FURSTENBERG ENTROPY REALIZATIONS FOR VIRTUALLY FREE GROUPS AND LAMPLIGHTER GROUPS

YAIR HARTMAN AND OMER TAMUZ

ABSTRACT. Let (G, μ) be a discrete group with a generating probability measure. Nevo shows that if G has property (T) then there exists an $\varepsilon > 0$ such that the Furstenberg entropy of any (G, μ)-stationary space is either zero or larger than ε.

Virtually free groups, such as $SL_2(\mathbb{Z})$, do not have property (T). For these groups, we construct stationary actions with arbitrarily small, positive entropy. This construction involves building and lifting spaces of lamplighter groups. For some classical lamplighters, these spaces realize a dense set of entropies between zero and the Poisson boundary entropy.

CONTENTS

1. Introduction

Let G be a countable discrete group, and let μ be a generating probability measure. A G-space X with a probability measure ν is called a (G, μ)-stationary space if $\sum_g \mu(g)g\nu = \nu$. Hence a stationary measure ν is not in general G-invariant, but it is invariant "on average", when the average is taken over μ. An important invariant of stationary spaces is the Furstenberg entropy [8], given by

$$h_\mu(X, \nu) = \sum_{g \in G} \mu(g) \int_X -\log \frac{d\nu}{dg\nu}(x)dg\nu(x).$$

Despite the fact that stationary spaces have been studied for several decades now, few examples are known, and the theory of their structure and properties is still far from complete [10]. For example, it is in general not known which Furstenberg entropy values they may take; this problem is called the *Furstenberg entropy realization problem* [2, 18]. More specifically, it is not known which groups have an entropy gap:

Definition. (G, μ) has an *entropy gap* if it admits stationary spaces of positive Furstenberg entropy, and if there exists an $\varepsilon > 0$ such that the Furstenberg entropy of any ergodic (G, μ)-stationary space is either zero or greater than ε.

A group G has an *entropy gap* if (G, μ) has an entropy gap for every generating measure μ.

Nevo [17] shows that any group with Kazhdan's property (T) has an entropy gap. A natural conjecture is that groups without property (T) do not have an entropy gap, so that having an entropy gap is a characterization of property (T). We show that, indeed, a large class of discrete groups without property (T) do not have an entropy gap.

Theorem 1. *Let G be a finitely generated virtually free group. Let μ be a generating measure on G, with finite first moment. Then (G, μ) does not have an entropy gap.*

In particular, any finitely generated virtually free group does not have an entropy gap.

A virtually free group is a group that has a free group as a finite index subgroup. In particular, $SL_2(\mathbb{Z})$ is virtually free, and so does not have an entropy gap.

Bowen [2] introduces a new example of stationary spaces based on invariant random subgroups. We shall refer to these spaces as *Bowen spaces*. He uses some insights into their entropy to show that, for free groups with the uniform measure over the generators, any entropy between zero and the Poisson boundary entropy can be realized. We also realize entropies using Bowen spaces: to prove Theorem 1, we construct Bowen spaces of lamplighter groups, and lift them to Bowen spaces of virtually free groups. Using a recent result [11] that relates the entropies of the actions of groups and their finite index subgroups, we control the entropies of the lifted spaces, and show that they can be made arbitrarily small.

A natural stationary space of lamplighter groups is the limit configuration boundary (see Section 2.6.1), which, in some classical lamplighters, has been shown to

coincide with the Poisson boundary (see [6, 13, 15]). We denote by $h_{\mathrm{conf}}(G, \mu)$ its Furstenberg entropy. Our construction of Bowen spaces for lamplighter groups yields the following realization result.

Theorem 2. *Let $G = L \wr \Gamma$ be a finitely generated discrete lamplighter with base group Γ and lamps in L. Let μ be a generating measure on G with finite entropy, and such that its projected random walk on Γ has a trivial Poisson boundary. Then there exists a dense set $H \subseteq [0, h_{\mathrm{conf}}(G, \mu)]$ such that for each $h \in H$ there exists an ergodic (G, μ)-stationary space with Furstenberg entropy h.*

1.1. **Related results.** The Furstenberg entropy of any stationary space is bounded from above by the entropy of the Poisson boundary [9]. Kaimanovich and Vershik [14] show that when $H(\mu)$ is finite then the Furstenberg entropy of the Poisson boundary is equal to $h_{RW}(G, \mu)$, the random walk entropy of μ, defined by

$$h_{RW}(G, \mu) = \lim_{n \to \infty} \frac{1}{n} H(\mu^n).$$

Little is known about which entropy values between 0 and $h_{RW}(G, \mu)$ can be realized by ergodic stationary spaces. Note that any entropy in this range can be realized with a non-ergodic space that is a convex linear combination of the Poisson boundary and a trivial space.

When G is abelian, or more generally, virtually nilpotent, then the entropy of the Poisson boundary, and hence of any stationary space, vanishes for any μ [14]. Furstenberg shows that a group G is amenable if and only if there exists a generating measure μ such that the entropy of $\Pi(G, \mu)$ vanishes.

When G has Kazhdan's property (T), then Nevo [17] shows that for every admissible (generating, in the discrete case) μ there exists an $\varepsilon > 0$ such that the entropy of any (G, μ) stationary space is either zero or larger than ε. That is, G has an entropy gap.

Nevo and Zimmer [18] show that $PSL_2(\mathbb{R})$, and more generally, any simple Lie group with $\mathbb{R}$-rank ≥ 2 with a parabolic subgroup that maps onto $PSL_2(\mathbb{R})$, has infinitely many distinct realizable entropy values, for any admissible measure. No other guarantees are given regarding these values.

Finally, when G is a free group of rank $2 \leq n < \infty$ and μ is the uniform measure on its generators, Bowen [2] shows that any entropy between 0 and $h_{RW}(G, \mu)$ can be realized.

1.2. **Acknowledgments.** We are grateful to Yuri Lima for many useful discussions. We would also like to thank Uri Bader and Amos Nevo for motivating conversations.

The remainder of the paper is organized as follows. In Section 2 we give general definitions and notation, and in particular elaborate on Bowen spaces. In addition, we show that, for the purpose of entropy realization, it is possible to assume without loss of generality that μ is supported everywhere. In Section 3 we prove our realization result for lamplighters, Theorem 2, and in Section 4 we prove our main result, Theorem 1.

2. Preliminaries and notation

2.1. **Random walks on groups.** Let G be a discrete group, and let $\mathcal{P}(G)$ be the space of probability measures on G. $\mu \in \mathcal{P}(G)$ is a *generating measure* if G is the semigroup generated by its support. We assume henceforth that μ is generating.

Consider the case that G is finitely generated, and let $\mathcal{S}$ be a finite symmetric generating set of G. Define the *word length metric* of G w.r.t. $\mathcal{S}$ to be $|g|_{\mathcal{S}} = \min\{n | s_1 \cdots s_n = g, s_i \in \mathcal{S}\}$. The measure μ has *finite first moment* if

$$\sum_{g \in G} \mu(g)|g|_{\mathcal{S}} < \infty.$$

Since word length metrics induced by different finite generating sets are bi-lipschitz equivalent, the property of having finite first moment does not depend on the choice of $\mathcal{S}$.

If μ has finite first moment then it has finite entropy $H(\mu)$, given by

$$H(\mu) = \sum_{g \in G} -\mu(g) \log \mu(g).$$

For $n \in \mathbb{N}$, let X_n be i.i.d. random variables taking values in G, with law μ, and let $Z_n = X_1 \cdots X_n$. A μ random walk on G is a measure $\mathbb{P}$ on $\Omega = G^{\mathbb{N}}$, such that $(Z_1, Z_2, \ldots) \sim \mathbb{P}$.

2.2. **Stationary spaces and the Poisson boundary.** Let (X, ν) be a Lebesgue probability space, equipped with a measurable G-action $G \times X \xrightarrow{a} X$. We denote by $g\nu$ the measure defined by $(g\nu)(E) = \nu\left(g^{-1}E\right)$. (X, ν) is a (G, μ)-*stationary space* if

$$\mu * \nu = \sum_{g \in G} \mu(g)g\nu = \nu,$$

where $\mu * \nu$, the convolution of μ with ν, is the image of $\mu \times \nu$ under the action a. It follows from stationarity and the fact that μ is generating that ν and $g\nu$ are mutually absolutely continuous for all $g \in G$.

Let (X, ν) be a (G, μ)-stationary space, and let (Y, η) be a G-space. A measurable map $\pi : X \to Y$ is a G-*factor* if it is G-equivariant (i.e., if π commutes with the G-actions) and if $\pi_* \nu = \eta$. In this case (Y, η) is called a G-*factor* of (X, ν), and it follows that (Y, η) is also (G, μ)-stationary. If, in addition, π is an isomorphism of the probability spaces, then π is a G-*isomorphism*.

An important (G, μ)-stationary space is $\Pi(G, \mu)$, the *Poisson boundary* of (G, μ). The Poisson boundary can be defined as the Mackey realization [16] of the shift invariant sigma-algebra of the space of random walks $(\Omega, \mathbb{P})$ [14, 21], also known as the space of shift ergodic components of Ω. Furstenberg's original definition [9] used the Gelfand representation of the algebra of bounded μ-harmonic functions on G. For formal definitions see also Furstenberg and Glasner [10], or a survey by Furman [7].

G-factors of the Poisson boundary are stationary spaces called (G, μ)-*boundaries*; the Mackey realization of each G-invariant, shift invariant sigma-algebra is a (G, μ)-boundary. A different perspective is that a compact G-space (X, ν) is a (G, μ)-boundary if it is a (G, μ)-stationary space such that $\lim_n Z_n \nu$, in the weak* topology, is almost surely a point mass measure $\delta_x \in \mathcal{P}(X)$. The map $\mathbf{bnd}_X : \Omega \to X$ that assigns to $(Z_1, Z_2, \ldots)$ the point x is called the *boundary map* of (X, ν). For further

discussion and a definition of boundaries that is independent of topology see Bader and Shalom [1].

We shall also consider the Poisson boundary of general Markov chains, defined again as the space of ergodic components of the shift invariant sigma-algebra [12].

2.3. **Furstenberg entropy.** The *Furstenberg entropy* of a (G, μ)-stationary space (X, ν) is given by

$$h_\mu(X, \nu) = \sum_{g \in G} \mu(g) \int_X -\log \frac{d\nu}{dg\nu}(x) dg\nu(x).$$

Alternatively, it can be written as

$$h_\mu(X, \nu) = \mathbb{E}\left[D_{KL}(Z_1\nu||\nu)\right],$$

where D_{KL} denotes the *Kullback-Leibler divergence*. Since the latter decreases under factors, it follows that if (Y, η) is a G-factor of (X, ν), then $h_\mu(X, \nu) \geq h_\mu(Y, \eta)$.

A space (X, ν) is G-invariant if and only if $h_\mu(X, \nu) = 0$, and, in general, the Furstenberg entropy can be thought of as quantifying the μ-average deformation of ν by the G-action.

2.4. **The induced walk on finite index subgroups.** Let Γ be a finite index subgroup of G, and let $\tau = \min_n\{Z_n \in \Gamma\}$ be the Γ hitting time of the μ random walk. τ is almost surely finite, and so it is possible to define the hitting measure $\theta \in \mathcal{P}(\Gamma)$ as the law of Z_τ. The θ random walk on Γ is intimately related to the μ random walk on G. In particular, any (G, μ)-stationary space is also a (Γ, θ)-stationary space. Furthermore, Furstenberg [9] shows that the Poisson boundaries of the two walks are identical.

