} - f\|_2^2 \leq C_0 \cdot \log d \cdot s \sigma^2.$$

Specially, when $D = I$, that is for the standard compressed sensing, we derive similar result as in [5, Theorem 7.2].

4 Sparse noise

In this section, we consider model (1.9), where z is suitably bounded and e is sparse in terms of a tight frame Ω .

Theorem 4.1. *Let D be an arbitrary $n \times d$ tight frame and let A be an $m \times n$ matrix with elements a_{ij} drawn i.i.d according to $N(0, 1/m)$. Let $\|\Omega^*e\|_0 \leq s'$, where $\Omega \in \mathbb{R}^{m \times M}$ is a tight frame for \mathbb{R}^m . Suppose $m \geq C\delta^{-2}(s + s') \log((d + M)/(s + s'))$ for some fixed $\delta \in (0, 1/4)$ and constant C .*

(a) *Let λ obeys $\|W^*\Phi^*z\|_\infty \leq \lambda$. Then with high probability, the solution \hat{u}^{SADS} to (1.11) obeys*

$$\|\hat{f}^{SADS} - f\|_2 \leq \|\hat{u}^{SADS} - u\|_2 \leq \min_{1 \leq k \leq s} \left[C_0 \sqrt{k + s'} \lambda + C_1 \frac{\|D^*f - (D^*f)_{[k]}\|_1}{\sqrt{k + s'}} \right].$$

(b) Let μ obeys $\|W^*\Phi^*z\|_\infty \leq \mu/2$. Then with high probability, the solution \hat{u}^{SAL} to (1.12) obeys

$$\|\hat{f}^{SAL} - f\|_2 \leq \|\hat{u}^{SAL} - u\|_2 \leq \min_{1 \leq k \leq s} \left[C_2(1 + 2\|D^*D\|_{1,1})\sqrt{k + s'}\mu + C_3 \frac{\|D^*f - (D^*f)_{[k]}\|_1}{\sqrt{k + s'}} \right].$$

(c) Assume that $\|z\|_2 \leq \varepsilon$. Then with high probability, the solution \hat{u}^{SABP} to (1.10) obeys

$$\|\hat{f}^{SABP} - f\|_2 \leq \|\hat{u}^{SABP} - u\|_2 \leq C_4\varepsilon + C_5 \frac{\|D^*f - (D^*f)_{[s]}\|_1}{\sqrt{s + s'}}.$$

In the above, C_0, \dots, C_5 are small constants depending only on δ .

Remark 4.2. (a) From the proof of this theorem, one can see that such results can be extended to the more general class of Sub-Gaussian matrices and the case that e is nearly sparse in terms of Ω .

(b) In the case of $z = 0$ and $\|D^*f\|_0 \leq s$, the above theorem implies exact recovery (both f and e) via

$$\hat{u} = \arg \min_{\tilde{u} \in \mathbb{R}^{n+m}} \|W^*\tilde{u}\|_1 \quad \text{subject to} \quad \Phi\tilde{u} = y.$$

Specially, when $\Omega = I$, we derive similar result as in [37].

(c) By applying Lemma 2.2 (Since from the proof of this theorem, one can see that A satisfies the W -RIP) and (2.2) to the above theorem, one can get error estimates for the SABP, the SADS and the SALASSO in the case of $z \sim N(0, \sigma^2 I)$.

5 Proofs

We first recall some useful properties of a tight frame. Let D be an arbitrary $n \times d$ tight frame for \mathbb{R}^n , then

$$\|f\|_2^2 = \|D^*f\|_2^2 \quad \text{for all } f \in \mathbb{R}^n, \quad \text{and} \quad \|Dv\|_2 \leq \|v\|_2 \quad \text{for all } v \in \mathbb{R}^d.$$

Refer the readers to [18, Chapter 3] for details.

