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Functions and Their Basic Properties

Czesław Byliński
Warsaw University
Białystok

Summary. The definitions of the mode Function and the graph of a function are introduced. The graph of a function is defined to be identical with the function. The following concepts are also defined: the domain of a function, the range of a function, the identity function, the composition of functions, the 1-1 function, the inverse function, the restriction of a function, the image and the inverse image. Certain basic facts about functions and the notions defined in the article are proved.

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The articles [1] and [2] provide the notation and terminology for this paper.

We use the following convention: X, X_1, X_2, Y, Y_1, Y_2 are sets and p, x, x_1, x_2, y, y_1, y_2, z are sets.

Let X be a set. We say that X is function-like if and only if:

(Def. 1) For all x, y_1, y_2 such that $\langle x, y_1\rangle \in X$ and $\langle x, y_2\rangle \in X$ holds $y_1 = y_2$.

Let us observe that there exists a set which is relation-like and function-like.

A function is a function-like relation-like set.

One can check that every set which is empty is also function-like.

We follow the rules: f, g, h denote functions and R, S denote binary relations.

Next we state the proposition

(2)[1] Let F be a set. Suppose for every p such that $p \in F$ there exist x, y such that $\langle x, y\rangle = p$ and for all x, y_1, y_2 such that $\langle x, y_1\rangle \in F$ and $\langle x, y_2\rangle \in F$ holds $y_1 = y_2$. Then F is a function.

The scheme *GraphFunc* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists f such that for all x, y holds $\langle x, y\rangle \in f$ iff $x \in \mathcal{A}$ and $\mathcal{P}[x,y]$

provided the parameters meet the following requirement:

- For all x, y_1, y_2 such that $\mathcal{P}[x,y_1]$ and $\mathcal{P}[x,y_2]$ holds $y_1 = y_2$.

Let us consider f, x. The functor $f(x)$ yielding a set is defined as follows:

(Def. 4)[2] (i) $\langle x, f(x)\rangle \in f$ if $x \in \operatorname{dom} f$,

(ii) $f(x) = \emptyset$, otherwise.

One can prove the following propositions:

(8)[3] $\langle x, y\rangle \in f$ iff $x \in \operatorname{dom} f$ and $y = f(x)$.

[1] The proposition (1) has been removed.
[2] The definitions (Def. 2) and (Def. 3) have been removed.
[3] The propositions (3)–(7) have been removed.

(9) If $\operatorname{dom} f = \operatorname{dom} g$ and for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$, then $f = g$.

Let us consider f. Then $\operatorname{rng} f$ can be characterized by the condition:

(Def. 5) For every y holds $y \in \operatorname{rng} f$ iff there exists x such that $x \in \operatorname{dom} f$ and $y = f(x)$.

We now state two propositions:

(12)[4] If $x \in \operatorname{dom} f$, then $f(x) \in \operatorname{rng} f$.

(14)[5] If $\operatorname{dom} f = \{x\}$, then $\operatorname{rng} f = \{f(x)\}$.

Now we present two schemes. The scheme *FuncEx* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the parameters meet the following requirements:

- For all x, y_1, y_2 such that $x \in \mathcal{A}$ and $\mathcal{P}[x,y_1]$ and $\mathcal{P}[x,y_2]$ holds $y_1 = y_2$, and
- For every x such that $x \in \mathcal{A}$ there exists y such that $\mathcal{P}[x,y]$.

The scheme *Lambda* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\operatorname{dom} f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

One can prove the following propositions:

(15) If $X \neq \emptyset$, then for every y there exists f such that $\operatorname{dom} f = X$ and $\operatorname{rng} f = \{y\}$.

(16) If for all f, g such that $\operatorname{dom} f = X$ and $\operatorname{dom} g = X$ holds $f = g$, then $X = \emptyset$.

(17) If $\operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{rng} f = \{y\}$ and $\operatorname{rng} g = \{y\}$, then $f = g$.

