# Constructive Renormalization for $\Phi_2^4$ Theory with Loop Vertex Expansion

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**Abstract**

In this paper we construct the 2 dimensional Euclidean $\phi^4$ quantum field theory using the method of loop vertex expansion. We reproduce the results of standard constructive theory, for example the Borel summability of the Schwinger functions in the coupling constant. Our method should be also suitable for the future construction of Grosse-Wulkenhaar models on non-commutative space-time.

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# 1 Introduction

In QFT theory the most important quantities are the Green functions, which are the vacuum expectation values of products of physical fields, and contain

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all the physical information. To compute these Green functions, perturbation theory expands the partition function in a power series of the coupling constant.

However when actually computing each term of the series we typically get infinities. Renormalization is a technique to get rid of these infinities. It allows to compute the mass, charge and magnetic moment of elementary particles or form factors in a way which agreed very well with experiments. The modern interpretation of renormalization is the renormalization group of Wilson. It starts with a bare action which is valid at a very high energy scale and after many renormalization-group steps leads to an effective theory accessible to today's experiments. In good cases the theory is *renormalizable*, which means that the effective action has the same form as the original bare action, but with modified constants.

However there is a second type of divergences in perturbative quantum field theory which is often overlooked. The power series itself usually has zero radius of convergence. It is often stated in textbooks that perturbation theory is therefore expected only to be asymptotic to the true functions. However since any formal power series is asymptotic to infinitely many smooth functions, perturbative field theory alone doesn't give any well defined recipe to compute to arbitrary accuracy the physical quantities. In a deep sense it is not a theory at all. Therefore we need to extend perturbative renormalization into constructive renormalization [1].

Constructive renormalization considers as equally important the finiteness of the renormalized arbitrary coefficients of the perturbation series and the *summability* of the series. For the Fermionic case, the summation of the series is easier thanks to the Pauli's principle [2, 3], which translates into compensations between graphs within a given order of perturbation theory. But the Bosonic case is much harder, as Bosons can condense. We need to combine amplitudes from infinitely many different orders, as the compensations occur only between these different orders. Also the best we can hope for is only Borel summability [4]. The standard techniques for proving such Borel summability are cluster and Mayer expansions [5, 6]. These methods require to divide the space into cubes. This decomposition seems odd since it breaks the rotation invariance of the theory. What's more, the cluster expansion seems unsuited for non-local theories, for example, noncommutative QFTs like the Grosse-Wulkenhaar model.

Recently a new constructive Bosonic method has been invented, namely the loop vertex expansion (LVE) [7, 8, 9, 10]. In this method the role of vertices and propagators is exchanged and one gets a natural explicit summation of the amplitudes of different orders. The method gives direct access to thermodynamic quantities such as connected functions, and requires only a single use of a forest formula [11, 12] to compute connected quantities. It is a definite improvement over cluster and Mayer expansions, which require to use *twice* such a formula.

In this paper we include for the first time renormalization into the loop vertex expansion to construct the Euclidean $\phi^4$ theory in two dimensions. This is a first step on the road to a full construction of the noncommutative $\phi_2^{\star 4}$ [13, 14, 15] and ultimately the $\phi_4^{\star 4}$ Grosse-Wulkenhaar model [16, 17, 18, 19].

For details on the standard construction of commutative $\phi_2^4$ theory let's refer to [20, 1]. Recall that it was proved in [5] that the Schwinger functions of this model are the Borel sum of their perturbative expansion.

This paper is organized as follows. In section 2 we recall the notion of intermediate fields and the BKAR formula [11, 12]. We rewrite the $\phi_2^4$ theory in the language of the loop vertex expansion distinguishing more clearly the direct and the dual version of the LVE representation. We perform already a fraction of the renormalization, canceling the so-called leaf-tadpoles. In section 3 we recall why, according to [7, 8], the LVE can compute the thermodynamic limit for the model with ultraviolet cutoff without any renormalization; however this can be done only for a coupling constant smaller and smaller as the ultraviolet cutoff tends to infinity, and is therefore not enough. In section 4 we explain how to superimpose a standard cluster expansion on the LVE to decide exactly the volume occupied by an LVE term. The main section is Section 5, which implements a new expansion called the *cleaning expansion*. It uses the canonical cyclic ordering of the dual LVE representation to turn around any loop vertex term and "clean" it, canceling in this process any inner tadpoles met with their corresponding counter terms. This cleaning is continued until sufficiently many convergent factors have been gathered to pay for the *Nelson's bound* [1]. This must be done in each unit square actually occupied by the loop vertex term, and that's why we need the cluster expansion of Section 4. Note however than in contrast with the standard constructions of $\phi_2^4$ we *never* use at any time any Mayer expansion, since the thermodynamic quantities are computed right at the start by the LVE. In the last sections we check more explicitly the combinatorics of tadpole versus countertem compensation and of Borel summability of the ordinary perturbation theory.

We think that the methods of this paper could also apply to the construction of the $\phi^4$ theory on a $2d$ curved background. The method is particularly well suited to the case of the $\phi^4_2$ theory on a 2 dimensional compact Riemannian manifold, since in that finite volume case the cluster expansion of section 4 is unnecessary.

# 2 The Model and its Loop Vertex Expansion

## 2.1 The Model

The free bosonic $\phi^4_2$ theory has variance $C^{-1} = -\nabla^2 + m^2$. In 2 dimensions the corresponding covariance or propagator is formally given by the kernel

$$C(x,y) = \frac{1}{4\pi}\int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}}. \tag{1}$$

It diverges at $x = y$ so we need ultraviolet regularization, hence we define the covariance with ultraviolet cutoff $\Lambda$ as

$$C_\Lambda(x,y) = \frac{1}{4\pi}\int_{\Lambda^{-2}}^\infty \frac{d\alpha}{\alpha} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}}. \tag{2}$$

We define the tadpole term $T_\Lambda = C_\Lambda(x,x)$ as the value of the propagator at coinciding points. It diverges proportionally to $\log\Lambda$ as $\Lambda \to \infty$.

It is well known that only the tadpole graphs are divergent in $\phi^4_2$ theory and the renormalization reduces to the Wick ordering. The model with ultraviolet cutoff $\Lambda$ and infrared volume cutoff $\mathcal{V}$ is therefore defined as:

$$Z(\lambda,\Lambda,\mathcal{V}) = \int d\mu_{C^\Lambda}(\phi) e^{-\frac{\lambda}{2}\int_{\mathcal{V}} d^2x :\phi^4(x):} \tag{3}$$

where $d\mu_C$ is the normalized Gaussian measure with covariance $C$. The Wick ordering in $:\phi^4(x): \equiv \phi^4 - 6T_\Lambda\phi^2 + 3T_\Lambda^2$ is taken with respect to $C^\Lambda$ [20].

Writing the Wick product explicitly the partition function becomes:

$$\begin{aligned} Z &= \int d\mu_{C^\Lambda} e^{-\frac{\lambda}{2}\int_{\mathcal{V}} d^2x[\phi^4 - 6T_\Lambda\phi^2 + 3T_\Lambda^2]} \\ &= \int d\mu_{C^\Lambda} e^{-\frac{\lambda}{2}\int_{\mathcal{V}} d^2x[(\phi^2 - 3T_\Lambda)^2 - 6T_\Lambda^2]}. \end{aligned} \tag{4}$$

## 2.2 The Intermediate Field Representation

Introducing the intermediate field $\sigma$ and integrating out the terms that are quadratic in $\phi(x)$, we get

$$Z(\lambda, \Lambda, \mathcal{V}) = \int d\nu(\sigma) e^{3\lambda|\mathcal{V}|T_\Lambda^2 + \mathrm{Tr}_{\mathcal{V}}\left(3i\sqrt{\lambda}T_\Lambda\sigma - \frac{1}{2}\log[1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}]\right)}, \qquad (5)$$

where $d\nu(\sigma)$ is the ultralocal measure on $\sigma$ with covariance $\delta(x-y)$, and $\mathrm{Tr}_{\mathcal{V}}$ means integration over 2 dimensional volume $\mathcal{V}$ for each $\sigma$ field argument while keeping the cyclic ordering for the product of operators $C^{1/2}\sigma C^{1/2}$. The integral $\int$ inside the *log* function means spacial integration over $\sigma$ field which is a local operator and we shall omit writing them as long as no misunderstanding is caused.

Defining a new interaction vertex

$$\begin{aligned} & V(\lambda, \Lambda, \mathcal{V}, \sigma) = 3\lambda|\mathcal{V}|T_\Lambda^2 \\ + \quad & \mathrm{Tr}_{\mathcal{V}}\left(-\frac{1}{2}\log(1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}) + 3i\sqrt{\lambda}T_\Lambda\sigma\right), \end{aligned} \qquad (6)$$

The partition function can be written as:

$$Z(\lambda, \Lambda, \mathcal{V}) = \int d\nu(\sigma) e^{V(\lambda,\Lambda,\mathcal{V},\sigma)}. \qquad (7)$$

There are three basic propagators in the loop vertex expansion, the propagators $C$, the resolvents $R$, whose definition will be given in section 2.3 and the ultralocal propagators for the $\sigma$ fields. They are shown in Figure 1.

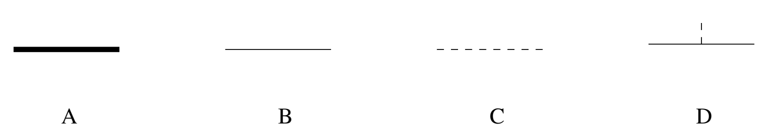


Figure 1: The basic propagators in LVE. $A$ stands for a pure propagator, $B$ for a a resolvent, $C$ for the ultralocal propagator for the $\sigma$ field. $D$ is the subtracted resolvent introduced further below.

Remark that the standard renormalization of $\phi_2^4$ involves canceling all tadpoles through Wick-ordering. But if we were to apply directly the LVE formalism at this stage, as in [8], tadpoles would appear in two different ways [10]. Leaves with a single propagator of type A are explicitly visible tadpoles, called leaf-tadpoles, see Figure 2 ; but there is a second kind of tadpoles hidden in the LVE. Indeed *consecutive* $\sigma$ fields hidden in any resolvent part of any given loop vertex can still contract together upon Gaussian integration of the $\sigma$ field. The corresponding tadpoles are called *inner tadpoles*. They are not explicitly visible after performing the LVE and are responsible for the non-trivial aspects of the $\phi_2^4$ renormalization in the LVE formalism. An inner tadpole and a leaf tadpole are shown in Figure 2.

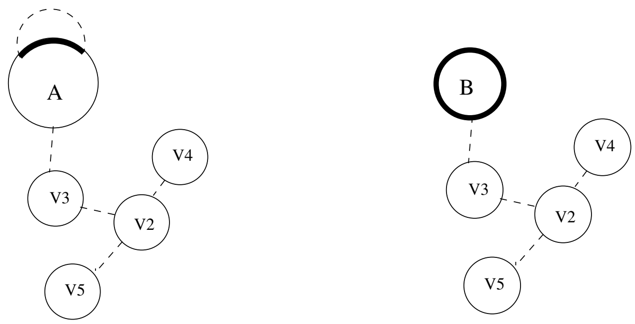


Figure 2: Inner tadpole A and leaf tadpole B. The bold lines mean the pure propagators while the ordinary lines mean the resolvents.

A very simple procedure can eliminate completely all leaf-tadpoles that are logarithmically divergent. Expanding the log function as:

$$\begin{aligned} & \log(1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}) \\ = \;& 2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2} + \log_2(1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}), \end{aligned}$$

where

$$\log_p(1+x) = \log(1+x) - \sum_{q=1}^{p-1}(-1)^{q-1}\frac{x^q}{q}, \tag{8}$$

we simplify the interaction vertex as:

$$\begin{aligned} V(\lambda,\Lambda,\mathcal{V},\sigma) &= CC(\lambda,\Lambda,\mathcal{V}) + CT(\lambda,\Lambda,\mathcal{V},\sigma) + W(\lambda,\Lambda,\mathcal{V},\sigma) \\ CC(\lambda,\Lambda,\mathcal{V}) &= 3\lambda|\mathcal{V}|T_\Lambda^2 \\ CT(\lambda,\Lambda,\mathcal{V},\sigma) &= 2i\sqrt{\lambda}T_\Lambda \mathrm{Tr}_{\mathcal{V}}\sigma \\ W(\lambda,\Lambda,\mathcal{V},\sigma) &= -\frac{1}{2}\mathrm{Tr}_{\mathcal{V}}\log_2(1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}). \end{aligned} \tag{9}$$

$CC$ is called the constant counter term, pictured as a single big black dot in Figure 3 part C; $CT$ is called the linear counter term, pictured as one of the two smaller black dots in part B of Figure 3, and $W$ is the (non-trivial) loop vertex.

This regrouping will forbid the formation of any leaf-tadpole in the LVE below. Remark that it also reduces the coefficient of the linear counter term which from an initial value of $3i\sqrt{\lambda}T_\Lambda\sigma$ becomes $2i\sqrt{\lambda}T_\Lambda\sigma$[1]. This remaining linear counter term is there to compensate all the *inner* tadpoles, in particular the ones that will appear in the cleaning expansions in section (5.2). This compensation is not obvious to explicitly perform and this is the main source of difficulty of this paper.

Let us now expand the exponential as $\sum_n \frac{V^n}{n!}$. To compute the connected function while avoiding an additional factor $n!$, we give a kind of $fictitious$ index $v$, $v = 1, \cdots, n$ to all the $\sigma$ fields of the vertex $V$. This means we consider $n$ different copies $\sigma_v$ of $\sigma$ with a degenerate Gaussian measure

$$d\nu(\{\sigma_v\}) = d\nu(\sigma_v) \prod_{v' \neq v}^{n} \delta(\sigma_v - \sigma_{v'}) d\sigma_{v'} \ , \tag{10}$$

whose covariance or ultralocal propagator

$$< \sigma_v(x), \sigma_{v'}(y) >= \delta(x-y) \tag{11}$$

does not depend on the fictitious indices $v$ and $v'$.