5.1 Proof of Lemma 2.2

Proof of Lemma 2.2. Note that from the definition of D -RIP, we have

$$\sqrt{1 - \delta_1} \|D_j\|_2 \leq \|AD_j\|_2 \leq \sqrt{1 + \delta_1} \|D_j\|_2 \leq \sqrt{1 + \delta_1}, \quad \forall j \in [d]. \quad (5.1)$$

Without loss of generality, we assume that $\|D_j\|_2 \neq 0$ for each $j \in [d]$. Then by (5.1), we have $\|AD_j\|_2 \neq 0$. Let $\omega_j = \frac{\langle AD_j, z \rangle}{\sigma \|AD_j\|_2}$. Then ω_j has Gaussian distribution $N(0, 1)$. By using the union

bound and then the inequality (5.1), we get

$$\begin{aligned}
& \mathbb{P}\left(\|D^*A^*z\|_\infty > \sigma\sqrt{2(1+\alpha)(1+\delta_1)\log d}\right) \\
& \leq \sum_{j=1}^d \mathbb{P}\left(|\omega_j|\|AD_j\|_2 > \sqrt{2(1+\alpha)(1+\delta_1)\log d}\right) \\
& \leq \sum_{j=1}^d \mathbb{P}\left(|\omega_j| > \sqrt{2(1+\alpha)\log d}\right) \\
& = d \cdot \mathbb{P}\left(|\omega_1| > \sqrt{2(1+\alpha)\log d}\right) \leq \frac{1}{d^\alpha\sqrt{(1+\alpha)\pi\log d}},
\end{aligned}$$

where the last step follows from the Gaussian tail probability bound that for a standard Gaussian variable V and any constant t , $\mathbb{P}(|V| > t) \leq 2t^{-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}$. It thus follows that

$$\begin{aligned}
& \mathbb{P}\left(\|D^*A^*z\|_\infty \leq \sigma\sqrt{2(1+\alpha)(1+\delta_1)\log d}\right) \\
& = 1 - \mathbb{P}\left(\|D^*A^*z\|_\infty > \sigma\sqrt{2(1+\alpha)(1+\delta_1)\log d}\right) \\
& \geq 1 - \frac{1}{d^\alpha\sqrt{(1+\alpha)\pi\log d}}.
\end{aligned}$$

□

5.2 Proof of Theorem 2.1

Proof of Theorem 2.1. The proof makes use of the ideas from [8, 15, 6, 10]. Let f and \hat{f}^{ADS} be as in the theorem, and let $T_0 = T$ denote the set of the s largest coefficients of D^*f in magnitude. Set $h = \hat{f}^{ADS} - f$ and observe that by the triangle inequality

$$\|D^*A^*Ah\|_\infty \leq \|D^*A^*(Af - y)\|_\infty + \|D^*A^*(A\hat{f}^{ADS} - y)\|_\infty \leq 2\lambda. \quad (5.2)$$

Since \hat{f}^{ADS} is a minimizer, one gets that

$$\|D^*f\|_1 \geq \|D^*\hat{f}^{ADS}\|_1.$$

That is

$$\|D_T^*f\|_1 + \|D_{T^c}^*f\|_1 \geq \|D_T^*\hat{f}^{ADS}\|_1 + \|D_{T^c}^*\hat{f}^{ADS}\|_1.$$

Thus

$$\|D_T^*f\|_1 + \|D_{T^c}^*f\|_1 \geq \|D_T^*f\|_1 - \|D_T^*h\|_1 + \|D_{T^c}^*h\|_1 - \|D_{T^c}^*f\|_1.$$

This implies

$$\|D_{T^c}^*h\|_1 \leq 2\|D_{T^c}^*f\|_1 + \|D_T^*h\|_1. \quad (5.3)$$

Next, we decompose the coordinates T_0^c into sets of size s in order of decreasing magnitude of $D_{T^c}^*h$. Denote these sets T_1, T_2, \dots , and for simplicity of notation set $T_{01} = T_0 \cup T_1$. Note that for each $j \geq 2$,

$$\|D_{T_j}^*h\|_2 \leq s^{1/2}\|D_{T_j}^*h\|_\infty \leq s^{-1/2}\|D_{T_{j-1}}^*h\|_1$$

and thus

$$\sum_{j \geq 2} \|D_{T_j}^* h\|_2 \leq \sum_{j \geq 1} s^{-1/2} \|D_{T_j}^* h\|_1 = s^{-1/2} \|D_{T_c}^* h\|_1. \quad (5.4)$$