It is shown in [11] that $\mathbb{E}[\tau] = [G : \Gamma]$, and that for any (G, μ)-stationary space (X, ν) it holds that

$$h_\theta(X, \nu) = [G : \Gamma] \cdot h_\mu(X, \nu). \tag{2.1}$$

2.5. **Bowen spaces.** In [2], Bowen introduces a novel example of stationary spaces, which we refer to as Bowen spaces. As we make extensive use of these spaces, we would like to motivate their definition and elaborate on it.

Let $(B, \nu) = \Pi(G, \mu)$ be the Poisson boundary of (G, μ), let $(Z_1, Z_2, \ldots)$ be a μ random walk on G, and let K be a normal subgroup of G with $G \xrightarrow{\varphi} K\backslash G$. Then $(KZ_1, KZ_2, \ldots)$ is a $\varphi_*\mu$ random walk on the group $K\backslash G$, which we call the induced random walk. $\Pi(K\backslash G, \varphi_*\mu)$, the Poisson boundary of $(K\backslash G, \varphi_*\mu)$, is a factor of $\Pi(G, \mu)$; the former is isomorphic to the ergodic components of the K action on the latter. $\Pi(K\backslash G, \varphi_*\mu)$ is therefore also a G-space, and, furthermore, a (G, μ)-boundary [1].

When K is not normal, we can still consider the induced Markov chain $(KZ_1, KZ_2, \ldots)$, which is, however, no longer a random walk on a group. The action of $g \in G$ on the μ random walk descends to

$$g(KZ_1, KZ_2, \ldots) = (gKZ_1, gKZ_2, \ldots) = (K^g gZ_1, K^g gZ_2, \ldots), \tag{2.2}$$

which maps the Markov chain on $K\backslash G$ starting from K to the chain on $K^g\backslash G$ starting from $K^g g$, where $K^g = gKg^{-1}$. Denote by $P_K^n(Kg, Kh)$ the transition probability from Kg to Kh in n steps of the induced chain.

Even though the induced Markov chain on $K\backslash G$ is not a random walk on a group, we can still consider its Poisson boundary, (B_K, ν_K), and a boundary map $(K\backslash G)^{\mathbb{N}} \to B_K$. As in the normal case, it is a factor of $\Pi(G, \mu)$. However, in this case (B_K, ν_K) does not admit a natural G-action; the induced action of $g \in G$ on (B_K, ν_K) maps it to $(B_{K^g}, \nu_{K^g g})$, where $\nu_{K^g g}$ is the measure on the Poisson boundary of the Markov chain $K^g \backslash G$ that starts at $K^g g$.

To build a G-space, Bowen considers a larger space, namely that of all Poisson boundaries of the form B_K. Denote by Sub_G the space of all subgroups of G, and denote

$$B(\mathrm{Sub}_G) = \{(K, x) \ : \ K \in \mathrm{Sub}_G, x \in B_K\}.$$

The G-action of Eq. 2.2 on Markov chains descends, via composition with the boundary map, to a G-action on $B(\mathrm{Sub}_G)$.

To construct a stationary measure over $B(\mathrm{Sub}_G)$, let $\lambda \in \mathcal{P}(\mathrm{Sub}_G)$ be an invariant random subgroup (IRS) - a measure on Sub_G that is invariant to conjugation. Let $\nu_\lambda \in \mathcal{P}(B(\mathrm{Sub}_G))$ be given by $d\nu_\lambda(K, x) = d\nu_K(x)d\lambda(K)$. This is the measure that gives the fiber above K the measure ν_K, with measure λ over the fibers.

Bowen shows that $(B(\mathrm{Sub}_G), \nu_\lambda)$ is (G, μ)-stationary, and is furthermore ergodic if λ is ergodic. We refer to this space as the *Bowen space* associated with λ.

By definition, the Furstenberg entropy of a Bowen space is given by

$$h_\mu(B(\mathrm{Sub}_G), \nu_\lambda) = \sum_{g \in G} \mu(g) \int_{B(\mathrm{Sub}_G)} -\log \frac{d\nu_\lambda}{dg\nu_\lambda}(K, x)dg\nu_\lambda(K, x).$$

Using $d\nu_\lambda(K, x) = d\nu_K(x)d\lambda(K)$ and the fact that $g\lambda = \lambda$ and $g\nu_K = \nu_{K^g g}$,

$$= \sum_{g \in G} \mu(g) \int_{\mathrm{Sub}_G} \int_{B_K} -\log \frac{d\nu_K}{d\nu_{Kg}}(x)d\nu_{Kg}(x)d\lambda(K).$$

Even though (B_K, ν_K) is not a (G, μ)-stationary space - in fact, not even a G-space - it will help us to define its (G, μ) Furstenberg entropy by

$$h_\mu(B_K, \nu_K) = \sum_{g \in G} \mu(g) \int_{B_K} -\log \frac{d\nu_K}{d\nu_{Kg}}(x)d\nu_{Kg}(x), \qquad (2.3)$$

so that

$$h_\mu(B(\mathrm{Sub}_G), \nu_\lambda) = \int_{\mathrm{Sub}_G} h_\mu(B_K, \nu_K)d\lambda(K). \qquad (2.4)$$

An alternative way to understand Eq. 2.4 is to regard $(\mathrm{Sub}_G, \lambda)$ as a G-factor of $(B(\mathrm{Sub}_G), \nu_\lambda)$. In general, if (Y, λ) is a G-factor of (X, ν), then it is possible to express the entropy of X as a sum of the entropy of Y and the average entropy of the fibers $X_y = \pi^{-1}(y)$:

$$h_\mu(X, \nu) = h_\mu(Y, \lambda) + \int_Y h_\mu(X_y, \nu_y)d\lambda(y), \qquad (2.5)$$

where

$$h_\mu(X_y, \nu_y) = \sum_{g \in G} \mu(g) \int_X -\log \frac{d\nu_{gy}}{dg\nu_y}(x)dg\nu_y(x). \qquad (2.6)$$

Here the measures on the fibers ν_y are defined by the disintegration $\nu = \int_Y \nu_y d\lambda(y)$. In our case, the fiber above $K \in \mathrm{Sub}_G$ is the Poisson boundary B_K, and so Eq. 2.6

becomes Eq. 2.3. Since λ is G-invariant, the entropy of $(\mathrm{Sub}_G, \lambda)$ vanishes, and so Eq. 2.5 becomes Eq. 2.4.

A useful property of $h_\mu(B_K, \nu_K)$ is that it is monotone in K: if $K \leq H$ then

$$h_\mu(B_K, \nu_K) \geq h_\mu(B_H, \nu_H). \tag{2.7}$$

This follows from the fact that (B_H, ν_H) is, in this case, a factor of (B_K, ν_K), and from the monotonicity of Kullback-Leibler divergence; $h_\mu(B_K, \nu_K)$ is the μ-expectation of $D_{KL}(\nu_{Kg} \| \nu_K)$.

Kaimanovich and Vershik [14] show that the Furstenberg entropy of the Poisson boundary is equal to the *random walk entropy*, given by

$$h_{RW}(G, \mu) = \lim_{n \to \infty} \frac{1}{n} H(Z_n),$$

where

$$H(Z_n) = -\sum_{g \in G} \mathbb{P}[Z_n = g] \log \mathbb{P}[Z_n = g] = H(\mu^n). \tag{2.8}$$

In this spirit, Bowen shows that the entropy of a Bowen space can also be written as

$$h_\mu(B(\mathrm{Sub}_G), \nu_\lambda) = \lim_{n \to \infty} \frac{1}{n} \int H(KZ_n) \, d\lambda(K) = \inf_n \frac{1}{n} \int H(KZ_n) \, d\lambda(K), \quad (2.9)$$

where

$$H(KZ_n) = -\sum_{Kg \in K \backslash G} \mathbb{P}[KZ_n = Kg] \log \mathbb{P}[KZ_n = Kg].$$

By the second equality of Eq. 2.9, the map $\lambda \mapsto h_\mu(B(\mathrm{Sub}_G), \nu_\lambda)$ is upper semi-continuous. It is not, however, continuous in general.

In the next section, in which we discuss lamplighter groups, we give some examples of a Bowen spaces.

2.5.1. *A general bound on the Radon-Nikodym derivatives of the Poisson boundaries of induced Markov chains.* The following general lemma, resembling one from Kaimanovich and Vershik [14], will be useful below.

Lemma 2.1. *For every $K \in \mathrm{Sub}_G$, ν_K-almost every $x \in B_K$ and every $g \in G$ such that $g, g^{-1} \in \mathrm{supp}\, \mu$ it holds that*

$$\frac{1}{\mu(g)} \geq \frac{d\nu_{Kg}}{d\nu_K}(x) \geq P_K(Kg, K) \geq \mu(g^{-1}).$$

Proof. Condition on the location of the Markov chain after taking n steps, starting at Kg. We get for ν_K-almost every $x \in B_K$,

$$1 = \frac{d\nu_{Kg}}{d\nu_{Kg}}(x) = \frac{d\sum_{Kh \in K \backslash G} P_K(Kg, Kh)\nu_{Kh}}{d\nu_{Kg}}(x) = \sum_{Kh \in K \backslash G} P_K(Kg, Kh) \frac{d\nu_{Kh}}{d\nu_{Kg}}(x). \tag{2.10}$$

Since each summand is positive, it follows that for all $g, h \in G$,

$$1 \geq P_K(Kg, Kh) \frac{d\nu_{Kh}}{d\nu_{Kg}}(x),$$

and in particular

$$\frac{d\nu_{Kg}}{d\nu_K}(x) \geq P_K(Kg, K).$$

Note that $P_K(Kg, K) \geq \mu\left(g^{-1}\right)$, by the definition of P_K.

Rewriting Eq. 2.10 as a sum over G, we get that

$$1 = \frac{d\nu_{Kg}}{d\nu_{Kg}}(x) = \frac{d\sum_{g \in G} \mu(g)\nu_{Kg}}{d\nu_K}(x) = \sum_{g \in G} \mu(g)\frac{d\nu_{Kg}}{d\nu_K}(x).$$

By the same argument above, it follows that

$$\frac{1}{\mu(g)} \geq \frac{d\nu_{Kg}}{d\nu_K}(x).$$

$\square$

2.6. **Lamplighter groups.** Let Γ and L be discrete groups. Let the compact configurations $C_C(L, \Gamma)$ be the group of all finitely supported functions $\Gamma \to L$; that is, if $f \in C_C(L, \Gamma)$ then f is equal to the identity of L for all but a finite number of elements of Γ. The group operation is pointwise multiplication:

$$[f_1 f_2](\gamma) = f_1(\gamma)f_2(\gamma),$$

and Γ acts on $C_C(L, \Gamma)$ by shifting:

$$[\gamma' f](\gamma) = f(\gamma\gamma').$$

The lamplighter group $G = L \wr \Gamma$ is equal to the semidirect product $C_C(L, \Gamma) \rtimes \Gamma$, so that the operation is

$$(f_1, \gamma_1) \cdot (f_2, \gamma_2) = (f_1(\gamma_1 f_2), \gamma_1 \gamma_2).$$

It follows that

$$(f, \gamma)^{-1} = (\gamma^{-1} f^{-1}, \gamma^{-1}).$$

We say that G has *base* Γ and *lamps* in L. We think of the first coordinate as the "lamp configuration" and of the second coordinate as the "position of the lighter".