(18) If $Y \neq \emptyset$ or $X = \emptyset$, then there exists f such that $X = \operatorname{dom} f$ and $\operatorname{rng} f \subseteq Y$.

(19) If for every y such that $y \in Y$ there exists x such that $x \in \operatorname{dom} f$ and $y = f(x)$, then $Y \subseteq \operatorname{rng} f$.

Let us consider f, g. We introduce $g \cdot f$ as a synonym of $f \cdot g$.

Let us consider f, g. One can check that $g \cdot f$ is function-like.

We now state several propositions:

(20) Let given h. Suppose for every x holds $x \in \operatorname{dom} h$ iff $x \in \operatorname{dom} f$ and $f(x) \in \operatorname{dom} g$ and for every x such that $x \in \operatorname{dom} h$ holds $h(x) = g(f(x))$. Then $h = g \cdot f$.

(21) $x \in \operatorname{dom}(g \cdot f)$ iff $x \in \operatorname{dom} f$ and $f(x) \in \operatorname{dom} g$.

(22) If $x \in \operatorname{dom}(g \cdot f)$, then $(g \cdot f)(x) = g(f(x))$.

(23) If $x \in \operatorname{dom} f$, then $(g \cdot f)(x) = g(f(x))$.

(25)[6] If $z \in \operatorname{rng}(g \cdot f)$, then $z \in \operatorname{rng} g$.

(27)[7] If $\operatorname{dom}(g \cdot f) = \operatorname{dom} f$, then $\operatorname{rng} f \subseteq \operatorname{dom} g$.

(33)[8] If $\operatorname{rng} f \subseteq Y$ and for all g, h such that $\operatorname{dom} g = Y$ and $\operatorname{dom} h = Y$ and $g \cdot f = h \cdot f$ holds $g = h$, then $Y = \operatorname{rng} f$.

Let us consider X. One can check that id_X is function-like.

Next we state several propositions:

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[4] The propositions (10) and (11) have been removed.

[5] The proposition (13) has been removed.

[6] The proposition (24) has been removed.

[7] The proposition (26) has been removed.

[8] The propositions (28)–(32) have been removed.

(34) $f = \mathrm{id}_X$ iff $\mathrm{dom}\, f = X$ and for every x such that $x \in X$ holds $f(x) = x$.

(35) If $x \in X$, then $\mathrm{id}_X(x) = x$.

(37)[9] $\mathrm{dom}(f \cdot \mathrm{id}_X) = \mathrm{dom}\, f \cap X$.

(38) If $x \in \mathrm{dom}\, f \cap X$, then $f(x) = (f \cdot \mathrm{id}_X)(x)$.

(40)[10] $x \in \mathrm{dom}(\mathrm{id}_Y \cdot f)$ iff $x \in \mathrm{dom}\, f$ and $f(x) \in Y$.

(42)[11] $f \cdot \mathrm{id}_{\mathrm{dom}\, f} = f$ and $\mathrm{id}_{\mathrm{rng}\, f} \cdot f = f$.

(43) $\mathrm{id}_X \cdot \mathrm{id}_Y = \mathrm{id}_{X \cap Y}$.

(44) If $\mathrm{rng}\, f = \mathrm{dom}\, g$ and $g \cdot f = f$, then $g = \mathrm{id}_{\mathrm{dom}\, g}$.

Let us consider f. We say that f is one-to-one if and only if:

(Def. 8)[12] For all x_1, x_2 such that $x_1 \in \mathrm{dom}\, f$ and $x_2 \in \mathrm{dom}\, f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$.

Next we state several propositions:

(46)[13] If f is one-to-one and g is one-to-one, then $g \cdot f$ is one-to-one.

(47) If $g \cdot f$ is one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{dom}\, g$, then f is one-to-one.

(48) If $g \cdot f$ is one-to-one and $\mathrm{rng}\, f = \mathrm{dom}\, g$, then f is one-to-one and g is one-to-one.