We obtain

$$\begin{aligned} Z(\lambda, \Lambda, \mathcal{V}) = \int d\nu(\{\sigma_v\}) \sum_{n=0}^{\infty} \frac{V^n}{n!} = \int d\nu(\sigma) \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{v=1}^{n} V_v(\sigma_v) \\ = \sum_{0}^{\infty} \frac{1}{n!} \int d\nu(\{\sigma_v\}) \prod_{v=1}^{n} \Big[ 3\lambda |\mathcal{V}| T_\Lambda^2 \\ + \mathrm{Tr}_{\mathcal{V}} \Big( -\frac{1}{2} \log_2 (1 + 2i\sqrt{\lambda} C^{1/2} \sigma_v C^{1/2}) + 2i\sqrt{\lambda} T_\Lambda \sigma_v \Big) \Big]. \end{aligned} \tag{12}$$

The real interesting quantities which do have limits as $\mathcal{V} \to \mathcal{R}^2$ are the connected functions, the simplest of all being the pressure $\frac{1}{|\mathcal{V}|} \ln Z(\lambda, \Lambda, \mathcal{V})$. Feynman graphs formally allow the computation of such quantities, but through a divergent expansion, as there are too many graphs ($(4n)!!$ at order $n$). As noticed in the introduction, the loop vertex expansion, hereafter

[1]Remark therefore that a third of the tadpoles are leaf-tadpoles and the majority, namely the remaining two-thirds, are inner-tadpoles.

called LVE, performs the same task of computing connected function but without introducing too many terms. Here it gives

**Theorem 2.1** (Loop Vertex Expansion)**.**

$$\frac{1}{\mathcal{V}} \log Z(\lambda, \Lambda, \mathcal{V}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} G_{\mathcal{T}} \tag{13}$$

$$G_{\mathcal{T}} = \Big\{ \prod_{\ell \in \mathcal{T}} \int d^2 x_\ell d^2 y_\ell \big[ \int_0^1 dw_\ell \big] \Big\} \int d\nu_{\mathcal{T}}(\{\sigma^v\}, \{w\})$$

$$\Big\{ \prod_{\ell \in \mathcal{T}} \big[ \delta(x_\ell - y_\ell) \frac{\delta}{\delta \sigma^{v(\ell)}(x_\ell)} \frac{\delta}{\delta \sigma^{v'(\ell)}(y_\ell)} \big] \Big\} \prod_{v=1}^{n} V_v,$$

*where*

- *each line $\ell$ of the tree joins two different loop vertices $V_{v(\ell)}$ and $V_{v'(\ell)}$ at points $x_\ell$ and $y_\ell$. These points are in fact identified through a $\delta(x_\ell - y_\ell)$ ultralocal $\sigma$ propagator*[2]*.*

- *the sum is over trees joining $n$ loop vertices. These trees have therefore $n-1$ lines, corresponding to $\sigma$ propagators.*

- *the normalized Gaussian measure $d\nu_{\mathcal{T}}(\{\sigma_v\}, \{w\})$ over the vector field $\sigma_v$ has a covariance*

$$< \sigma^v(x), \sigma^{v'}(y) >= \delta(x-y) w^{\mathcal{T}}(v, v', \{w\}), \tag{14}$$

*which depends on the "fictitious" indices. Here $w^{\mathcal{T}}(v, v', \{w\})$ equals 1 if $v = v'$, and equals the infimum of the $w_\ell$ for $\ell$ running over the unique path from $v$ to $v'$ in $\mathcal{T}$ if $v \neq v'$.*

The proof is a completely standard consequence of the BKAR formula and is given in detail in [7, 8]. The BKAR formula rewrites $Z$ as a sum over forests. The key observation is the factorization of the forests contributions according to their connected parts, which are the trees of the forest. This allows to compute $\log Z$ as the sum of the *same* contributions, but indexed by

[2] Our convention is that we have a single derivation for each line joining two loop vertices; an other convention would be to have two derivations but add a factor 1/2 in the formula.

trees. Here like in the computation of connected functions through connected Feynman graphs, Joyal theory of species [21] is the mathematical explanation behind the scene. The $d\nu_{\mathcal{T}}$ measure is well-defined since the matrix $w^{\mathcal{T}}$ is positive.

The n-point Euclidean Green's functions, where $n = 2p$ is an even number, can also be written in the LVE representation [8]. They could be derived eg by introducing source fields. The partition function with source fields reads:

$$Z(\Lambda, j(x)) = \int d\mu_{C^\Lambda} e^{-\frac{\lambda}{2}\int_{\mathcal{V}} d^2x[(\phi^2-3T_\Lambda)^2-6T_\Lambda^2]+j(x)\phi(x)} \tag{15}$$

The connected n-point functions are given by:

$$S(x_1, \cdots, x_n) = \frac{1}{(2p)!}\frac{\partial^{2p}}{\partial j(x_1)\cdots\partial j(x_{2p})}\frac{1}{\mathcal{V}}\log Z(\Lambda, j(x))|_{j=0}, \tag{16}$$

and we have a similar following theorem for the connected Schwinger's function:

**Theorem 2.2.**

$$S^c = \sum_\pi \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\mathcal{T}}\Big\{\prod_{\ell\in\mathcal{T}}\int d^2x_\ell d^2y_\ell\big[\int_0^1 dw_\ell\big]\Big\}$$
$$\int d\nu_{\mathcal{T}}(\{\sigma^v\},\{w\})\Big\{\prod_{\ell\in\mathcal{T}}\big[\delta(x_\ell - y_\ell)\frac{\delta}{\delta\sigma^{v(\ell)}(x_\ell)}\frac{\delta}{\delta\sigma^{v'(\ell)}(y_\ell)}\big]\Big\}$$
$$\prod_{v=1}^{n} V_v \prod_{r=1}^{p} C_R(\sigma_r, x_{\pi(r,1)}, x_{\pi(r,2)}), \tag{17}$$

*where the sum over $\pi$ runs over the parings of the $2p$ external variables into pairs $(x_{\pi(r,1)}, x_{\pi(r,2)})$, $r = 1, \cdots, p$, the tree $\mathcal{T}$ now joins the $n$ vertices and the $p$ resolvents $C_R$ (whose explicit form is given in formula (22) below).*

For simplicity we now treat only the vacuum connected function, as the more general case could be obtained easily by introducing external resolvents.

## 2.3 The first terms

Remark that the constant part $CC$ in (9) cannot bear any derivation. Hence it can appear only in the empty tree $\mathcal{T} = \emptyset$. The linear counter term can bear only one derivative hence it can appear only as a leaf of $\mathcal{T}$, see Figure 3. All other terms have at least one regular resolvent loop.

We want to develop therefore further each $G_{\mathcal{T}}$ according to how many leaves of $\mathcal{T}$ are counter terms and how many are (nontrivial) loop vertices $W$. Let us first treat the two first terms of the expansion, corresponding to $\mathcal{T}$ the empty tree and the tree with a single line, since they are special.

We now explicit a bit better the first terms $Z_1$ and $Z_2$ of the expansion (13) to show the compensation of the special $CC$ counter term.

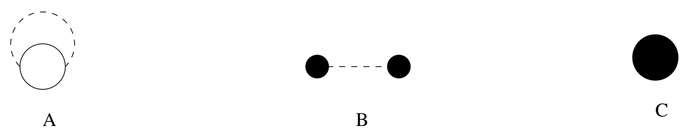


Figure 3: The lowest order graphs. Graph C represents the counterterm $3\lambda T_\Lambda^2$.

We expand first the $n = 1$ $Z_1$ term in (13) as

$$\begin{aligned} Z_1 &= \int d\nu(\sigma)\Big(3\lambda|\mathcal{V}|T_\Lambda^2 + \mathrm{Tr}_{\mathcal{V}}\big(-\frac{1}{2}\log_2(1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2})\big)\Big) \\ &= 2\lambda|\mathcal{V}|T_\Lambda^2 + \int d\nu(\sigma)\mathrm{Tr}_{\mathcal{V}}\big(-\frac{1}{2}\log_4(1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2})\big). \end{aligned} \tag{18}$$

The $n = 2$ term, $Z_2$, corresponds to a single tree with two leaves that can be either counter terms $CT$ or loop vertices $W$. In hopefully transparent notations:

$$Z_2 = Z_{2;CT,CT} + 2Z_{2;CT,W} + Z_{2;W,W}. \tag{19}$$

The contribution with two counterterms is

$$Z_{2,CT,CT} = \frac{1}{2}\int d\nu(\sigma) Tr_{\mathcal{V}}(2i\sqrt{\lambda}T_\Lambda\sigma)^2 = -2\lambda|\mathcal{V}|T_\Lambda^2, \tag{20}$$

hence cancels out exactly as expected with the $2\lambda|\mathcal{V}|T_\Lambda^2$ term in (18). So we find that the divergent terms or order $\lambda$ cancel. The divergent terms in $2Z_{2;CT,W}$ and $Z_{2;W,W}$ are of orders $\lambda^2$ or higher.

In this way we have checked how the first counterterm, which corresponds to a tadpole, cancel, and written the remainder in a form which is similar to the more ordinary LVE terms at the order of $\lambda^2$ and higher, so that they can be treated in the same way by the cleaning expansion below.

#### 2.3.1 Direct Resolvent Representation

To write down in a compact notation of the of the correlation function it is convenient to introduce further notations. We define the resolvent $R$ and the subtracted resolvent $\hat{R}$ through

$$\begin{aligned} R'(\sigma) &= \frac{1}{1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}}, \;\; \hat{R}'(\sigma) = \frac{1}{1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}} - 1 \\ D'(x) &= 2iC^{1/2}(.,x)C^{1/2}(x,.), \end{aligned} \tag{21}$$

and the full dressed resolvents and subtracted resolvents

$$C'_R = C^{1/2}R'C^{1/2}, \;\; \hat{C}'_R = C^{1/2}\hat{R}'C^{1/2}. \tag{22}$$

The dressed resolvent is pictured as a $B$ line in Figure 1 and the dressed subtracted resolvent is pictured as drawing D of the same figure.

The result of deriving once with respect to $\sigma$ a (non trivial) loop vertex $W$ gives

$$\frac{\partial}{\partial\sigma(x)}[-\frac{1}{2}\mathrm{Tr}_{\mathcal{V}}\log_2(1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2})] = -i\sqrt{\lambda}\hat{C}'_R(\sigma,x,x). \tag{23}$$

The result of deriving twice or more gives

$$\begin{aligned} &\frac{\partial}{\partial\sigma(x_1)}\cdots\frac{\partial}{\partial\sigma(x_p)}[-\frac{1}{2}\mathrm{Tr}_{\mathcal{V}}\log_2(1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2})] \\ = &-\frac{1}{2}(2i\sqrt{\lambda})^p(-1)^{p-1}\sum_\tau C'_R(x_1,x_{\tau(2)},\sigma)\cdots C'_R(x_{\tau(p)},x_1,\sigma) \end{aligned} \tag{24}$$

with $p \geq 2$ and the sum over $\tau$ is over the $(p-1)!$ permutations of $[2,...p]$. Hence only the first derivation can lead to a trace with a single $\hat{C_R}$ operator.

Let's index the leaves of a tree with a letter $f$. A decorated tree $\mathcal{T}$ is an ordinary tree plus an index $c_f$ with values 0 or 1 for each leaf $f$ telling whether the leaf is an ordinary loop vertex ($c_f = 1$) or a counter term ($c_f = 0$). If we call $F$ the set of leaves we can rewrite the outcome of the action of the

$$\prod_{\ell\in\mathcal{T}}[\delta(x_\ell - y_\ell)\frac{\delta}{\delta\sigma^{v(\ell)}(x_\ell)}\frac{\delta}{\delta\sigma^{v'(\ell)}(y_\ell)}] \tag{25}$$

operator in (13) as

$$\log Z(\lambda, \Lambda, \mathcal{V}) = a_1 + a_2 + \sum_{\mathcal{T} \text{ decorated tree with } n \geq 3 \text{ vertices}} G_{\mathcal{T}}$$

$$G_{\mathcal{T}} = \prod_{\ell \in \mathcal{T}} (-\lambda)^{n-1} \int_{\mathcal{V}} d^2 x_\ell \prod_{f \in F, c_f = 0} (2T_\Lambda)$$

$$\prod_{f \in F, c_f = 1} \mathrm{Tr}_{\mathcal{V}} D'(x_{\ell_f}) \hat{R}'(\sigma) \prod_{v \notin F} \mathrm{Tr}_{\mathcal{V}} \vec{\prod_{\ell \in v}} D'(x_{\ell_f}) R'(\sigma), \tag{26}$$

where $a_1$ and $a_2$ are the finite remainder terms of order $n = 1$ and $n = 2$ in the expansion, respectively, the $\vec{\prod}$ means that one has to take the ordered product of the operators along the loop vertex.

## 2.4 Graphic Representations

### 2.4.1 Direct Representation

We introduce two equivalent graphic representations for the LVE. For the first representation we defined $\mathcal{T}$ as the spanning tree of a given term $G_{\mathcal{T}}$ in the LVE. We remark that the loop vertex $V_v$ is the sum of the nontrivial loop vertex $W_v = \mathrm{Tr}_{\mathcal{V}} \left( -\frac{1}{2} \log_2 (1 + 2i\sqrt{\lambda} C^{1/2} \sigma_v C^{1/2}) \right)$ the linear counter term $CT_v = 2iT_\Lambda \sigma_v$ also called the counter term, and the constant counter term $CC = 3\lambda |\mathcal{V}| T_\Lambda^2$.

The direct representation (see Figure 4) pictures a term in the LVE as a decorated tree between loop vertices. This tree joins together the derived loop vertices $\prod_{\ell | v(\ell) = v} \frac{\delta}{\delta \sigma^{v(\ell)}(x_\ell)} V_v$ for $v \in W$, $|W| \leq n$. When the tree is non empty each such loop vertex bears at least one derivation hence is a cyclic product of ordinary $\phi^4$ resolvent lines of the $B$ type in Figure 1 [8]. The lines of the tree are of type $C$ in Figure 1 hence should not be confused with the resolvent lines forming the loop vertices, that's why we picture them as dashed lines. They are labeled by an index $\ell \in \mathcal{T}$, and correspond to $\sigma$ propagators that join the different loop vertices.