2.6.1. *The limit configuration boundary.* There exists a natural group homomorphism $\pi : L \wr \Gamma \to \Gamma$ defined by $\pi(f, \gamma) = \gamma$. We denote $\overline{g} = \pi(g)$, and $\overline{\mu} = \pi_*\mu$. Thus the μ random walk on $L \wr \Gamma$ induces a $\overline{\mu}$ random walk on Γ. When μ has finite first moment, and when the $\overline{\mu}$ random walk on Γ is transient, Erschler [6] shows that the "value of each lamp stabilizes":

Theorem (Erschler). *Let $(Z_1, Z_2, \ldots)$ be a μ random walk on a finitely generated $G = L \wr \Gamma$, let μ have finite first moment, and let the $\overline{\mu}$ random walk on Γ be transient. Denote $\omega_n = (f_n, \gamma_n)$. Then there exists a map $\mathbf{conf} : \Omega \to L^\Gamma$ such that for every $\gamma \in \Gamma$*

$$\mathbf{conf}(\omega_1, \omega_2, \ldots)(\gamma) = \lim_{n \to \infty} f_n(\gamma)$$

$\mathbb{P}$-*almost everywhere.*

In particular, each limit $\lim_n f_n(\gamma)$ exists almost surely.

The space of functions L^Γ admits the natural left Γ-action. As such, it is a (G, μ)-stationary space when equipped with the measure $\mathbf{conf}_*\mathbb{P}$. Since $\mathbf{conf}$ is shift invariant, $(L^\Gamma, \mathbf{conf}_*\mathbb{P})$ is a (G, μ)-boundary, which we shall refer to as the *limit configuration boundary*. Erschler's theorem equivalently implies that under the claim hypotheses, this boundary has positive entropy, which we denote by $h_{\mathrm{conf}}(G, \mu)$.

2.6.2. *Some Bowen spaces of lamplighters.* In accordance with the definition of $C_C(L, \Gamma)$ as the set of finitely supported functions from Γ to L, let $C_C(L, S)$ be the set of finitely supported functions from Γ to L, which are supported on $S \subseteq \Gamma$. Let K_S be the subgroup of G defined by

$$K_S = \{(f, e_\Gamma) \in G \,:\, f \in C_C(L, S)\},$$

where e_Γ is the identity of Γ. The conjugation of K_S by an element $g = (f, \gamma)$ of G amounts to a shift of S by γ, as we show in the next claim.

Claim 2.2. *Let $(f, \gamma) \in G$. Then $K_S^{(f,\gamma)} = K_{\gamma S}$.*

Proof. By definition

$$K_S^{(f,\gamma)} = \{(f, \gamma)(g, e_\Gamma)(f, \gamma)^{-1} \,:\, \operatorname{supp} g \subseteq S\}.$$

Since

$$(f, \gamma)(g, e_\Gamma)(f, \gamma)^{-1} = (f(\gamma g), \gamma)(\gamma^{-1} f^{-1}, \gamma^{-1}) = (f(\gamma g) f^{-1}, e_\Gamma),$$

it follows that

$$K_S^{(f,\gamma)} = \{g \,:\, \operatorname{supp} g \subseteq \gamma S\} = K_{\gamma S}.$$

$\square$

We call a measure on the subsets of Γ a *percolation* measure. The map that assigns the subgroup $K_S < G$ to each $S \subseteq \Gamma$ maps percolation measures to measures on Sub_G. It follows from Claim 2.2 above that if a percolation measure λ is Γ-invariant, then the associated measure on Sub_G is an IRS and therefore G-invariant. Likewise, if λ is Γ-ergodic, then the associated IRS, which we also call λ, is G-ergodic.

Recall that given an ergodic IRS λ, the associated Bowen space $(B(\mathrm{Sub}_G), \nu_\lambda)$ is also ergodic. We shall use, for our purposes of entropy realization, Bowen spaces built from ergodic percolations on Γ.

As a motivating example, consider the canonical lamplighter $G = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d$. Let E be the set of even elements in $\mathbb{Z}^d$, and let O be its complement, or the set of odd elements. The subgroups K_E and K_O are, respectively, the finite configurations supported on the even positions and on the odd positions. By Claim 2.2, conjugation of either of these groups by any element of G either leaves it invariant or maps it to the other. It follows that $\lambda = \frac{1}{2}\delta_{K_E} + \frac{1}{2}\delta_{K_O}$ is an ergodic IRS, and that $(B(\mathrm{Sub}_G), \nu_\lambda)$ is an ergodic Bowen space. This space consists of two fibers, which are the Poisson boundaries of the induced Markov chains on $K_E \backslash G$ and on $K_O \backslash G$.

Informally, Eq. 2.8 states that the entropy of the Poisson boundary of the random walk on G is equal to the exponential growth rate of the support of Z_n. Intuitively, the growth rate of the support of $K_E Z_n$, which "mods out" the even lamps, should

be half that of the support of Z_n, since the random walk entropy of the projected random walk on Γ vanishes. Therefore, by Eq. 2.9, the entropy of $(B(\mathrm{Sub}_G), \nu_\lambda)$ can be expected to equal half that of the Poisson boundary. By the same intuition, if we choose an IRS in which K includes each lamp independently with probability $1 - p$, then we expect that the entropy of the associated Bowen space would be p times $h_{RW}(G, \mu)$, the entropy of the Poisson boundary, and that therefore any entropy in $[0, h_{RW}(G, \mu)]$ can be realized. We are not able to show this, and instead resort to a more elaborate construction which only realizes a dense set of entropies (see Section 3).

For more on invariant random subgroups of lamplighters see [3].

2.7. Digression: the Radon-Nikodym compact is not necessarily a boundary.

The Radon-Nikodym factor $rn : X \to \mathbb{R}^G$ assigns to almost every point x in a (G, μ)-stationary space (X, ν) the function $f_x(g) = \frac{dg\nu}{d\nu}(x)$. Since this factor commutes with G, its image, called the Radon-Nikodym compact of (X, ν), is also a stationary space, which Kaimanovich and Vershik show to have the same entropy as (X, ν) [14]. An equivalent definition is given by Nevo and Zimmer [18].

Using the example above of the Bowen space associated with $\lambda = \frac{1}{2}\delta_{K_E} + \frac{1}{2}\delta_{K_O}$, we show that the Radon-Nikodym compact of $(B(\mathrm{Sub}_G), \nu_\lambda)$ is not a (G, μ)-boundary, in apparent contradiction to Proposition 3.6 in [14]. Note that counterexamples in Lie groups appear in [18], but these do not contradict the statement of the said Proposition, since it is made for discrete groups only.

To see this, note first that $\nu_\lambda(K_E \times B_{K_E}) = \nu_\lambda(K_O \times B_{K_O}) = \frac{1}{2}$, and that furthermore $g\nu_\lambda(K_E \times B_{K_E}) = g\nu_\lambda(K_O \times B_{K_O}) = \frac{1}{2}$ for all $g \in G$. Hence $\lim_n Z_n\nu$ cannot be a point mass, and $(B(\mathrm{Sub}_G), \nu_\lambda)$ is not a G-boundary. In fact, it is easy to see that $\lim_n Z_n\nu = \frac{1}{2}\delta_{(K_E, x_E)} + \frac{1}{2}\delta_{(K_O, x_O)}$, for randomly drawn $x_E \in B_{K_E}$ and $x_O \in B_{K_O}$, with probability one.

We show that its Randon-Nikodym compact is also not a boundary by showing that, under the Radon-Nikodym factor, the images of (K_E, x_E) and (K_O, x_O) are in general different. This shows that the limiting distributions of the Radon-Nikodym compacts are also not point masses. Indeed,

$$[rn(K_E, x_E)](g) = \frac{dg\nu_\lambda}{d\nu\lambda}(K_E, x_E) = \frac{d\nu_{K_E^g g}}{d\nu_{K_E}}(x_E),$$

and similarly for $rn(K_O, x_O)$. To see that $rn(K_E, x_E) \neq rn(K_E, x_E)$, let $g = (\delta_e, e)$ be the element of g that leaves the walker in place and changes only the state of the lamp at the origin. Then $K_E^g g = K_E$, and so $\nu_{K_E^g g} = \nu_{K_E}$, and it follows that $[rn(K_E, x_E)](g) = 1$. However $K_O^g g = K_O g \neq K_O$, and so $[rn(K_O, x_O)](g) \neq 1$, for at least some values of x_O.

2.8. The support of μ.

Given a generating measure $\mu \in \mathcal{P}(G)$, we construct in this section a measure η supported everywhere on G (and hence also generating) such that any G-space (X, ν) is (G, μ)-stationary if and only if it is (G, η)-stationary, and furthermore $h_\eta(X, \nu) = h_\mu(X, \nu)$. For our purposes of entropy realization, this will allow us to assume, without loss of generality, that μ has full support, which will simplify our proofs.

We first show that if the Poisson boundaries of (G, μ) and (G, η) coincide then so do their stationary spaces. For this, we use the characterization of the Poisson boundary via harmonic functions. Indeed, $\Pi(G, \mu) = \Pi(G, \eta)$ if and only if every

bounded μ-harmonic function is η-harmonic, and vice versa. We next construct a measure η that has the same Poisson boundary as μ.

Lemma 2.3. *Let $\mu, \eta \in \mathcal{P}(G)$ be two generating measures such a bounded function $h : G \to \mathbb{R}$ is μ-harmonic if and only if it is η-harmonic. Then a G-space (X, ν) is μ-stationary if and only if it is η-stationary.*

Proof. Let (X, ν) be μ-stationary, and let A be an arbitrary ν-measurable set. Then $h(g) = g\nu(A)$ is μ-harmonic. By the claim hypothesis it is also η-harmonic, and therefore

$$\sum_{g \in G} \eta(g) h(g) = h(e).$$

Hence

$$\sum_{g \in G} \eta(g) g\nu(A) = \nu(A),$$

and since this holds for any A we have that $\eta * \nu = \nu$, and (X, ν) is η-stationary. The other direction follows by symmetry. $\square$

Let α be a measure over the non-negative integers such that $\alpha(1) \neq 0$. Let $\{T_n\}_{n=1}^{\infty}$ be i.i.d. random variables with law α, and let $\tau_n = \sum_{i=1}^{n} T_i$. Then the distribution of τ_n is η^n, where

$$\eta = \sum_{n=0}^{\infty} \alpha(n) \mu^n,$$

and μ^n denotes the convolution of μ with itself n times, or the distribution of n steps of a μ random walk. Since $\alpha(1) \neq 0$ then η is also generating.

Claim 2.4. *A bounded function $h : G \to \mathbb{R}$ is μ-harmonic if and only if it is η-harmonic.*

Proof. It is easy to show that any bounded μ-harmonic function is also μ^n-harmonic, and therefore any μ-harmonic function is also η-harmonic, since η is a linear combination of convolution powers of μ (see, e.g., [14]).

To see the converse, let h be a bounded η-harmonic function on G. Let $(Z_1, Z_2, \ldots)$ be a μ random walk on G starting from g, and denote by $\mathbb{E}_g[\cdot]$ the expectation on its probability space. Then $(Z_{\tau_1}, Z_{\tau_2}, \ldots)$ is a coupled η random walk on G, also starting from g. Let $M = \lim_n h(Z_{\tau_n})$; note that $h(Z_{\tau_n})$ is a bounded martingale w.r.t. the filtration $\mathcal{F}_n = \sigma(Z_{\tau_1}, \ldots, Z_{\tau_n})$, and therefore M is well defined.