(49) f is one-to-one iff for all g, h such that $\mathrm{rng}\, g \subseteq \mathrm{dom}\, f$ and $\mathrm{rng}\, h \subseteq \mathrm{dom}\, f$ and $\mathrm{dom}\, g = \mathrm{dom}\, h$ and $f \cdot g = f \cdot h$ holds $g = h$.

(50) If $\mathrm{dom}\, f = X$ and $\mathrm{dom}\, g = X$ and $\mathrm{rng}\, g \subseteq X$ and f is one-to-one and $f \cdot g = f$, then $g = \mathrm{id}_X$.

(51) If $\mathrm{rng}(g \cdot f) = \mathrm{rng}\, g$ and g is one-to-one, then $\mathrm{dom}\, g \subseteq \mathrm{rng}\, f$.

(52) id_X is one-to-one.

(53) If there exists g such that $g \cdot f = \mathrm{id}_{\mathrm{dom}\, f}$, then f is one-to-one.

One can verify that there exists a function which is empty.
Let us observe that every function which is empty is also one-to-one.
One can verify that there exists a function which is one-to-one.
Let f be an one-to-one function. Observe that $f^{\smile}$ is function-like.
Let us consider f. Let us assume that f is one-to-one. The functor f^{-1} yields a function and is defined by:

(Def. 9) $f^{-1} = f^{\smile}$.

The following propositions are true:

(54) Suppose f is one-to-one. Let g be a function. Then $g = f^{-1}$ if and only if the following conditions are satisfied:

(i) $\mathrm{dom}\, g = \mathrm{rng}\, f$, and

(ii) for all y, x holds $y \in \mathrm{rng}\, f$ and $x = g(y)$ iff $x \in \mathrm{dom}\, f$ and $y = f(x)$.

(55) If f is one-to-one, then $\mathrm{rng}\, f = \mathrm{dom}(f^{-1})$ and $\mathrm{dom}\, f = \mathrm{rng}(f^{-1})$.

(56) If f is one-to-one and $x \in \mathrm{dom}\, f$, then $x = f^{-1}(f(x))$ and $x = (f^{-1} \cdot f)(x)$.

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[9] The proposition (36) has been removed.
[10] The proposition (39) has been removed.
[11] The proposition (41) has been removed.
[12] The definitions (Def. 6) and (Def. 7) have been removed.
[13] The proposition (45) has been removed.

(57) If f is one-to-one and $y \in \operatorname{rng} f$, then $y = f(f^{-1}(y))$ and $y = (f \cdot f^{-1})(y)$.

(58) If f is one-to-one, then $\operatorname{dom}(f^{-1} \cdot f) = \operatorname{dom} f$ and $\operatorname{rng}(f^{-1} \cdot f) = \operatorname{dom} f$.

(59) If f is one-to-one, then $\operatorname{dom}(f \cdot f^{-1}) = \operatorname{rng} f$ and $\operatorname{rng}(f \cdot f^{-1}) = \operatorname{rng} f$.

(60) Suppose f is one-to-one and $\operatorname{dom} f = \operatorname{rng} g$ and $\operatorname{rng} f = \operatorname{dom} g$ and for all x, y such that $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} g$ holds $f(x) = y$ iff $g(y) = x$. Then $g = f^{-1}$.

(61) If f is one-to-one, then $f^{-1} \cdot f = \mathrm{id}_{\operatorname{dom} f}$ and $f \cdot f^{-1} = \mathrm{id}_{\operatorname{rng} f}$.

(62) If f is one-to-one, then f^{-1} is one-to-one.

(63) If f is one-to-one and $\operatorname{rng} f = \operatorname{dom} g$ and $g \cdot f = \mathrm{id}_{\operatorname{dom} f}$, then $g = f^{-1}$.

(64) If f is one-to-one and $\operatorname{rng} g = \operatorname{dom} f$ and $f \cdot g = \mathrm{id}_{\operatorname{rng} f}$, then $g = f^{-1}$.