Set $u_{01} = D_{T_{01}}^* h / \|DD_{T_{01}}^* h\|_2$ and $u_j = D_{T_j}^* h / \|DD_{T_j}^* h\|_2$ for each $j \geq 2$. Then $\|Du_{01}\|_2 = 1$ and $\|Du_j\|_2 = 1$ for each $j \geq 2$. We then obtain that

$$\begin{aligned} & \frac{\langle ADD_{T_{01}}^* h, ADD_{T_j}^* h \rangle}{\|DD_{T_{01}}^* h\|_2 \|DD_{T_j}^* h\|_2} = \langle ADu_j, ADu_{01} \rangle = \frac{1}{4} \{ \|ADu_j + ADu_{01}\|_2^2 - \|ADu_j - ADu_{01}\|_2^2 \} \\ & \geq \frac{1}{4} \{ (1 - \delta_{3s}) \|Du_j + Du_{01}\|_2^2 - (1 + \delta_{3s}) \|Du_j - Du_{01}\|_2^2 \} \\ & = \langle Du_j, Du_{01} \rangle - \frac{\delta_{3s}}{2} \{ \|Du_j\|_2^2 + \|Du_{01}\|_2^2 \} = \langle Du_j, Du_{01} \rangle - \delta_{3s}. \end{aligned}$$

It thus follows that

$$\begin{aligned} \langle Ah, ADD_{T_{01}}^* h \rangle &= \langle ADD_{T_{01}}^* h, ADD_{T_{01}}^* h \rangle + \sum_{j \geq 2} \langle ADD_{T_j}^* h, ADD_{T_{01}}^* h \rangle \\ &\geq (1 - \delta_{3s}) \|DD_{T_{01}}^* h\|_2^2 - \delta_{3s} \|DD_{T_{01}}^* h\|_2 \sum_{j \geq 2} \|DD_{T_j}^* h\|_2 + \sum_{j \geq 2} \langle DD_{T_j}^* h, DD_{T_{01}}^* h \rangle. \end{aligned}$$

By applying the equality

$$\sum_{j \geq 2} \langle DD_{T_j}^* h, DD_{T_{01}}^* h \rangle = \langle h - DD_{T_{01}}^* h, DD_{T_{01}}^* h \rangle = \|D_{T_{01}}^* h\|_2^2 - \|DD_{T_{01}}^* h\|_2^2,$$

we get

$$\begin{aligned} \langle Ah, ADD_{T_{01}}^* h \rangle &\geq \|D_{T_{01}}^* h\|_2^2 - \delta_{3s} \|DD_{T_{01}}^* h\|_2^2 - \delta_{3s} \|DD_{T_{01}}^* h\|_2 \sum_{j \geq 2} \|DD_{T_j}^* h\|_2 \\ &\geq (1 - \delta_{3s}) \|D_{T_{01}}^* h\|_2^2 - \delta_{3s} \|D_{T_{01}}^* h\|_2 \sum_{j \geq 2} \|D_{T_j}^* h\|_2. \end{aligned}$$

Substituting the inequality (5.4) into the above inequality, we derive

$$\langle Ah, ADD_{T_{01}}^* h \rangle \geq (1 - \delta_{3s}) \|D_{T_{01}}^* h\|_2^2 - s^{-1/2} \delta_{3s} \|D_{T_{01}}^* h\|_2 \|D_{T_c}^* h\|_1.$$

Besides, by using the holder inequality and (5.2), we have

$$\langle Ah, ADD_{T_{01}}^* h \rangle = \langle D^* A^* Ah, D_{T_{01}}^* h \rangle \leq \|D^* A^* Ah\|_\infty \|D_{T_{01}}^* h\|_1 \leq 2\lambda\sqrt{2s} \|D_{T_{01}}^* h\|_2.$$

Now combining the above two inequalities and by an easy computation, we can derive

$$\|D_{T_{01}}^* h\|_2 \leq \frac{2\lambda\sqrt{2s} + s^{-1/2} \delta_{3s} \|D_{T_c}^* h\|_1}{1 - \delta_{3s}}. \quad (5.5)$$

It thus follows that

$$\|D_T^* h\|_1 \leq \sqrt{s} \|D_T^* h\|_2 \leq \sqrt{s} \|D_{T_{01}}^* h\|_2 \leq \frac{2\sqrt{2}\lambda s + \delta_{3s} \|D_{T_c}^* h\|_1}{1 - \delta_{3s}}.$$