To see that h is also μ-harmonic, note that M is measurable in the sigma-algebra generated by the union of the following two sigma-algebras: $\sigma(Z_1, Z_2, \ldots)$ and the shift-invariant sigma-algebra of $\sigma(\tau_1, \tau_2, \ldots)$. However, the latter is trivial, as it is the shift-invariant sigma-algebra of an aperiodic, irreducible random walk on $\mathbb{Z}^+$. Hence M is measurable in $\sigma(Z_1, Z_2, \ldots)$, and so $h'(Z_n) = \mathbb{E}[M|Z_n]$ is μ-harmonic. But $h'(Z_{\tau_n}) = \mathbb{E}[M|Z_{\tau_n}] = h(Z_{\tau_n})$, so h is μ-harmonic. $\square$

We have thus, by Lemma 2.3, shown that a G-space (X, ν) is μ-stationary if and only if it is η-stationary. Furthermore, it is easy to show [14] that

$$h_{\mu^n}(X, \nu) = n \cdot h_\mu(X, \nu),$$

and therefore

$$h_\eta(X, \nu) = h_\mu(X, \nu) \sum_{n=0}^{\infty} n\alpha(n).$$

If we choose α so that $\sum_{n=0}^{\infty} n\alpha(n) = 1$, then we have that $h_\eta(X, \nu) = h_\mu(X, \nu)$.

To summarize, we have shown that μ, η share the same stationary spaces, and that furthermore the set of entropies that can be realized using (G-ergodic) stationary spaces for μ and η are identical. Additionally, it is straightforward to show that if μ has finite first moment then so does η. Therefore, for the purposes of entropy realization for finite first moment measures, μ and η are equivalent.

The advantage of η is that it is supported everywhere on G. We will henceforth assume, without loss of generality, that μ is supported everywhere, which will simplify our proofs. Note also that if μ is supported everywhere then so are its hitting measures on finite index subgroups, which we discuss in Section 2.4. Hence all the measures we will concern ourselves with will be assumed to be supported everywhere.

3. Entropy realization for lamplighter groups

In this section we prove Theorem 2. To this end, we will prove the following more general proposition, of which the theorem will be a direct consequence. This proposition will also be useful to us later.

In Section 3.2 we introduce the boundary (B_ℓ, ν_ℓ) of lamplighter groups, which is an extension of the limit configuration boundary. We denote its entropy by $h_\ell(G, \mu)$, and so

$$h_{\mathrm{conf}}(G, \mu) \leq h_\ell(G, \mu) \leq h_{RW}(G, \mu).$$

An interesting question is to understand when these numbers are all equal. In some cases this is known to be true (see Section 3.4), and furthermore, the authors are not aware of any counterexample.

Proposition 3.1. *Let $G = L \wr \Gamma$ be a finitely generated discrete lamplighter with base group Γ and lamps in L. Then there exists a family of G-ergodic invariant random subgroups $\{\lambda_{p,m} : p \in (0,1), m \in \mathbb{N}\}$, such that, for every generating measure $\mu \in \mathcal{P}(G)$ with finite entropy, and such that the projected random walk on Γ has a trivial Poisson boundary, it holds that*

$$\lim_{m \to \infty} h_\mu(B(\mathrm{Sub}_G), \nu_{\lambda_{p,m}}) = p \cdot h_\ell(G, \mu).$$

We proceed by deducing Theorem 2 from Proposition 3.1, before proving the proposition itself.

Proof of Theorem 2. The statement of Proposition 3.1 is stronger than that of the theorem, since $h_{\mathrm{conf}}(G, \mu) \leq h_\ell(G, \mu)$, and since the family $\{\lambda_{p,m}\}$ is universal, in the sense that it can be used to realize entropy densely for any finite entropy generating measure on G. □

3.1. **Proof of Proposition 3.1.** Let $G = L \wr \Gamma$ be a lamplighter group, and let μ be a generating measure. Recall that we denote by π the projection $G \to \Gamma$ defined by $\pi(f, \gamma) = \gamma$, and denote $\overline{\mu} = \pi_* \mu$. We assume that the $\overline{\mu}$ random walk on Γ has a trivial Poisson boundary. It follows that Γ is amenable.

Recall (Section 2.6) the definition of G as the set $\Gamma \times C_C(L,\Gamma)$, where $C_C(L,\Gamma)$ is the set of finitely supported functions from Γ to L, and recall that for S a subset of Γ, $C_C(L,S)$ is the set of finitely supported functions supported on S. Finally $K_S < G$ is the subgroup of finite configurations supported on S, with the walker in the origin:

$$K_S = \{(f, e_\Gamma) \in G \ : \ f \in C_C(L,S)\}.$$

A percolation measure λ on Γ is a measure on subsets of Γ. In Section 2.6 above we showed how any G-invariant ergodic λ can be associated, via the map that assigns the subgroup K_S to the set S, with an ergodic Bowen space $(B(\mathrm{Sub}_G), \nu_\lambda)$.

To prove Proposition 3.1 we first, for any $p \in [0,1]$, construct an ergodic percolation measure λ on subsets S of Γ such that, with probability close to p, S excludes a large neighborhood of the origin, and with probability close to $1 - p$, S includes a large neighborhood of the origin. Hence, the associated IRS has the property that K_S, with high probability, either includes or excludes all the lamps in a large neighborhood of the origin.

Given a percolation measure λ on Γ, we say that "$\gamma \in \Gamma$ is open" (or closed) to signify the event that γ is (or is not) an element of the subset drawn from λ. Likewise, we say that "$S \subseteq \Gamma$ is open" when all $\gamma \in S$ are open, and that "S is closed" when all $\gamma \in S$ are closed.

Lemma 3.2. *Let Γ be a discrete amenable group. Then there exists a family of percolation measures $\{\lambda_{p,m} \ : \ p \in [0,1], m \in \mathbb{N}\}$ that satisfy the following conditions.*

(1) $\lambda_{p,m}$ is a Γ-invariant ergodic measure for all $p \in [0,1]$ and $m \in \mathbb{N}$.

(2) For all $\gamma \in \Gamma$, $p \in [0,1]$ and $m \in \mathbb{N}$ it holds that

$$\lambda_{p,m}(\gamma \ is \ closed) = p$$

(3) For any finite $S \subset \Gamma$ and all $p \in (0,1)$ it holds that

$$\lim_{m \to \infty} \lambda_{p,m}(S \ is \ open \cup S \ is \ closed) = 1.$$

We prove this lemma in Appendix A below. For a related result see [4].

The limit $\lim_m \lambda_{p,m}$ is the non-ergodic percolation $\lambda_p = p\delta_\emptyset + (1-p)\delta_\Gamma$. Clearly, the Furstenberg entropy of the associated Bowen space is $p \cdot h_{RW}(G,\mu)$. This is the basic intuition behind Proposition 3.1. However, λ_p is not ergodic, and the map $\lambda \mapsto h_\mu(B(\mathrm{Sub}_G), \nu_\lambda)$ is not continuous, and therefore the proof of Proposition 3.1 requires some additional work.

Consider two events, namely that K_S either includes or excludes all the lamps in a large neighborhood of the origin; by Lemma 3.2 above, the union of these events nearly covers the probability space.

Consider first the case that K_S includes all the lamps in a large neighborhood of the origin. Then in the $K_S \backslash G$ Markov chain, we "mod out by the lamps of S", so that the states of the Markov chain do not includes the lamp configuration around the origin. Therefore, this Markov chain resembles, for the first few steps, the projected $\overline{\mu}$ random walk on the base group Γ, and therefore the entropy $h_\mu(B_K, \nu_K)$ could be expected to be low. Conversely, when K_S excludes all the lamps in a large neighborhood of the origin, the $K_S \backslash G$ chain includes all the information about the lamps around the origin, and therefore, in the first few steps, resembles the μ random walk on G, and thus $h_\mu(B_K, \nu_K)$ could be expected to have entropy that is close to that of the Poisson boundary, or at least that of the limit configuration

boundary. This intuition is formalized in the following two lemmas, which we prove below.

Lemma 3.3. *Let $\{S_r\}_{r=1}^{\infty}$ be a sequence of cofinite subsets of Γ such that $\lim_r S_r = \emptyset$, in the topology of convergence on finite sets. Then there exists an $h_\ell(G, \mu) > 0$ such that*

$$\frac{1}{n} \lim_{r \to \infty} h_{\mu^n}\left(B_{K_{S_r}}, \nu_{K_{S_r}}\right) = h_\ell(G, \mu),$$

for all $n \geq 1$.

Lemma 3.4. *Let $\{S_r\}_{r=1}^{\infty}$ be a sequence of finite subsets of Γ such that $\lim_r S_r = \Gamma$, in the topology of convergence on finite sets. Then*

$$\lim_{n \to \infty} \frac{1}{n} \lim_{r \to \infty} h_{\mu^n}\left(B_{K_{S_r}}, \nu_{K_{S_r}}\right) = 0.$$

The following corollary is a direct consequence of these two lemmas.

Corollary 3.5. *For every $\varepsilon > 0$ there exists a finite set $S \subset \Gamma$ and $n \in \mathbb{N}$ such that both*

$$\frac{1}{n} h_{\mu^n}\left(B_{K_S}, \nu_{K_S}\right) \leq \varepsilon \qquad \text{and} \qquad \frac{1}{n} h_{\mu^n}\left(B_{K_{S^c}}, \nu_{K_{S^c}}\right) \geq h_\ell(G, \mu) - \varepsilon,$$

where S^c is the complement of S in Γ.

We are now ready to prove Proposition 3.1. The idea of the proof is as follows. By Corollary 3.5, for finite S large enough, the entropy of (B_{K_S}, ν_{K_S}), the Poisson boundary of the induced Markov chain on $K_S \backslash G$, is close to zero whenever the event "S is open" occurs. On the other hand, if the event "S is closed" occurs, then the entropy is close to $h_\ell(G, \mu)$. Now, using Lemma 3.2, we can find a percolation measure such that the event "S is open" occurs with probability almost $1 - p$ and "S is closed" occurs with probability almost p. It follows that the entropy is close to $p \cdot h_\ell(G, \mu)$.

Proof of Proposition 3.1. Fix $p \in (0, 1)$. Let $\{\lambda_{p,m}\}$ be the set of measures defined in Lemma 3.2, and let $(B(\mathrm{Sub}_G), \nu_{\lambda_{p,m}})$ be the Bowen space associated with $\lambda_{p,m}$; we here identify the percolation measure $\lambda_{p,m}$ with the associated IRS measure, and denote the latter too by $\lambda_{p,m}$.

We shall prove the claim by showing that for every $\varepsilon > 0$, and for every $\mu \in \mathcal{P}(G)$ that satisfies the conditions of the claim, it holds that for m large enough

$$p \cdot h_\ell(G, \mu) - \varepsilon \leq h_\mu(B(\mathrm{Sub}_G), \nu_{\lambda_{p,m}}) \leq p \cdot h_\ell(G, \mu) + \varepsilon.$$

Let $\varepsilon > 0$ and let $\mu \in \mathcal{P}$ satisfy the conditions of the claim. Denote $h_\ell = h_\ell(G, \mu)$, and let S be a finite subset of Γ such that

$$\frac{1}{n} h_{\mu^n}\left(B_{K_S}, \nu_{K_S}\right) \leq \varepsilon \qquad \text{and} \qquad \frac{1}{n} h_{\mu^n}\left(B_{K_{S^c}}, \nu_{K_{S^c}}\right) \geq h_\ell(G, \mu) - \varepsilon,$$

where S^c is the complement of S in Γ. The existence of this set is guaranteed by Corollary 3.5. Set $\varepsilon' = \varepsilon(1 + h_\ell)$ and apply Lemma 3.2 to S to find $m \in \mathbb{N}$ large enough such that

$$(1 - p) - \varepsilon' \leq \lambda_{p,m}(S \text{ is open}) \leq 1 - p$$

and

$$p - \varepsilon' \leq \lambda_{p,m}(S \text{ is closed}) \leq p.$$

Recall that the μ^n Furstenberg entropy of $(B(\mathrm{Sub}_G), \nu_{\lambda_{p,m}})$ is given by

$$h_{\mu^n}\left(B(\mathrm{Sub}_G), \nu_{\lambda_{p,m}}\right) = \int_{\mathrm{Sub}_G} h_{\mu^n}(B_K, \nu_K) d\lambda_{p,m}(K).$$

We shall integrate separately over the event that S is open $(\mathcal{A}_1)$, the event that S is closed $(\mathcal{A}_2)$ and the complements of these events $(\mathcal{A}_3)$ and bound the integral over $\mathcal{A}_1$ and $\mathcal{A}_3$ from above and over $\mathcal{A}_2$ from both above and below.