A *leaf* of the tree is an "extremal" vertex, i.e. a vertex of coordination 1. Leaves will play a particular role in the LVE above because of renormalization and the presence of counterterms, absent in [8]. In developing (13), two different types of leaves occur. The ordinary ones are loop vertices, bearing a single $\sigma$ propagator and we label them by an index $v_0 \in S$ (as "simple loops"). But there are also counterterms leaves, which we picture as small black disks, labeled by an index $b \in B$. These counterterms correspond to the fact that the $2i\sqrt{\lambda}T_\Lambda\sigma_v$ term in (13), which is linear, can bear only one $\frac{\partial}{\partial\sigma}$ derivation. Hence their value is $2i\sqrt{\lambda}T_\Lambda$. The loop vertices which are not leaves have coordination at least 2 and are indexed by a different index, $v_1 \in W - S$, where $W$ is the set of loop vertices (hence excluding the counterterms).

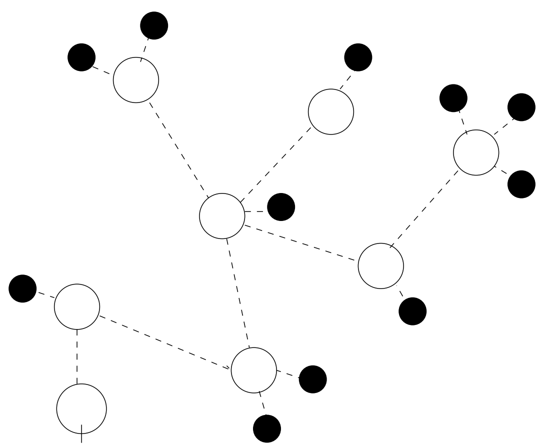

Figure 4: A tree of loop vertices with $|W| = 8$, $|B| = 11$, $|n| = 19$. The dash lines are the propagators for the $\sigma$ fields while the ordinary lines are the fully dressed resolvents which contain the resolvents and pure propagators. The circle with a bar means the leaf term $\hat{C}_R$.

Hence a term in the LVE direct representation is pictured as a cactus, decorated with an arbitrary number of black counterterms.

For $n \geq 2$ we have relations, such as $|\mathcal{T}| = n - 1$, $|B| + |V| = n$.

#### 2.4.2 Dual Representation

In the dual representation (see Figure 5), the $\sigma$ propagators are replaced by their dual, still pictured as propagators. Since a LVE term is *planar*, this notion of duality is globally well-defined. We define

- $\mathcal{C}$ as the cycle in the dual representation of the LVE corresponding to $G$ (see below).
- $\bar{\mathcal{T}}$ as the tree dual to $\mathcal{T}$ (see below).

A LVE dual term for a decorated tree $\{\mathcal{T}, B\}$ of the direct representation corresponds to a cycle $\mathcal{C}$ made of objects forming a set $\mathcal{O}$. There is a dual tree $\bar{\mathcal{T}}$ pairings half lines plus dual decorations $\bar{B}$. Hence it is a fairly complicated combinatoric object which we shall describe in loose terms first.

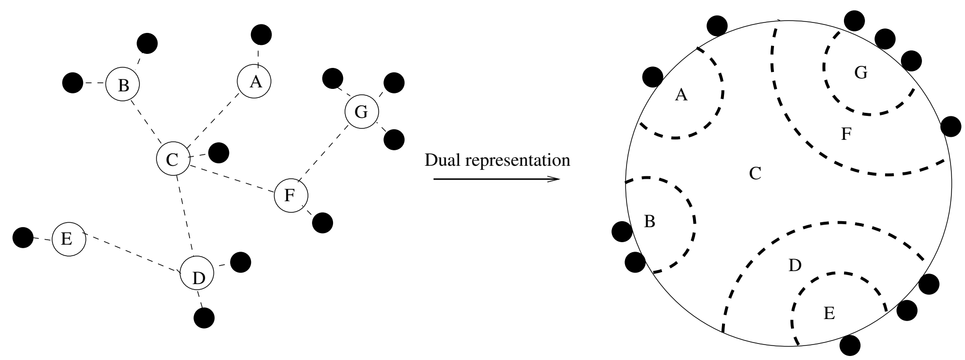


Figure 5: The dual graph of the tree of loop vertices. The regions A, B, E and G are in the right figure are the corresponding leaves of the left one. The ordinary dash lines for the $\sigma$ propagators on the LHS are drawn as bold dash line for the dual graph.

This dual object first consists of a single huge loop vertex, called the *cycle* $\mathcal{C}$ of the tree. This cycle contains all the ordinary resolvent lines of all the loop vertices of the direct representation, but read *in the cyclic order obtained by turning around the tree*. These full $\phi$-resolvent lines should be again carefully distinguished from the dual tree lines $\bar{\ell}$ of $\bar{\mathcal{T}}$, each of which corresponds to a line $\ell$ of the direct picture. To better distinguish the two pictures, we picture the dual tree lines of the dual representation $\bar{\ell}$ as bold dash lines rather than dotted lines. The important fact reflecting the tree character of $\mathcal{T}$ is that these bold dash lines $\bar{\ell}$ when drawn inside the disk $\mathcal{D}$ bounded by the cycle $\mathcal{C}$ *cannot cross*. Hence they divide this disk $\mathcal{D}$ into different connected regions. Each connected region $\bar{v} \in \bar{V}$ correspond to a single loop vertex $v \in V$ of the direct expansion, namely the one made of the lines forming the boundary of the region. This explains how the two pictures, dual of each other, are equivalent and how anyone can be reconstructed from the other.

Counterterms decorations have also to be added to this basic dual picture. They are pictured as arbitrarily many "black dots" $\bar{b} \in \bar{B}$ decorating the ordinary full lines of the cycle.

Remark that since the lines joining a counterterm to a loop vertex are omitted in the dual representation, we have $|\bar{\mathcal{T}}| = |\mathcal{T}| - |B|$.

The simple loop vertices of $S$, namely the loop vertices which were leaves in $\mathcal{T}$, can still be identified in this dual representation: they are indeed the minimal connected regions $\bar{s} \in \bar{S} \subset \bar{V}$ of the disk, namely those bounded by a single line $\bar{\ell}$.

This second representation is very interesting as it gives a canonical (up to an orientation choice) cyclic ordering of all ingredients occurring in a LVE term.

In this dual picture the measure $d\nu$ correspond to the following rule: the weakening factor between a $\sigma_{\bar{v}}$ and a $\sigma_{\bar{v}'}$ is the infimum of the $w$ parameters of the lines $\bar{\ell}$ that have to be *crossed* to join the two regions $\bar{v}$ and $\bar{v}'$.

In the set $\mathcal{O}$ there are four types of different objects which can occur when we follow the cycle $\mathcal{C}$, which have different values and need to be carefully distinguished:

- Black vertices $\bar{b} \in \bar{B}$
- Half-dash lines. They are labeled by an index $h \in H$; obviously $H$ has even cardinal, as $|H| = 2|\bar{\mathcal{T}}|$. The wavy lines $\bar{\ell} = (h_\ell, h_{\ell'})$ form a pairing on this set.
- Simple leaves i.e resolvent lines sandwiched between two consecutive objects in $H$ which are paired. They form a set $\bar{S}$ and are indexed by an index $\bar{v}_0$.
- Non-leaf resolvent lines, ie resolvent lines not in $\bar{S}$. They form a set $\mathcal{L}$, indexed by an index $\bar{v}_1$.

Hence $\mathcal{O} = \bar{B} \cup H \cup \bar{S} \cup \mathcal{L}$.

We have $|\bar{B}| = |B|$, $|\bar{S}| = |S|$, $|\bar{V}| = |V|$ and we can check that a contribution with tree $\bar{\mathcal{T}}$ and decorations $\bar{B}$ of $n$ vertices is of order $O(\lambda^{n-1})$, as we have $n-1 = |\mathcal{T}| = |\bar{B}| + |\bar{\mathcal{T}}|$.

We can also check

$$|\bar{S}| + |\bar{B}| + |\mathcal{L}| = 2(n-1); \quad |S| + |\mathcal{L}| = |\bar{B}| + |H|. \tag{27}$$

### 2.5 Resolvent Representations

It is convenient to introduce further notations to write down in more compact form the contribution, or amplitude of a tree in the LVE.

The beauty of the LVE representation is that all the various traces of the loop vertices in the direct representation give rise to a *single* trace in the dual representation. This is the fundamental observation which made the representation suited for constructive matrix and non commutative field theory [7].

Consider a dual tree $\{\mathcal{C}, \mathcal{O}, \bar{\mathcal{T}}\}$ made of the set of objects $\mathcal{O} = \bar{B} \cup H \cup \bar{S} \cup \mathcal{L}$ cyclically ordered according to $\mathcal{C}$, together with the pairing rules for the wavy lines in $H$ encoded in $\bar{\mathcal{T}}$.

We need a label $u$ to describe the various object met when turning around the cycle $\mathcal{C}$. To every counterterm in $\bar{b} \in \bar{B}$ is associated a position $x_b$ and to every half wavy line $h \in H$ a position $x_{\bar{\ell}(h)}$, the same for the two ends of any $\bar{\ell} \in \bar{\mathcal{T}}$ .

Then to each object $u \in \mathcal{O}$ is associated an operator $P_u$, with value

$$P_u = (-iT_\Lambda) D'(x_{\bar{b}}) \text{ if } u = \bar{b} \in \bar{B}, \tag{28}$$
$$P_u = D'(x_{\bar{\ell}(h)}) \text{ if } u = h \in H, \tag{29}$$
$$P_u = \hat{R}'(\sigma_{\bar{v}_0(\bar{\ell})}) \text{ if } u = \bar{v}_0 \in \bar{S}, \tag{30}$$
$$P_u = R'(\sigma_{\bar{v}_1(\bar{\ell})}) \text{ if } u = \bar{v}_1 \in \mathcal{L}. \tag{31}$$

Then in the dual resolvent representation we have:

$$\log Z(\lambda, \Lambda, \mathcal{V}) = R_2 + \sum_{\bar{\mathcal{T}} \text{ decorated cycle with } l \geq 3 \text{ lines}} G_{\bar{\mathcal{T}}}$$
$$G_{\bar{\mathcal{T}}} = (-\lambda)^{n-1} \int_{\mathcal{V}} [\prod_{\bar{\ell}} d^2 x_{\bar{\ell}} \prod_{\bar{b} \in \bar{B}} d^2 x_{\bar{b}}] \ \mathrm{Tr}_{\mathcal{V}} \left\{ \vec{\prod}_{u \in \mathcal{O}} P_u \right\} \tag{32}$$

where $R_2$ is the finite remainder terms up to 2nd order expansion and $\vec{\prod}$ means that one has to take the ordered product of the operators along the cycle $\mathcal{C}$.

# 3 Thermodynamic Limit at fixed UV cutoff

To perform the thermodynamic limit one has to divide by the volume to quotient out the translation invariance. This is an almost trivial but subtle "global gauge fixing" step which consists in fixing a preferred root line $\ell_0$ (hence a former $\phi^4$ vertex) in the tree $\mathcal{T}$ at the origin. The pressure has then the expansion:

$$
\begin{aligned}
&\frac{1}{|\mathcal{V}|}\log Z(\lambda,\Lambda,\mathcal{V}) = \int d\nu(\sigma)\mathrm{Tr}_{\mathcal{V}}\big(-\frac{1}{2}\log_4(1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2})\big) \\
+ &\int d\nu(\sigma_v)2i\sqrt{\lambda}T_\Lambda \hat{R}' + \int d\nu(\sigma_v\sigma_{v'})\hat{R}'\hat{R}' \\
+ &\sum_{n=3}^{\infty}\frac{1}{n!}\sum_{\mathcal{T}\text{ with } n \text{ vertices}} G_{\mathcal{T},\ell_0}, \\
&G_{\mathcal{T},\ell_0} = \Big\{\prod_{\ell\in\mathcal{T}}\int d^2x_\ell d^2y_\ell\big[\int_0^1 dw_\ell\big]\Big\}\int d\nu_{\mathcal{T}}(\{\sigma^v\},\{w\}) \\
&\Big\{\prod_{\ell\in\mathcal{T}}\big[\frac{1}{2}\delta(x_\ell-y_\ell)\frac{\delta}{\delta\sigma^{v(\ell)}(x_\ell)}\frac{\delta}{\delta\sigma^{v'(\ell)}(y_\ell)}\big]\Big\}\prod_{v=1}^{n}V_v|_{x_{\ell_0}=0}
\end{aligned} \tag{33}
$$

**Theorem 3.1.** *This expansion (33) for the pressure $P_\Lambda$ is absolutely convergent and defines an analytic function in the half-disk $\mathcal{D}_\Lambda - \{\lambda \mid \Re(\lambda^{-1}) \geq K\log\Lambda$, where $K$ is a large constant.*

**Proof**

This is just an application of the techniques of [7, 8] which we recall briefly.

- The number of trees is $n^{n-2}$,
- each resolvent $R$ is bounded by 1 in the disk,
- each resolvent $\hat{R}$ is bounded by 2 in the disk,
- each decorating counterterm is bounded by $\log\Lambda$.

So the connected function is bounded by

$$
P_\Lambda \leq \sum_n \frac{n^{n-2}}{n!}2^{n-1}\lambda^n \ln\Lambda^n \leq \sum(\lambda K)^n\log\Lambda^n. \tag{34}
$$

which is a convergent geometric series as long as $\Re(\lambda^{-1}) \geq K\log\Lambda$

□

However the radius of convergence goes to zero as $\Lambda \to \infty$. This divergence comes from the linear counterterms that are unbounded. To improve the bounds we *must* combine several different terms in the LVE in order to compensate *many* of these counterterms with the inner tadpoles that can be generated when contracting the $\sigma$ fields hidden in the resolvents. However we cannot compensate *all* these tadpoles because that would generate the full ordinary renormalized perturbative series hence lead to a divergent series.