Substituting the above inequality to (5.3) and by an easy calculation, we can obtain

$$\|D_{T^c}^* h\|_1 \leq \frac{2(1 - \delta_{3s})\|D_{T^c}^* f\|_1 + 2\sqrt{2}\lambda s}{1 - 2\delta_{3s}}. \quad (5.6)$$

Now we are ready to give the error estimates. Note that

$$\|h\|_2 = \|D^* h\|_2 \leq \|D_{T_0^1}^* h\|_2 + \sum_{j \geq 2} \|D_{T_j}^* h\|_2.$$

Introducing (5.4) and (5.5) to the above, we get

$$\|h\|_2 \leq \frac{2\lambda\sqrt{2s} + s^{-1/2}\|D_{T^c}^* h\|_1}{1 - \delta_{3s}}.$$

By applying (5.6), we derive

$$\|h\|_2 \leq \frac{4\sqrt{2s}\lambda}{1 - 2\delta_{3s}} + \frac{2\|D_{T^c}^* f\|_1}{(1 - 2\delta_{3s})\sqrt{s}}.$$

Repeating the above argument for each $1 \leq k < s$, one can prove that

$$\|h\|_2 \leq \frac{4\sqrt{2k}\lambda}{1 - 2\delta_{3k}} + \frac{2\|D_{T^c}^* f\|_1}{(1 - 2\delta_{3k})\sqrt{k}}.$$

Now the proof can be finished by noting that $\delta_{3k} \leq \delta_{3s}$ for $1 \leq k \leq s$. □

5.3 Proof of Theorem 2.5

We first introduce the following well-known lemma, see for example [10, Lemma 3.11]. It gives the minimax risk for estimating the vector $x \in \mathbb{R}^s$ from the data $y \in \mathbb{R}^m$ and the linear model

$$y = \Phi x + z, \quad (5.7)$$

where $\Phi \in \mathbb{R}^{m \times s}$ and $z \sim N(0, \sigma^2 I_m)$.

Lemma 5.1. *Let Φ, x, y, z follow the linear model (5.7) and that $\lambda_i(\Phi^* \Phi)$ be the eigenvalues of the matrix $\Phi^* \Phi$. Then*

$$\inf_{\hat{x}} \sup_{x \in \mathbb{R}^s} \mathbb{E} \|\hat{x} - x\|_2^2 = \sigma^2 \text{trace}((\Phi^* \Phi)^{-1}) = \sum_i \frac{\sigma^2}{\lambda_i(\Phi^* \Phi)},$$

where the infimum is over all measurable functions $\hat{x}(y)$ of y . In particular, if one of the eigenvalues vanishes, then the minimax risk is unbounded.

Proof of Theorem 2.5. Note that we have

$$\begin{aligned}
\inf_{\hat{f}} \sup_{\|D^*f\|_0 \leq s} \mathbb{E} \|\hat{f} - f\|_2^2 &\geq \inf_{\hat{f}} \sup_{D^*f \in \Sigma_{T_0}} \mathbb{E} \|\hat{f} - f\|_2^2 \\
&= \inf_{\hat{f}} \sup_{D^*f \in \Sigma_{T_0}} \mathbb{E} \|D^*\hat{f} - D^*f\|_2^2 \\
&\geq \inf_{\hat{f}} \sup_{D^*f \in \Sigma_{T_0}} \mathbb{E} \|D_{T_0}^*\hat{f} - D_{T_0}^*f\|_2^2.
\end{aligned} \tag{5.8}$$

