NONCOMMUTATIVE GRASSMANIAN OF CODIMENSION TWO HAS COHERENT COORDINATE RING

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Abstract. A noncommutative Grassmanian $A = \mathrm{NGr}(m, n)$ is introduced by Efimov, Luntz, and Orlov in *Deformation theory of objects in homotopy and derived categories III: Abelian categories* as a noncommutative algebra associated to an exceptional collection of $n - m + 1$ coherent sheaves on $\mathbb{P}^n$. It is a graded Calabi–Yau $\mathbb{Z}$-algebra of dimension $n - m + 1$. We show that this algebra is coherent provided that the codimension $d = n - m$ of the Grassmanian is two. According to *op. cit.*, this gives a t-structure on the derived category of the coherent sheaves on the noncommutative Grassmanian.

The proof is quite different from the recent proofs of the coherence of some graded 3-dimensional Calabi–Yau algebras and is based on properties of a PBW-basis of the algebra A.

1. Introduction

Recall that a connected $\mathbb{N}$-graded algebra of the form $A = A_0 \oplus A_1 \oplus \dots$ such that A_0 is a copy of the basic field k is called *regular* if it has finite global dimension (say, d) and satisfies the following Gorenstein property:

$$\mathrm{Ext}^{\,i}_A(k, k) \cong \begin{cases} k^*[l] \text{ for some } l \in \mathbb{Z}, & i = d \\ 0, & i \neq d. \end{cases}$$

The same notion of regularity is extend (following Bondal and Polishchuk [BP]) to a slightly more general case of a $\mathbb{Z}$-algebra A, see Subsection 2.1 below.

Regular algebras play the roles of coordinate rings of noncommutative projective spaces in a version of noncommutative projective geometry [Po, BVdB] which generalizes the well-known approach of Artin and Zhang [AZ]. Namely, suppose that a regular algebra of global dimension d is graded coherent. Consider the quotient category $\mathrm{qgr}\, A = \mathrm{cmod}\, A/\mathrm{tors}\, A$ of the category $\mathrm{cmod}\, A$ of finitely presented (=graded coherent) right graded A-modules by its subcategory $\mathrm{tors}\, A$ of finite-dimensional modules. This category $\mathrm{qgr}\, A$ plays the role of the category of coherent sheaves on a noncommutative $(d-1)$-dimensional projective space.

Here we are interested in the case $d = 3$, that is, in the case of noncommutative planes. The famous classification of 3-dimensional regular algebras A of polynomial growth is obtained Artin and Shelter [AS]. Particularly, they have shown that these algebras are Noetherian (hence, coherent). It is not hard to construct also non-Noetherian 3-dimensional regular algebras (e.g., one may follow the approach of [AS, Sections 2 and 3]). In contrast, it is often not easy to prove that

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such an algebra is coherent. However, there are important examples for which the coherence is established. These are the octonion algebra of P. Smith [S], the Yang–Mills algebra introduced by Movshev and Swartz [MS] which is coherent by a theorem of Herscovich [H] (the Yang–Mils algebra introduced by Connes and Dobios–Violette [CDV] is a particular case of it), and 3-Calabi–Yau algebras which are Ore extensions of 2-Calaby–Yau ones by He, Oystaeyen, and Zhang [HOZ].

In all these cases, the coherence property is proved using the same lemma [P08, Prop. 3.2]. It states that if a non-trivial two-sided ideal I in a graded algebra A is free as a left module and the quotient algebra A/I is right Noetherian, then A is graded coherent. So, the known examples of coherent regular algebras are, in a sense, extensions of Noetherian algebras along free modules.

In this paper, we prove the coherence property of another 3-dimensional regular algebra. In contrast to the previous cases, it seems that the approach based on the above lemma fails for this algebra.[1] This algebra is introduced by Efimov, Luntz, and Orlov [ELO] under the name *Noncommutative Grassmanian* (see Subsection 2.2 below for an explicit definition). According to [ELO, Section 7], a noncommutative Grassmanian $\mathrm{NGr}(m,n)$ 'is a true noncommutative moduli space of the structure sheaves $\mathcal{O}_{\mathbb{P}(W)} \in D^b_{coh}(\mathbb{P}^n)$', where $W \subset k^n$ runs the vector subspaces of dimension m. The connection with structure sheaves is based on the theory of helices in derived categories by A.Bondal and A.Polishchuk [Bon, BP]. The construction of [ELO] gives a description (by constructing a t-structure) of the derived category of $\operatorname{qgr}\mathrm{NGr}(m,n)$ provided that the algebra $\mathrm{NGr}(m,n)$ of coherent. It is pointed out in [ELO, Remark 8.23] that $\mathrm{NGr}(m,n)$ is coherent if the codimension $d = n-m$ of the Grassmanian is equal to 1 (since the algebra $\mathrm{NGr}(m,n)$ has global dimension 2 in this case). The first non-trivial case is the Grassmanian of codimension $d = 2$, when the algebra has global dimension three. The main result of this paper is the following.

