

Nested Lattice Codes for Gaussian Two-Way Relay Channels

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Abstract

In this paper, we consider a Gaussian two-way relay channel (GTRC), where two sources exchange messages with each other through a relay. We assume that there is no direct link between sources, and all nodes operate in full-duplex mode. By utilizing nested lattice codes for the uplink (i.e., MAC phase), and structured binning for the downlink (i.e., broadcast phase), we propose two achievable schemes. Scheme 1 is based on “compute and forward” scheme of [1] while scheme 2 utilizes two different lattices for source nodes based on a three-stage lattice partition chain. We show that scheme 2 can achieve capacity region at the high signal-to-noise ratio (SNR). Regardless all channel parameters, the achievable rate of scheme 2 is within 0.2654 bit from the cut-set outer bound for user 1. For user 2, the proposed scheme achieves within 0.167 bit from the outer bound if channel coefficient is larger than one, and achieves within 0.2658 bit from the outer bound if channel coefficient is smaller than one. Moreover, sum rate of the proposed scheme is within 0.334 bits from the sum capacity. These gaps for GTRC are the best gap-to-capacity results to date.

I. INTRODUCTION

In this paper, we study a two-way relay channel, where two nodes exchange their messages with each other via a relay. This can be considered as, e.g., two mobile users communicate to each other via the access point in a WLAN [2] or a satellite which enables data exchange between several earth stations where there is no direct link between the stations.

Two-way or bi-directional communication between two nodes without a relay was first proposed and studied by Shannon in [3]. In this setup, two nodes want to exchange messages with each other, and act as transmitters and receivers at the same time. For this setup, the capacity region is not known in general and only inner and outer bounds on the capacity region are obtained in the literature.

The two-way relay channel, also known as the bi-directional relay channel, consists of two nodes communicating with each other in a bi-directional manner via a relay. This setup was first introduced in [4]; later studied in [4], [5], [6] and an approximate characterization of the capacity region of the Gaussian case is derived.

The traditional relaying protocols require four channel uses to exchange the data of two nodes whereas the two-way relaying protocol in [4] only needs two phases to achieve bidirectional communication between the two nodes. The first phase is referred to as the multiple access (MAC) phase, and the second phase is referred to as the broadcast (BRC) phase. In the MAC phase, both nodes transmit their messages to the relay node which decodes them. In the BRC phase, the relay combines the data from both nodes and broadcasts the combined data back to both nodes. For this phase, there exist several strategies for the processing at the relay node, e.g., an amplify-and-forward (AF) strategy [4], a decode-and-forward (DF) strategy [4], [7], or a compress-and-forward (CF) strategy [8].

The AF protocol is a simple scheme, which amplifies the signal transmitted from both nodes and retransmit it to them, and unlike the DF protocol, no decoding at the relay is performed. In the two-way AF relaying strategy, the signals at the relay are actually combined on the symbol level. Due to amplification of noise, its performance degrades at low signal-to-noise ratios

(SNR). The two-way DF relaying strategy was proposed in [4], where the relay decodes the received data bits from the both nodes. Since the decoded data at the relay can be combined on the symbol level or on the bit level, there has been different data combining schemes at the relay for the two-way DF relaying strategy: superposition coding, network coding, and lattice coding [2]. In the superposition coding scheme, applied in [4], the data from the two nodes are combined on the symbol level, where the relay sends the linear sum of the decoded symbols from both nodes. Shortly after the introduction of the two-way relay channel, its connection to network coding [9] was observed and investigated. The network coding schemes combine the data from nodes on the bit level using the XOR operation, see e.g., [10], [11], [12], [13], [14], [15]. Lattice coding uses modulo addition in a multi-dimensional space and utilizes nonlinear operations for combining the data. Applying lattice coding in two-way relaying systems was considered in, e.g., [16], [17], [18]. In general, as in CF or partial DF relaying strategies, the relay node need not to decode the source messages, but only need to pass sufficient information to the destination nodes.

In this paper, we focus on the Gaussian TRC (GTRC) as shown in Fig. 1, where nodes 1 and 2 want to communicate with each other at rates R_1 and R_2 , respectively, while a relay node facilitates the communication between them. TRC with a direct link between nodes 1 and 2 is studied in [19], [20], [21], [22], [23]. Here, we consider a GTRC without a direct link between nodes 1 and 2. Our model is an extension of [18] and it is essentially the same as those considered in [5], [24], [17], [25], [26], [27]. Similar to [18], [17], we apply the nested lattice coding. For a comprehensive review on lattices and their performance analysis, we refer the reader to the study of [28], [29], [30]. Nested lattice codes have been shown to be capacity achieving for the AWGN channel [31], [32], AWGN broadcast channel [32] and the AWGN multiple access channel [1]. Song and Devroye [33], using list decoding technique, show that lattice codes can achieve the DF rate of single source, single destination Gaussian relay channels with one or more relays. It is also shown that the lattice CF scheme, which exploits a lattice Wyner-Ziv binning, for the Gaussian relay channel achieves the same rate as the Cover-El Gamal CF achievable rate [33]. Nazer and Gastpar by introducing compute-and-forward scheme obtain significantly higher rates between users than previous results in the literature over a relay network [1]. By utilizing a deterministic approach [34] and a simple AF or a particular superposition coding strategy at the relay, [6] achieves full-duplex GTRC capacity region within 3 bits from the cut-set bound for all values of channel gains. A general GTRC is considered in [17], and it is shown that regardless of all channel parameters, the capacity region for each user is achievable within $\frac{1}{2}$ bit and the sum rate is within $\log \frac{3}{2}$ bits from the sum capacity, which was the best gap-to-capacity. Communication over a symmetric GTRC when all source and relay nodes have the same transmit powers and noise variances, is considered in [35]. It is shown that lattice codes and lattice decoding at the relay node can achieve capacity region at the high SNR. Lim et.al, using noisy network coding for GTRC, show that the achievable rate of their scheme for each user is within $\frac{1}{2}$ bit from the cut-set bound. [36]. The achievable sum rate in [36] is within 1 bit from the cut-set bound.

Here, we apply nested lattice codes at the relay to obtain linear combination of messages. By comparing with the cut-set outer bound, we show that the achievable rate region, regardless of all channel parameters, is within 0.2654 bit from the outer bound for user 1. For user 2, the proposed scheme achieves within 0.167 bit from the outer bound if the channel coefficient is larger than one, and achieves within 0.2658 bit from the outer bound if channel coefficient (g) is smaller than one. The achievable sum rate by the proposed scheme is within 0.334 bit from the cut-set bound. Thus, the proposed scheme outperforms [17], [6], [36] and the resulting gap is the best gap-to-capacity result to date.

The remainder of the paper is organized as follows. We present the preliminaries of lattice codes and the channel model in Section II. In Section III, we introduce two achievable coding schemes. Scheme 1 is based on compute-and-forward strategy while scheme 2 utilizes two different lattices for source nodes based on a three-stage lattice partition chain. In Section IV, we analyze the gap between the achievable rate region and the cut-set outer bound for each user. Numerical results and performance comparison between two schemes and the outer bound are presented in Section V. Section VI concludes the paper.

II. PRELIMINARIES: LATTICES AND CHANNEL MODEL

A. Notations and Channel Model

Throughout the paper, random variables and their realizations are denoted by capital and small letters, respectively. \mathbf{x} stands for a vector of length n , (x_1, x_2, \dots, x_n) . Also, $\|\cdot\|$ denotes the Euclidean norm, and all logarithms are with respect to base 2.

In this paper, we consider a Gaussian two-way relay channel (GTRC), with two sources that exchange messages through a relay. We assume that there is no direct link between the sources and all nodes operate in a full-duplex mode. The system model is depicted in Fig. 1. Communication process takes place in two phases: MAC phase BRC phase, which are described in the following:

- **MAC phase:** In this phase, first the input message to both encoders, W_1, W_2 , are mapped to

$$X_i^{(t)} = f_i(W_i, Y_i^{t-1}), \quad \text{for } i = 1, 2$$

where f_i is the encoder function at node i and $Y_i^{t-1} = \{Y_i^{(1)}, Y_i^{(2)}, \dots, Y_i^{(t-1)}\}$ is the set of past channel outputs at node i . Without loss of generality, we assume that both transmitted sequences $X_1^{(t)}, X_2^{(t)}$ are average-power limited to $P > 0$, i.e.,

$$\frac{1}{n} \sum_{t=1}^n \mathbb{E} \left[|X_i^{(t)}|^2 \right] \leq P, \quad \text{for } i = 1, 2. \quad (1)$$

Both nodes send their signals to the relay. The received signal at the relay at time t is specified by

$$Y_R^{(t)} = X_1^{(t)} + \sqrt{g}X_2^{(t)} + Z_R^{(t)},$$

where $X_1^{(t)}$ and $X_2^{(t)}$ are the signals transmitted from node 1 and node 2 at time t , respectively. \sqrt{g} denotes the channel gain between node 2 and the relay and all other channel gains are assumed to be one. $Z_R^{(t)}$ represents an independent identically distributed (i.i.d.) Gaussian random variable with mean zero and variance N_R , which models an additive white Gaussian noise (AWGN) at the relay.

- **BRC phase:** During the broadcast phase, the relay node processes the received signal and retransmits the combined signals back to both nodes, i.e., the relay communicates signal X_R to both nodes 1 and 2. Since the relay has no messages of its own, the relay signal at time t , $X_R^{(t)}$, is a function of the past relay inputs, i.e.,

$$X_R^{(t)} = f_R(Y_R^{t-1}),$$

where f_R is an encoding function at the relay and $Y_R^{t-1} = \{Y_R^{(1)}, Y_R^{(2)}, \dots, Y_R^{(t-1)}\}$ is a sequence of past relay inputs. The power constraint at the relay is given by

$$\frac{1}{n} \sum_{t=1}^n \mathbb{E} \left[|X_R^{(t)}|^2 \right] \leq P_R.$$

The received signal at each node at time t is given by

$$Y_i^{(t)} = X_R^{(t)} + Z_i^{(t)}, \quad \text{for } i = 1, 2$$

where $X_R^{(t)}$ is the transmitted signal from the relay node at time t and $Z_i^{(t)}$ represents an i.i.d AWGN with zero mean and variance N_i .