For each f such that $D^*f \in \Sigma_{T_0}$, we rewrite the original model $y = Af + z$ as $y = AD_{T_0}v + z$, where $v \in \mathbb{R}^s$ and $z \sim N(0, \sigma^2 I_m)$. Since we have $\Sigma_{T_0} \subset \{D^*\tilde{f} : \tilde{f} \in \mathbb{R}^n\}$ and that $D_{T_0}^*\hat{f}(y)$ is measurable of y , we get

$$\inf_{\hat{f}} \sup_{D^*f \in \Sigma_{T_0}} \mathbb{E} \|D_{T_0}^*\hat{f} - D_{T_0}^*f\|_2^2 \geq \inf_{\hat{v}} \sup_{v \in \mathbb{R}^s} \mathbb{E} \|\hat{v} - v\|_2^2, \tag{5.9}$$

where $v \in \mathbb{R}^s$, $AD_{T_0}v, y, z$ follow the linear model $y = AD_{T_0}v + z$, $z \sim N(0, \sigma^2 I_m)$, and the infimum of the last term is over all measurable functions $\hat{v}(y)$ of y . Note that from the definition of D -RIP, for all $v \in \mathbb{R}^s$, we have

$$\|AD_{T_0}v\|_2^2 \leq (1 + \delta_s) \|D_{T_0}v\|_2^2 \leq (1 + \delta_s) \|v\|_2^2.$$

It thus follows that

$$\lambda_{\max}(D_{T_0}^*A^*AD_{T_0}) \leq 1 + \delta_s. \tag{5.10}$$

By using Lemma 5.1 and (5.10), we have

$$\inf_{\hat{v}} \sup_{v \in \mathbb{R}^s} \mathbb{E} \|\hat{v} - v\|_2^2 = \sum_i \frac{\sigma^2}{\lambda_i(D_{T_0}^*A^*AD_{T_0})} \geq \frac{1}{1 + \delta_s} s \cdot \sigma^2. \tag{5.11}$$

Introducing (5.9) and (5.11) to (5.8), we derive

$$\inf_{\hat{f}} \sup_{\|D^*f\|_0 \leq s} \mathbb{E} \|\hat{f} - f\|_2^2 \geq \frac{1}{1 + \delta_s} s \cdot \sigma^2.$$

□

5.4 Proof of Theorem 2.7

We begin by introducing the following lemma, see [10, Lemma 3.14].

Lemma 5.2. *Suppose that x, y, Φ, z follow the linear model (5.7) with $z \sim N(0, \sigma^2 I)$. Then*

$$\inf_{\hat{x}} \sup_{x \in \mathbb{R}^s} \mathbb{P} \left(\|\hat{x} - x\|_2^2 \geq \frac{1}{2\|\Phi\|^2} s \cdot \sigma^2 \right) \geq 1 - e^{-\frac{s}{16}}.$$

Proof of Theorem 2.7. From the definition of tight frame, we have

$$\begin{aligned}
& \sup_{\|D^*f\|_0 \leq s} \mathbb{P} \left(\|\hat{f} - f\|_2^2 \geq \frac{1}{2(1 + \delta_s)} s \cdot \sigma^2 \right) \\
&= \sup_{\|D^*f\|_0 \leq s} \mathbb{P} \left(\|D^*\hat{f} - D^*f\|_2^2 \geq \frac{1}{2(1 + \delta_s)} s \cdot \sigma^2 \right) \\
&\geq \sup_{D^*f \in \Sigma_{T_0}} \mathbb{P} \left(\|D^*\hat{f} - D^*f\|_2^2 \geq \frac{1}{2(1 + \delta_s)} s \cdot \sigma^2 \right) \\
&\geq \sup_{D^*f \in \Sigma_{T_0}} \mathbb{P} \left(\|D_{T_0}^*\hat{f} - D_{T_0}^*f\|_2^2 \geq \frac{1}{2(1 + \delta_s)} s \cdot \sigma^2 \right) \\
&\geq \sup_{D^*f \in \Sigma_{T_0}} \mathbb{P} \left(\|D_{T_0}^*\hat{f} - D_{T_0}^*f\|_2^2 \geq \frac{1}{2\|AD_{T_0}\|^2} s \cdot \sigma^2 \right),
\end{aligned}$$

where we have used (5.10) for the last step. Note that $D_{T_0}^*\hat{f}(y)$ is measurable of y since $\hat{f}(y)$ is measurable. Then, with the assumption $\Sigma_{T_0} \subset \{D^*\tilde{f} : \tilde{f} \in \mathbb{R}^n\}$, we get