We have therefore to settle for a compromise in the compensation of inner tadpoles. This compromise is provided by the cleaning expansion, which is a systematic expansion along the cycle $\mathcal{C}$ but with a *stoppping rule*. This stopping rule stipulates that if enough non-tadpole $\phi$-propagators have been generated at high enough energy we should stop and retransform the remaining counterterms in the rest of the loop $\mathcal{C}$ into an oscillating exponential of the form $e^{2i\sqrt{\lambda}\sigma T_\Lambda}$ type, which can be then bounded by 1. But this backward step however comes with some price: it generates a factor $e^{+\lambda T_\Lambda^2 |\mathcal{V}|}$. However we can taylor our stopping rule to precisely ensure that enough good factors have been generated to pay for this kind of bad term. This is our implementation in this context of the so-called *Nelson's bound* [1].

Unfortunately Nelson's bound is exponential in the volume. This is why we need now to perform a cluster expansion which selects the volume truly occupied by an LVE term. This auxiliary cluster expansion is not necessary for Grosse-Wulkenhaar models or in the case of the $\phi_2^4$ theory on a compact Riemannian surface, for which the LVE is therefore more natural.

# 4 The cluster expansion

In this section we consider the connected function in a finite large volume $\mathcal{V}$ made of a union of unit volume squares, and we test which region of $\mathcal{V}$ is really occupied by the $\sigma$ fields in a given LVE term. For this purpose we need a standard cluster expansion on the regular propagator $C$; the ultralocal propagator for $\sigma$ cannot couple distinct squares (see Figure 6).

The formalism is uniform in the limit $\mathcal{V} \to \mathbb{R}^2$. Consider a 2 dimensional finite lattice of unit squares $\mathcal{D} = \{\Delta_0, \cdots, \Delta_{|\mathcal{V}|-1}\} \in \mathcal{V}$ centered on $(1/2, 1/2) + \mathbb{Z}^2$ of finite volume $\mathcal{V}$, where $|\mathcal{V}|$ is an integer. This is convenient since there is in $\mathcal{D}$ a unique square $\Delta_0 = [-1/2, 1/2] \times [-1/2, 1/2]$ centered at the origin. The set of pairs of different squares $b_{ij} = \{\Delta_i, \Delta_j\}$, $\Delta_i \neq \Delta_j$ of $\mathcal{D}$ is called $\mathbb{B}$.

Define $\chi_{\Delta_i}(x)$ as the characteristic function of the square $\Delta_i \in \mathcal{D}$ as

$$\chi_{\Delta_i}(x) = \begin{cases} 1, \text{ if } x \in \Delta_i \\ 0, \text{ otherwise} \end{cases} \tag{35}$$

We can use once more the BKAR forest formula on any LVE term $G_{\mathcal{T},\ell_0}$. The idea is to test whether any $C$ propagator in $G_{\mathcal{T},\ell_0}$ is a tree link between different squares or is a loop.

However here a technical subtlety occurs. It is simpler to perform the cluster expansion on the $C$ propagators; not on the $C^{1/2}$. But this can be done using the $D$ and $R$ rather than $D'$ and $R'$ representation. Recall we can write any loop vertex either as a trace of $D$ and $R$ operators or as the same trace with $D'$ and $R'$ operators.

Using the non-symmetric representation without prime we consider the $C$ operators as matrix-valued operators between cubes:

$$C_{\Delta,\Delta'}(x,y) = \chi_{\Delta}(x) C(x,y) \chi_{\Delta'}(y) \tag{36}$$

and we apply the forest formula to $G_{\mathcal{T},\ell_0}$ that is we interpolate the off-diagonal terms[3].

The result is

$$G_{\mathcal{T},\ell_0} = \sum_{\mathcal{T}',\Gamma \supset \Delta_0} G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma} \tag{37}$$

where $\Gamma$ is any finite set of squares in $\mathcal{D}$ and $\mathcal{T}'$ is a spanning tree joining these squares.

The formula for $G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma}$ is a bit heavy since it is similar to (13). Let us rather describe it in plain words. In $G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma}$ for each link $\ell' \in \mathcal{T}'$ between squares $\Delta_{\ell'}$ and $\Delta'_{\ell'}$ we have an explicit propagator $C$ pulled in the numerator of the cycle $\mathcal{C}$ with one end in $\Delta_{\ell'}$ and the other in $\Delta'_{\ell'}$. Furthermore there exists a weakening parameter $w'_{\ell'}$ for each link $\ell'$ in $\mathcal{T}'$ integrated from 0 to 1. Finally any other remaining propagator $C$ in $G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma}$ is a function of these $w$'s through the infimum formula between end points, namely $C(x,y)$ for $x \in \Delta$ and $y \in \Delta'$ has to be multiplied by the infimum of the parameters $w'_{\ell'}$ for $\ell'$ in the unique path of $\mathcal{T}'$ joining $\Delta$ to $\Delta'$.

Let us answer a few natural questions at this point:

[3] Technically the BKAR formula is written for pairs on a *finite set* and this is why we take $\mathcal{V}$ finite; but the infinite volume limit is uniform.

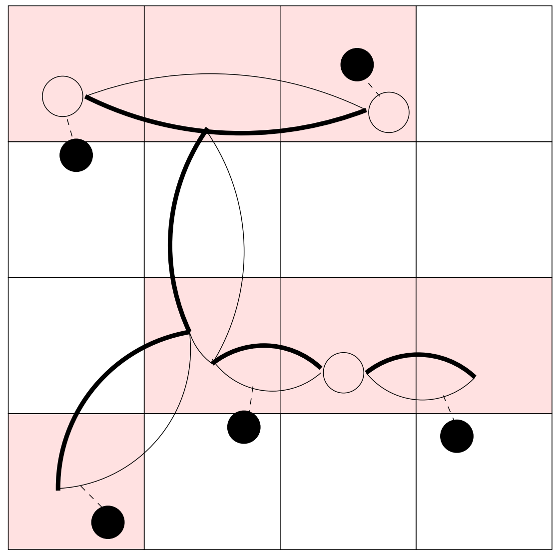

Figure 6: An LVE which spreads over a set $\Gamma$. The ultralocal $\sigma$ propagators that connect the loops are omitted. Each loop has several counterterms attached. The tree $\mathcal{T}'$ is shown in bold.

- Why does the BKAR formula gives directly a result indexed by trees rather than forests? This is because the cycle $\mathcal{C}$ is connected and the $\sigma$ measure is ultralocal. The vertex $\ell_0$ was fixed at the origin, hence $\Gamma$ must contain $\Delta_0$. The BKAR formula a priori leads to a result indexed by forests on $\mathcal{D}$, but every forest which is not a tree in this case gives zero contribution because the cycle cannot jump from a square to another without using propagators[4], as these are the only non-local terms in $G_{\mathcal{T},\ell_0}$.

- What about the sigma field measure after the volume has been restricted to $\Gamma$? It can be replaced by a measure which is a white noise on $\Gamma$ only, that is a Gaussian measure $d\nu_\Gamma(\sigma)$ of propagator

$$\chi_\Gamma(x)\delta(x-y)\chi_\Gamma(y). \tag{38}$$

In other words after the cluster expansion has delimited the volume occupied by all ends of $C$ propagators, the $\sigma$ field must also live only

[4]Here the pernickety reader could worry whether our characteristic functions include the boundary of the squares or not, but this is a zero measure hence irrelevant subproblem.

in that volume, and can be freely replaced by $\chi_\Gamma \sigma$. This is because expanding resolvents, $\sigma$ fields can be situated only at ends of propagators and this expansion is easily shown Borel-summable at least in any finite volume with a finite ultraviolet cutoff, hence it determines the support properties of $\sigma$. This is an important point for the later bounds and for Nelson's argument.

- What about the convergence of the expansion? Is the new sum over trees $\mathcal{T}'$ not going to jeopardize the combinatoric of this convergence? Again the answer is no. The true formula behind the BKAR formula is an ordered formula in which the tree $\mathcal{T}'$ can be created in any arbitrary order. The corresponding factorial $|\mathcal{T}'|!$ which comes from the choice of resolvents to be derived is then automatically canceled by the simplex integral over the ordered $w'$s parameters.

- What about the sum over $\Gamma$, hence over the cluster? Any line $\ell$ of $\mathcal{T}'$ is associated to an explicit $C$ propagator between the two different squares $\Delta_\ell, \Delta'_\ell$. Hence in the estimate of $G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma}$ we have always tree exponential decay. More precisely defining $\tau(\Gamma)$ as the minimal length of any tree connecting $\Gamma$ we can extract a factor $e^{-c\tau(\Gamma)}$ from the bound on $|G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma}|$. This exponential decay ensures that we could sum over the position of all the squares in $\Gamma$, starting from the leaves of the tree until we arrive at the root $\Delta_0$ which is fixed. Remark also that the exponential tree decay allows to absorb, through the so called "volume effect", any finite product of the factorials of the coordination number of the tree $\mathcal{T}'$. Indeed large coordination number emanating from one square always lead to many propagators traveling a large distance see eg [1], chapter III.1, Lemma III.1.3. Such factorials easily occur if one uses relatively sloppy bounds.

Remark that as we obtained already connected functions by LVE, we don't follow this cluster expansion with a Mayer expansion as in the traditional method of constructive physics. This is the main advantages of the LVE. We turn now to the main expansion of this paper.

# 5 The Cleaning Expansion

## 5.1 Multiscale analysis

In this section we shall for simplicity study the connected vacuum amplitudes in unit volume. The Schwinger functions and the infinite volume limit will be considered in following sections.

It is convenient to put $\Lambda = M^{j_{max}}$, where $M > 1$ is a constant, $j_{max} \in \mathbb{N}$ and the ultraviolet limit is $j_{max} \to \infty$, so that we can slice the propagator according to renormalization group slices as $C^\Lambda = \sum_{j=0}^{j_{max}}$ with:

$$C_j(x,y) = \int_{M^{-2j}}^{M^{-2j+2}} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}} \frac{d\alpha}{\alpha} \leq K e^{-cM^j|x-y|}. \tag{39}$$

$K$ and $c$ are generic names for inessential constants, respectively large and small.

Considering the square root decomposition of $C$ with a middle point $x$

$$D'(y,x,z) = 2iC^{1/2}(y,x)C^{1/2}(x,z). \tag{40}$$

we have an associated matrix-like decomposition of $D$ as

$$D'_{jk}(x) \equiv 2iC_j^{1/2}(.\ ,x)C_k^{1/2}(x,\ .). \tag{41}$$

Similarly the resolvent $R$ writes

$$R'_{jk}(\sigma,x) \equiv \frac{1}{1+2i\sqrt{\lambda}C_j^{1/2}(.\ ,x)\sigma(x)C_k^{1/2}(x,\ .)}. \tag{42}$$

The multiplication rule of two $D_{ij}(x)$ operators reads:

$$\begin{aligned} \int dzdy' D'_{jk}(y,x,z)D'_{lm}(y',x',z') &= \int dzdy'(2i\sqrt{\lambda})^2\delta_{kl}\delta(z-y')C_j^{1/2}(y,x) \\ &\quad C_k^{1/2}(x,z)C_l^{1/2}(y',x')C_m^{1/2}(x',z') \qquad (43) \\ &= -4\lambda C_j^{1/2}(y,x)C_k(x,x')C_m^{1/2}(x',z'), \end{aligned}$$

so that the integral over $z$ and $y'$ reconstructs $C_k$ in the middle[5].

[5]Remark that such integrals over middle propagators points are never restricted to a finite volume $\mathcal{V}$, but always performed over all $\mathbb{R}^2$. Only end points of $C$'s are affected by our volume cutoff, if any.

It is convenient to split these matrix-valued operators according to the largest of their two indices:

$$
\begin{aligned}
D' &= \sum_j D'_j, \quad D'_j = \sum_{k,l \text{ such that } \sup(k,l)=j} D'_{kl}, \\
R' &= \sum_j R'_j, \quad R'_j = \sum_{k,l \text{ such that } \sup(k,l)=j} R'_{kl}.
\end{aligned} \tag{44}
$$

We define also the less symmetric operators:

$$
\begin{aligned}
D(x) &= 2iC(x,.), \qquad (45)\\
R(\sigma) &= \frac{1}{1+2i\sqrt{\lambda}\sigma C}, \quad \hat{R}(\sigma) = \frac{1}{1+2i\sqrt{\lambda}\sigma C} - 1,
\end{aligned}
$$

and the full resolvent

$$
\mathcal{R} = DR, \quad \hat{\mathcal{R}} = D\hat{R}. \tag{46}
$$

The multiscale representation of these operators reads:

$$
D_{\bar{j}} = \sum_{k \le j} D_k, \quad R_{\bar{j}} = \sum_{k \le j} R_k, \tag{47}
$$

and

$$
D_k(x,\sigma) = D_k(x) \cdot \sigma(x). \tag{48}
$$

While the less symmetric representation is more suitable for the cleaning expansion, the symmetric one is more suitable for proving the bounds of the resolvents. In this section we use only the unsymmetric representation of the resolvents (45). Then one may ask the following question: after the weakening parameters $w$ have been introduced, is it possible to return to the square root formulation of $G$ in terms of $D'$ and $R'$ operators? This is a clever question and the good news is that the answer is yes. It is good to return to the $D'$ and $R'$ operators rather than keeping the not-hermitian $D$ and $R$ operators because the norm estimates are clearer with these square roots; it is obvious that $\|(1+iH)\| \le 1$ if $H$ is Hermitian, but if $H$ is not Hermitian one could worry. Fortunately it is possible to reconstruct square roots after the $w'$ parameters have been added, simply because the matrix of the $w'$ parameters between the squares of $\Gamma$, completed by a 1 on the diagonal, is *positive* symmetric, hence has itself a positive square root. The problem is studied at length in eg section 4.2 of [22], so we refer the reader to that paper.