Node 1, based on the received sequence, \mathbf{Y}_1 , and its own message, W_1 , makes an estimate of the other message, W_2 , as

$$\hat{W}_2 = \psi_1(\mathbf{Y}_1, W_1),$$

where ψ_1 is a decoding function for at node 1. Decoding at node 2 is performed in a similar way. The average probability of error is defined as

$$P_e = \Pr \left\{ \hat{W}_1 \neq W_1 \text{ or } \hat{W}_2 \neq W_2 \right\}.$$

A rate pair (R_1, R_2) of non-negative real values is achievable if there exists encoding and decoding functions with $P_e \rightarrow 0$ as $n \rightarrow \infty$ [37]. The capacity region of the GTRC is the convex closure of the set of achievable rate pairs (R_1, R_2) .

Note that the model depicted in Fig. 1 is referred to as the symmetric model if all source and relay nodes have the same transmit powers, noise variances, i.e., $P = P_R$ and $N_1 = N_2 = N_R$.

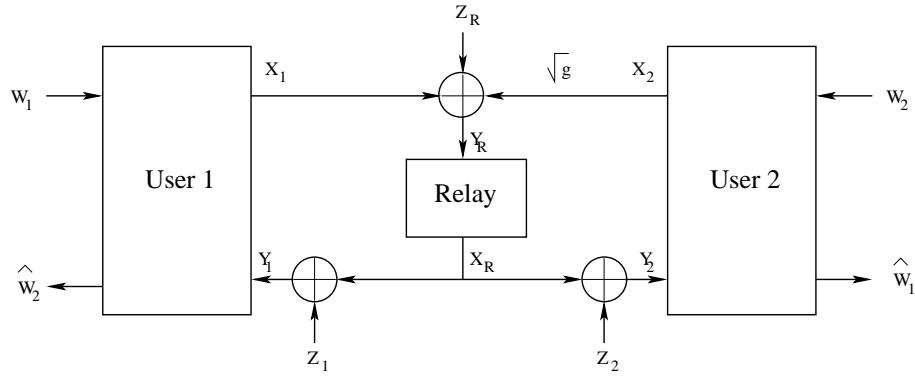


Figure 1. System Model. Two communication phases for a Gaussian two-way relay channel: MAC phase and Broadcast phase.

B. Lattice Definitions

Here, we provide some necessary definitions on lattices and nested lattice codes [31], [1], [38].

Definition 1. (Lattice): An n -dimensional lattice Λ is a set of points in Euclidean space \mathbb{R}^n such that, if $\mathbf{x}, \mathbf{y} \in \Lambda$, then $\mathbf{x} + \mathbf{y} \in \Lambda$, and if $\mathbf{x} \in \Lambda$, then $-\mathbf{x} \in \Lambda$. A lattice Λ can always be written in terms of a generator matrix $\mathbf{G} \in \mathbb{Z}^{n \times n}$ as

$$\Lambda = \{\mathbf{x} = \mathbf{z}\mathbf{G} : \mathbf{z} \in \mathbb{Z}^n\},$$

where \mathbb{Z} represents integers.

Definition 2. (Quantizer): The *nearest neighbor quantizer* $\mathcal{Q}(\cdot)$ associated with the lattice Λ is

$$\mathcal{Q}_\Lambda(\mathbf{x}) = \arg \min_{\mathbf{l} \in \Lambda} \|\mathbf{x} - \mathbf{l}\|.$$

Definition 3. (Voronoi Region): The *fundamental Voronoi region* of a lattice Λ is set of points in \mathbb{R}^n closest to the zero codeword, i.e.,

$$\mathcal{V}_0(\Lambda) = \{\mathbf{x} \in \mathbb{R}^n : \mathcal{Q}(\mathbf{x}) = 0\}.$$

Definition 4. (Moments): $\sigma^2(\Lambda)$ which is called the second moment of lattice Λ is given by

$$\sigma^2(\Lambda) = \frac{1}{n} \frac{\int_{\mathcal{V}(\Lambda)} \|\mathbf{x}\|^2 d\mathbf{x}}{\int_{\mathcal{V}(\Lambda)} d\mathbf{x}}. \quad (2)$$

and the *normalized second moment* of lattice Λ is

$$G(\Lambda) = \frac{\sigma^2(\Lambda)}{[\int_{\mathcal{V}(\Lambda)} d\mathbf{x}]^{\frac{2}{n}}} = \frac{\sigma^2(\Lambda)}{V^{\frac{2}{n}}},$$

where $V = \int_{\mathcal{V}(\Lambda)} d\mathbf{x}$ is the Voronoi region volume, i.e., $V = \text{Vol}(\mathcal{V})$.

Definition 5. (Modulus): The *modulo- Λ operation* with respect to lattice Λ is defined as

$$\mathbf{x} \bmod \Lambda = \mathbf{x} - \mathcal{Q}(\mathbf{x}),$$

that maps \mathbf{x} into a point in the fundamental Voronoi region.

Definition 6. (Quantization Goodness or Rogers-good): A sequence of lattices $\Lambda^{(n)} \subseteq \mathbb{R}^n$ is good for mean-squared error

(MSE) quantization if

$$\lim_{n \rightarrow \infty} G(\Lambda^{(n)}) = \frac{1}{2\pi e}.$$

The sequence is indexed by the lattice dimension n . The existence of such lattices is shown in [39], [28].

Definition 7. (AWGN channel coding goodness or Poltyrev-good): Let \mathbf{Z} be a length- $i.i.d$ Gaussian vector, $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \sigma_Z^2 \mathbf{I}_n)$. The volume-to-noise ratio of a lattice is given by

$$\mu(\Lambda, \epsilon) = \frac{(\text{Vol}(\mathcal{V}))^{2/n}}{\sigma_Z^2},$$

where σ_Z^2 is chosen such that $\Pr\{\mathbf{Z} \notin \mathcal{V}\} = \epsilon$ and \mathbf{I}_n is an $n \times n$ identity matrix. A sequence of lattices is $\Lambda^{(n)}$ Poltyrev-good if

$$\lim_{n \rightarrow \infty} \mu(\Lambda^{(n)}, \epsilon) = 2\pi e, \quad \forall \epsilon \in (0, 1)$$

and, for fixed volume-to-noise ratio greater than $2\pi e$, $\Pr\{\mathbf{Z} \notin \mathcal{V}^n\}$ decays exponentially in n .

Poltyrev showed that sequences of such lattices exist [40]. The existence of a sequence of lattices $\Lambda^{(n)}$ which are good in both senses (i.e., simultaneously are Poltyrev-good and Rogers-good) has been shown in [28].

Definition 8. (Nested Lattices): A lattice $\Lambda^{(n)}$ is said to be nested in lattice $\Lambda_1^{(n)}$ if $\Lambda^{(n)} \subseteq \Lambda_1^{(n)}$. $\Lambda^{(n)}$ is referred to as the coarse lattice and $\Lambda_1^{(n)}$ as the fine lattice.

(Nested Lattice Codes): A nested lattice code is the set of all points of a fine lattice $\Lambda_1^{(n)}$ that are within the fundamental Voronoi region \mathcal{V} of a coarse lattice $\Lambda^{(n)}$,

$$\mathcal{C} = \{\Lambda_1 \cap \mathcal{V}\}.$$

The rate of a nested lattice code is

$$R = \frac{1}{n} \log |\mathcal{C}| = \frac{1}{n} \log \frac{\text{Vol}(\mathcal{V})}{\text{Vol}(\mathcal{V}_1)}.$$

The existence of nested lattices where the coarse lattice as well as the fine lattice are good in both senses has also been shown in [1], [41]. An interesting property of these codes is that any integer combinations of transmitted codewords are themselves codewords.

In the following, we present a key property of dithered nested lattice codes.

Lemma 1. The Crypto Lemma [30], [31] Let \mathbf{V} be a random vector with an arbitrary distribution over \mathbb{R}^n . If \mathbf{D} is independent of \mathbf{V} and uniformly distributed over \mathcal{V} , then $(\mathbf{V} + \mathbf{D}) \bmod \Lambda$ is also independent of \mathbf{V} and uniformly distributed over \mathcal{V} .

Proof: See Lemma 2 in [30]. ■

III. NESTED LATTICE CODES

A. Relay Strategies

In this Section, we introduce two strategies for processing and transmitting at the relay. In both schemes, we recover a linear combination of messages instead of separate recovery of messages at the relay.

1) *Scheme 1: Compute-and-Forward:* The compute-and-forward strategy is proposed in [1]. In this scheme, the goal is recover an integer linear combination of codewords, \mathbf{V}_1 and \mathbf{V}_2 , i.e., we estimate

$$\mathbf{V}_1 + a\mathbf{V}_2, \tag{3}$$

where $a \in \mathbb{Z}$. Since the transmitted sequences are from lattice codes, it guarantee that any integer linear combination of the codewords is a codeword. However, at the receiver, the received signal which is a linear combination of the transmitted codewords is no longer integer since the channel coefficients are real (or complex). Also, the received signal is corrupted by

noise. As a solution, Nazer and Gastpar [1] propose to scale the received signal by a factor such that the obtained vector is made as close as possible to an integer linear combination of the transmitted codewords.

To reach to this goal, for n large enough, we assume that there exist three lattices $\Lambda_1^{(n)}$, $\Lambda_2^{(n)}$ and $\Lambda^{(n)}$ such that $\Lambda^{(n)} \subseteq \Lambda_1^{(n)} \subseteq \Lambda_2^{(n)}$. $\Lambda_1^{(n)}$ and $\Lambda_2^{(n)}$ are both Poltyrev-good and Rogers-good and $\Lambda^{(n)}$ is Rogers-good with the second moment

$$\sigma^2(\Lambda) = P.$$

We denote the Voronoi regions of $\Lambda_1^{(n)}$, $\Lambda_2^{(n)}$ and $\Lambda^{(n)}$ with \mathcal{V}_1 , \mathcal{V}_2 and \mathcal{V} , respectively.