$$\begin{aligned}
& \sup_{\|D^*f\|_0 \leq s} \mathbb{P} \left(\|\hat{f} - f\|_2^2 \geq \frac{1}{2(1 + \delta_s)} s \cdot \sigma^2 \right) \\
&\geq \sup_{D^*f \in \Sigma_{T_0}} \mathbb{P} \left(\|D_{T_0}^*\hat{f} - D_{T_0}^*f\|_2^2 \geq \frac{1}{2\|AD_{T_0}\|^2} s \cdot \sigma^2 \right) \\
&\geq \inf_{\hat{v}} \sup_{v \in \mathbb{R}^s} \mathbb{P} \left(\|\hat{v} - v\|_2^2 \geq \frac{1}{2\|AD_{T_0}\|^2} s \cdot \sigma^2 \right) \\
&\geq 1 - e^{-\frac{s}{16}},
\end{aligned}$$

where the last step follows from Lemma 5.2. □

5.5 Proof of Theorem 3.1

Proof of Theorem 3.1. The proof is similar to that of Theorem 2.1. Set $h = \hat{f}^{AL} - f$. We will prove the following two inequalities:

- $\|D^*A^*(A\hat{f}^{AL} - y)\|_\infty \leq \mu\|D^*D\|_{1,1}$.
- $\|D_{T^c}^*h\|_1 \leq 3\|D_T^*h\|_1 + 4\|D_{T^c}^*f\|_1$.

With these two inequalities and the assumptions of this theorem, a similar approach as that for Theorem 2.1 would lead to our results.

For convenience, we denote \mathcal{L} as the function

$$\mathcal{L}(\tilde{f}) = \frac{1}{2}\|(A\tilde{f} - y)\|_2^2 + \mu\|D^*\tilde{f}\|_1,$$

in which $\mu = 4\sigma\sqrt{2\log d}$. The subdifferential $\partial\mathcal{F}$ of a real valued convex lower semicontinuous function $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}$ is the multifunction defined by

$$\partial\mathcal{F}(f_0) = \left\{ g \in \mathbb{R}^n \mid \forall \tilde{f} \in \mathbb{R}^n, \mathcal{F}(\tilde{f}) \geq \mathcal{F}(f_0) + \langle g, \tilde{f} - f_0 \rangle \right\}.$$

Note that f_0 is a minimum of \mathcal{F} if and only if $0 \in \partial\mathcal{F}(f_0)$. The subdifferential of $\mathcal{L}(\hat{f}^{AL})$ is

$$\partial\mathcal{L}(\hat{f}^{AL}) = \left\{ A^*(A\hat{f}^{AL} - y) + \mu Dv \mid v \in \mathbb{R}^d : v_i = \text{sgn}(D_i^* \hat{f}^{AL}) \text{ if } D_i^* \hat{f}^{AL} \neq 0 \text{ and } |v_i| \leq 1 \text{ otherwise} \right\}.$$

Hence there exists $v \in \mathbb{R}^d$ such that $\|v\|_\infty \leq 1$ satisfying

$$A^*(A\hat{f}^{AL} - y) + \mu Dv = 0.$$

Now we get

$$\|D^* A^*(A\hat{f}^{AL} - y)\|_\infty = \mu \|D^* Dv\|_\infty \leq \mu \|D^* D\|_{\infty, \infty} = \mu \|D^* D\|_{1,1}.$$

Since \hat{f}^{AL} is the minimizer to (1.3), we have

$$\frac{1}{2} \|A\hat{f}^{AL} - y\|_2^2 + \mu \|D^* \hat{f}^{AL}\|_1 \leq \frac{1}{2} \|Af - y\|_2^2 + \mu \|D^* f\|_1.$$

Plug in $y = Af + z$ and rearrange terms to give

$$\frac{1}{2} \|Ah\|_2^2 + \mu \|D^* \hat{f}^{AL}\|_1 \leq \langle Ah, z \rangle + \mu \|D^* f\|_1.$$

From the definition of tight frame, and then by using the holder inequality and the assumption $\|D^* A^* z\|_\infty \leq \mu/2$, we have

$$\begin{aligned} \langle Ah, z \rangle + \mu \|D^* f\|_1 &= \langle D^* h, D^* A^* z \rangle + \mu \|D^* f\|_1 \leq \|D^* h\|_1 \|D^* A^* z\|_\infty + \mu \|D^* f\|_1 \\ &\leq \mu/2 \|D^* h\|_1 + \mu \|D^* f\|_1. \end{aligned}$$