Theorem 1.1. *The noncommutative Grassmanian algebra* $\mathrm{NGr}(m,n)$ *is coherent provided that* $n - m = 2$.

The paper is organized as follows. In Section 2 we briefly remind necessary facts about $\mathbb{Z}$-algebras and, in particular, about the noncommutative Grassmanian algebra $\mathrm{NGr}(m,n)$. In Subsection 3.1, we note that the $\mathbb{Z}$–algebra $A = \mathrm{NGr}(m,n)$ with $n - m = 2$ is 3-periodic, so, its properties are essentially the same as the properties of the corresponding algebra $\hat{A}$ over a triangle quiver. We immediately calculate here the Hilbert series of $\hat{A}$. This obviously gives also the Hilbert series of A. In Subsection 3.2, we show that A is a PBW algebra as a 6-periodic $\mathbb{Z}$-algebra (in particular, it admits a quadratic Gröbner basis of relations). Note that we do not know if $\hat{A}$ is PBW or not. In Proposition 3.5, we prove that A satisfies a property of bounded processing [P01], that is, the structure of the multiplication of paths in this quiver algebra is essentially depend only on bounded segments of the multipliers. We recall necessary definitions in Subsection 3.3. Using a result of [P01], we finally deduce in Corollary 3.8 that the algebra A is coherent. A stronger consequence of bounded processing is briefly discussed in Remark 3.9.

[1] At least, for this algebra A calculations of Hilbert series for several ideals I such that the quotient algebra A/I is Noetherian by natural reasons show that I cannot be projective as a left module.

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2. Background and notations

2.1. **$\mathbb{Z}$-algebras.** Recall that a $\mathbb{Z}$-algebra is a path algebra (with relations) over the infinite line quiver $\dots \to -1 \to 0 \to 1 \to 2 \to \dots$ with multiple arrows. We refer the reader to [BP] or [PP, Ch. 4, Sect. 9–10] for the basic definitions of quadratic, Koszul and PBW $\mathbb{Z}$–algebras. We say that a $\mathbb{Z}$-algebra $A = \oplus_{i,j\in\mathbb{Z}}$ (where $A_{i,j} = 0$ for $i < j$ and $A_{i,i} = k$) is *regular* of dimension d if each irreducible module $k_i = P_i/\oplus_{j>i} A_{i,j}$ has global dimension d (where $P_i = \oplus_{j\geq i} A_{i,j}$ is the corresponding projective module) and the Exts of these modules satisfy an analogous Gorenstein condition [BP, Sect. 4]

$$\operatorname{Ext}^i(k_s, P_t) \cong \begin{cases} k^* \text{ if } t = s + l, & i = d, \\ 0, & \text{otherwise} \end{cases}$$

for some $l \in \mathbb{Z}$. Note that regular algebras are also called *AS-regular*.

2.2. **Noncommutative Grassmanian.** A noncommutative Grassmanian is defined by Efimov, Lunts, and Orlov [ELO, Part 3] as a noncommutative scheme associated to the following algebra. Given two positive integers $m < n$ and an n-dimensional vector space V, let $A = A^{m,V}$ be a quadratic $\mathbb{Z}$-algebra with $A_{ij} = k$ (a basic field) and generators

$$A_{i,i+1} = \begin{cases} \Lambda^d V, & i|(d+1), \\ V^*, & \text{otherwise,} \end{cases}$$

where $d = n - m$. The quadratic relations of A are defined via the natural exact sequences

$$0 \to \Lambda^{d-1} V \to A_{i+1,i+2} \otimes A_{i,i+1} \to A_{i,i+2} \to 0 \text{ for } (d+1)|i, i+1$$

and

$$0 \to \Lambda^2 V^* \to A_{i+1,i+2} \otimes A_{i,i+1} \to A_{i,i+2} \to 0, \text{ otherwise.}$$