Encoding: We choose two codebooks \mathcal{C}_1 and \mathcal{C}_2 , such that

$$\begin{aligned} \mathcal{C}_1 &= \{\Lambda_1 \cap \mathcal{V}\}, \\ \mathcal{C}_2 &= \{\Lambda_2 \cap \mathcal{V}\}. \end{aligned}$$

Now, for each input node i , the message set $\{1, \dots, 2^{nR_i}\}$ is arbitrarily one-to-one mapped onto \mathcal{C}_i . We also define two random dither vectors $\mathbf{D}_i \sim \text{Unif}(\mathcal{V})$ for $i = 1, 2$. Dither vectors are independent of each other and also independent of the message of each node and the noise. Dither \mathbf{D}_i is known to both the input nodes and the relay node. To transmit a message, node i chooses $\mathbf{V}_i \in \mathcal{C}_i$ associated with the message and sends

$$\mathbf{X}_i = [\mathbf{V}_i - \mathbf{D}_i] \bmod \Lambda.$$

Note that by the crypto lemma, \mathbf{X}_i is uniformly distributed over \mathcal{V} and independent of \mathbf{V}_i . Thus, the average transmit power at node is equal to $\sigma^2(\Lambda) = P$, so that the power constraint is satisfied.

Decoding: Upon receiving \mathbf{Y}_R , the relay node computes

$$\begin{aligned} \mathbf{Y}_{dR} &= [\alpha \mathbf{Y}_R + \mathbf{D}_1 + a \mathbf{D}_2] \bmod \Lambda, \\ &= [\alpha \mathbf{X}_1 + \alpha \sqrt{g} \mathbf{X}_2 + \alpha \mathbf{Z}_R + \mathbf{D}_1 + a \mathbf{D}_2] \bmod \Lambda, \\ &\stackrel{(a)}{=} [\mathbf{V}_1 + a \mathbf{V}_2 + \alpha \mathbf{X}_1 - (\mathbf{V}_1 - \mathbf{D}_1) + \alpha \sqrt{g} \mathbf{X}_2 - a(\mathbf{V}_2 - \mathbf{D}_2) + \alpha \mathbf{Z}_R] \bmod \Lambda, \\ &\stackrel{(b)}{=} [\mathbf{V}_1 + a \mathbf{V}_2 + (\alpha - 1) \mathbf{X}_1 + (\alpha \sqrt{g} - a) \mathbf{X}_2 + \alpha \mathbf{Z}_R] \bmod \Lambda, \\ &= [\mathbf{T} + \mathbf{Z}_{\text{eff}}] \bmod \Lambda, \end{aligned}$$

where (a) follows by adding and subtracting the term $\mathbf{V}_1 + a \mathbf{V}_2$, and (b) follows from the distributive law of modulo- Λ operation [29], [1], i.e.,

$$[[\mathbf{X} \bmod \Lambda] + \mathbf{Y}] \bmod \Lambda = [\mathbf{X} + \mathbf{Y}] \bmod \Lambda,$$

$\mathbf{X}, \mathbf{Y} \in \mathbb{R}^n$. The effective noise is given by

$$\mathbf{Z}_{\text{eff}} = (\alpha - 1) \mathbf{X}_1 + (\alpha \sqrt{g} - a) \mathbf{X}_2 + \alpha \mathbf{Z}_R,$$

and

$$\mathbf{T} = [\mathbf{V}_1 + a \mathbf{V}_2] \bmod \Lambda.$$

Since $\mathbf{V}_1, \mathbf{V}_2, \mathbf{X}_1, \mathbf{X}_2$ are independent, and also independent of \mathbf{Z}_R , and using Crypto lemma, \mathbf{Z}_{eff} is independent of \mathbf{V}_1 and \mathbf{V}_2 and thus independent of \mathbf{T} . The relay aims to recover \mathbf{T} from \mathbf{Y}_{dR} instead of recovering \mathbf{V}_1 and \mathbf{V}_2 individually. Due to the lattice chain, i.e., $\Lambda^{(n)} \subseteq \Lambda_1^{(n)} \subseteq \Lambda_2^{(n)}$, \mathbf{T} is a point from Λ_2^n . To get an estimate of \mathbf{T} , this vector is quantized onto Λ_2^n modulo the lattice Λ^n :

$$\begin{aligned} \hat{\mathbf{T}} &= \mathcal{Q}_{\Lambda_2}(\mathbf{Y}_{dR}) \bmod \Lambda, \\ &= \mathcal{Q}_{\Lambda_2}(\mathbf{T} + \mathbf{Z}_{\text{eff}}) \bmod \Lambda, \end{aligned}$$

where \mathcal{Q}_{Λ_2} denotes the nearest neighbor lattice quantizer associated with Λ_2 . Thus, the decoding error probability at the relay vanishes as $n \rightarrow \infty$ if

$$\Pr(\mathbf{Z}_{\text{eff}} \notin \mathcal{V}_2) \rightarrow 0. \quad (4)$$

By using Lemma 8 in [1], we know that the density of \mathbf{Z}_{eff} can be upper bounded by the density of $\mathbf{Z}_{\text{eff}}^*$, which is an *i.i.d* zero-mean Gaussian vector whose variance approaches

$$N_{eq} = \left((\alpha\sqrt{g} - a)^2 + (\alpha - 1)^2 \right) P + \alpha^2 N_R,$$

as $n \rightarrow \infty$. From Definition 7, this means that $\Pr(\mathbf{Z}_{\text{eff}}^* \notin \mathcal{V}_2) \rightarrow 0$ so long as the volume-to-noise ratio satisfies

$$\mu(\Lambda_2) = \frac{(\text{Vol}(\mathcal{V}_2))^{2/n}}{N_{eq}} > 2\pi e.$$

Therefore, for the volume of each Voronoi region, we have:

$$\text{Vol}(\mathcal{V}_i) > (2\pi e N_{eq})^{n/2} \quad i = 1, 2. \quad (5)$$

For the volume of the fundamental Voronoi region of $\Lambda^{(n)}$, we have:

$$\text{Vol}(\mathcal{V}) = \left(\frac{P}{G(\Lambda)} \right)^{n/2}. \quad (6)$$

Now, by using (5) and (6) and definition of the rate of a nested lattice code, we can achieve the following rate for each node:

$$R_i < \frac{1}{2} \log \left(\frac{P}{G(\Lambda) 2\pi e N_{eq}} \right), \quad i = 1, 2.$$

Since Λ is Rogers-good, $G(\Lambda) 2\pi e \rightarrow 1$ as $n \rightarrow \infty$. Thus,

$$R_i < \frac{1}{2} \log \left(\frac{P}{\left((\alpha\sqrt{g} - a)^2 + (\alpha - 1)^2 \right) P + \alpha^2 N_R} \right), \quad i = 1, 2. \quad (7)$$

Now, we choose α as the minimum mean-square error (MMSE) coefficient that minimizes the variance of the effective noise, N_{eq} . Thus, we get

$$\alpha_{\text{MMSE}} = \frac{(a\sqrt{g} + 1)P}{(g + 1)P + N_R}. \quad (8)$$

By inserting (7) in (8), we can achieve any rate satisfying

$$R_i < \frac{1}{2} \log \left(\frac{P}{\left((\alpha_{\text{MMSE}}\sqrt{g} - a)^2 + (\alpha_{\text{MMSE}} - 1)^2 \right) P + \alpha_{\text{MMSE}}^2 N_R} \right). \quad (9)$$

i.e., it is possible to decode \mathbf{T} within arbitrarily low error probability, if the coding rates of the nested lattice codes associated with the lattice partition $\Lambda^{(n)} \subseteq \Lambda_1^{(n)}$ and $\Lambda^{(n)} \subseteq \Lambda_2^{(n)}$ satisfy (9). In this scheme, we must only obtain an integer linear combination of messages. Since we want to obtain an estimate of $\mathbf{V}_1 + a\mathbf{V}_2$, a should be an integer. On the other hand, we aim to obtain higher achievable rates as much as we can. To reach this goal, we choose a as the closest integer to \sqrt{g} i.e., $a = \lceil \sqrt{g} \rceil$.

2) *Scheme 2: Our proposed scheme:* Let us first consider a theorem that is a key to our code construction.

Theorem 1. [42] For any $P_1 \geq P_2 \geq 0$, a sequence of n -dimensional lattice partition chains $\Lambda_C^{(n)}/\Lambda_2^{(n)}/\Lambda_1^{(n)}$, i.e., $\Lambda_1^{(n)} \subseteq \Lambda_2^{(n)} \subseteq \Lambda_C^{(n)}$, exists that satisfies the following properties:

- $\Lambda_1^{(n)}$ and $\Lambda_2^{(n)}$ are simultaneously Rogers-good and Poltyrev-good while $\Lambda_C^{(n)}$ is Poltyrev-good.
- For any $\epsilon > 0$, $P_i - \epsilon \leq \sigma^2(\Lambda_i^{(n)}) \leq P_i$, $i \in \{1, 2\}$ for sufficiently large n .

- The coding rate of the nested lattice code associated with the lattice partition $\Lambda_C^{(n)}/\Lambda_1^{(n)}$ is

$$R_1 = \frac{1}{n} \log(|\mathcal{C}_1|) = \frac{1}{n} \log\left(\frac{\text{Vol}(\mathcal{V}_1)}{\text{Vol}(\mathcal{V}_c)}\right) = \gamma + o_n(1),$$

where $\mathcal{C}_1 = \{\Lambda_C^{(n)} \cap \mathcal{V}_1\}$ and $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$. The coding rate of the nested lattice code associated with $\Lambda_C^{(n)}/\Lambda_2^{(n)}$ is given by

$$R_2 = \frac{1}{n} \log(|\mathcal{C}_2|) = \frac{1}{n} \log\left(\frac{\text{Vol}(\mathcal{V}_2)}{\text{Vol}(\mathcal{V}_c)}\right) = R_1 + \frac{1}{2} \log\left(\frac{P_2}{P_1}\right),$$

where $\mathcal{C}_2 = \{\Lambda_C^{(n)} \cap \mathcal{V}_2\}$.