(65) If f is one-to-one, then $(f^{-1})^{-1} = f$.

(66) If f is one-to-one and g is one-to-one, then $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$.

(67) $(\mathrm{id}_X)^{-1} = \mathrm{id}_X$.

Let us consider f, X. One can verify that $f \restriction X$ is function-like.
One can prove the following propositions:

(68) $g = f \restriction X$ iff $\operatorname{dom} g = \operatorname{dom} f \cap X$ and for every x such that $x \in \operatorname{dom} g$ holds $g(x) = f(x)$.

(70)[14] If $x \in \operatorname{dom}(f \restriction X)$, then $(f \restriction X)(x) = f(x)$.

(71) If $x \in \operatorname{dom} f \cap X$, then $(f \restriction X)(x) = f(x)$.

(72) If $x \in X$, then $(f \restriction X)(x) = f(x)$.

(73) If $x \in \operatorname{dom} f$ and $x \in X$, then $f(x) \in \operatorname{rng}(f \restriction X)$.

(76)[15] $\operatorname{dom}(f \restriction X) \subseteq \operatorname{dom} f$ and $\operatorname{rng}(f \restriction X) \subseteq \operatorname{rng} f$.

(82)[16] If $X \subseteq Y$, then $f \restriction X \restriction Y = f \restriction X$ and $f \restriction Y \restriction X = f \restriction X$.

(84)[17] If f is one-to-one, then $f \restriction X$ is one-to-one.

Let us consider Y, f. Observe that $Y \upharpoonleft f$ is function-like.
One can prove the following propositions:

(85) $g = Y \upharpoonleft f$ if and only if the following conditions are satisfied:
 (i) for every x holds $x \in \operatorname{dom} g$ iff $x \in \operatorname{dom} f$ and $f(x) \in Y$, and
 (ii) for every x such that $x \in \operatorname{dom} g$ holds $g(x) = f(x)$.

(86) $x \in \operatorname{dom}(Y \upharpoonleft f)$ iff $x \in \operatorname{dom} f$ and $f(x) \in Y$.

(87) If $x \in \operatorname{dom}(Y \upharpoonleft f)$, then $(Y \upharpoonleft f)(x) = f(x)$.

(89)[18] $\operatorname{dom}(Y \upharpoonleft f) \subseteq \operatorname{dom} f$ and $\operatorname{rng}(Y \upharpoonleft f) \subseteq \operatorname{rng} f$.

(97)[19] If $X \subseteq Y$, then $Y \upharpoonleft (X \upharpoonleft f) = X \upharpoonleft f$ and $X \upharpoonleft (Y \upharpoonleft f) = X \upharpoonleft f$.

[14] The proposition (69) has been removed.
[15] The propositions (74) and (75) have been removed.
[16] The propositions (77)–(81) have been removed.
[17] The proposition (83) has been removed.
[18] The proposition (88) has been removed.
[19] The propositions (90)–(96) have been removed.

(99)[20] If f is one-to-one, then $Y \restriction f$ is one-to-one.

Let us consider f, X. Then $f^\circ X$ can be characterized by the condition:

(Def. 12)[21] For every y holds $y \in f^\circ X$ iff there exists x such that $x \in \operatorname{dom} f$ and $x \in X$ and $y = f(x)$.

One can prove the following propositions:

(117)[22] If $x \in \operatorname{dom} f$, then $f^\circ\{x\} = \{f(x)\}$.

(118) If $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$, then $f^\circ\{x_1, x_2\} = \{f(x_1), f(x_2)\}$.

(120)[23] $(Y \restriction f)^\circ X \subseteq f^\circ X$.

(121) If f is one-to-one, then $f^\circ(X_1 \cap X_2) = f^\circ X_1 \cap f^\circ X_2$.

(122) If for all X_1, X_2 holds $f^\circ(X_1 \cap X_2) = f^\circ X_1 \cap f^\circ X_2$, then f is one-to-one.