We decompose also the counterterm $T_\Lambda$ as:

$$T_\Lambda = \sum_{j \le jmax} T_j, \quad T_{\bar{j}} = \sum_{k \le j} T_k, \tag{49}$$

where

$$T_k = \int d^2x C_k(x,x). \tag{50}$$

In this case the formula of the loop vertex expansion becomes:

**Theorem 5.1** (Multi Scale Loop Vertex Expansion)**.**

$$\log Z(\lambda, \Lambda, \mathcal{V}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} G_{\mathcal{T}} \tag{51}$$

$$G_{\mathcal{T}} = \sum_{j=j_{min}}^{j_{max}} \Big\{ \prod_{\ell \in \mathcal{T}} \int d^2x_\ell d^2y_\ell \big[ \int_0^1 dw_\ell \big] \Big\} \int d\nu_{\mathcal{T}}(\{\sigma^v\}, \{w\})$$

$$\Big\{ \prod_{\ell \in \mathcal{T}} \big[ \frac{1}{2} \delta(x_\ell - y_\ell) \frac{\delta}{\delta \sigma^{v(\ell)}(x_\ell)} \frac{\delta}{\delta \sigma^{v'(\ell)}(y_\ell)} \big] \Big\} \prod_{v=1}^{n} V_v^j,$$

*the notations being straightforward and almost identical to the ones of Theorem 2.1.*

## 5.2 Overview of the Cleaning Expansion

We turn now to the heart of our paper, namely to the cleaning expansion. It develops perturbation theory, and combines different LVE terms in order to compensate any inner tadpole with other LVE's which have counter terms at the exact position of these tadpoles.

In a Feynman graph let's call *good line* at scale $j$ or in short $j$-line any $C_j$ propagator which is not part of a tadpole. Conversely a $j$-inner-tadpole is a line $C_j$ enclosed between two $\sigma$-half propagators, hence the scale $j$ part of an inner tadpole. The key observation on which the cleaning expansion rests is that when a sufficiently large number of $j$-lines of high enough $j$ has been produced, we can stop the expansion, re-detach the counter terms from the LVE, and pay the corresponding bad estimate. This is the analog in our context of Nelson's argument.

A characteristic feature of the cleaning expansion is to use the natural canonical cyclic ordering on any LVE provided by its dual representation

(32). We start from an arbitrary origin of the cycle which in fact can be chosen as the preferred root point at the origin in (33). Then we use a Taylor formula with integral remainder which forces potential tadpole or non-tadpole propagators to appear, in their natural order "along the cycle $\mathcal{C}$". This formula works along the cycle[6]. We compensate the tadpoles produced, if any, with the other LVE terms with appropriate counterterms at these exact tadpoles positions. This compensation is called the "cleaning".

We cannot clean forever, as this would develop the full perturbation series and ultimately diverge. But we stop and write a Taylor integral remainder when enough cleaning, depending on the scale, has succeeded. If we were not to use the canonical cyclic ordering of the LVE, it would be difficult to find the corresponding "weakening parameters" rule for that Taylor remainder term. But fortunately the cyclic ordering[7] solves nicely this problem! The beginning of the cycle $\mathcal{C}$ is explicitly and fully cleaned of potential tadpoles; the rest of the cycle $\mathcal{C}$ has *no weakening parameters* on the remaining potential tadpoles!

The cleaning is done scale by scale along the cycle, starting from $j_{max}$ towards scale 0. We expand the $\sigma$ fields and contract them, either detecting tadpoles or detecting perturbation lines of scale $j$ which *are not tadpole lines*. If at any scale we find a Taylor remainder term (i.e. the cycle contained more than $a.j$ not-tadpoles lines of scale $j$), we stop the expansion and *don't test lower scales*. There are two kinds of $j$-lines in the cleaning expansion: the ones which belong to *crossing* sigma propagators like in Figure 7 and the ones which appear into *nesting* sigma propagators (see Figure 8 ). In both cases the line is a $j$-line because we are sure it cannot belong to a tadpole[8].

Each crossing $j$-line of scale $j$ could be simply bounded by

$$\int d^2xC_j(x,y) = \int_{M^{-2j}}^{M^{-2j+2}} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}}\ \frac{d\alpha}{\alpha} \leq Ke^{-cM^j|x-y|} \sim KM^{-2j}, \quad (52)$$

[6]Here the expansion is not canonical (but only in a very slight way) since the two natural orderings along the cycle could be used.

[7]Remark that this cyclic ordering is a new feature of the LVE without any analog in the Feynman graphs or in the usual constructive tools, and that it is also the reason why the LVE can build constructively matrix models.

[8]This is a simplifying feature of the ordinary $\phi^4_2$ theory; in the $GW_4$ theory we expect that nesting situations potentially require renormalization, because they lead to planar graphs; only crossings give good factors. Hence nesting lines should not be counted in the stopping rule. Nevertheless the expansion should still converge as nesting lines lead to planar graphs, which are very few compared to the non-planar ones and lead to convergent series.

and each nesting $j$-line is bounded by the integration of the full resolvent exactly in the same way than a crossing $j$-line

$$
\begin{aligned}
|\int d^2x \mathcal{R}_j(\sigma, x, y)| &= 2|\int d^2x R_j(\sigma) C_j(x,y)| \leqslant 2\int d^2x |R_j(\sigma)| \cdot |C_j(x,y)| \\
&\leqslant 2\int d^2x |C_j(x,y)| \sim KM^{-2j}, \qquad (53)
\end{aligned}
$$

where we have used the fact that $|R_j(\sigma)| < 1$.

Hence for any $j$-line, the amplitude (after bounding all remaining resolvents and oscillating factors of the $e^{ia\sigma}$ type by constants as in (74)-(77) will be that of an *ordinary graph* with that $j$-line, hence it will provide a small factor $KM^{-2j}$ [1].

A growing number of such compensations must be performed as the scale $j_{max} \to \infty$, because of the worsening of "Nelson's bound" with the ultraviolet cutoff [1].

Hence the result is indexed by new LVE's with a dividing scale $j_0$. No uncompensated tadpoles of scale higher than $j_0$ remain. Among tadpoles of scale $j_0$, the first $a.j_0$ along the cycle are compensated, and the next ones are not. Finally all potential tadpoles of scale lower than $j_0$ are uncompensated[9].

The reason for this rule will become apparent in section 6.1. Consider the theory in a single unit square $\Delta$. After this cleaning expansion, if the dividing scale is $j_0$ we have accumulated at least $a.j_0$ good factors $M^{-cj_0}$ from the power counting of non-tadpole lines with scale $j_0$. This allows to pay both for a sloppy bound on a reexponentiated $e^{2iTr_\Delta \sqrt{\lambda} T_{\bar{j}} \sigma}$ oscillating term (see (9) for the origin of that term) and for the large combinatoric of the expansion. This is the exact analog for the LVE of Nelson's argument as explained in [1].

This ends up the story for the theory in a single square. But in the infinite volume case, Nelson's bound has to be paid in each unit volume square $\Delta$ *actually occupied by the LVE term*. That's why we had to introduce the cluster expansion of section 4. Once knowing the exact finite set $\Gamma$ of squares occupied by the LVE, we need to repeat the cleaning operation once for each square in $\Gamma$. This is conceptually easy and clear, but the notations become heavy and obscure the argument. Hence we define below the cleaning

[9]There is a last term with no dividing scale, or by convention a kind of dividing scale -1; it is the one where the cleaning succeeded at all scales. That one has no longer any tadpole at any scale.

expansion only in a single unit square for simplicity and let the generalization to several squares to the reader.

## 5.3 The stopping rule

The expansion should be based on developing first only the resolvent of highest scale and then the lower ones. It uses the formula

$$
\begin{aligned}
R^j(\sigma) &= \frac{1}{1+\sum_{k\leq j} D^k(\sigma)} = \frac{1}{1+\sum_{k<j} D^k\sigma + D^j\sigma} \\
&= R^{j-1}(\sigma)\frac{1}{1+R^{j-1}(\sigma)D^j\sigma}.
\end{aligned} \tag{54}
$$

We then decompose the amplitude of the dual tree according to the scale of each object and we can write

$$
G_{\bar{\mathcal{T}},\ell_0} = \sum_{j=0}^{j_{max}} G^j_{\bar{\mathcal{T}}} = (-\lambda)^{n-1}\sum_{j}^{j_{max}} \int_{\mathcal{V}} \prod_{\bar{\ell}} d^2x_{\bar{\ell}} \prod_{\bar{b}\in\bar{B}} d^2x_{\bar{b}} Tr_{\mathcal{V}} \Big\{ \vec{\prod}_{u\in\mathcal{O}} P^j_u \Big\} \tag{55}
$$

where the $\vec{\prod}$ means that we take the ordered product of operators along the cycle $\mathcal{C}$ and $\ell_0$ is the marked point where we start the expansion.

In this way an algebraic expansion step of the resolvent consists in writing:

$$
R^j(\sigma) = R^{j-1}(\sigma) - R^{j-1}(\sigma)\cdot\sigma\cdot D^j \; R^j(\sigma) \tag{56}
$$

for any scale index $j$.

We call formula (56) the cleaning expansion for the resolvent in that we factorize the pure propagator and the remaining resolvents where other possible $\sigma$ fields are hidden. We shall use the integration by parts to contract the $\sigma$ fields generated with other one hidden in the resolvent. More explicitly, we use the two formulae to generate either an inner tadpole or a crossing (see also graph 7):

$$
\int d\mu(\sigma)\sigma R^j(\sigma) = \int d\mu(\sigma)\frac{\partial}{\partial\sigma}R^j(\sigma) = -R^j(\sigma)D^{\bar{j}}R^j(\sigma). \tag{57}
$$

and

$$
R^j = \frac{1}{1+\int D^{\bar{j}}\;\sigma} = 1 - \int D^{\bar{j}}\;\sigma R^j(\sigma). \tag{58}
$$

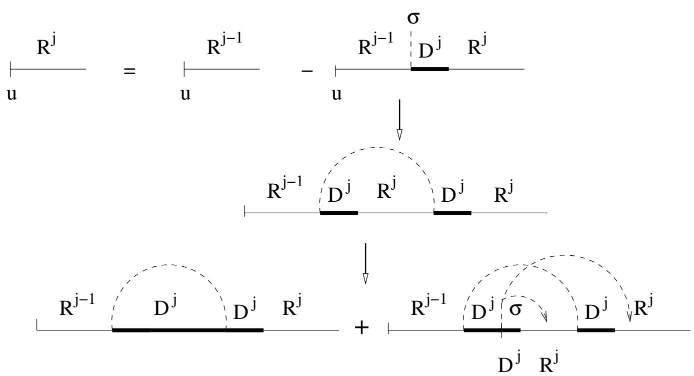


Figure 7: A typicle process of cleaning expansion. The LHS of the last line stand for an innertadpole, while the R.H.S could be either a crossing or a nesting line. Here u is the marked point.

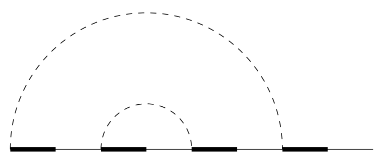

Figure 8: A graph of LVE with two nesting lines

We start the cleaning expansion for the resolvent with formula (56) which is on the *right* side of an arbitrary marked point ($\ell_0$ for example), and use the integration by parts for the $\sigma$ fields clockwisely to generate $j$-lines or tadpoles. In the latter case we search for the corresponding term with a counterterm instead of the tadpole and perform the cancellation; the result is exactly 0. In conclusion, we stop the expansion whenever we generate an inner tadpole and we compensate it by the graph that has the same structure except that a tadpole is replaced by a counter term. This cancelation is exact, so that no inner tadpoles should appear in the renormalized graph. See section 7.2.

On the other hand we could gain a convergent factor $M^{-j}$ for each $j$-line. But for a given scale $j$ we should not generate an arbitrary number of crossings, since otherwise the expansion would diverge. More precisely, we start from the resolvents of scale $j_{max}$. Each time a $j_{max}$-line is generated, we add 1 to a counter. We stop the expansion until the number of $j_{max}$-lines reaches $N_{j_{max}} = aj_{max}$. In that case we have gained a convergent factor at least $M^{-j^2_{max}} \leq e^{-j^2_{max}}$ for $M > e$. Otherwise, hence if we couldn't generate

$N_{j_{max}}$ $j_{max}$-lines. we turn to the expansion of the resolvents of scale $j_{max}-1$ and so on.

The combinatoric factor coming from a maximal number of $N_j = aj$ crossings in the loop vertex expansion reads:

$$N_j! \sim e^{N_j \ln N_j} \sim e^{aj(\ln j + \log a)}. \tag{59}$$

This term is not dangerous as it is easily bounded by $e^{-j^2}$, which is what the $N_j$ $j$-lines provide.

The cleaning expansion is conceptually clear but the explicit mathematical notations are heavy, like in any "conditional expansion" where the steps depends on the previous choices. So we propose to consider the case of a relatively simple loop $\mathcal{C}$ of the dual representation with only *two* resolvent $R(x_1, x_2)$ $R(y_1, y_2)$ to explain the cleaning expansion, see Figure 9. It corresponds to an $n = 2$ term in the LVE. The reader can convince himself easily that this example generalizes to an arbitrary number of resolvents in $\mathcal{C}$, with just heavier notations.

## 5.4 An Example

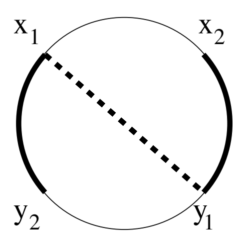


Figure 9: An example for the cleaning expansion. The ordinary line means the resolvent while the thick line means the pure propagator. The bold dash line is the ultralocal $\sigma$ propagator in the dual representation and corresponds to the term $\delta(x_1 - y_1)$.