Proof: The proof of theorem is given in [42]. ■

In the following, by applying a lattice-based coding scheme, we obtain achievable rate region at the relay. Suppose that there exist three lattices $\Lambda_1^{(n)}$, $\Lambda_2^{(n)}$ and $\Lambda_3^{(n)} = \sqrt{g}\Lambda_2^{(n)}$, which are Rogers-good (i.e., $\lim_{n \rightarrow \infty} G(\Lambda_i^{(n)}) = \frac{1}{2\pi e}$, for $i = 1, 2, 3$), and Poltyrev-good with the following second moments

$$\sigma^2(\Lambda_1) = P, \text{ and } \sigma^2(\Lambda_3) = gP,$$

and a lattice $\Lambda_C^{(n)}$ which is Poltyrev-good with

$$\begin{aligned} \Lambda_1^{(n)} &\subseteq \Lambda_C^{(n)}, \\ \Lambda_3^{(n)} &\subseteq \Lambda_C^{(n)}. \end{aligned}$$

Encoding: To transmit both messages, we construct the following codebooks:

$$\begin{aligned} \mathcal{C}_1 &= \{\Lambda_C \cap \mathcal{V}_1\}, \\ \mathcal{C}_2 &= \left\{ \frac{\Lambda_C}{\sqrt{g}} \cap \mathcal{V}_2 \right\}. \end{aligned}$$

Then node i chooses $\mathbf{V}_i \in \mathcal{C}_i$ associated with the message W_i and sends

$$\mathbf{X}_i = [\mathbf{V}_i - \mathbf{D}_i] \bmod \Lambda_i,$$

where \mathbf{D}_1 and \mathbf{D}_2 are two independent dithers that are uniformly distributed over Voronoi regions \mathcal{V}_1 and \mathcal{V}_2 , respectively. Dithers are known at the source nodes and the relay. Due to the crypto-lemma, \mathbf{X}_i is uniformly distributed over \mathcal{V}_i and independent of \mathbf{V}_i . Thus, the average transmit power of node i is equal to P , and the power constraint is met.

Decoding: At the relay node, based on the channel output that is given by

$$\mathbf{Y}_R = \mathbf{X}_1 + \sqrt{g}\mathbf{X}_2 + \mathbf{Z}_R, \tag{10}$$

we estimate $\mathbf{V}_1 + \sqrt{g}\mathbf{V}_2$. Depend on the value of g , we consider two cases:

Case (I): $g \geq 1$: Based on Theorem 1, we can find two lattices, $\Lambda_1^{(n)}$ and $\Lambda_3^{(n)}$, such that $\Lambda_3^{(n)} \subseteq \Lambda_1^{(n)}$. With this selection of lattices, the relay node performs the following operation:

$$\begin{aligned} \mathbf{Y}_{dR} &= [\alpha\mathbf{Y}_R + \mathbf{D}_1 + \sqrt{g}\mathbf{D}_2] \bmod \Lambda_1 \\ &= [\alpha\mathbf{X}_1 + \alpha\sqrt{g}\mathbf{X}_2 + \alpha\mathbf{Z}_R + \mathbf{D}_1 + \sqrt{g}\mathbf{D}_2] \bmod \Lambda_1 \\ &= [\mathbf{V}_1 + \sqrt{g}\mathbf{V}_2 + \alpha\mathbf{X}_1 - (\mathbf{V}_1 - \mathbf{D}_1) + \alpha\sqrt{g}\mathbf{X}_2 - \sqrt{g}(\mathbf{V}_2 - \mathbf{D}_2) + \alpha\mathbf{Z}_R] \bmod \Lambda_1 \\ &\stackrel{(c)}{=} [[\mathbf{V}_1 + \sqrt{g}\mathbf{V}_2] \bmod \Lambda_3 + (\alpha - 1)\mathbf{X}_1 + \sqrt{g}(\alpha - 1)\mathbf{X}_2 + \alpha\mathbf{Z}_R] \bmod \Lambda_1 \\ &= [\mathbf{T} + \mathbf{Z}_{\text{eff}}] \bmod \Lambda_1, \end{aligned} \tag{11}$$

where (c) follows from $\Lambda_3^{(n)} \subseteq \Lambda_1^{(n)}$ and $\Lambda_3^{(n)} = \sqrt{g}\Lambda_2^{(n)}$ and the distributive law of modulo- Λ operation. The effective noise is given by

$$\mathbf{Z}_{\text{eff}} = [(\alpha - 1)\mathbf{X}_1 + \sqrt{g}(\alpha - 1)\mathbf{X}_2 + \alpha\mathbf{Z}_R] \bmod \Lambda_1,$$

and

$$\mathbf{T} = [\mathbf{V}_1 + \sqrt{g}\mathbf{V}_2] \bmod \Lambda_3.$$

Due to the dithers, the vectors $\mathbf{V}_1, \mathbf{V}_2, \mathbf{X}_1, \mathbf{X}_2$ are independent, and also independent of \mathbf{Z}_R . Therefore, \mathbf{Z}_{eff} is independent of \mathbf{V}_1 and \mathbf{V}_2 . From the crypto-lemma, it follows that \mathbf{T} is uniformly distributed over $\sqrt{g}\mathcal{C}_2$ and independent of \mathbf{Z}_{eff} [42].

The problem of finding the optimum value for α when the lattice dimension goes to infinity, reduces to obtain the value of α that minimizes the effective noise variance. Hence, by minimizing variance of \mathbf{Z}_{eff} , we obtain

$$\alpha_{\text{MMSE}} = \frac{(g+1)P}{(g+1)P + N_R}. \quad (12)$$

The relay attempts to recover \mathbf{T} from \mathbf{Y}_{d_R} instead of recovering \mathbf{V}_1 and \mathbf{V}_2 individually. The method of decoding is minimum Euclidean distance lattice decoding [31], [43], [40], which finds the closest point to \mathbf{Y}_{d_R} in $\Lambda_C^{(n)}$. Thus, the estimate of \mathbf{T} is given by,

$$\hat{\mathbf{T}} = \mathcal{Q}_{\Lambda_C}(\mathbf{Y}_{d_R}).$$

Then, from the type of decoding, the probability of decoding error is given by

$$\begin{aligned} P_e &= \Pr\{\hat{\mathbf{T}} \neq \mathbf{T}\} \\ &= \Pr\{\mathbf{Z}_{\text{eff}} \notin \mathcal{V}_C\}. \end{aligned}$$

Now, we have the following theorem which bounds the error probability.

Theorem 2. For the described lattice partition chain and any rate R_1 satisfying

$$R_1 < R_1^* = \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N_R} \right) \right]^+,$$

the error probability under minimum Euclidean distance lattice decoding is bounded by

$$P_e = e^{-n(E_P(2^{2(R_1^* - R_1)})) - o_n(1)},$$

where $E_P(\cdot)$ is the Poltyrev exponent, which is given by [40]

$$E_P(x) = \begin{cases} \frac{x}{2} - \frac{1}{2}(x - 1 - \ln x), & 1 \leq x \leq 2 \\ \frac{1}{2}(1 + \ln \frac{x}{4}), & 2 \leq x \leq 4 \\ \frac{x}{8}, & x \geq 4 \end{cases} \quad (13)$$

and $[x]^+ = \max(0, x)$.

Proof: The proof of theorem is similar to the proof of theorem 3 in [42] and removed here. ■

Since $E_P(x) > 0$ for $x > 1$, the error probability vanishes as $n \rightarrow \infty$ if $R_1 < R_1^*$. Thus, by Theorem 1 and Theorem 2, the error probability at the relay node vanishes if

$$R_1 \leq \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N_R} \right) \right]^+, \quad (14)$$

$$R_2 \leq \left[\frac{1}{2} \log \left(\frac{g}{g+1} + \frac{gP}{N_R} \right) \right]^+. \quad (15)$$

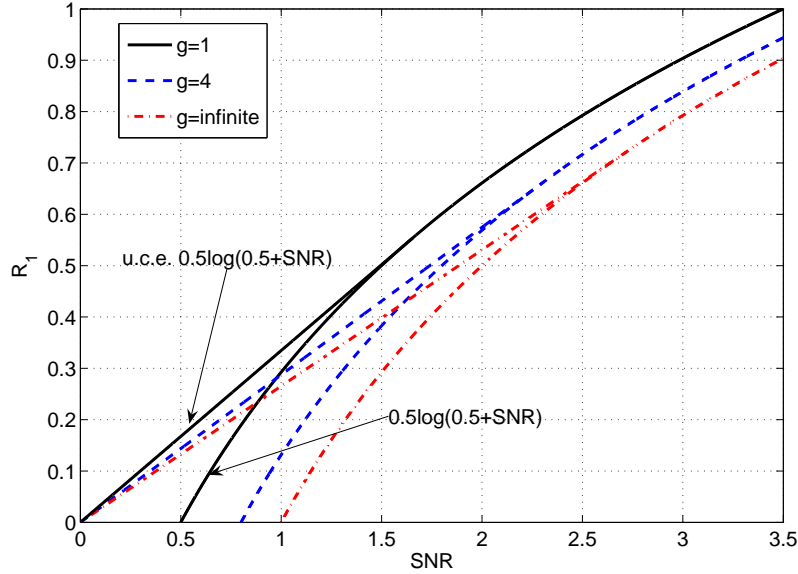


Figure 2. Comparison of the achievable rate R_1 for different values of g .

Clearly, using a time sharing argument the following rates can be achieved:

$$R_1 \leq \text{u.c.e} \left\{ \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N_R} \right) \right]^+ \right\},$$

$$R_2 \leq \text{u.c.e} \left\{ \left[\frac{1}{2} \log \left(\frac{g}{g+1} + \frac{gP}{N_R} \right) \right]^+ \right\},$$

where u.c.e is the upper convex envelope with respect to $\frac{P}{N_R}$.

At low SNR, i.e., $\text{SNR} \leq \frac{g}{g+1}$, pure (infinite dimensional) lattice-strategies cannot achieve any positive rates for R_1 as shown in Fig. 2. Hence, time sharing is required between the point $\text{SNR} = 0$ and SNR^* , which is a solution of the following equation:

$$f(\text{SNR}) = \frac{df(\text{SNR})}{d\text{SNR}} \text{SNR},$$

where $f(x) = \frac{1}{2} \log \left(\frac{1}{g+1} + x \right)$. We also evaluate numerically the achievable rates for R_1 with lattice strategies for different values of g . As we observe, with increasing g , the achievable rate with lattice scheme decreases. As it is shown in Fig. 2, the maximum difference between two extreme cases ($g = 1$ and $g = \infty$) is 0.1218 bit.