By the definitions of these events, for every $K_{S_1} \in \mathcal{A}_1$ it holds that S is a subset of S_1, and for every $K_{S_2} \in \mathcal{A}_2$ it holds that the complement of S is a subset of S_2. Hence, by Corollary 3.5, and by the monotonicity of $h_\mu(B_K, \nu_K)$ (Eq. 2.7), we have that

$$\frac{1}{n} \int_{\mathcal{A}_1} h_{\mu^n}(B_K, \nu_K) d\lambda_{p,m}(K) \leq \frac{1}{n} \int_{\mathcal{A}_1} h_{\mu^n}(B_{K_S}, \nu_{K_S}) d\lambda_{p,m}(K)$$
$$\leq \varepsilon \cdot \lambda_{p,m}(\mathcal{A}_1)$$
$$\leq \varepsilon' \cdot (1 - p). \tag{3.1}$$

and

$$\frac{1}{n} \int_{\mathcal{A}_2} h_{\mu^n}(B_K, \nu_K) d\lambda_{p,m}(K) \geq \frac{1}{n} \int_{\mathcal{A}_2} h_{\mu^n}(B_{K_S}, \nu_{K_S}) d\lambda_{p,m}(K)$$
$$\geq (h_\ell - \varepsilon) \cdot \lambda_{p,m}(\mathcal{A}_2)$$
$$\geq (h_\ell - \varepsilon') \cdot (p - \varepsilon'). \tag{3.2}$$

By Claim 3.7, and using the monotonicity of Kullback-Leibler divergence, $h_{\mu^n}(B_K, \nu_K)$ is uniformly bounded by $n \cdot h_\ell$. Hence

$$\frac{1}{n} \int_{\mathcal{A}_3} h_{\mu^n}(B_K, \nu_K) d\lambda_{p,m}(K) \leq h_\ell \cdot \lambda_{p,m}(\mathcal{A}_3) \leq h_\ell \cdot \varepsilon', \tag{3.3}$$

as $\lambda_{p,m}(\mathcal{A}_3) \leq \varepsilon'$. Likewise

$$\frac{1}{n} \int_{\mathcal{A}_2} h_{\mu^n}(B_K, \nu_K) d\lambda_{p,m}(K) \leq h_\ell \cdot \lambda_{p,m}(\mathcal{A}_2) \leq h_\ell \cdot (1 - p). \tag{3.4}$$

Collecting terms and substituting $\varepsilon = \varepsilon'/(1 + h_\ell)$, we get that

$$\frac{1}{n} \int_{\mathrm{Sub}_G} h_{\mu^n}(B_K, \nu_K) d\lambda_{p,m}(K) \in [p \cdot h_\ell - \varepsilon, p \cdot h_\ell + \varepsilon].$$

But the left hand side is equal to $h_\mu(B(\mathrm{Sub}_G), \lambda_{p,m})$, and so the proof is complete. $\square$

3.2. The boundary (B_ℓ, ν_ℓ) and a proof of Lemma 3.3.

To prove Lemma 3.3, we introduce the boundary (B_ℓ, ν_ℓ) of the (G, μ) random walk. We show that this boundary is an extension of the limit configuration boundary, and as such has positive entropy $h_\ell(G, \mu) = h_\mu(B_\ell, \nu_\ell) \geq h_{\mathrm{conf}}(G, \mu)$. We then show that when S excludes a large neighborhood of lamps around the origin then $h_\mu(B_{K_S}, \nu_{K_S})$ is close to $h_\ell(G, \mu)$. While it may be the case that (B_ℓ, ν_ℓ) is, in fact, equal to $\Pi(G, \mu)$, the Poisson boundary of (G, μ), we are not able to show this in full generality (see the discussion below in Section 3.4).

Let S be a cofinite subset of Γ. A state of the μ Markov chain $K_S \backslash G$ can be thought of as the pair consisting of the finite configuration of the lamps outside S, and the position of the lighter.

Let (B_{K_S}, ν_{K_S}) be the Poisson boundary of the μ Markov chain on $K_S \backslash G$. Note that B_{K_S} is not a G-space, but it is a factor of $\Pi(G, \mu)$, and as such we can think of it as a sub-sigma-algebra $\mathcal{F}_S$ of the shift invariant sigma-algebra on Ω, which, however, is not G-invariant. We define $\mathcal{F}_\ell$ as the sigma-algebra generated by the union of all these sigma-algebras:

$$\mathcal{F}_\ell = \sigma \left(\bigcup_{\text{cofinite } S} \mathcal{F}_S \right).$$

Note that while $\mathcal{F}_S$ is not G-invariant, $\mathcal{F}_\ell$ is, since $g = (f, \gamma)$ acts on $\mathcal{F}_S$ by shifting it to $\mathcal{F}_{\gamma S}$. Therefore $\mathcal{F}_\ell$ is G-invariant and shift invariant, and therefore its Mackey realization, which we denote by (B_ℓ, ν_ℓ), is a (G, μ)-boundary. As such, it is a factor of $\Pi(G, \mu)$. Denote its entropy by $h_\ell(G, \mu) = h_\mu(B_\ell, \nu_\ell)$.

Since the limit configuration boundary is generated by cylinders of the final states of finite sets of lamps, then it is a factor of B_ℓ. The following claim is a consequence of these definitions.

Claim 3.6.

$$h_\ell(G, \mu) \geq h_{\text{conf}}(G, \mu).$$

Let $S_r \subset \Gamma$ be a sequence of cofinite subsets such that $\lim_r S_r = \emptyset$. Consider the sequence of subgroups $K_r = K_{S_r}$, and let $\mathcal{F}_r$ be the sigma-algebra of the Poisson boundary of the μ Markov chain on $K_r \backslash G$.

Claim 3.7.

$$\mathcal{F}_\ell = \sigma \left(\bigcup_{r=1}^{\infty} \mathcal{F}_r \right).$$

Proof. Denote $\mathcal{F}_\infty = \sigma \left(\cup_{r=1}^{\infty} \mathcal{F}_r \right)$, and recall that $\mathcal{F}_\ell = \sigma \left(\cup_{|S| < \infty} \mathcal{F}_S \right)$. Since $\mathcal{F}_r = \mathcal{F}_{S_r}$, where S_r is finite, it follows that $\mathcal{F}_r \subseteq \mathcal{F}_\ell$, and so $\cup_{r=1}^{\infty} \mathcal{F}_r \subseteq \mathcal{F}_\ell$ and

$$\mathcal{F}_\infty = \sigma \left(\cup_{r=1}^{\infty} \mathcal{F}_{S_r} \right) \subseteq \mathcal{F}_\ell.$$

Conversely, note that for each finite $S \subset \Gamma$ there exists an r such that S is a subset of the complement in Γ of S_r, since $\lim_r S_r \to \emptyset$. Hence $\mathcal{F}_S \subseteq \mathcal{F}_{S_r} \subseteq \mathcal{F}_\infty$, $\cup_{|S| < \infty} \mathcal{F}_S \subseteq \mathcal{F}_\infty$, and the claim follows. $\qquad\square$

Let $\mathbf{bnd}_\ell : \Omega \to B_\ell$ be the boundary map associated with B_ℓ, let $\Omega_r = (K_r \backslash G)^{\mathbb{N}}$, let $\mathbf{bnd}_r : \Omega_r \to B_{K_r}$ be the boundary map of the induced Markov chain on $K_r \backslash G$, and let $\pi_r : B_\ell \to B_{K_r}$ be the natural factor. Then π_r is similar to G-equivariant maps, in the sense that $\pi_{r*} g \nu_\ell = \nu_{K_r g}$ for all $g \in G$.

Claim 3.8. *For a fixed $g \in G$, and ν_ℓ-almost every $b \in B_\ell$,*

$$\lim_{r \to \infty} \frac{d\nu_{K_r g}}{d\nu_{K_r}}(\pi_r(b)) = \frac{dg\nu_\ell}{d\nu_\ell}(b).$$

Proof. Since

$$\mathbb{P}\left[Z_1 = g | \mathbf{bnd}_r(Z_1, Z_2, \ldots) = \pi_r(b) \right] = \mathbb{P}\left[Z_1 = g \right] \frac{d\nu_{K_r g}}{d\nu_{K_r}}(b)$$

and

$$\mathbb{P}\left[Z_1 = g | \mathbf{bnd}_\ell(Z_1, Z_2, \ldots) = b \right] = \mathbb{P}\left[Z_1 = g \right] \frac{dg\nu_\ell}{d\nu_\ell}(b),$$

it is enough to show that for ν_ℓ-almost every $b \in B_\ell$,

$$\lim_{r \to \infty} \mathbb{P}\left[Z_1 = g | \mathbf{bnd}_r(Z_1, Z_2, \ldots) = \pi_r(b)\right] = \mathbb{P}\left[Z_1 = g | \mathbf{bnd}_\ell(Z_1, Z_2, \ldots) = b\right].$$

This, however, is a consequence of Claim 3.7, and the claim follows. $\qquad\square$

Claim 3.9. *For a fixed $g \in G$,*

$$\lim_{r \to \infty} D_{KL}\left(\nu_{K_r g} || \nu_{K_r}\right) = D_{KL}\left(g\nu_\ell || \nu_\ell\right).$$

Proof. Since $\nu_{K_r} = \pi_{r*}\nu_\ell$ we get that

$$\begin{aligned}
D_{KL}\left(\nu_{K_r g^{-1}} || \nu_{K_r}\right) &= \int_{B_{K_r}} -\log \frac{d\nu_{K_r g}}{d\nu_{K_r}}(x) d\nu_{K_r}(x) \\
&= \int_{B_\ell} -\log \frac{d\nu_{K_r g}}{d\nu_{K_r}}(\pi_r(b)) d\nu_\ell(b).
\end{aligned}$$

By Claim 3.8, the functions $f_r(b) = -\log \frac{d\nu_{K_r g}}{d\nu_{K_r}}(\pi_r(b))$ converge pointwise to $f(b) = -\log \frac{dg\nu_\ell}{d\nu_\ell}(b)$. Hence the claim will follow by the dominated convergence theorem, provided that we can show that the functions f_r are uniformly bounded. By Lemma 2.1 we have that

$$\mu\left(g^{-1}\right) \le \frac{d\nu_{K_r g}}{d\nu_{K_r}}(x) \le \frac{1}{\mu(g)}. \qquad (3.5)$$

Since μ is without loss of generality supported everywhere (see Section 2.2), $\mu(g) > 0$ and $\mu\left(g^{-1}\right) > 0$, and so

$$-\log\mu\left(g^{-1}\right) \ge -\log\frac{d\nu_{K_r g}}{d\nu_{K_r}}(x) \ge -\log\frac{1}{\mu\left(g\right)}.$$

Therefore the functions f_r are uniformly bounded. $\qquad\square$

Proof of Lemma 3.3. We prove the lemma for $n = 1$, and note that for arbitrary n the proof follows by the same argument, since $h_{\mu^n}(B_\ell, \nu_\ell) = n \cdot h_\ell(G, \mu)$.