Obviously, A is $(d+1)$-periodic (that is, $A_{i,j}$ is naturally isomorphic to $A_{i+d+1,j+d+1}$). By [ELO, Prop. 8.18], A isomorphic to the automorphism $\mathbb{Z}$-algebra of the helix generated by the exceptional collection of $(d+1)$ coherent sheaves on $\mathbb{P}(k^n)$

$$E = (\mathcal{O}_{\mathbb{P}(k^n)}(m-n), \dots, \mathcal{O}_{\mathbb{P}(k^n)}(-1), \mathcal{O}_{\mathbb{P}(k^n)}).$$

It follows from [BP] that A is Koszul and Gorenstein of global dimension $d+1$.

Note that A is a so-called graded Calabi–Yau algebra of dimension 3 (in the sense of [Bock]). This follows from the same property of the corresponding algebra $\hat{A}$, see Subsection 3.1 below.

It is pointed out in [ELO, Remark 8.23] that the description of the derived category of $\operatorname{QMod} A$ ("quasicoherent sheaves" on the noncommutative Grassmanian) can the transferred to $D^b(\operatorname{qmod} A)$ (derived category of the "coherent sheaves") provided that the category $\operatorname{qmod} A$ is Abelian, that is, A is coherent. Namely, in this case a t-structure on the derived category of the finite-dimensional modules over a finite-dimensional algebra $\oplus_{1\leq i,j\leq n} A_{i,j}$ constructed in [ELO, Section 8] induces a t-structure on $D^b(\operatorname{qmod} A)$.

3. The results

3.1. **Algebra $\hat{A}$ and its Hilbert series.** Since A is $(d+1)$-periodic (in terms of [BP, Section 4], it is geometric of period $d+1$), its categories of graded modules (all , finitely generated, finitely presented, finite dimensional...) are equivalent to the ones of the algebra $\hat{A} = \oplus_{i=0}^{d} \oplus_{j\in\mathbb{Z}} A_{ij}$ considered as a path algebra over a cyclic quiver of length $d+1$ with multiple arrows. It follows that $\hat{A}$ is Koszul and Gorenstein of global dimension $d+1$ as well as A is. The components of $\hat{A}$ are indexed by the pairs (i,j) with $i \in \mathbb{Z}/(d+1)\mathbb{Z}$ and $j \in \mathbb{Z}$, where elements of $\hat{A}_{i,j}$ are considered as path from i to $(j-i)\mathrm{mod}\,(d+1)$. The surjection $A \to \hat{A}$ is induced by the natural surjection of quiver algebras.

In our case $d=2$, the algebra $\hat{A}$ is a path algebra (=quiver algebra with relations) over the quiver $Q: 0 \to 1 \to 2 \to 0$ where the arrows are multiple, namely, with $n(n-1)/2$ arrows $0 \to 1$, n arrows $1 \to 2$ and n arrows $2 \to 0$. Note that it follows from [Bock, Theorem 3.1] that $\hat{A}$ is a graded Calabi–Yau algebra of dimension 3. In the notations of Subsection 3.2 below, the corresponding superpotential is equal to a cyclic element represented by

$$\sum_{t\in 3\mathbb{Z}} \sum_{i,j\in[1..n], i\neq j} x_i^{t-1} e_{ij}^t x_j^{t+1}.$$

Recall that the Hilbert series $H_{\hat{A}}$ of the graded path algebra $\hat{A}$ over the above quiver with 3 vertices is a 3×3 matrix $H_{\hat{A}} = (h_{ij})$ over $\mathbb{Z}[[t]]$ defined by

$$h_{ij} = \sum_{m\in\mathbb{Z}} (\text{the number of paths } i \to j \text{ of length } m) t^m = \sum_{n\in Z} t^{3n+j-i} \dim \hat{A}_{i,3n+j}.$$

Proposition 3.1. *The Hilbert series $H_{\hat{A}}$ of the path algebra $\hat{A}$ is*

$$H_{\hat{A}} = \begin{pmatrix} 1-t^3 & \binom{n}{2}t & nt^2 \\ nt^2 & 1-t^3 & -nt \\ -nt & \binom{n}{2}t^2 & 1-t^3 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1+\frac{4-5n^2+n^4}{4}t^3 & \frac{n(n-1)}{2}t & n\left(n^2-n-2\right)t^2/2 \\ \frac{n(n+1)}{2}t^2 & 1+\frac{4-5n^2+n^4}{4}t^3 & nt \\ nt & n\frac{n^2-n-2}{2}t^2 & 1+\frac{2+n^4-n^3-4n^2}{2}t^3 \end{pmatrix} + O(t^4).$$