We start the cleaning expansion from the highest scale $j_{max}$ and downwards until we gain enough convergent factors. So in this example we shall use the following formula for each resolvent:

$$R^{j_{max}}(\sigma) = R^{j_{max}-1}(\sigma) - R^{j_{max}-1}(\sigma)\sigma D^{j_{max}}\ R^{j_{max}}(\sigma). \tag{60}$$

The amplitude for this graph reads:

$$
\begin{aligned}
G &= \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 R^{j_{max}}(\sigma, x_1, x_2) C^{j_{max}}(x_2, y_1) \\
& \quad R^{j_{max}}(\sigma, y_1, y_2) C^{j_{max}}(y_2, x_1).
\end{aligned} \tag{61}
$$

We choose $x_1$ to be the fixed marked point and start the cleaning expansion from the resolvent $R^{j_{max}}(x_1, x_2)$. We have

$$
\begin{aligned}
& R^{j_{max}}(\sigma, x_1, x_2) = R^{j_{max}-1}(\sigma, x_1, x_2) \qquad (62) \\
- & \int d^2z_1 d^2z_2 R^{j_{max}-1}(\sigma, x_1, z_1)\sigma(z_1) D^{j_{max}}(z_1, z_2) R^{j_{max}}(\sigma, z_2, x_2).
\end{aligned}
$$

So the amplitude reads:

$$
\begin{aligned}
G &= \sum_{j}^{j_{max}} \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 R^{j_{max}-1}(x_1, x_2) C^{j_{max}}(x_2, y_1) \\
& \quad R^{j_{max}}(y_1, y_2) C^{j_{max}}(y_2, x_1) \delta(x_1 - y_1) \; - \sum_{j}^{j_{max}} \int d\mu(\sigma) \\
& \quad \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2z_1 d^2z_2 R^{j_{max}-1}(x_1, z_1) \sigma(z_1) D^{j_{max}}(z_1, z_2) \\
& \quad R^{j_{max}}(z_2, x_2) C^{j_{max}}(x_2, y_1) R^{j_{max}}(y_1, y_2) C^{j_{max}}(y_2, x_1) \delta(x_1 - y_1).
\end{aligned} \tag{63}
$$

We shall first forget the terms of scale $j_{max} - 1$ and consider only the terms of scale $j_{max}$, which gives the main contribution to the correlation function. And we to go back to the terms of scale $j_{max} - 1$ or lower ones in the renormalization process. So we forget the first term in (63).

Since there is a single $\sigma$ field in the numerator of the second term of (63), we perform the the integration by parts and second term reads:

$$
\begin{aligned}
- & \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2z_1 C^{j_{max}}(y_2, x_1) D^{j_{max}}(z_1, z_2) \\
& \frac{\delta}{\delta\sigma(z_1)} [R^{j_{max}-1}(\sigma, x_1, z_1) R^{j_{max}}(\sigma, z_2, x_2) R^{j_{max}}(\sigma, y_1, y_2)] C^{j_{max}}(x_2, y_1)
\end{aligned}
$$

Again we ignore the case of $\frac{\delta}{\delta\sigma(z_1)} R^{j_{max}-1}(\sigma, x_1, z_1)$ which is of lower scale and the case $\frac{\delta}{\delta\sigma(z_1)} R^{j_{max}}(\sigma, y_1, y_2)$, which means that we generate a crossing line whose amplitude is small and convergent.

So the main contribution to (64) is:

$$
\begin{aligned}
- & \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2z_1 d^2z_2 C^{j_{max}}(y_2, x_1) R^{j_{max}-1}(\sigma, x_1, z_1) \\
& D^{j_{max}}(z_1, z_2) \frac{\delta}{\delta\sigma(z_1)} [R^{j_{max}}(\sigma, z_2, x_2)] R^{j_{max}}(\sigma, y_1, y_2) C^{j_{max}}(x_2, y_1) \\
= & \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2z_1 d^2z_2 d^2w_1 d^2w_2 C^{j_{max}}(y_2, x_1) \\
& R^{j_{max}-1}(\sigma, x_1, z_1) D^{j_{max}}(z_1, z_2) R^{j_{max}}(\sigma, z_2, w_1) D^{\bar{j}_{max}}(w_1, w_2) \delta(w_1 - z_1) \\
& R^{j_{max}}(\sigma, w_2, x_2) C^{j_{max}}(x_2, y_1) R^{j_{max}}(y_1, y_2) C^{j_{max}}(y_2, x_1) \delta(x_1 - y_1). \quad (64)
\end{aligned}
$$

We use the following formula for the resolvent $R^{j_{max}}(\sigma, z_2, w_1)$ which is sandwidged by two pure propagator:

$$
R^j(z_2, w_1) = [\frac{1}{1+\sigma D^{\bar{j}}}]\ (z_2, w_1) = \delta(z_2, w_1) - [\sigma D^{\bar{j}} R^j]\ (z_2, w_1). \qquad (65)
$$

Then (64) reads:

$$
\begin{aligned}
& \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2z_1 d^2w_2 C^{j_{max}}(y_2, x_1) R^{j_{max}-1}(\sigma, x_1, z_1) \\
& D^{j_{max}}(z_1, z_1) D^{\bar{j}}(z_1, w_2) R^{j_{max}}(\sigma, w_2, x_2) C^{j_{max}}(x_2, y_1) R^{j_{max}}(y_1, y_2) \\
& C^{j_{max}}(y_2, x_1) \delta(x_1 - y_1) \\
- & \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2z_1 d^2z_2 d^2w_1 d^2w_2 d^2w_3 C^{j_{max}}(y_2, x_1) \\
& R^{j_{max}-1}(\sigma, x_1, z_1) D^{j_{max}}(z_1, z_2) \delta(w_1 - z_1) \sigma(z_2) D^{\bar{j}}(z_2, w_3) R^{j_{max}}(\sigma, w_3, w_1) \\
& D^{\bar{j}}(z_1, w_2) R^{j_{max}}(\sigma, w_2, x_2) C^{j_{max}}(x_2, y_1) R^{j_{max}}(y_1, y_2) \delta(x_1 - y_1), \qquad (66)
\end{aligned}
$$

In the first term an inner tadpole of scale $j_{max}$ is generated (see also Figure 10) which should be compensated by a counter term of the same scale. Now we consider the second term. We still perform integration by parts for the intemediate field $\sigma(z_2)$. If $\sigma(z_2)$ acts on $R^{j_{max}}(\sigma, w_3, w_1)$, we shall repeat the analysis as before to see whether this would generate an inner tadpole. If $\sigma(z_2)$ acts on $R^{j_{max}}(\sigma, w_2, x_2)$, then this generates a crossing and we gain a convergent factor $M^{-2j_{max}}$. There is also the third possibility that the $\sigma$ field acts on the resolvent $R^{j_{max}}$ whthout generating a tadpole nor a crossing. We call this line a nesting line. There could be infinite number of such "nesting" lines. But the nesting lines are not dangerous. For each of them we gain a convergent factor $M^{-2j_{max}}$.

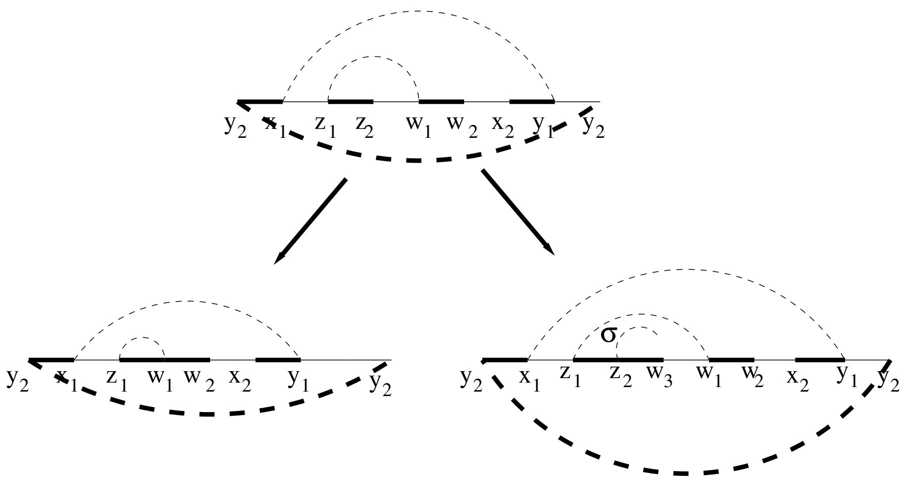


Figure 10: The graphs corresponding to formulae (64) and (66). We omit the scales of the corresponding resolvents and pure propagators. The bold dash line means we identify the two ends of the line so as to make it a loop. The intermediate field $\sigma(z_2)$ of the second line should be contracted with other ones hidden in the other resolvents. If it hits the resolvent $R^j(w_3, w_1)$, then this generates a nesting line; if it hits the resolvents $R^j(w_2, x_2)$ or $R^j(y_1, y_1)$ by crossing the lines $\{z_1, w_1\}$ or $\{x_1, y_1\}$, this would generate a crossing line.

## 5.5 Reexponentiation of Remaining Counterterms

All inner tadpoles have been canceled against the appropriate counterterms in the cleaned part of the dual graph. But there might still be arbitrary number of counterterms in the uncleaned part and they are divergent. Instead of canceling all of them, we reexponentiate them by using the properties of Gaussian measure and integration by parts. We consider first of all the case of the connected function in a unit square and then consider the general case.

The general formula for the remainder term of order $N$ reads:

$$A^N_{\bar{\mathcal{T}}}|_{|\bar{\mathcal{T}}|=N+1} = \prod_{l,l'\in\bar{\mathcal{T}}} \int d\nu(w,\sigma) \int_0^1 dw_{l'} \lambda^N \prod \hat{R}_l(\sigma) \prod R_{l'}(\sigma) \prod_{m=1}^{N-n} (T_\Lambda)^m. \quad (67)$$

where we have used the fact that the weakening factors for the counterterm leaves equal to one. There are only weakening factors for the propagators between different loop vertices.

Now we consider the unrenormalized amplitude

$$G = \int d\nu(\sigma, w) \prod_{l,l' \in \bar{\mathcal{T}}} \lambda^{|\bar{\mathcal{T}}|} [\hat{R}_l(\sigma)][R_{l'}(\sigma)] e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda}. \tag{68}$$

We use the formula

$$\int d\nu(w,\sigma) f(\sigma) g(\sigma) = e^{\frac{1}{2}\frac{\partial}{\partial\sigma(x)} C(x,x',w) \frac{\partial}{\partial\sigma(x')}} f(\sigma) g(\sigma))|_{\sigma=0}, \tag{69}$$

where $C(x,x',w)$ is the covariance that may or may not depend on the weakening factor $w$. Hence

$$\begin{aligned}
G &= \int d\nu(w,\sigma) \\
&\quad \sum_{N=0}^{\infty} \frac{1}{N!} [\frac{1}{2}\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}]^N \{ \prod_{l,l' \in \bar{\mathcal{T}}} \int_0^1 dw_{l'} [\hat{R}_l(\sigma) R_{l'}(\sigma, l')] e^{\int 2i\sigma T_\Lambda}]\} \\
&= \int d\nu(w,\sigma) \sum_{N=N_1+N_2+N_3}^{\infty} \sum_{N_1}^{\infty}\sum_{N_2}^{\infty}\sum_{N_3}^{\infty} \frac{1}{N!}\frac{N!}{N_1!N_2!N_3!} \prod_{l,l' \in \bar{\mathcal{T}}} \int_0^1 dw_{l'} \\
&\quad [(\frac{1}{2})^{N_1+N_2} (\frac{\partial}{\partial\sigma})^{N_2} \{\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}\}^{N_1}] [\hat{R}_l(\sigma) R_{l'}(\sigma, w_{l'})] \\
&\quad \{(\frac{\partial}{\partial\sigma'})^{N_2} [\frac{1}{2}\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}]^{N_3} e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda}\}|_{\sigma=0} \\
&= \int d\nu(w,\sigma) \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\sum_{N_3=0}^{\infty} \prod_{l,\ l' \in \bar{\mathcal{T}}} \int_0^1 dw_{l'} \frac{1}{N_1!N_2!} (\frac{1}{2})^{N_1+N_2} \\
&\quad [(\frac{\partial}{\partial\sigma})^{N_2} \{\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}\}^{N_1}] [\hat{R}_l(\sigma) R_{l'}(\sigma)] \\
&\quad \{(\frac{\partial}{\partial\sigma'})^{N_2} \frac{1}{N_3!} [\frac{1}{2}\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}]^{N_3} e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda}\}|_{\sigma=0}.
\end{aligned} \tag{70}$$

While the $N_1$ and $N_2$ derivations generate connected terms, the last derivatives generate $N_3$ disconnected terms, see Figure 3.

We sum over the $N_3$ non-connected terms and we have:

$$\begin{aligned}
G &= \int d\nu(w,\sigma) \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\sum_{N_3=0}^{\infty} \prod_{l,\ l' \in \bar{\mathcal{T}}} \int_0^1 dw_{l'} \frac{1}{N_1!N_2!} (\frac{1}{2})^{N_1+N_2} \\
&\quad \{(\frac{\partial}{\partial\sigma})^{N_2} [\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}]^{N_1} \hat{R}_l(\sigma) R_{l'}(\sigma)]\} \{(\frac{\partial}{\partial\sigma'})^{N_2} e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda}\} e^{-2\lambda T_\Lambda^2} \\
&= A_{\bar{\mathcal{T}}} e^{-2\lambda T_\Lambda^2}.
\end{aligned} \tag{71}$$

Here we have used the fact that the weakening factor for each counterterm is 1. Hence

$$A_{\bar{\mathcal{T}}} = \int d\nu(\sigma, w) \prod_{l,l' \in \bar{\mathcal{T}}} \lambda^{|\bar{\mathcal{T}}|} \hat{R}_l(\sigma) R_{l'}(\sigma) e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda} e^{2\lambda T_\Lambda^2}. \tag{72}$$

## 5.6 The Nelson's Bound

After the resummation of the remaining counterterms we can finally apply bounds to any individual term in the cleaned, reexponentiated expansion The bad factor $e^{2\lambda T_\Lambda^2}$ in (72) will be compensated by the convergent factors generated by the crossings and nesting lines. More precisely, we have

$$e^{-a j_{max}^2} \cdot j_{max}! \cdot e^{2\lambda T_\Lambda^2} \sim e^{-a j_{max}^2 + j_{max} \ln j_{max} + 2\lambda j_{max}^2} < 1 \tag{73}$$

as long as we choose $a$ properly, for example, let $a > 3\lambda$. We have used the fact that $T_\Lambda \sim j_{max}$. This resummation process is shown in Figure 11.