Note that by the monotonicity of Kullback-Leibler divergence, $D_{KL}\left(\nu_{K_r g} || \nu_{K_r}\right) \le D_{KL}\left(g\nu_\ell || \nu_\ell\right)$. Note also that the finiteness of $h_\ell(G, \mu)$ implies that $D_{KL}\left(g\nu_\ell || \nu_\ell\right)$ is μ-integrable, as function of g. Hence, by the dominated convergence theorem,

$$\begin{aligned}
\lim_{r \to \infty} h_\mu(B_{K_r}, \nu_{K_r}) &= \lim_{r \to \infty} \sum_{g \in G} \mu(g) D_{KL}(\nu_{K_r g} || \nu_{K_r}) \\
&= \sum_{g \in G} \mu(g) \lim_{r \to \infty} D_{KL}(\nu_{K_r g} || \nu_{K_r}) \\
&= \sum_{g \in G} \mu(g) D_{KL}(g\nu_\ell || \nu_\ell) \\
&= h_\ell(G, \mu)
\end{aligned}$$

$\qquad\square$

3.3. **Proof of Lemma 3.4.** Let $\{S_r\}_{r=1}^{\infty}$ be a sequence of subsets of Γ with $\lim_r S_r = \Gamma$, consider the sequence of subgroups $K_r = K_{S_r}$, and consider the induced Markov chains on $K_r \backslash G$. As $r \to \infty$, we are "modding out by more and more lamps", and so the Markov chains resemble more and more closely the projected walk on Γ itself, which has zero entropy, since it has a trivial Poisson boundary. Indeed, in this section we prove that the entropies $h_\mu(B_{K_r}, \nu_{K_r})$ convergence to zero.

Recall (Sec 2.5) that $P_K(Kg, Kh)$ is the transition probability from Kg to Kh in the induced Markov chain on $K \backslash G$. Recall also that the projection $\pi : G \to \Gamma$ is defined by $\pi(f, \gamma) = \gamma$, and that we denote $\overline{g} = \pi(g)$ and $\overline{\mu} = \pi_* \mu$.

Claim 3.10. *For all $g, h \in G$ it holds that*

$$\lim_{r \to \infty} P_{K_r}(K_r, K_r g) = \overline{\mu}(\overline{g}).$$

It follows directly that

$$\lim_{r \to \infty} P_{K_r}^n(K_r, K_r g) = \overline{\mu}^n(\overline{g}).$$

Proof. Recall that $C_C(L, S)$ is the set of finite lamp configurations supported on S. By definition,

$$P_{K_r}(K_r, K_r g) = \sum_{k \in K_r} \mu(kg)$$

$$= \sum_{f \in C_C(L, S_r)} \mu((f, e_\Gamma)g)$$

Observe that $\Gamma = K_\Gamma \backslash G$, and that K_Γ is a normal subgroup in G. Hence

$$\overline{\mu}(\overline{g}) = \sum_{f \in C_C(L, \Gamma)} \mu((f, e_\Gamma)g)$$

$$= \sum_{f \in C_C(L, \Gamma)} \mu(g(f, e_\Gamma)).$$

Now, since $\lim_r S_r = \Gamma$, it follows that $\lim_r C_C(L, S_r) = C_C(L, \Gamma)$ and hence

$$\lim_{r \to \infty} P_{K_r}(K_r, K_r g) = \lim_{r \to \infty} \sum_{f \in C_C(L, S_r)} \mu((f, e_\Gamma)g)$$

$$= \sum_{f \in C_C(L, \Gamma)} \mu((f, e_\Gamma)g)$$

$$= \overline{\mu}(\overline{g}).$$

$\square$

Proof of Lemma 3.4. For fixed $n, r \in \mathbb{N}$,

$$h_{\mu^n}(B_{K_r}, \nu_{K_r}) = \sum_{g \in G} \mu^n(g) \int_{B_{K_r}} -\log \frac{d\nu_{K_r}}{d\nu_{K_r g}}(b) d\nu_{K_r g}(b)$$

$$\leq \sum_{g \in G} \mu^n(g) \int_{B_{K_r}} -\log P_{K_r}^n(K_r, K_r g) d\nu_{K_r g}(b)$$

$$= \sum_{g \in G} \mu^n(g) \cdot -\log P_{K_r}^n(K_r, K_r g),$$

where the inequality is an application of Lemma 2.1.

By Claim 3.10, $\lim_r P^n_{K_r}(K_r, K_r g) = \overline{\mu}(\overline{g})$. By $0 \leq -\log P^n_{K_r}(K_r, K_r g) \leq -\log \mu^n(g)$ and the finiteness of $H(\mu)$ we can use the dominated convergence theorem to arrive at

$$\frac{1}{n} \lim_{r \to \infty} h_{\mu^n}(B_{K_r}, \nu_{K_r}) \leq \frac{1}{n} \sum_{g \in G} \mu^n(g) \cdot - \log \overline{\mu}^n(\overline{g})$$

$$= \frac{1}{n} \sum_{\gamma \in \Gamma} -\overline{\mu}^n(\gamma) \log \overline{\mu}^n(\gamma)$$

Taking the limit as n tends to infinity we get that

$$\lim_{n \to \infty} \frac{1}{n} \lim_{r \to \infty} h_{\mu^n}(B_{K_r}, \nu_{K_r}) \leq \lim_{n \to \infty} \frac{1}{n} H(\overline{\mu}^n) = 0,$$

where the final equality is a again a consequence of the fact that the $\overline{\mu}$ random walk on Γ has a trivial Poisson boundary [14]. □

3.4. **Dense entropy realization for some lamplighter groups.** The boundary B_ℓ is an extension of the limit configuration boundary and a factor of the Poisson boundary. It follows that when the limit configuration boundary is equal to the Poisson boundary than so is B_ℓ, and $h_\ell = h_{RW}$.

Thus, a direct consequence of Theorem 2 is the following theorem.

Theorem 3.11. *Let $G = L \wr \Gamma$ with non-trivial L, let μ generate G, and let the (G, μ) limit configuration boundary equal its Poisson boundary. Then there exists a dense set $H \subseteq [0, h_{RW}(G, \mu)]$ such that for every $h \in H$ there exists a (G, μ) ergodic Bowen space (X, ν) with $h_\mu(X, \nu) = h$.*

The relation between the Poisson boundary and the limit configuration boundary is a subject of active research. Kaimanovich [13] shows that the Poisson boundary coincides with the limit configuration boundary on $((\mathbb{Z}/2\mathbb{Z}) \wr Z^d, \mu)$ when μ has a first moment and the projected random walk on $\mathbb{Z}^d$ has a drift. Erschler [6] shows that these boundaries are equal for $((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^d, \mu)$, when $d \geq 5$ and μ has with finite third moment. An additional equality result on non-amenable base groups is given by Karlsson and Woess [15].

4. No entropy gap for virtually free groups

In this section we prove our main result, Theorem 1, which states that when G is virtually free and μ has finite first moment then (G, μ) does not have an entropy gap.

The general idea is to "lift" the no entropy gap result from lamplighters to virtually free groups. First, we establish this result for free groups in Section 4.1, and then lift it to finite index supergroups.

To relate the stationary actions of a group and a finite index subgroup, we consider the hitting measure on the subgroup. In Section 4.2, we show that measures with finite first moment have hitting measures with finite first moment.

Then, in Section 4.3 we discuss a standard construction which lifts IRS measures from finite index subgroups, and hence also lifts the associated Bowen spaces. For the case that the finite index subgroup is normal, we relate, in Section 4.4, the entropies these Bowen spaces. Finally, in Section 4.5, we bring these ideas together to prove a no entropy gap result for virtually free groups.

4.1. **From lamplighters to free groups.** In this section we show that as extensions of lamplighters, free groups admit a result that is parallel to Proposition 3.1, the stronger version of Theorem 2 that we proved above.

The following claim is standard.

Claim 4.1. *Let G be finitely generated, and let $G \xrightarrow{\varphi} Q$ be a group homomorphism onto Q. If $\mu \in \mathcal{P}(G)$ has finite first moment then $\varphi_* \mu \in \mathcal{P}(Q)$ has finite first moment.*

Lemma 4.2. *Let G be finitely generated, let $G \xrightarrow{\varphi} Q$ be a group homomorphism onto Q, and let $\mu \in \mathcal{P}(G)$. Let $(B(\mathrm{Sub}_Q), \nu_\lambda)$ be an ergodic $(Q, \varphi_* \mu)$ Bowen space. Then $(B(\mathrm{Sub}_G), \nu_{\varphi^{-1}\lambda})$ is an ergodic (G, μ) Bowen space, and furthermore*

$$h_\mu(B(\mathrm{Sub}_G), \nu_{\varphi^{-1}\lambda}) = h_{\varphi_* \mu}(B(\mathrm{Sub}_Q), \nu_\lambda).$$

Proof. Note that G acts naturally on Sub_Q through φ. Hence Sub_Q is a G-space. Since $\varphi^{-1} : \mathrm{Sub}_Q \to \mathrm{Sub}_G$ is G-equivariant, then $(\mathrm{Sub}_G, \varphi_*^{-1}\lambda)$ is a G-factor of $(\mathrm{Sub}_Q, \lambda)$. Since the latter is invariant and ergodic, it follows that the former is too, and hence is an ergodic G IRS. Furthermore, since $\varphi(\varphi^{-1}(K)) = K$, the two spaces are G-isomorphic.

The same can also be said for the spaces of induced random walks on $K \backslash Q$ and $\varphi^{-1}(K) \backslash G$, and therefore the induced Markov chains are also isomorphic. Finally, since, as a map between $(K \backslash Q)^{\mathbb{N}} \to (\varphi^{-1}(K) \backslash G)^{\mathbb{N}}$, φ^{-1} is shift invariant, it follows that (B_K, ν_K) is G-isomorphic to $(B_{\varphi^{-1}(K)}, \nu_{\varphi^{-1}(K)})$, and so the Bowen spaces $(B(\mathrm{Sub}_Q), \nu_{\varphi_* \lambda})$ and $(B(\mathrm{Sub}_G), \nu_{\varphi^{-1}\lambda})$ are G-isomorphic.

To see the equality in entropies, note that in general, every $(Q, \varphi_* \mu)$-stationary space is also (G, μ)-stationary, and

$$h_\mu(X, \nu) = h_{\varphi_* \mu}(X, \nu).$$

$\square$

Consider the canonical lamplighter $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^3$. Since any generating random walk on $\mathbb{Z}^3$ is transient (see., e.g., Proposition 3.20 in [20]), by Erschler [6], for any finite first moment μ, it holds that $h_\ell((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^3, \mu) \geq h_{\mathrm{conf}}((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^3, \mu) > 0$. Therefore, Claim 4.1 and Lemma 4.2, together with Proposition 3.1, yield the following proposition.

Proposition 4.3. *Let G be a finitely generated extension of $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^3$, with $\varphi : G \to (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^3$ the quotient map. Then there exists a family of G-ergodic invariant random subgroups $\{\lambda_{p,m} : p \in (0,1), m \in \mathbb{N}\}$ such that, for every generating measure $\mu \in \mathcal{P}(G)$ with finite first moment it holds that*

$$\lim_{m \to \infty} h_\mu(B(\mathrm{Sub}_G), \nu_{\lambda_{p,m}}) = p \cdot h_\ell(\varphi G, \varphi_* \mu),$$

where $h_\ell(\varphi G, \varphi_* \mu) > 0$.

Since $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^3)$ can be generated as a group by a set of four generators, this holds for F_n, with $n \geq 4$.