Case (II): $g < 1$: By using Theorem 1, we can choose two lattices $\Lambda_1^{(n)}$ and $\Lambda_3^{(n)}$ such that $\Lambda_1^{(n)} \subseteq \Lambda_3^{(n)}$. The relay calculates $\mathbf{Y}_{d_R} = [\alpha \mathbf{Y}_R + \mathbf{D}_1 + \sqrt{g} \mathbf{D}_2] \bmod \Lambda_3$. The equivalent channel is given by

$$\begin{aligned} \mathbf{Y}_{d_R} &= [\alpha \mathbf{X}_1 + \alpha \sqrt{g} \mathbf{X}_2 + \alpha \mathbf{Z}_R + \mathbf{D}_1 + \sqrt{g} \mathbf{D}_2] \bmod \Lambda_3 \\ &= [\mathbf{V}_1 + \sqrt{g} \mathbf{V}_2 + \alpha \mathbf{X}_1 - (\mathbf{V}_1 - \mathbf{D}_1) + \alpha \sqrt{g} \mathbf{X}_2 - \sqrt{g} (\mathbf{V}_2 - \mathbf{D}_2) + \alpha \mathbf{Z}_R] \bmod \Lambda_3 \\ &\stackrel{(d)}{=} [\mathbf{V}_1 + \sqrt{g} \mathbf{V}_2 + (\alpha - 1) \mathbf{X}_1 + \sqrt{g} (\alpha - 1) \mathbf{X}_2 + \alpha \mathbf{Z}_R] \bmod \Lambda_3 \\ &= [\mathbf{T} + \mathbf{Z}_{\text{eff}}] \bmod \Lambda_3, \end{aligned} \tag{16}$$

where (d) follows from $\Lambda_1^{(n)} \subseteq \Lambda_3^{(n)}$ and $\Lambda_3^{(n)} = \sqrt{g}\Lambda_2^{(n)}$ and the distributive law of modulo- Λ operation. The effective noise is given by

$$\mathbf{Z}_{\text{eff}} = [(\alpha - 1)\mathbf{X}_1 + \sqrt{g}(\alpha - 1)\mathbf{X}_2 + \alpha\mathbf{Z}_R] \bmod \Lambda_3,$$

and

$$\mathbf{T} = [\mathbf{V}_1 + \sqrt{g}\mathbf{V}_2] \bmod \Lambda_1.$$

Due to the dithers, the vectors $\mathbf{V}_1, \mathbf{V}_2, \mathbf{X}_1, \mathbf{X}_2$ are independent, and also independent of \mathbf{Z}_R . Therefore, \mathbf{Z}_{eff} is independent of \mathbf{V}_1 and \mathbf{V}_2 . From the crypto-lemma, it follows that \mathbf{T} is uniformly distributed over \mathcal{C}_1 and independent of \mathbf{Z}_{eff} [42]. In order to achieve the maximal rate, the optimal MMSE factor is used, i.e.,

$$\alpha = \alpha_{\text{MMSE}} = \frac{(g+1)P}{(g+1)P + N_R}. \quad (17)$$

Similar to case $g \geq 1$, instead of recovering \mathbf{V}_1 and \mathbf{V}_2 separately, the relay recovers \mathbf{T} . Again, the decoding method is minimum Euclidean distance lattice decoding, which finds the closest point to \mathbf{Y}_{d_R} in $\Lambda_C^{(n)}$. Thus, the estimate of \mathbf{T} is given by,

$$\hat{\mathbf{T}} = \mathcal{Q}_{\Lambda_C}(\mathbf{Y}_{d_R}).$$

Theorem 3. *For the described lattice partition chain, if*

$$R_2 < R_2^* = \left[\frac{1}{2} \log \left(\frac{g}{g+1} + \frac{gP}{N_R} \right) \right]^+,$$

the error probability under minimum Euclidean distance lattice decoding is bounded by

$$P_e = e^{-n(E_P(2^{2(R_2^* - R_2)})) - o_n(1)},$$

where $E_P(\cdot)$ is the Poltyrev exponent given in (13).

Proof: The proof of theorem is similar to the proof of theorem 3 in [42] and removed here. ■

Here also the error probability vanishes as $n \rightarrow \infty$ if $R_2 < R_2^*$ since $E_P(x) > 0$ for $x > 1$. Thus, by Theorem 1 and Theorem 3, the error probability at the relay vanishes if

$$R_1 \leq \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N_R} \right) \right]^+,$$

$$R_2 \leq \left[\frac{1}{2} \log \left(\frac{g}{g+1} + \frac{gP}{N_R} \right) \right]^+,$$

Clearly, using a time sharing argument the following rates can be achieved:

$$R_1 \leq \text{u.c.e} \left\{ \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N_R} \right) \right]^+ \right\}, \quad (18)$$

$$R_2 \leq \text{u.c.e} \left\{ \left[\frac{1}{2} \log \left(\frac{g}{g+1} + \frac{gP}{N_R} \right) \right]^+ \right\}, \quad (19)$$

where u.c.e is the upper convex envelope with respect to $\frac{P}{N}$.

For $\text{SNR} \leq \frac{1}{g(g+1)}$, pure (infinite dimensional) lattice-strategies cannot achieve any positive rate for R_2 as shown in Fig. 3. Hence, time sharing is required between the point $\text{SNR} = 0$ and SNR^* , which is a solution of the following:

$$f(\text{SNR}) = \frac{df(\text{SNR})}{d\text{SNR}} \text{SNR},$$

where $f(x) = \frac{1}{2} \log \left(\frac{g}{g+1} + gx \right)$. We also evaluate numerically the achievable rate R_2 of lattice strategy for different values of g . As we see with decreasing g , the achievable rate with lattice scheme is decreased.

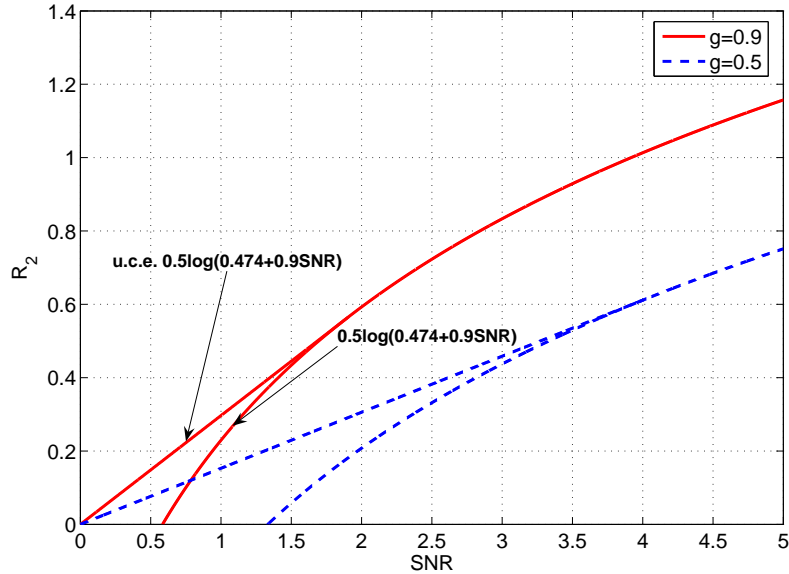


Figure 3. Comparison of the achievable symmetric rate for different values of g

B. Broadcast Phase

We assume that the relay can recover the linear combination of both messages correctly, i.e., there is no error in the MAC phase, $\hat{\mathbf{T}} = \mathbf{T}$. The relay attempts to broadcast a message such that each node can recover the other node's message based on both the received signal from the relay node and the available side information at each node, i.e., its own message. For the decoding at node 1 and node 2, we can use jointly typical decoding or lattice based scheme. Here, we apply jointly typical decoding. We consider scheme 2; decoding for scheme 1 is similar to scheme 2. For scheme 2, we also assume that $g \geq 1$. Under this assumption, we have $R_2 \geq R_1$. Now, we generate 2^{nR_2} -sequences with each element *i.i.d.* according to $\mathcal{N} \sim (0, P_R)$. These sequences form a codebook \mathcal{C}_R . We assume that there is a one-to-one correspondence between \mathcal{C}_2 and \mathcal{C}_R .

Let us denote the relay codeword by $\mathbf{X}_R(\mathbf{T}')$. Based on $\mathbf{Y}_2 = \mathbf{X}_R + \mathbf{Z}_2$, node 2 estimates the relay message \mathbf{T}' as $\hat{\mathbf{T}}'_2$ if a unique codeword $\mathbf{X}_R(\hat{\mathbf{T}}'_2) \in \mathcal{C}_{R,2}$ exists such that $(\mathbf{X}_R(\hat{\mathbf{T}}'_2), \mathbf{Y}_2)$ are jointly typical, where

$$\mathcal{C}_{R,2} = \left\{ \mathbf{X}_R(\mathbf{T}') : \mathbf{T}' = [\mathbf{V}_1 + \sqrt{g}\mathbf{v}_2] \bmod \Lambda_3, \mathbf{v}_2 \in \mathcal{C}_2 \right\}.$$

Note that $|\mathcal{C}_{R,2}| = 2^{nR_1}$. Now, by using the knowledge of \mathbf{V}_2 and $\hat{\mathbf{T}}'_2$, node 2 estimates the message of node 1 as:

$$\hat{\mathbf{V}}_1 = [\hat{\mathbf{T}}'_2 - \sqrt{g}\mathbf{v}_2] \bmod \Lambda_3.$$

From the argument of random coding and jointly typical decoding [37], we get

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{P_R}{N_2} \right). \quad (20)$$

Similarly, at node 1, we get

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{P_R}{N_1} \right). \quad (21)$$

Now, we summarize our results for both schemes in the following two theorem:

Theorem 4. For the Gaussian two-way relay channel, the following rate region is achievable:

$$R_1 \leq \min \left(\frac{1}{2} \log \left(\frac{P}{\left((\alpha_{\text{MMSE}} \sqrt{g} - a)^2 + (\alpha_{\text{MMSE}} - 1)^2 \right) P + \alpha_{\text{MMSE}}^2 N_R} \right), \frac{1}{2} \log \left(1 + \frac{P_R}{N_2} \right) \right), \quad (22)$$

$$R_2 \leq \min \left(\frac{1}{2} \log \left(\frac{P}{\left((\alpha_{\text{MMSE}} \sqrt{g} - a)^2 + (\alpha_{\text{MMSE}} - 1)^2 \right) P + \alpha_{\text{MMSE}}^2 N_R} \right), \frac{1}{2} \log \left(1 + \frac{P_R}{N_1} \right) \right), \quad (23)$$

where $\alpha_{\text{MMSE}} = \frac{(a\sqrt{g}+1)P}{(g+1)P+N_R}$ and a is the closest integer to \sqrt{g} i.e., $a = \lceil \sqrt{g} \rceil$.