It thus follows that

$$\mu \|D^* \hat{f}^{AL}\|_1 \leq \frac{1}{2} \|Ah\|_2^2 + \mu \|D^* \hat{f}^{AL}\|_1 \leq \mu/2 \|D^* h\|_1 + \mu \|D^* f\|_1.$$

This gives

$$\|D^* \hat{f}^{AL}\|_1 \leq \|D^* h\|_1/2 + \|D^* f\|_1.$$

Now a similar argument as that for (5.3) leads to

$$\|D_{T^c}^* h\|_1 \leq 3\|D_T^* h\|_1 + 4\|D_{T^c}^* f\|_1. \quad (5.12)$$

Now we sketch the important steps of the proof. Similar to (5.2), we have

$$\|D^* A^* Ah\|_\infty \leq \|D^* A^*(Af - y)\|_\infty + \|D^* A^*(A\hat{f}^{AL} - y)\|_\infty \leq c_0 \mu,$$

where $c_0 = 1/2 + \|D^* D\|_{1,1}$. With the above inequality, a similar argument as that for (5.5) gives

$$\|D_{T_0^1}^* h\|_2 \leq \frac{c_0 \mu \sqrt{2s} + s^{-1/2} \delta_{3s} \|D_{T^c}^* h\|_1}{1 - \delta_{3s}}. \quad (5.13)$$

It thus follows that

$$\|D_T^* h\|_1 \leq \sqrt{s} \|D_T^* h\|_2 \leq \sqrt{s} \|D_{T_{01}}^* h\|_2 \leq \frac{\sqrt{2}c_0\mu s + \delta_{3s} \|D_{T^c}^* h\|_1}{1 - \delta_{3s}}.$$

Substituting the above inequality to (5.12) and by an easy calculation, we can obtain

$$\|D_{T^c}^* h\|_1 \leq \frac{4(1 - \delta_{3s}) \|D_{T^c}^* f\|_1 + 3\sqrt{2}c_0\mu s}{1 - 4\delta_{3s}}. \quad (5.14)$$

Using (5.13), (5.4) and then applying (5.13), we get

$$\|h\|_2 = \|D^* h\|_2 \leq \|D_{T_{01}}^* h\|_2 + \sum_{j \geq 2} \|D_{T_j}^* h\|_2 \leq \frac{c_0\mu\sqrt{2s} + s^{-1/2} \|D_{T^c}^* h\|_1}{1 - \delta_{3s}} \leq \frac{4\sqrt{2s}c_0\mu}{1 - 4\delta_{3s}} + \frac{4 \|D_{T^c}^* f\|_1}{(1 - 4\delta_{4s})\sqrt{s}},$$

which leads to the result.

Repeating the above argument for each $1 \leq k < s$, one can finish the proof. \square

5.6 Proof of Theorem 4.1

We introduce the following result, see [37, Lemma 1]. As shown in [37], such results can be extended with different constants to the more general class of Sub-Gaussian matrices.

Lemma 5.3. *Let A be an $m \times n$ matrix with elements a_{ij} drawn i.i.d according to $N(0, 1/m)$ and let $\Phi = [A, I]$. Then for every $v \in \mathbb{R}^{m+n}$,*

$$\mathbb{P}(|\|\Phi v\|_2^2 - \|v\|_2^2| \geq 2\delta \|v\|_2^2) \leq 3e^{-m\delta^2/8}, \quad \delta \in (0, 1). \quad (5.15)$$

Proof of Theorem 4.1. Under the assumptions of the theorem, by Lemma 5.3, we have that for every $v \in \mathbb{R}^{m+n}$, (5.15) holds. Using a standard covering argument as in [3] (also [49]), one can prove that with probability exceeding $1 - 3e^{-C_2 m}$, Φ satisfies the W -RIP of order $s+s'$ with constant δ . Then, the conclusions follow from Theorem 2.1, Theorem 3.1, (1.4) and that

$$\|W^* u - (W^* u)_{[s+s']}\|_1 \leq \|D^* f - (D^* f)_{[s]}\|_1 + \|\Omega^* e - (\Omega^* e)_{[s']}\|_1.$$

\square

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