4.2. **Hitting measures and finite first moments.**

Lemma 4.4. *Let G be a finitely generated group, and let $\Gamma \leq G$ with $[G : \Gamma] < \infty$. If $\mu \in \mathcal{P}(G)$ has finite first moment, then the hitting measure $\theta \in \mathcal{P}(\Gamma)$ also has finite first moment.*

We prove this lemma in Appendix B.

By Proposition 4.3 and Lemma 4.4 we conclude the following. Let G be a group with F_n as a finite index subgroup. Let $\mu \in \mathcal{P}(G)$ be a finite first moment generating measure, and consider its hitting measure $\theta \in \mathcal{P}(F_n)$. Then (F_n, θ) has no entropy gap. In the next sections, we use this fact to prove our no entropy gap result for (G, μ).

4.3. **Lifting Bowen spaces from lattices.** The following construction applies to a more general settings, where G is a locally compact group and Γ is a lattice in G (see, e.g., [19]). That is, there exists a G-invariant measure $\eta \in \mathcal{P}(G/\Gamma)$.

Let $\lambda \in IRS(\Gamma)$. Then λ is Γ-invariant but not, in general, G-invariant. Note, however, that if $g_1\Gamma = g_2\Gamma$ then there exists a $\gamma \in \Gamma$ such that $g_1 = g_2\gamma$. Hence $g_1\lambda = g_2\gamma\lambda = g_2\lambda$. Therefore, the G-action on λ is constant on cosets of Γ, and the measure $(g\Gamma)\lambda$ is well defined for every $g\Gamma \in G/\Gamma$.

Denote by $\eta * \lambda$ the measure

$$\eta * \lambda = \int_{G/\Gamma} (g\Gamma)\lambda \, d\eta(g\Gamma).$$

The following claim is straightforward.

Claim 4.5. *If λ is an ergodic IRS of Γ then $\eta * \lambda$ is an ergodic IRS of G.*

Let Γ be a finite index subgroup of G, and let θ be the hitting measure on Γ of the μ random walk on G. Let λ be a Γ IRS, so that $(B(\mathrm{Sub}_\Gamma), \nu_\lambda)$ is a (Γ, θ) Bowen space. It follows that $(B(\mathrm{Sub}_G), \nu_{\eta*\lambda})$ is a (G, μ) Bowen space. Since Γ is finite index in G, every (G, μ)-stationary space is also a (Γ, θ)-stationary space [9]. In particular, $(B(\mathrm{Sub}_G), \nu_{\eta*\lambda})$ is also a (Γ, θ)-stationary space. Furthermore, (G, μ) and (Γ, θ) share the same Poisson boundary (B, ν), and so there is no ambiguity in referring to the measure $\nu_{\eta*\lambda}$, when considering $(B(\mathrm{Sub}_G), \nu_{\eta*\lambda})$ as either a (G, μ) Bowen space or a (Γ, θ) Bowen space.

Note that when Γ is normal in G then $\eta * \lambda$ is supported on subgroups of Γ. In this case $(B(\mathrm{Sub}_\Gamma), \nu_{\eta*\lambda})$ is both a (G, μ) and a (Γ, θ) Bowen space. It may be the case that it is ergodic with respect to the G action, but not with respect to the Γ action.

4.4. **The entropy of Bowen spaces lifted from finite index normal subgroups.** We now return to consider discrete groups. In particular, let G be a discrete group with $\Gamma \lhd G$ a finite index normal subgroup. Let μ be a generating measure on G, and let θ be the hitting measure on Γ. Denote by θ_g the measure on Γ given by $\theta_g(\gamma) = \theta(\gamma^g)$. Note that if θ has finite first moment then so does θ_g. Note also that the entropy of a random variable drawn from θ_g^n is independent of g, and so $h_{RW}(\Gamma, \theta_g)$ is also independent of g. It follows that the entropy of $\Pi(\Gamma, \theta_g)$ is independent of g.

Let (B, ν) be the Poisson boundary of (Γ, θ). It follows from the definitions that $(B, g^{-1}\nu)$ is (Γ, θ_g)-stationary, and that furthermore $h_{\theta_g}(B, g^{-1}\nu) = h_\theta(B, \nu) = h_{RW}(\Gamma, \theta) = h_{RW}(\Gamma, \theta_g)$. Finally, if $\lim_n Z_n\nu$ is a point mass, then $\lim_n Z_n^{g^{-1}} g^{-1}\nu = \lim_n gZ_n\nu$ is also a point mass, and so $(B, g^{-1}\nu)$ is a (Γ, θ_g)-boundary. As a maximum entropy boundary, it is the Poisson boundary of (Γ, θ_g).

Let λ be a Γ IRS, and let $(B(\mathrm{Sub}_\Gamma), \nu_\lambda)$ be the associated (Γ, θ) Bowen space. Then it follows from the discussion above that $(B(\mathrm{Sub}_\Gamma), (g^{-1}\nu)_\lambda)$ is the associated (Γ, θ_g) Bowen space.

We are now ready to present the following result, which relates the entropy of a lifted Bowen space $(B(\mathrm{Sub}_\Gamma), \nu_{\eta*\lambda})$, to the entropies of the original space, with respect to the different conjugated measures θ_g.

Lemma 4.6. *Let G be a discrete group with $\Gamma \lhd G$, $[G : \Gamma] < \infty$. Let μ be a generating measure on G, and let θ be the hitting measure on Γ.*
Let $(B(\mathrm{Sub}_\Gamma), \nu_\lambda)$ be a Γ Bowen space. Then

$$h_\theta(B(\mathrm{Sub}_\Gamma), \nu_{\eta*\lambda}) = \frac{1}{[G : \Gamma]} \sum_{g\Gamma \in G/\Gamma} h_{\theta_g}\left(B(\mathrm{Sub}_\Gamma), \left(g^{-1}\nu\right)_\lambda\right)$$

and

$$h_\mu(B(\mathrm{Sub}_\Gamma), \nu_{\eta*\lambda}) = \frac{1}{[G : \Gamma]^2} \sum_{g\Gamma \in G/\Gamma} h_{\theta_g}\left(B(\mathrm{Sub}_\Gamma), \left(g^{-1}\nu\right)_\lambda\right).$$

Proof. $(B(\mathrm{Sub}_\Gamma), \nu_{\eta*\lambda})$ is both a (G, μ) and a (Γ, θ) Bowen space. Its θ-entropy is given by Eq. 2.9 as

$$h_\theta(B(\mathrm{Sub}_\Gamma), \nu_{\eta*\lambda}) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathrm{Sub}_\Gamma} H\left(KZ_n\right) d(\eta * \lambda)(K),$$

where $(Z_1, Z_2, \ldots)$ is here a θ random walk on Γ. We can now rewrite this as

$$h_\theta(B(\mathrm{Sub}_\Gamma), \nu_{\eta*\lambda}) = \lim_{n\to\infty} \frac{1}{n} \frac{1}{[G : \Gamma]} \sum_{g\Gamma \in G/\Gamma} \int_{\mathrm{Sub}_\Gamma} H\left(KZ_n\right) d(g\lambda)(K)$$

$$= \lim_{n\to\infty} \frac{1}{n} \frac{1}{[G : \Gamma]} \sum_{g\Gamma \in G/\Gamma} \int_{\mathrm{Sub}_\Gamma} H\left(K^g Z_n\right) d\lambda(K).$$

Note that

$$H\left(K^g Z_n\right) = H\left(gKg^{-1}Z_n\right) = H\left(Kg^{-1}Z_n g\right) = H\left(KZ_n^{g^{-1}}\right),$$

and so

$$h_\theta(B(\mathrm{Sub}_\Gamma), \nu_{\eta*\lambda}) = \lim_{n\to\infty} \frac{1}{n} \frac{1}{[G : \Gamma]} \sum_{g\Gamma \in G/\Gamma} \int_{\mathrm{Sub}_\Gamma} H\left(KZ_n^{g^{-1}}\right) d\lambda(K). \qquad (4.1)$$

By another application of Eq. 2.9 we have that

$$h_{\theta_g}\left(B(\mathrm{Sub}_\Gamma), (g^{-1}\nu)_\lambda\right) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathrm{Sub}_\Gamma} H\left(KZ_n^{g^{-1}}\right) d\lambda(K).$$

Applying this to Eq. 4.1 yields

$$h_\theta(B(\mathrm{Sub}_\Gamma), \nu_{\eta*\lambda}) = \frac{1}{[G : \Gamma]} \sum_{g\Gamma \in G/\Gamma} h_{\theta_g}\left(B(\mathrm{Sub}_\Gamma), \left(g^{-1}\nu\right)_\lambda\right).$$

Finally, we apply Eq. 2.1, which states that the ratio between the (Γ, θ) entropy and the (G, μ) entropy is $[G : \Gamma]$. This yields

$$h_\mu(B(\mathrm{Sub}_\Gamma), \nu_{\eta*\lambda}) = \frac{1}{[G : \Gamma]^2} \sum_{g\Gamma \in G/\Gamma} h_{\theta_g}\left(B(\mathrm{Sub}_\Gamma), \left(g^{-1}\nu\right)_\lambda\right).$$

$\square$

4.5. **No entropy gap for virtually free groups.**

Proof of Theorem 1. Let G be a finitely generated discrete group, and let G have a free group of rank n as a finite index subgroup. If $n = 1$ then G is virtually $\mathbb{Z}$ and it follows that $\Pi(G, \mu)$ is trivial, for any μ. In particular, G has no entropy gap.

Consider then the case that $n \geq 2$.

We claim that there exists a finite index subgroup Γ in G that is a free group of rank ≥ 4, and is furthermore normal in G: If G has F_2 or F_3 as a finite index subgroup then it must also have a higher rank free group F_n as a finite index subgroup. Now, let Γ be the normal core of F_n in G. Then Γ is a finite index subgroup of G, and, as a finite index subgroup of F_n, it is also free, and of rank $\geq n$. Denote by φ a projection from Γ to $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^3$.

Let μ be a generating probability measure on G, and let θ denote the hitting measure on F_n. Since μ has finite first moment by the claim hypothesis, it follows from Lemma 4.4 that the conjugated measure θ_g has finite first moment, for any g.

Let $\{\lambda_{p,m}\}_{m=1}^{\infty}$ be a sequence of invariant random subgroups of Γ, such that for any generating probability measure ζ on Γ with finite first moment it holds that $\lim_m h_\zeta \left(B(\mathrm{Sub}_\Gamma), \nu_{\lambda_{p,m}} \right) = p \cdot h_\ell(\varphi\Gamma, \varphi_*\zeta) > 0$, as guaranteed by Proposition 4.3. Then by Lemma 4.6 above

$$ h_\mu(B(\mathrm{Sub}_\Gamma), \nu_{\eta * \lambda_{p,m}}) = \frac{1}{[G : \Gamma]^2} \sum_{g\Gamma \in G/\Gamma} h_{\theta_g} \left(B(\mathrm{Sub}_\Gamma), \left(g^{-1}\nu\right)_{\lambda_{p,m}} \right). $$

Taking the limits of both sides yields

$$ \lim_{m \to \infty} h_\mu(B(\mathrm{Sub}_\Gamma), \nu_{\eta * \lambda_{p,m}}) = \frac{p}{[G : \Gamma]^2} \sum_{g\Gamma \in G/\Gamma} h_\ell(\varphi\Gamma, \varphi_*\theta_g). $$

Note that by considering only the addend for which $g\Gamma = \Gamma$, it follows that

$$ \lim_{m \to \infty} h_\mu(B(\mathrm{Sub}_\Gamma), \nu_{\eta * \lambda_{p,m}}) \geq \frac{p}{[G : \Gamma]^2} h_\ell(\varphi\Gamma, \varphi_*\theta), $$

and in particular for m large enough the entropy is strictly positive.