Proof: The proof of theorem follows from the achievable rate-region of scheme 1 at the relay (9) and achievable rates at nodes 1 and 2 (20), (21). ■

Theorem 5. For Gaussian two-way relay channel, the following rate region is achievable:

$$R_1 \leq \min \left(u.c.e \left\{ \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N_R} \right) \right]^+ \right\}, \frac{1}{2} \log \left(1 + \frac{P_R}{N_2} \right) \right), \quad (24)$$

$$R_2 \leq \min \left(u.c.e \left\{ \left[\frac{1}{2} \log \left(\frac{g}{g+1} + \frac{gP}{N_R} \right) \right]^+ \right\}, \frac{1}{2} \log \left(1 + \frac{P_R}{N_1} \right) \right). \quad (25)$$

Proof: The proof of theorem follows from the achievable rate-region of scheme 2 at the relay (18),(19) and achievable rates at nodes 1 and 2 (20), (21). ■

IV. OUTER BOUND

By using the cut-set bound, an outer bound for a TRC can be derived. If a rate pair (R_1, R_2) is achievable for a general TRC, then

$$R_1 \leq \min \{ I(\mathbf{X}_1; \mathbf{Y}_R, \mathbf{Y}_2 | \mathbf{X}_R, \mathbf{X}_2), I(\mathbf{X}_1, \mathbf{X}_R; \mathbf{Y}_2 | \mathbf{X}_2) \}, \quad (26)$$

$$R_2 \leq \min \{ I(\mathbf{X}_2; \mathbf{Y}_R, \mathbf{Y}_1 | \mathbf{X}_R, \mathbf{X}_1), I(\mathbf{X}_2, \mathbf{X}_R; \mathbf{Y}_1 | \mathbf{X}_1) \}, \quad (27)$$

where the minimization is over a joint probability $p(x_1, x_2, x_R)$. If we evaluate the outer bound for the GTRC, all the terms will be minimized by the product distribution $p(x_1, x_2, x_R) = p(x_1)p(x_2)p(x_R)$, where $p(x_1)$, $p(x_2)$ and $p(x_R)$ are Gaussian pdfs with zero means and variances of P_1 , P_2 and P_R , respectively. Therefore, from (26) and (27), one can get [26]

$$R_1 \leq \min \left(\frac{1}{2} \log \left(1 + \frac{P}{N_R} \right), \frac{1}{2} \log \left(1 + \frac{P_R}{N_2} \right) \right), \quad (28)$$

$$R_2 \leq \min \left(\frac{1}{2} \log \left(1 + \frac{gP}{N_R} \right), \frac{1}{2} \log \left(1 + \frac{P_R}{N_1} \right) \right). \quad (29)$$

Although there is a gap between the outer region and the achievable rate-region, we show that the gap vanishes at high SNR and hence the capacity region is completely determined in this limit of high SNR.

Corollary 1. At high SNR, the capacity region of the GTRC, is given by

$$R_1 \leq \min \left(\frac{1}{2} \log \left(1 + \frac{P}{N_R} \right), \frac{1}{2} \log \left(1 + \frac{P_R}{N_2} \right) \right) - o(1), \quad (30)$$

$$R_2 \leq \min \left(\frac{1}{2} \log \left(1 + \frac{gP}{N_R} \right), \frac{1}{2} \log \left(1 + \frac{P_R}{N_1} \right) \right) - o(1), \quad (31)$$

where $o(1) \rightarrow 0$ as $\text{SNR} \rightarrow \infty$.

Proof: Using (11) and (16) with $\alpha = 1$ and taking Λ_1 and Λ_3 to be lattices that are Poltyrev-good and Rogers-good with

the second moment equals to P and gP , respectively, we get the following achievable rate region at the relay:

$$\begin{aligned} R_1 &\leq \frac{1}{2} \log \left(\frac{P}{N} \right), \\ R_2 &\leq \frac{1}{2} \log \left(\frac{gP}{N} \right). \end{aligned}$$

Now, by using (20) and (21), we can achieve the following rate region for GTRC:

$$R_1 \leq \min \left(\frac{1}{2} \log \left(\frac{P}{N} \right), \frac{1}{2} \log \left(1 + \frac{P_R}{N_2} \right) \right), \quad (32)$$

$$R_2 \leq \min \left(\frac{1}{2} \log \left(\frac{gP}{N} \right), \frac{1}{2} \log \left(1 + \frac{P_R}{N_1} \right) \right). \quad (33)$$

By evaluating the outer bound in (28), and (29) for $\text{SNR} \rightarrow \infty$, we observe that we can approach the outer bound at high SNR and the proof is complete. \blacksquare

A. Calculating the gap

In the following, we bound the gap between the outer region, given in (28), (29) and the achievable region in (24), (23). It is easy to show that the gap for the symmetric model, i.e., $P_R = P$ and $N_1 = N_2 = N_R \triangleq N$ is always greater than the gap in other cases. Thus, we focus on the symmetric model. For the symmetric model, the outer bound on the capacity region is given by:

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{P}{N} \right), \quad (34)$$

$$R_2 \leq \min \left(\frac{1}{2} \log \left(1 + \frac{gP}{N} \right), \frac{1}{2} \log \left(1 + \frac{P}{N} \right) \right), \quad (35)$$

and the achievable rate region is simplified as:

$$R_1 \leq \text{u.c.e} \left\{ \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N} \right) \right]^+ \right\}, \quad (36)$$

$$R_2 \leq \min \left(\text{u.c.e} \left\{ \left[\frac{1}{2} \log \left(\frac{g}{g+1} + \frac{gP}{N} \right) \right]^+ \right\}, \frac{1}{2} \log \left(1 + \frac{P}{N} \right) \right). \quad (37)$$

To obtain the gap between the outer bound and the achievable region, first we bound the gap for R_1 .

1) *Gap for R_1* : Let us define this gap, denoted by $\xi(P, g, N)$, as the following:

$$\xi(P, g, N) \triangleq \frac{1}{2} \log \left(1 + \frac{P}{N} \right) - \text{u.c.e} \left\{ \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N} \right) \right]^+ \right\}.$$

In the following theorem, we provide an upper bound on $\xi(P, g, N)$.

Theorem 6. *Let x^* be the solution of the equation $\frac{x}{x+g+1} = \log_e \left(x + \frac{1}{g+1} \right)$. For any P, g, N , the gap is bounded by $\xi(P, g, N)$*

$$\xi(P, g, N) \leq \max \left\{ \frac{1}{2} \log \left(x^* + \frac{1}{g+1} \right) - \frac{x^* - \frac{g}{g+1}}{2 \ln 2 \left(x^* + \frac{1}{g+1} \right)}, \frac{1}{2} \log \left(1 + \frac{\frac{g}{g+1}}{x^* + \frac{1}{g+1}} \right) \right\}. \quad (38)$$

Proof: Let us define $x \triangleq \frac{P}{N}$.

$$\xi(P, g, N) = \frac{1}{2} \log(1+x) - \text{u.c.e} \left\{ \left[\frac{1}{2} \log \left(x + \frac{1}{g+1} \right) \right]^+ \right\} \triangleq \tilde{\xi}(x).$$

Now, we calculate the upper convex envelope with respect to x . First, we compute the line through the origin which is tangent

to $\frac{1}{2} \log \left(x + \frac{1}{g+1} \right)$:

$$y = \frac{x}{2 \ln 2 \left(x^* + \frac{1}{g+1} \right)},$$

where x^* is a solution of the following equation

$$\frac{x}{x + \frac{1}{g+1}} = \log_e \left(x + \frac{1}{g+1} \right).$$

Therefore, we get

$$u.c.e \left\{ \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N} \right) \right]^+ \right\} = \begin{cases} \frac{1}{2} \log \left(x + \frac{1}{g+1} \right) & x \geq x^* \\ \frac{x}{2 \ln 2 \left(x^* + \frac{1}{g+1} \right)} & 0 \leq x \leq x^* \end{cases},$$

a) For $x \geq x^*$: $\tilde{\xi}(x)$ is given by

$$\tilde{\xi}(x) = \frac{1}{2} \log \left(\frac{x+1}{x + \frac{1}{g+1}} \right) = \frac{1}{2} \log \left(1 + \frac{\frac{g}{g+1}}{x + \frac{1}{g+1}} \right). \quad (39)$$

Since $\tilde{\xi}(x)$ is decreasing with respect to x , hence $\tilde{\xi}(x)$ is maximized for $x = x^*$.

b) For $0 \leq x \leq x^*$: $\tilde{\xi}(x)$ is given by

$$\tilde{\xi}(x) = \frac{1}{2} \log(1+x) - \frac{x}{2 \ln 2 \left(x^* + \frac{1}{g+1} \right)}.$$

The maximum of $\tilde{\xi}(x)$ occurs at $x_m = x^* - \frac{g}{g+1}$, hence we get

$$\tilde{\xi}(x) \leq \tilde{\xi}(x_m) = \frac{1}{2} \log \left(x^* + \frac{1}{g+1} \right) - \frac{x^* - \frac{g}{g+1}}{2 \ln 2 \left(x^* + \frac{1}{g+1} \right)}. \quad (40)$$

Combining (39) and (40) completes the proof. ■

2) *Gap for R_2* : To obtain the gap for R_2 , we consider two cases:

Case 1: $g < 1$: In this case, the gap between the outer bound (35) and the achievable rate in (37) is defined as the following:

$$\eta_1(P, g, N) = \frac{1}{2} \log \left(1 + \frac{gP}{N} \right) - u.c.e \left\{ \left[\frac{1}{2} \log \left(\frac{g}{g+1} + \frac{gP}{N} \right) \right]^+ \right\}.$$