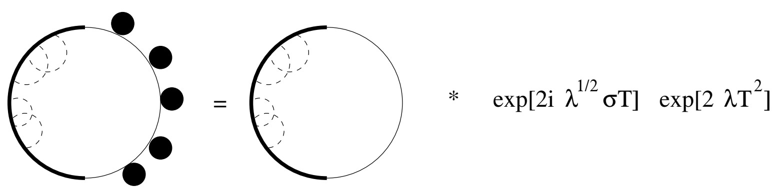


Figure 11: A sketch of the resummation of the counter terms.

So we have

$$\begin{aligned} |G_{\bar{\mathcal{T}}}|_{|T|=n-1} \quad < \quad & \int d\nu(\sigma, w) \prod_{l,l' \in \mathcal{T}} |\lambda|^{n-1} |\hat{R}_l(\sigma)|\ |R_{l'}(\sigma, w_{l'})|\ |e^{\int 2i\sigma T_\Lambda}| \\ \times \quad & e^{2\lambda T_\Lambda^2} e^{-a j_{max}^2} \leqslant (K|\lambda|)^{n-1}. \end{aligned} \tag{74}$$

where $K$ is an arbitrary constant.

In a finite volume $\mathcal{V}$ this is slightly modified into

$$G_T = \int d\nu(\sigma, w) \prod_{l,l' \in \bar{\mathcal{T}}} \lambda^{|\bar{\mathcal{T}}|} \hat{R}_l(\sigma) R_{l'}(\sigma, w_{l'}) e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda} e^{|\mathcal{V}| 2\lambda T_\Lambda^2}. \tag{75}$$

The amplitude is now divergent due to the term $e^{|\mathcal{V}|2\lambda T_\Lambda^2}$. The connected function for a cluster $\Gamma$ could be written as:

$$G(\lambda, w) = \sum_{\Delta \in \Gamma} G_\Delta(\lambda, w), \quad \Gamma = \{\Delta_1, \cdots, \Delta_k\} \subseteq \mathcal{V}. \tag{76}$$

In each square $\Delta$ there could be an arbitrary number of $\sigma$ fields coming from the resolvents (or none). For each square occupied by the $\sigma$ field attached with $2\sqrt{\lambda}T_j$ the expansion generated $aj$ crossings, where $j$ runs from $j_{max}$ until $\sum_j^{j_{max}} aj > 2\lambda j_{max}^2$, so as to compensate the factor $2\lambda T_\Lambda^2$. Eventually as the $\sigma$ fields will visit all the squares in $\mathcal{V}$ and we could compensate the factor $e^{2\lambda T_\Lambda^2 \mathcal{V}}$. As we have exponential decay between different squares in the lattice $\mathcal{V}$ we could sum over all squares without generating any divergent factor.

Then we have again

$$
\begin{aligned}
|G^{ren}_{\mathcal{T}}|_{|\bar{\mathcal{T}}|=n} \leqslant \sum_{\Delta\in T(\Gamma)} (\lambda)^{n-1} \int d\nu(\sigma) \prod_i |\hat{R}_l(\sigma)||e^{\int d^2x 2i\sigma T_\Lambda}|e^{-\tau(T)} \\
\leqslant \quad (2|\lambda|K)^{n-1}. \qquad (77)
\end{aligned}
$$

where $|\hat{R}(\sigma)| \leq 2$, $K$ is an arbitrary constant.

Summing over all trees of loop vertices, we have:

$$
\begin{aligned}
|G_n| \leqslant \frac{(2K|\lambda|)^{n-1}}{(n-1)!} \sum_T \sum_\tau \prod_{i=1}^{k_v} \int d\nu(\sigma) \prod_{l\in T} |\hat{R}_l(\sigma)||e^{\int d^2x 2i\sigma T_\Lambda}| \\
\leq \quad (2K|\lambda|)^{n-1} \frac{1}{(n-1)!} \sum_T (k-1)!. \qquad (78)
\end{aligned}
$$

By Cayley's theorem the sum over tree gives exactly $\frac{n!}{(k-1)!}$ and this cancels all the factors in above formula.

Then for the connected vacuum function we have:

$$
P(\lambda, \mathcal{V}) = \frac{1}{|\mathcal{V}|} \log Z(\lambda, \mathcal{V}) = \sum_n G_n, \qquad (79)
$$

with $|G_n| \leq (2K|\lambda|)^{n-1}$. Hence $P(\lambda, \mathcal{V})$ is bounded as long as $|\lambda| < \frac{1}{2K}$.

# 6 The Renormalization

In this section we check more explicitly the combinatoric of cancellation of inner tadpoles with counterterms. We know that this must work and that the compensation leads to no remainder. Indeed tadpoles are *exactly* local and the renormalized amplitudes for graphs with tadpoles vanish *exactly* in the $\phi_2^4$ theory. But it is interesting to see how the corresponding combinatorics precisely occurs in the context of the loop vertex expansion. We establish three lemmas for this cancelation, in increasing order of complexity.

## 6.1 A Single Loop Vertex

**Definition 6.1.** *We define the primary divergent graph of order $n$ as a trivial tree made of a single loop vertex with no crossings or overlaps. More precisely, each $\sigma$ field in the loop vertex can then contract only either with one of its nearest neighbor or with the counterterm.*

An example of primary divergent graphs and a sketch of the renormalization process is shown in Figure 12. Here the graph $d$ in this Figure means that there are 3 counterterms attached to the loop vertex. These 3 counterterms could be at any positions and we need to consider all the possibilities in the calculation. So is the case for the other graphs.

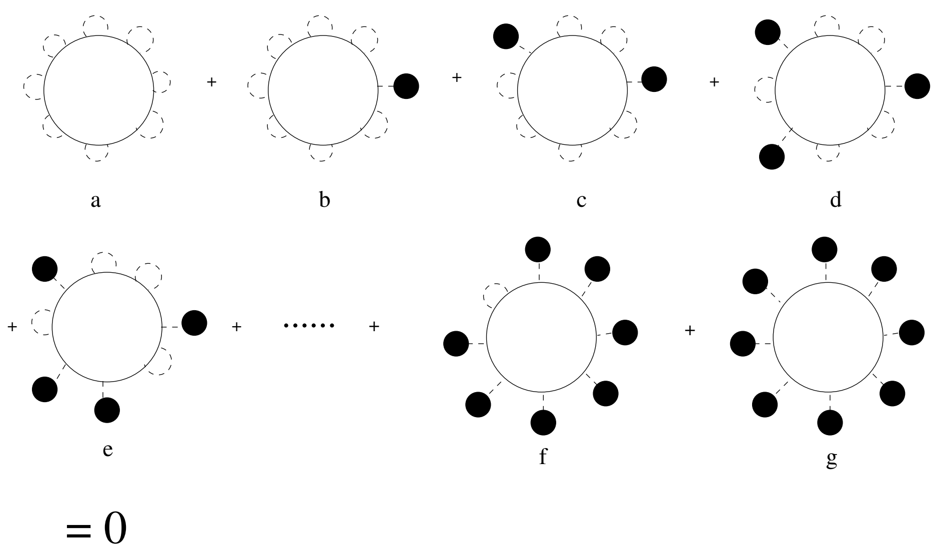


Figure 12: A sketch of renormalization of primary divergent graph or order 8.

To cancel this graph at order $n$ we need to consider all the counterterms up to order $n$ and from scale $j_{max}$ to $j_{min}$.

Now we consider the amplitude of a primary divergent vacuum graph $G$ of order $\lambda^n$ with $k$ counterterms attached. In this case we have a single loop vertex with $2n - k$ $\sigma$ fields to contract with k counterterms. The amplitude of $G$ reads:

$$\Gamma_G = -\frac{1}{2}\sum_{k=0}^{n} \lambda^n \int d\mu(\sigma) \frac{(-1)^{2n-k+1}}{(k+1)!} \int d^2x_1 d^2x_2 \cdots d^2x_n d^2x_{n+1} \cdots d^2x_k$$

$$\times \sum_{i_1,i_2,\cdots i_N=j_{min},\ N=2n-k}^{j_{max}} D_{i_1,i_2}(x_1,x_2,\sigma) D_{i_2,i_3}(x_2,x_3,\sigma) \cdots D_{i_N,i_1}(x_N,x_1,\sigma)$$

$$\times \prod_{i=1}^{k} (2i \sum_{j=j_{min}}^{j_{max}} \int T^j \sigma(x_i)), \tag{80}$$

with $x_{n+1} = x_1$ and $D_{i,i+1}(x_i, x_{i+1}, \sigma) = D_{i,i+1}(x_i, x_{i+1})\ \sigma(x_{i+1})$. We can renormalize it by using the multi-scale analysis. We start with the highest energy scale, where all the propagators and counter terms have the energy scale $j_{max}$, cancel the tadpole at this scale with the counterterm and then we go to the scale $j_{max} - 1$ and so on. But in our case as the cancelation between the tadpoles and counters are *exact* for all energy scales, we omit the index $j$ during the renormalization process.

Here we have two lemmas concerning the combinatorics of the Wick contractions and the sum of the amplitudes of trivial planar graphs. The first lemma is about combinatorics:

**Lemma 6.1.** *The number of planar divergent graph of order $n$ (whose amplitude is proportional to $\lambda^n$) with $k$ counterterm attached reads:*

$$f_n^k = \frac{(2n-k)C_n^k}{n} \times k! = \frac{(2n-k)(n-1)!}{(n-k)!}. \tag{81}$$

*Proof.* After the contractions of the vertex within itself or contraction with the counterterm, there are exactly $n$ objects which could be either counterterms or inner tadpoles. Then we consider all the combinations of these objects forming a graph. The first position among the $(2n-k)$ ones is special, as this breaks the cyclic ordering. We could first of all choose a line for attaching one counterterm. There are $(2n-k)$ possibilities. Then there are $n-1$ objects left including $k-1$ counterterms. The possibilities of having $k-1$ counterterm attached is $C_{k-1}^{n-1} \times (k-1)!$. By choosing the counterterm we considered also all possibilities of choosing tadpoles.

So the possibilities of having $k$ counterterms attached at order $n$ reads:

$$\begin{aligned} P_n^k &= (2n-k)\times C_{n-1}^{k-1}\times(k-1)! = (2n-k)\times\frac{(n-1)!(k-1)!}{(k-1)!(n-1-k+1)!} \\ &= \frac{(2n-k)C_n^k}{n}\times k!. \end{aligned} \tag{82}$$

□

The following lemma summarizes the fact that renormalization in this leaves no remainder:

**Lemma 6.2.** *The sum of the amplitudes of planar divergent graphs, under the constraint that each line should contract with a nearest neighbor, at arbitrary scale $j$ of order $\lambda^n$ vanishes exactly:*

$$\begin{aligned} G_n &= \int dx_i \frac{1}{2n}\mathrm{Tr}\prod_{i=1}^n C_i(x_i-x_{i+1})(A-B)^n \qquad (83)\\ &= \frac{1}{2n}[C^j(x_1-x_2)C^j(x_2-x_3)\times\cdots C^j(x_n-x_1)](A-B)^n = 0. \end{aligned}$$

*where*

$$A = -2\sqrt{\lambda}T^j = B, \tag{84}$$

*and* $\mathrm{Tr}\prod C^j(x_i-x_{i+1})$ *means the product of all the propagators of the graph according to the cyclic order.*

*Proof.* For a planar divergent graph with k counterterm attached at order n, the amplitude reads:

$$\begin{aligned} G_n^k &= \frac{1}{(k+1)!}C_{k+1}^1(-\frac{1}{2})\frac{(-1)^{(2n-k+1)}}{2n-k}\mathrm{Tr}\prod C(x_i-x_{i+1})A^{n-k}y^kP_n^k \\ &= \frac{1}{2n}\frac{(-1)^k}{k!}\mathrm{Tr}\prod_{i=1}^n C(x_i-x_{i+1})[C_n^kA^{n-k}B^kk!] \\ &= \frac{1}{2n}\mathrm{Tr}\prod_i C(x_i-x_{i+1})[C_n^kA^{n-k}(-B)^k]. \end{aligned} \tag{85}$$

Then the sum of the amplitude reads:

$$\frac{1}{2n}\mathrm{Tr}\prod_{i=1}^n C(x_i-x_j)\sum_{k=0}^n[C_n^kA^{n-k}(-B)^k] = \frac{1}{2n}\mathrm{Tr}\prod_{i=1}^n C(x_i-x_{i+1})(A-B)^n = 0, \tag{86}$$

as we have $A = B = -2\sqrt{\lambda}T^j$. □

## 6.2 A Single Loop Vertex with Crossings

In this subsection we consider the renormalization of the divergent graphs with crossings, see Figure 13.

**Lemma 6.3.** *The divergent graphs with crossings are canceled by the corresponding graphs with counterterms.*

*Proof.* Let us consider a divergent graph with order $n$ $n-m$ crossings and $k$ decorations by counterterms. We note the convergent crossing part by $Q_{m-n}$. The combinatoric factor of such graph reads:

$$P_m^k = (2m-k)\times C_n^k \times k!. \tag{87}$$

The idea of the proof is the same as the case of primary divergent graphs.

The amplitude of such a graph reads:

$$\begin{aligned} G_m^k &= \frac{1}{(k+1)!}C_{k+1}^1(-\frac{1}{2})\frac{(-1)^{(2m-k+1)}}{2m-k}\mathrm{Tr}\prod_{i=1}^n C(x_i-x_{i+1})A^{n-k}B^kP_n^k \\ &\times \; Q_{m-n}(\lambda^{m-n}) \\ &= \frac{1}{2}\frac{(-1)^k}{k!}\mathrm{Tr}\prod_{i=1}^n C(x_i-x_{i+1})[C_n^kA^{n-k}B^kk!]\times Q_{m-n}(\lambda^{m-n}) \\ &= \frac{1}{2}\mathrm{Tr}\prod_i C(x_i-x_{i+1})[C_n^kA^{n-k}(-B)^k]Q_{m-n}(\lambda^{m-n}), \end{aligned} \tag{88}$$

and the sum reads:

$$\begin{aligned} &\mathrm{Tr}\prod_{i=1}^n C(x_i-x_{i+1})\sum_{k=0}^n[C_n^kA^{n-k}(-B)^k]Q_{m-n} \\ = \; &\mathrm{Tr}\prod_{i=1}^n C(x_i-x_j)(A-B)^nQ_{m-n}(\lambda^{m-n}) = 0, \end{aligned} \tag{89}$$

as we have $A = B = -2\sqrt{\lambda}T_j$. Remark that this lemma holds also for the case where there are still resolvents in the loop vertex and they could appear anywhere in the vertex. This is also the case for the crossings. The only requirement is that their positions should be fixed in all graphs so that the combinatorics lemma is still valid. □

Some typical such graphs are shown in Figure 13. Figure 13 shows also the process of the renormalization.