On the other hand, $h_\ell(\varphi\Gamma, \varphi_*\theta_{g_i}) \leq h_{RW}(\Gamma, \theta_{g_i}) = h_{RW}(\Gamma, \theta)$, and so

$$ \lim_{m \to \infty} h_\theta(B(\mathrm{Sub}_\Gamma), \nu_{\eta * \lambda_{p,m}}) \leq \frac{p}{[G : \Gamma]} h_{RW}(\Gamma, \theta), $$

For every $\varepsilon > 0$ there exists a $0 \leq p \leq 1$ such that $\frac{p}{[G:\Gamma]} h_{RW}(\Gamma, \theta) < \varepsilon$. Therefore, for large enough m, we get that

$$ 0 < h_\mu(B(\mathrm{Sub}_\Gamma), \nu_{\eta * \lambda_{p,m}}) < \varepsilon. $$

Hence for each $\varepsilon > 0$ there exists an ergodic (G, μ) stationary space with positive entropy that is less than ε. We conclude that (G, μ) has no entropy gap for any finite first moment measure μ.

$\square$

Appendix A. Long range percolations on amenable groups

Proof of Lemma 3.2. Let $\{F_m\}_{m=1}^{\infty}$ be a Følner sequence in Γ. For each $0 \leq p \leq 1$ and $m \in \mathbb{N}$ we construct a corresponding percolation measure $\lambda_{p,m}$ as follows. Let $q = (1-p)^{1/|F_m|}$, and let α be the ergodic i.i.d. percolation measure on Γ with parameter q, so that under α each element γ is open w.p. q. Let γ be open under

$\lambda_{p,m}$ if and only if all of the elements in γF_m were open under α. Let $Q \sim \alpha$ and $R \sim \lambda_{p,m}$, so that $\gamma \in R$ if only if $\gamma F_m \subseteq Q$.

$\lambda_{p,m}$ is clearly Γ-invariant. It is ergodic, since it is a factor of α. Furthermore,

$$\mathbb{P}\left[\gamma \text{ is open}\right] = \mathbb{P}\left[\gamma \in R\right] = \mathbb{P}\left[\gamma F_m \subseteq Q\right] = q^{|\gamma F_m|} = q^{|F_m|} = 1 - p.$$

Let S be a finite subset of Γ, and let $\gamma_0, \gamma \in S$. We would like to show that for any ε there exists an m large enough for which it holds that the probability that one is open and the other not is at most $\varepsilon/|S|$. This, by the union bound, will establish the claim. Assume without loss of generality that $\gamma_0 = e$.

Since S is finite, for each δ, there exists m large enough such that $|F_m \triangle \gamma F_m| < \delta|F_m|$ for all $\gamma \in S$, by the definition of a Følner sequence. Choose m large enough so that $\delta < \varepsilon/|S|$ and also $\delta < \frac{\log(1-\varepsilon/|S|)}{\log(1-p)}$, or $(1-p)^\delta > 1 - \varepsilon/|S|$.

Consider first the case that e is open in $\lambda_{p,m}$. Then

$$\mathbb{P}\left[\gamma \in R | e \in R\right] = \mathbb{P}\left[\gamma F_m \subseteq Q | F_m \subseteq Q\right]$$
$$= \mathbb{P}\left[F_m \cap \gamma F_m \subseteq Q, F_m \setminus \gamma F_m \subseteq Q | F_m \subseteq Q\right].$$

Since α is i.i.d.,

$$= \mathbb{P}\left[F_m \setminus \gamma F_m \subseteq Q\right] = q^{|F_m \setminus \gamma F_m|} > (1-p)^\delta > 1 - \varepsilon/|S|.$$

Consider now the case that e is closed in $\lambda_{p,m}$. Then there exists an element $h \in F_m \setminus Q$. Hence, by Γ-invariance, with probability greater than $|F_m \cap \gamma F_m|/|F_m|$, this h belongs to $(F_m \cap \gamma F_m) \setminus Q$, and in particular to $\gamma F_m \setminus Q$. By definition, this implies that γ is also closed in $\lambda_{p,m}$. Hence

$$\mathbb{P}\left[\gamma \notin R | e \notin R\right] > \frac{|F_m \cap \gamma F_m|}{|F_m|} > 1 - \delta > 1 - \varepsilon/|S|.$$

$\square$

Appendix B. Hitting measures and finite first moments

Lemma (4.4). Let G be a finitely generated group, and let $\Gamma \leq G$ with $[G : \Gamma] < \infty$. If $\mu \in \mathcal{P}(G)$ has finite first moment, then the hitting measure $\theta \in \mathcal{P}(\Gamma)$ also has finite first moment.

By definition, θ is of finite first moment if

$$\sum_{\gamma \in \Gamma} \theta(\gamma)|\gamma|_{\mathcal{S}} < \infty \tag{B.1}$$

where $\mathcal{S}$ is some finite symmetric generating set of Γ. Since finite index subgroups are quasi-isometric to the group, it is enough to check the condition in Eq. B.1 for a word length metric that is induced by a word length metric of G, or, equivalently, for $\mathcal{S}$ a finite symmetric generating set of G.

Consider the μ random walk $(Z_1, Z_2, \ldots)$ on G. Fix $\mathcal{S}$, a finite symmetric generating set of G, and denote $|g| = |g|_{\mathcal{S}}$. Let $L_n = |Z_n|$. The first moment of μ can be written as $\mathbb{E}[L_1]$, and so $C_1 = \mathbb{E}[L_1] < \infty$.

Denote by τ be the Γ-hitting time of the μ random walk, and recall (Sec 2.4) that θ, the hitting measure, is the law of Z_τ. Then the first moment of the hitting measure θ is $\mathbb{E}[L_\tau]$. We therefore need to show that $\mathbb{E}[L_\tau] < \infty$.

Let $M_n = nC_1 - L_n$. We want to apply the optional stopping time theorem on M_n. For that we prove the following claim.

Claim B.1. M_n *is a submartingale w.r.t. the filtration* $\sigma(Z_1, \ldots, Z_n)$*, and*

$$\mathbb{E}\left[|M_{n+1} - M_n||Z_1, \ldots, Z_n\right] \leq 2C_1.$$

Proof. By symmetry and the triangle inequality we have that $|gh| \leq |g| + |h|$. Now,

$$\mathbb{E}\left[L_{n+1}|Z_1, \ldots, Z_n\right] = \sum_{g \in G} \mu(g)|Z_n g|$$

$$\leq \sum_{g \in G} \mu(g)(|Z_n| + |g|)$$

$$= L_n + C_1,$$

and so

$$\mathbb{E}\left[M_{n+1}|Z_n\right] \geq M_n.$$

Therefore M_n is indeed a submartingale. To prove the bound, note that

$$\mathbb{E}\left[|M_{n+1} - M_n||Z_1, \ldots, Z_n\right] = \mathbb{E}\left[|L_n - L_{n+1} + C_1||Z_1, \ldots, Z_n\right]$$

$$\leq C_1 + \mathbb{E}\left[|L_{n+1} - L_n||Z_n\right].$$

By the triangle inequality and the symmetry of $\mathcal{S}$ it follows that

$$= C_1 + \sum_{g \in G} \mu(g) \left||Z_n g| - |Z_n|\right|$$

$$\leq C_1 + \sum_{g \in G} \mu(g)|g|$$

$$= 2C_1.$$

$\square$

Proof of Lemma 4.4. In general, the index of Γ in G is equal to the expected hitting time [11], and so $\mathbb{E}\left[\tau\right] = [G : \Gamma] < \infty$. It follows by Theorem (7.5) in [5] that because M_n is a submartingale satisfying the condition of Claims B.1, then

$$\mathbb{E}\left[M_\tau\right] \geq \mathbb{E}\left[M_1\right] = 0.$$

Hence

$$\mathbb{E}\left[\tau C_1 - L_\tau\right] \geq 0$$

and since $\mathbb{E}\left[\tau\right] = [G : \Gamma]$ then

$$\mathbb{E}\left[L_\tau\right] \leq [G : \Gamma]C_1 < \infty.$$

$\square$

REFERENCES

[1] U. Bader and Y. Shalom, *Factor and normal subgroup theorems for lattices in products of groups*, Invent. Math. **163** (2006), no. 2, 415–454.

[2] L. Bowen, *Random walks on coset spaces with applications to furstenberg entropy*, Arxiv preprint arXiv:1008.4933 (2010).

[3] L. Bowen, R. Grigorchuk, and R. Kravchenko, *Invariant random subgroups of the lamplighter group*, arXiv preprint arXiv:1206.6780 (2012).

[4] A. Connes and B. Weiss, *Property T and asymptotically invariant sequences*, Israel Journal of Mathematics **37** (1980), no. 3, 209–210.

[5] R. Durrett, *Probability: theory and examples*, second, Cambridge University Press, 1996.

[6] A. Erschler, *Poisson–furstenberg boundary of random walks on wreath products and free metabelian groups*, Commentarii Mathematici Helvetici **86** (2011), no. 1, 113–143.

[7] A. Furman, *Random walks on groups and random transformations*, Handbook of dynamical systems **1A** (2002), 931–1014.

[8] H. Furstenberg, *Noncommuting random products*, Transactions of the American Mathematical Society **108** (1963), no. 3, 377–428.

[9] ———, *Random walks and discrete subgroups of Lie groups*, Advances in Probability and Related Topics **1** (1971), 1–63.

[10] H. Furstenberg and E. Glasner, *Stationary dynamical systems*, Dynamical numbers—interplay between dynamical systems and number theory, 2010, pp. 1–28. MR2762131

[11] Y. Hartman, Y. Lima, and O. Tamuz, *An abramov formula for stationary spaces of discrete groups*, arXiv preprint arXiv:1204.5414 (2012).

[12] V.A. Kaimanovich, *Measure-theoretic boundaries of markov chains, 0-2 laws and entropy*, Proceedings of the conference on harmonic analysis and discrete potential theory, 1992, pp. 145–180.

[13] ———, *The poisson formula for groups with hyperbolic properties*, Annals of Mathematics-Second Series **152** (2000), no. 3, 659–692.

[14] V.A. Kaimanovich and A.M. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Probab. **11** (1983), no. 3, 457–490.

[15] A. Karlsson and W. Woess, *The poisson boundary of lamplighter random walks on trees*, Geometriae Dedicata **124** (2007), no. 1, 95–107.

[16] G.W. Mackey, *Point realizations of transformation groups*, Illinois J. Math **6** (1962), no. 2, 327–335.

[17] A. Nevo, *The spectral theory of amenable actions and invariants of discrete groups*, Geometriae Dedicata **100** (2003), no. 1, 187–218.

[18] A. Nevo and R.J. Zimmer, *Rigidity of furstenberg entropy for semisimple lie group actions*, Annales Scientifiques de l´École Normale Supérieure **33** (2000), no. 3, 321–343.

[19] G. Stuck and R.J. Zimmer, *Stabilizers for ergodic actions of higher rank semisimple groups*, Annals of Mathematics-Second Series **139** (1994), no. 3, 723–748.

[20] W. Woess, *Random walks on infinite graphs and groups*, Vol. 138, Cambridge university press, 2000.

[21] R.J. Zimmer, *Amenable ergodic group actions and an application to poisson boundaries of random walks*, Journal of Functional Analysis **27** (1978), no. 3, 350–372.

WEIZMANN INSTITUTE OF SCIENCE, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, POB 26, 76100, REHOVOT, ISRAEL.

E-mail address: `yair.hartman, omer.tamuz@weizmann.ac.il`