Theorem 7. Let x^* be the solution of the equation $\frac{x}{x + \frac{1}{g+1}} = \log_e \left(gx + \frac{g}{g+1} \right)$. For any P, g, N , the gap for R_2 is bounded by

$$\eta_1(P, g, N) \leq \max \left\{ \frac{1}{2} \log \left(gx^* + \frac{g}{g+1} \right) - \frac{gx^* - \frac{1}{g+1}}{2 \ln 2 \left(gx^* + \frac{g}{g+1} \right)}, \frac{1}{2} \log \left(\frac{gx^* + 1}{gx^* + \frac{g}{g+1}} \right) \right\}. \quad (41)$$

Proof: Let $x \triangleq \frac{P}{N}$. Then the gap is

$$\eta_1(P, g, N) = \frac{1}{2} \log(gx + 1) - u.c.e \left\{ \left[\frac{1}{2} \log \left(gx + \frac{g}{g+1} \right) \right]^+ \right\} \triangleq \tilde{\eta}_1(x).$$

Here, again we calculate the upper convex envelope. In a similar way, we obtain

$$u.c.e \left\{ \left[\frac{1}{2} \log \left(gx + \frac{g}{g+1} \right) \right]^+ \right\} = \begin{cases} \frac{1}{2} \log \left(gx + \frac{g}{g+1} \right) & x \geq x^* \\ \frac{x}{2 \ln 2 \left(x^* + \frac{1}{g+1} \right)} & 0 \leq x \leq x^* \end{cases},$$

where x^* is the solution of the following equation

$$\frac{x}{x + \frac{1}{g+1}} = \log_e \left(gx + \frac{g}{g+1} \right).$$

Now, we calculate $\tilde{\eta}_1(x)$ as follows:

a) For $x \geq x^*$: $\tilde{\eta}_1(x)$ is given by

$$\tilde{\eta}_1(x) = \frac{1}{2} \log \left(\frac{gx + 1}{gx + \frac{g}{g+1}} \right) = \frac{1}{2} \log \left(1 + \frac{\frac{1}{g+1}}{gx + \frac{g}{g+1}} \right).$$

Since $\tilde{\eta}_1(x)$ is decreasing with x , hence $\tilde{\eta}_1(x)$ is maximized at $x = x^*$.

b) For $0 \leq x \leq x^*$: $\tilde{\eta}_1(x)$ is given by

$$\tilde{\eta}_1(x) = \frac{1}{2} \log(gx + 1) - \frac{x}{2 \ln 2 \left(x^* + \frac{1}{g+1} \right)}.$$

The maximum of $\tilde{\eta}_1(x)$ occurs at $x^* - \frac{1}{g(g+1)}$, hence, we get

$$\tilde{\eta}_1(x) \leq \tilde{\eta}_1 \left(x^* - \frac{1}{g(g+1)} \right) = \frac{1}{2} \log \left(gx^* + \frac{g}{g+1} \right) - \frac{gx^* - \frac{1}{g+1}}{2 \ln 2 \left(gx^* + \frac{g}{g+1} \right)}.$$

■

Case II: $g > 1$: Let x^* be the solution of the equation $\frac{x}{x + \frac{1}{g+1}} = \log_e \left(gx + \frac{g}{g+1} \right)$. Then, if there is no intersection between the straight line $y = \frac{x}{2 \ln 2 \left(x^* + \frac{1}{g+1} \right)}$ and the curve $\frac{1}{2} \log \left(1 + \frac{P}{N} \right)$, the outer bound in (34) meet the achievable rate R_2 in (36). Thus, the gap is zero. In other wise, we evaluate the gap as follows:

$$\eta_2(P, g, N) \triangleq \frac{1}{2} \log \left(1 + \frac{P}{N} \right) - u.c.e \left\{ \left[\frac{1}{2} \log \left(\frac{g}{g+1} + \frac{gP}{N} \right) \right]^+ \right\}. \quad (42)$$

If SNR is fixed, then $\eta_2(P, g, N)$ is a decreasing function of g ; thus $\eta_2(P, g, N)$ is maximized at $g = 1$. In the following theorem, we bound $\eta_2(P, g, N)$ for $g = 1$.

Theorem 8. Let x^* be the solution of the equation $\frac{x}{x + \frac{1}{2}} = \log_e \left(x + \frac{1}{2} \right)$. For any P and N , the gap for R_2 is bounded by

$$\eta_2(P, 1, N) \leq 0.167$$

Proof: Let $x \triangleq \frac{P}{N}$. We can rewrite (42) as

$$\eta_2(P, g, N) = \frac{1}{2} \log(1 + x) - u.c.e \left\{ \left[\frac{1}{2} \log \left(\frac{1}{2} + x \right) \right]^+ \right\} \triangleq \tilde{\eta}_2(x).$$

a) For $x \geq x^*$: $\tilde{\eta}_2(x)$ is given by

$$\tilde{\eta}_2(x) = \frac{1}{2} \log \left(\frac{1+x}{\frac{1}{2} + x} \right).$$

Since $\tilde{\eta}_2(x)$ is decreasing with respect to x , $\tilde{\eta}_2(x)$ is maximized at $x = x^* \cong 1.655$.

b) For $0 \leq x \leq x^*$: $\tilde{\eta}_2(x)$ is given by

$$\tilde{\eta}_2(x) = \frac{1}{2} \log(1 + x) - \frac{x}{2 \ln 2 \left(x^* + \frac{1}{2} \right)}.$$

The maximum of $\tilde{\eta}_2(x)$ occurs at $x^* - \frac{1}{2}$, hence, we get

$$\tilde{\eta}_2(x) \leq \tilde{\eta}_2 \left(x^* - \frac{1}{2} \right).$$

The Theorem holds since $\tilde{\eta}_2(x^*) \leq \tilde{\eta}_2 \left(x^* - \frac{1}{2} \right)$.

■

3) Maximum gap for R_1 , R_2 and $R_1 + R_2$:

Theorem 9. For $g < 1$, the outer bound in (34) is within 0.2654 bit from the achievable rate in (36) for user 1. In addition, the outer bound of (35) is within 0.2658 bit from the achievable rate in (37) for user 2.

Proof: The gap $\xi(P, g, N)$ in (38) is an increasing function of g , hence this gap is maximized for $g = \infty$. Thus, the maximum of gap for R_1 is 0.2654 bit. For $g < 1$, the gap in (41) is decreasing with respect to g . Therefore, this gap is maximized as $g \rightarrow 0$. By evaluating (41) for $g \rightarrow 0$, we observe that the maximum gap is 0.2658 bit. ■

Theorem 10. For $g > 1$, the outer bound in (34) is within 0.2654 bit from the achievable rate, given (36) for user 1. For user 2, the outer bound in (35) is within 0.167 bit from the achievable rate in (37).

Proof: The gap in (38) is an increasing function with respect to g ; hence this gap is maximized for $g = \infty$. Thus, the maximum of the gap for R_1 is 0.2654 bit. Now, by using Theorem 8, we observe that the maximum of the gap for R_2 is 0.167 bit. ■

Theorem 11. For $g < 1$, the sum-rate gap of our proposed scheme (i.e., scheme 2) is within 0.334 of the outer bound.

Proof: We have

$$\zeta(P, g, N) = (R_1 + R_2)_O - (R_1 + R_2)_I = (R_{1O} - R_{1I}) + (R_{2O} - R_{2I}),$$

By using (38) and (41), we get

$$\begin{aligned} \zeta_1(P, g, N) \leq & \left\{ \max \left\{ \frac{1}{2} \log \left(x^* + \frac{1}{g+1} \right) - \frac{x^* - \frac{g}{g+1}}{2 \ln 2 \left(x^* + \frac{1}{g+1} \right)}, \frac{1}{2} \log \left(1 + \frac{\frac{g}{g+1}}{x^* + \frac{1}{g+1}} \right) \right\} + \right. \\ & \left. \max \left\{ \frac{1}{2} \log \left(gy^* + \frac{g}{g+1} \right) - \frac{gy^* - \frac{1}{(g+1)}}{2 \ln 2 \left(gy^* + \frac{g}{g+1} \right)}, \frac{1}{2} \log \left(\frac{gy^* + 1}{gy^* + \frac{g}{g+1}} \right) \right\} \right\}, \end{aligned} \quad (43)$$

where x^* is the solution of the equation $\frac{x}{x + \frac{1}{g+1}} = \log_e \left(x + \frac{1}{g+1} \right)$ and y^* is the solution of the equation $\frac{y}{y + \frac{1}{g+1}} = \log_e \left(gy + \frac{g}{g+1} \right)$. Now, by evaluating the gap (43), which is an increasing function with respect to g , we observe that the maximum gap is 0.334 bit (For more detail, see Section V). ■

Theorem 12. For $g > 1$, the sum-rate gap of our proposed scheme (i.e., scheme 2) is within 0.334 of the outer bound.

Proof: For the sum-rate gap, we have

$$\zeta_2(P, g, N) = (R_1 + R_2)_O - (R_1 + R_2)_I = (R_{1O} - R_{1I}) + (R_{2O} - R_{2I}).$$

The first term, using (38), can be calculated as

$$R_{1O} - R_{1I} \leq \max \left\{ \frac{1}{2} \log \left(x^* + \frac{1}{g+1} \right) - \frac{x^* - \frac{g}{g+1}}{2 \ln 2 \left(x^* + \frac{1}{g+1} \right)}, \frac{1}{2} \log \left(1 + \frac{\frac{g}{g+1}}{x^* + \frac{1}{g+1}} \right) \right\}, \quad (44)$$

where x^* is a solution of the equation $\frac{x}{x + \frac{1}{g+1}} = \log_e \left(x + \frac{1}{g+1} \right)$. For evaluating the second term, we consider two cases:

1) There is no intersection between the straight line $y = \frac{x}{2 \ln 2 \left(x + \frac{1}{g+1} \right)}$ and the curve $\frac{1}{2} \log \left(1 + \frac{P}{N} \right)$, where x^* is the solution of the equation $\frac{x}{x + \frac{1}{g+1}} = \log_e \left(gx + \frac{g}{g+1} \right)$. For this case, the outer bound (34) meet the achievable rate R_2 in (36). Thus, the gap for the second term is zero and it is sufficient to calculate the gap for the first term. The gap in (44) is increasing with g

and is maximized for $g = \infty$. With evaluating (44) for $g = \infty$, we observe that the maximum gap is 0.2654 bit.