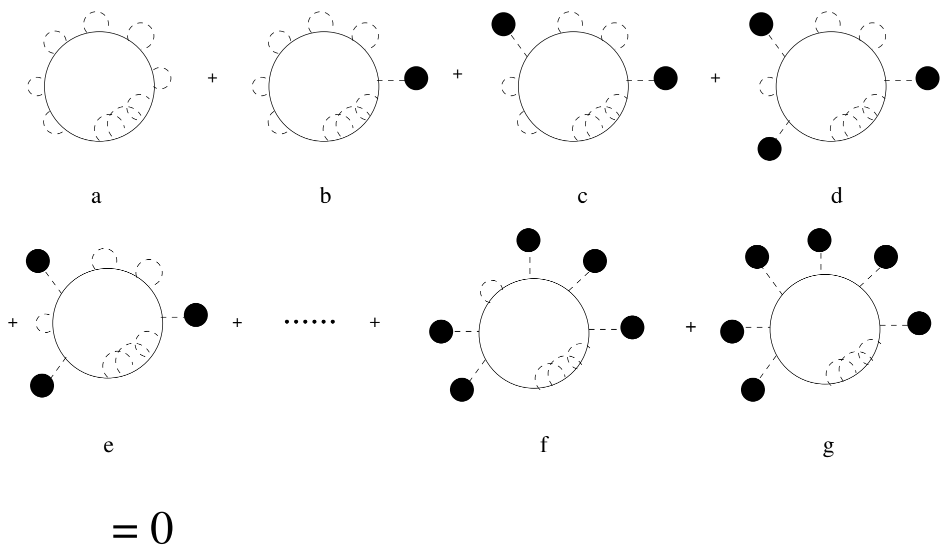


Figure 13: A sketch of renormalization of divergent graph of order 9 with crossings. There could be an arbitrary number of uncleaned resolvents in the loop vertex.

### 6.3 The General Case

A typical tree of loop vertices with divergent part, crossing and counterterms is shown in the Figure 14.

For the cancelation of the divergences in the tree of loop vertices we go back to the direct representation, as the amplitude for each vertices factorizes. This is the key point for the renormalization of the divergent general tree.

For each divergent vertex we cancel it by the counterterms as shown in last section, which means we use different trees to cancel each divergent term, each tree having exactly the same structure as the remainder vertices of the divergent term.

## 7 Borel summability

Let us introduce the $N$-th order Taylor remainder operator $R^N$ which acts on a function $f(\lambda)$ through

$$R^N f = f(\lambda) - \sum_{n=0}^{N} a_n \lambda^n = \lambda^{N+1} \int_0^1 \frac{(1-t)^N}{N!} f^{(N+1)}(t\lambda) dt. \tag{90}$$

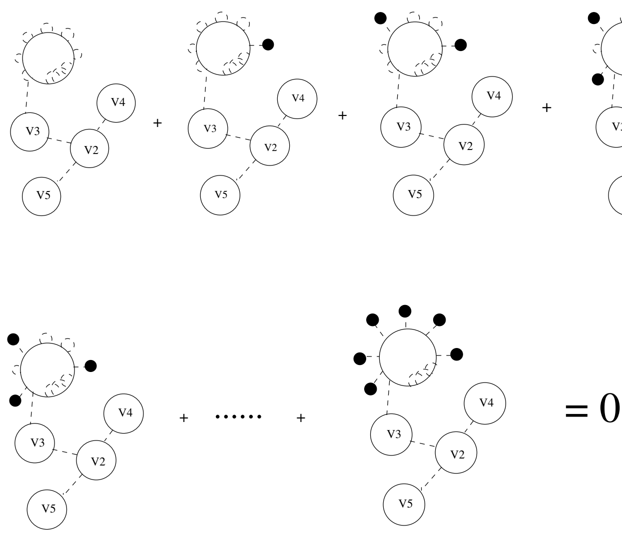


Figure 14: A sketch of renormalization of a tree of loop vertices. The vertices $v_i$ are all complex objects like the one drawn explicitly.

**Theorem 7.1.** *(Nevanlinna)[4]*

*A series $\sum_{n=0} \frac{a_n}{n!}\lambda^n$ is Borel summable to the function $f(\lambda)$ if the following conditions are met:*

- *For some rational number $k > 0$, $f(\lambda)$ is analytic in the domain $C_R^1 = \{\lambda \in C : \Re\lambda^{-1} > R^{-1}\}$.*
- *The function $f(\lambda)$ admits $\sum_{n=0}^{\infty} a_n\lambda^n$ as a strong asymptotic expansion to all orders as $|\lambda| \to 0$ in $C_R$ with uniform estimate in $C_R^1$:*

$$\left|R^N f\right| \leqslant AB^N N! |\lambda|^{N+1}. \tag{91}$$

*where $A$ and $B$ are some constants.*

*Then the Borel transform of order $k$ reads:*

$$B_f(u) = \sum_{n=0}^{\infty} \frac{a_N}{N!} u^N, \tag{92}$$

*it is holomorphic for $|u| < B^{-1}$, it admits an analytic continuation to the strip $\{u \in C : |\Im u| < R, \Re u > 0\}$ and for $0 \leqslant R$, one has*

$$f(\lambda) = \frac{1}{\lambda}\int_0^{\infty} B_f(u) exp[-(u/\lambda)](u/\lambda) du. \tag{93}$$

**Theorem 7.2.** *The perturbation series of the connected function for $\phi_2^4$ theory is Borel summable.*

**Proof** From the last section and last theorem we know that the analyticity domain for $\lambda$ is $\mathrm{Re}\lambda > 0$ which means

$$-\frac{\pi}{4} \leq \mathrm{Arg}\sqrt{\lambda} \leq \frac{\pi}{4} \tag{94}$$

and $|\lambda| < \frac{1}{2K}$.

Then we have for each resolvent

$$|\mathcal{R}| = |\frac{1}{1+2i\sqrt{\lambda}C\sigma}| \leq \sqrt{2}, \tag{95}$$

and for the leaf tadpole

$$|\hat{\mathcal{R}}| = |\frac{1}{1+2i\sqrt{\lambda}C\sigma} - 1| \leq \sqrt{2}+1, \tag{96}$$

since the covariance $C$ and the field $\sigma$ are real.

However, since $-\frac{\pi}{4} \leq \mathrm{Arg}\sqrt{\lambda} \leq \frac{\pi}{4}$ in the Borel plane of $\lambda$, the term $e^{2i\sqrt{\lambda}\sigma T_\Lambda}$ could be written as:

$$e^{2i\sqrt{\lambda}\sigma} = e^{2i|\sqrt{\lambda}|\sigma\cos\theta T_\Lambda} e^{-2|\sqrt{\lambda}|\sigma\sin\theta T_\Lambda}, \tag{97}$$

where $\theta = \mathrm{Arg}\sqrt{\lambda}$ and $|\sqrt{\lambda}|$ is the norm of $\sqrt{\lambda}$. This term cannot be bounded simply by 1, since the second term in (97) is not oscillating but could diverge for negative values of $\sigma$.

So before taking the norm for this problematic term we rewrite the $\sin\theta$ term as:

$$\int d\mu\sigma e^{-1/2\int d^2x\sigma^2} e^{-2\int d^2x|\sqrt{\lambda}|\sin\theta\sigma T_\Lambda} = \int d\mu\sigma e^{-1/2\int d^2x(\sigma+2|\sqrt{\lambda}|T_\Lambda)+2\mathcal{V}\sin^2\theta|\lambda|T_\Lambda^2}, \tag{98}$$

and bound the $\cos\theta$ term simply by 1.

The term $e^{2\mathcal{V}\sin^2\theta|\lambda|T_\Lambda^2}$ could diverge at worst as $e^{\mathcal{V}|\lambda|T_\Lambda^2}$ where $\theta = \pm\pi/4$. But this is not dangerous since we could still choose the coefficient $a$ in the convergent factor $e^{-a\mathcal{V}j_{max}^2} \sim e^{-a\mathcal{V}T_\Lambda^2}$ that we have gained from the $j$-lines (crossings and/or nesting) in the occupied volume $\mathcal{V}$.

Hence we still have

$$|G_N| = \int d\nu(\sigma, w) \prod_{l,l'\in\mathcal{T}} |\lambda^N||\hat{\mathcal{R}}_l(\sigma)||\mathcal{R}_{l'}(\sigma, w_{l'})|$$
$$\cdot \quad |e^{\int i\cos\theta\sigma T_\Lambda}|e^{|\mathcal{V}|(2+1)\lambda T_\Lambda^2}e^{-a\mathcal{V}T_\Lambda^2} \leq (K|\lambda|)^N. \tag{99}$$

We expand the connected function up to order $N$ in $\lambda$ by an explicit Taylor formula with integral remainder (90) followed by explicit Wick contractions. We have

$$|\sum_{n=N+1}^{\infty} G_n| < |\lambda|^{N+1}K^N(2N)!! \leq |\lambda|^{N+1}K^N(N)! \tag{100}$$

which leads to the $K^N N!$ term in formula (91), where $K$ is an arbitrary positive constant times the possible factors $\sqrt{2}$ from $\mathcal{R}$ and $\sqrt{2}+1$ from $\hat{\mathcal{R}}$.

Hence this proves the theorem. $\square$.

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# References

[1] V. Rivasseau, From Perturbative to Constructive Renormalization, Princeton University Press, 1991.

[2] A. Lesniewski, Effective Action for the Yukawa$_2$ Quantum Field Theory, Commun. Math. Phys. **108**, 437 (1987).

[3] A. Abdesselam and V. Rivasseau, Explicit fermionic tree expansions arXiv:cond-mat/9712055, Lett. Math. Phys. Vol. 44, Number 1, 77-88 (1998).

[4] A. D. Sokal, "An Improvement Of Watson's Theorem On Borel Summability," J. Math. Phys. **21**, 261 (1980).

[5] J.P. Eckmann, J. Magnen and R. Sénéor, Decay properties and Borel summability for the Schwinger functions in $P(\phi)_2$ theories, *Comm. Math. Phys.* **39**, 251 (1975).

[6] J. Magnen and R. Sénéor, Phase space cell expansion and Borel summability for the Euclidean $\phi_3^4$ theory, *Comm Math. Phys.* **56**, 237 (1977).

[7] V. Rivasseau, “Constructive Matrix Theory,” JHEP **0709** (2007) 008 [arXiv:0706.1224 [hep-th]].

[8] J. Magnen and V. Rivasseau, Constructive field theory without tears, Ann. Henri Poincaré **9** 403-424 (2008), arXiv:0706.2457[math-ph].

[9] V. Rivasseau and Zhituo Wang, “Loop Vertex Expansion for $\Phi^{2k}$ Theory in Zero Dimension,” *Journal of Mathematical Physics*, **51** 092304 (2010), arXiv:1003.1037 [math-ph].

[10] V. Rivasseau and Zhituo Wang, “How are Feynman graphs resumed by the Loop Vertex Expansion?”, arXiv:1006.4617 [math-ph].

[11] D. Brydges and T. Kennedy, Mayer expansions and the Hamilton-Jacobi equation, Journal of Statistical Physics, **48**, 19 (1987).

[12] A. Abdesselam and V. Rivasseau, “Trees, forests and jungles: A botanical garden for cluster expansions,” in Constructive Physics, Lecture Notes in Physics 446, Springer Verlag, 1995, arXiv:hep-th/9409094.

[13] H. Grosse and R. Wulkenhaar, “Renormalisation of phi**4 theory on noncommutative R**2 in the matrix JHEP **0312**, 019 (2003) [arXiv:hep-th/0307017].

[14] J. Zahn, “Divergences in quantum field theory on the noncommutative two-dimensional Minkowski space with Grosse-Wulkenhaar potential,” Annales Henri Poincaré **12**, 777-804, (2011), arXiv:1005.0541 [hep-th].

[15] Zhituo Wang, “Construction of 2-dimensional Grosse-Wulkenhaar Model,” arXiv:1104.3750 [math-ph].

[16] H. Grosse and R. Wulkenhaar, “Renormalisation of phi**4 theory on noncommutative R**4 in the matrix base,” Commun. Math. Phys. **256**, 305 (2005) [arXiv:hep-th/0401128].

[17] H. Grosse and R. Wulkenhaar, “Progress in solving a noncommutative quantum field theory in four dimensions,” arXiv:0909.1389 [hep-th].

[18] V. Rivasseau, F. Vignes-Tourneret and R. Wulkenhaar, “Renormalization of noncommutative phi**4-theory by multi-scale analysis,” Commun. Math. Phys. **262**, 565 (2006) [arXiv:hep-th/0501036].

[19] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, "Vanishing of beta function of non commutative phi(4)**4 theory to all orders," Phys. Lett. B **649** (2007) 95 [arXiv:hep-th/0612251].

[20] B. Simon, "The $P(\Phi)_2$ Euclidean (Quantum) Field Theory," *Princeton University Press, princeton 1974, 392 P.(Princeton Series In Physics)*

[21] F. Bergeron, G. Labelle and P. Leroux, Combinatorial Species and Tree-like Structures (Encyclopedia of Mathematics and its Applications), Cambridge University Press (1997).

[22] M. Disertori and V. Rivasseau, "Continuous constructive fermionic renormalization," Annales Henri Poincare **1**, 1 (2000) [arXiv:hep-th/9802145].