(123) If f is one-to-one, then $f^\circ(X_1 \setminus X_2) = f^\circ X_1 \setminus f^\circ X_2$.

(124) If for all X_1, X_2 holds $f^\circ(X_1 \setminus X_2) = f^\circ X_1 \setminus f^\circ X_2$, then f is one-to-one.

(125) If X misses Y and f is one-to-one, then $f^\circ X$ misses $f^\circ Y$.

(126) $(Y \restriction f)^\circ X = Y \cap f^\circ X$.

Let us consider f, Y. Then $f^{-1}(Y)$ can be characterized by the condition:

(Def. 13) For every x holds $x \in f^{-1}(Y)$ iff $x \in \operatorname{dom} f$ and $f(x) \in Y$.

We now state a number of propositions:

(137)[24] $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$.

(138) $f^{-1}(Y_1 \setminus Y_2) = f^{-1}(Y_1) \setminus f^{-1}(Y_2)$.

(139) $(R \restriction X)^{-1}(Y) = X \cap R^{-1}(Y)$.

(142)[25] $y \in \operatorname{rng} R$ iff $R^{-1}(\{y\}) \neq \emptyset$.

(143) If for every y such that $y \in Y$ holds $R^{-1}(\{y\}) \neq \emptyset$, then $Y \subseteq \operatorname{rng} R$.

(144) For every y such that $y \in \operatorname{rng} f$ there exists x such that $f^{-1}(\{y\}) = \{x\}$ iff f is one-to-one.

(145) $f^\circ f^{-1}(Y) \subseteq Y$.

(146) If $X \subseteq \operatorname{dom} R$, then $X \subseteq R^{-1}(R^\circ X)$.

(147) If $Y \subseteq \operatorname{rng} f$, then $f^\circ f^{-1}(Y) = Y$.

(148) $f^\circ f^{-1}(Y) = Y \cap f^\circ \operatorname{dom} f$.

(149) $f^\circ(X \cap f^{-1}(Y)) \subseteq f^\circ X \cap Y$.

(150) $f^\circ(X \cap f^{-1}(Y)) = f^\circ X \cap Y$.

(151) $X \cap R^{-1}(Y) \subseteq R^{-1}(R^\circ X \cap Y)$.

(152) If f is one-to-one, then $f^{-1}(f^\circ X) \subseteq X$.

[20] The proposition (98) has been removed.
[21] The definitions (Def. 10) and (Def. 11) have been removed.
[22] The propositions (100)–(116) have been removed.
[23] The proposition (119) has been removed.
[24] The propositions (127)–(136) have been removed.
[25] The propositions (140) and (141) have been removed.

(153) If for every X holds $f^{-1}(f^{\circ}X) \subseteq X$, then f is one-to-one.

(154) If f is one-to-one, then $f^{\circ}X = (f^{-1})^{-1}(X)$.

(155) If f is one-to-one, then $f^{-1}(Y) = (f^{-1})^{\circ}Y$.

(156) If $Y = \operatorname{rng} f$ and $\operatorname{dom} g = Y$ and $\operatorname{dom} h = Y$ and $g \cdot f = h \cdot f$, then $g = h$.

(157) If $f^{\circ}X_1 \subseteq f^{\circ}X_2$ and $X_1 \subseteq \operatorname{dom} f$ and f is one-to-one, then $X_1 \subseteq X_2$.

(158) If $f^{-1}(Y_1) \subseteq f^{-1}(Y_2)$ and $Y_1 \subseteq \operatorname{rng} f$, then $Y_1 \subseteq Y_2$.

(159) f is one-to-one iff for every y there exists x such that $f^{-1}(\{y\}) \subseteq \{x\}$.

(160) If $\operatorname{rng} R \subseteq \operatorname{dom} S$, then $R^{-1}(X) \subseteq (R \cdot S)^{-1}(S^{\circ}X)$.

References

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