2) If there is an intersection between the straight line $y = \frac{x}{2 \ln 2 (x^* + \frac{1}{g+1})}$ and the curve $\frac{1}{2} \log(1 + \frac{P}{N})$, where x^* is the solution of the equation $\frac{x}{x + \frac{1}{g+1}} = \log_e(gx + \frac{g}{g+1})$, we evaluate the following gap:

$$\xi(P, g, N) \leq \max \left\{ \frac{1}{2} \log \left(x^* + \frac{1}{g+1} \right) - \frac{x^* - \frac{g}{g+1}}{2 \ln 2 \left(x^* + \frac{1}{g+1} \right)}, \frac{1}{2} \log \left(1 + \frac{\frac{g}{g+1}}{x^* + \frac{1}{g+1}} \right) \right\} + \xi'(P, g, N),$$

where

$$\xi'(P, g, N) = \frac{1}{2} \log \left(1 + \frac{P}{N} \right) - u.c.e \left\{ \left[\frac{1}{2} \log \left(\frac{g}{g+1} + \frac{gP}{N} \right) \right]^+ \right\}.$$

The gap is decreasing with g . Thus, it is maximized for $g = 1$. By evaluating the gap for $g = 1$, we get

$$\xi(P, g, N) \leq 0.334.$$

■

V. DISCUSSION AND NUMERICAL RESULT

First, we assume that \sqrt{g} is an integer number. Then, the achievable rate region of scheme 1 can be calculated as:

$$\begin{aligned} R_1 &< \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N_R} \right) \right]^+ \\ R_2 &< \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N_R} \right) \right]^+, \end{aligned}$$

and the achievable rate region of scheme 2 is given by:

$$\begin{aligned} R_1 &\leq \left[\frac{1}{2} \log \left(\frac{1}{g+1} + \frac{P}{N} \right) \right]^+, \\ R_2 &\leq \left[\frac{1}{2} \log \left(\frac{g}{g+1} + \frac{gP}{N} \right) \right]^+. \end{aligned}$$

As we see, if g is an integer number and larger than one, the achievable rate region of scheme 2 is larger than that of scheme 1. But for $g = 1$, both schemes have the same performance as the proposed scheme in [35]. For other integer values of g i.e., $g = 2, 3, 4, \dots$, scheme 2 have better performance than scheme 1. This comparison is depicted in Fig.4 for $g = 3$ and in Fig.5 for $g = 10$.

In Table I and Table II, we provide the gap between the achievable rate of user 1 (R_1) and the outer bound, given in (38). We observe that when channel gain increases, the gap increases and the maximum gap is 0.2654 bit. For $g = 10$, we compare the achievable rate (R_1) of scheme 2 and the outer bound in Fig.6. As SNR increases, the gap decreases.

In Table III and for $g < 1$, we compare the gap between the achievable rate for user 2 (R_2) and the outer bound, given in (41). As we observe, when the channel gain decreases, the gap increases and the maximum gap is 0.2658 bit.

Table I
THE GAP BETWEEN THE OUTER BOUND AND THE ACHIEVABLE RATE FOR USER 1, R_1 FOR $g > 1$

Channel Gain (g)	1	4	9	16	25	64	100	inf
Gap (bits)	0.167	0.2361	0.2497	0.2547	0.2573	0.2621	0.2637	0.2654

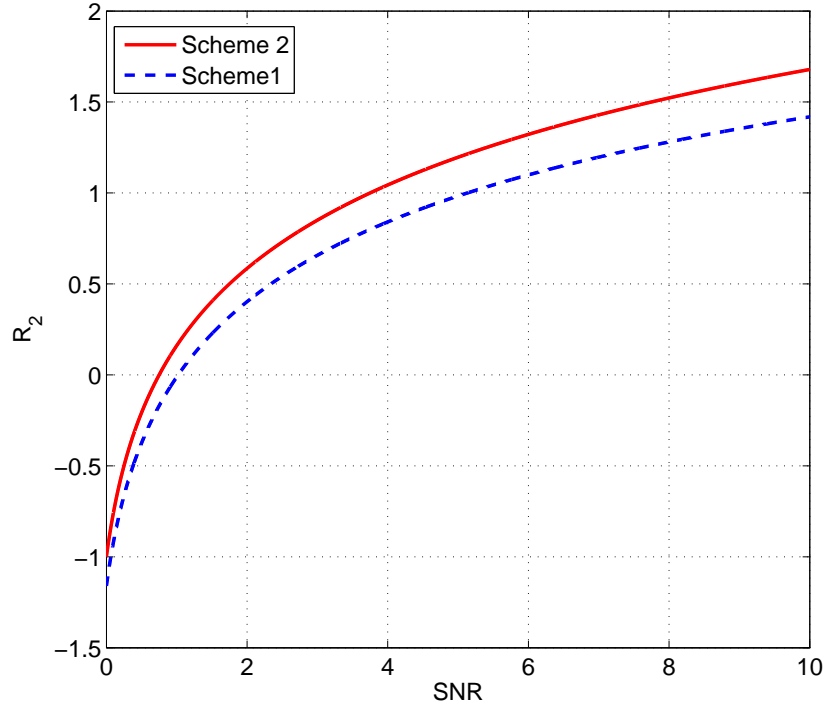


Figure 4. Comparison of the achievable rates of scheme 1 and scheme 2 for user 1, R_2 , in the symmetric GTRC. The channel gain is $g = 3$.

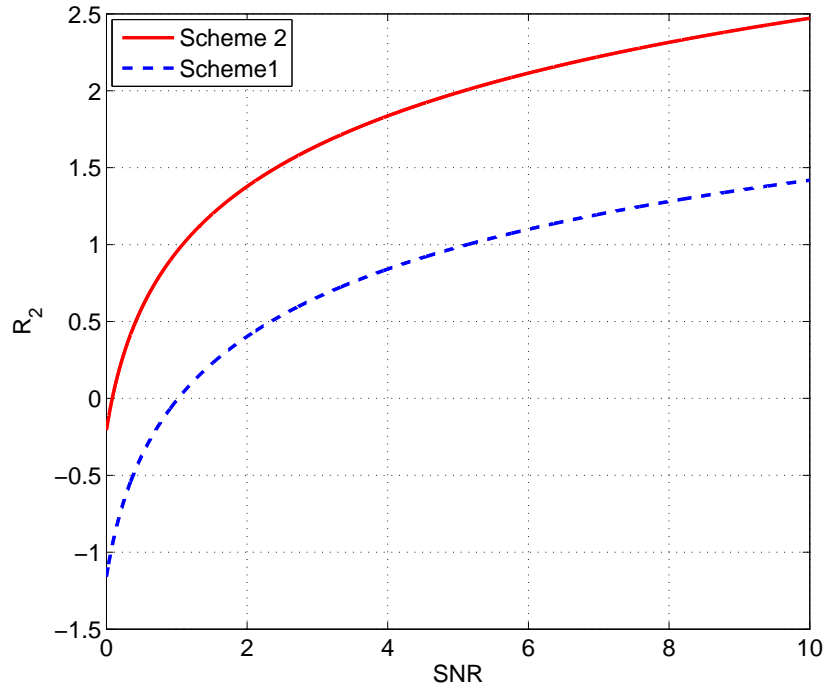


Figure 5. Comparison of the achievable rates of scheme 1 and scheme 2 for user 2, R_2 , in the symmetric GTRC. The channel gain is $g = 10$.

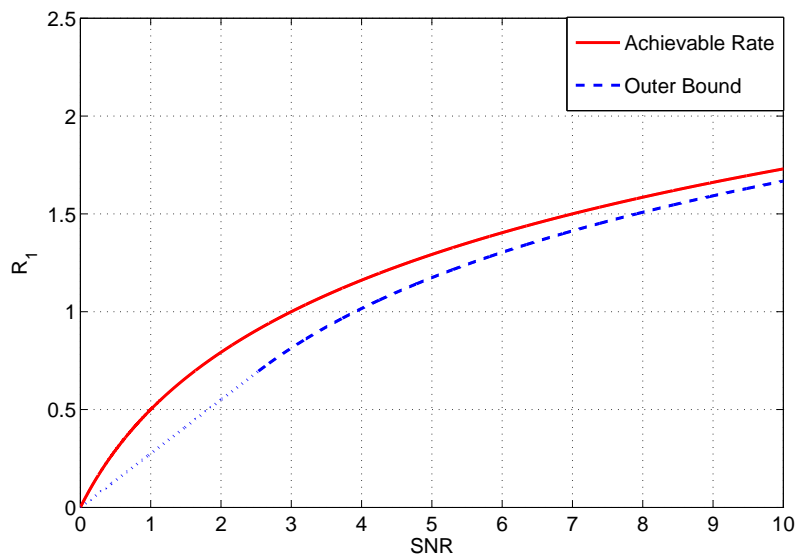


Figure 6. Comparison of the achievable rate of scheme 2 for R_1 and the outer bound in the symmetric GTRC. The channel gain is $g = 10$.

Table II
THE GAP BETWEEN THE OUTER BOUND AND THE ACHIEVABLE RATE FOR USER 1, R_1 FOR $g < 1$

Channel Gain (g)	1	0.7	0.6	0.5	0.3	0.1	0.001	0.0001
Gap (bits)	0.167	0.146	0.136	0.1252	0.09497	0.045	0.000686	0.0001611

Table III
THE GAP BETWEEN THE OUTER BOUND AND THE ACHIEVABLE RATE FOR USER 2, R_2

Channel Gain (g)	1	0.7	0.6	0.5	0.3	0.1	0.001	0.0001
Gap (bits)	0.167	0.1872	0.195	0.2038	0.224	0.2498	0.2651	0.2658

VI. CONCLUSION

In this paper, a Gaussian two way relay channel (GTRC) is considered. By using nested lattice-based coding scheme, we obtain two achievable rate regions for this channel and characterize the theoretical gap between the achievable rate region and the outer bound. We show that the gap is less than 0.2658 bit for each user, which is best gap-to-capacity result to date. We also show that, at high SNR, the lattice based coding scheme can achieve the capacity region.

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