Proof. Note that the Koszul dual algebra of $\hat{A}$ has the following components:

$$\hat{A}^!_{ii} = k \text{ for all } i = 0,1,2,$$

$$\hat{A}^!_{i,i+1} = (\hat{A}_{i,i+1})^* \approx \begin{cases} \left(\Lambda^2 V\right)^*, & i=0, \\ V, & i=1,2, \end{cases}$$

$$\hat{A}^!_{i,i+2} = (\text{relations of } A \text{ in} A_{i+1,i+2}\otimes A_{i,i+1})^* \approx \begin{cases} \left(\Lambda^{d-1}V\right)^* = V^*, & i=0,2 \\ \left(\Lambda^2 V^*\right)^*, & i=1, \end{cases}$$

and, by the Gorenstein property,

$$\hat{A}^!_{i,i+3} \approx k.$$

All other components of $\hat{A}^!$ vanish.

It follows that the Hilbert series of $\hat{A}^!$ is the following matrix of order $d+1=3$

$$H_{\hat{A}^!}(t) = \begin{pmatrix} 1+t^3 & \binom{n}{2}t & nt^2 \\ \binom{n}{2}t^2 & 1+t^3 & nt \\ nt & nt^2 & 1+t^3 \end{pmatrix}$$

Since $\hat{A}$ is Koszul, we have $H_{\hat{A}}(t)H_{\hat{A}}(-t) = \mathrm{Id}$, thus,

$$H_{\hat{A}}(t) = H_{\hat{A}}(-t)^{-1}.$$

□

3.2. **PBW property.** Let us fix a pair of dual bases $x = \{x_1, \dots, x_n\}$ in V^* and $e = \{e_1, \dots, e_n\}$ in V. Given a finite sequence $\alpha = (\alpha_1, \dots, \alpha_s)$, we denote by e_α the product $e_{\alpha_1} \wedge \dots \wedge e_{\alpha_s} \in \Lambda^s V$, so that the set $e = \{e_{ij} | i < j\}$ is a basis of $\Lambda^2 V$. For $t \in \mathbb{Z}$, we denote by $x^t = \{x_1^t, \dots, x_n^t\}$ and $e^t = \{e_{ij}^t | i < j\}$ the corresponding bases of $A_{t,t+1}$ in the cases $3|t$ and $3 \not| t$, respectively. We will sometimes omit the upper indexes for elements of these bases.

In this notations, the relations of the algebra A of the grading component $(t, t+2)$ has the following form (where for $i > j$ we use the sign $e_{ij} := -e_{ji}$):

$$\begin{array}{lll} f_i = f_i^t := \sum\limits_{j \neq i} e_{ij}^t x_j^{t+1}, & i = 1..n, & t \in 3\mathbb{Z}, \\ c_{ij} = c_{ij}^t := x_i^t x_j^{t+1} - x_j^t x_i^{t+1}, & 1 \le i < j \le n, & t \in 3\mathbb{Z}+1, \\ g_i = g_i^t := \sum\limits_{j \neq i} x_j^t e_{ji}^{t+1}, & i = 1..n, & t \in 3\mathbb{Z}+2. \end{array} \tag{3.1}$$

Let us fix orderings of the elements of the bases x^t and e^t in the following (6-periodic) way:

$$\begin{array}{lll} e_{ij} > e_{kl} & \text{if } i < k \text{ or } i = k, j < l & \text{for } t \in 6\mathbb{Z}, \\ e_{ij} > e_{kl} & \text{if } i > k \text{ or } i = k, j > l & \text{for } t \in 6\mathbb{Z}+3, \\ x_1 < \dots < x_n & & \text{for } t \in 6\mathbb{Z} \pm 1, \\ x_1 > \dots > x_n & & \text{for } t \in 6\mathbb{Z} \pm 2. \end{array}$$

Let us introduce the reverse lexicographical order on the paths of the quiver Q, that is, for two paths $v = v_1 \dots v_t$ and $w = w_1 \dots w_t$ of length t with the same head and the same tail we set $v < w$ iff $v_j < w_j, v_{j+1} = w_{j+1}, \dots$ and $v_t = w_t$ for some j.

Proposition 3.2. *The above quadratic relations of the algebra $\hat{A}$ form a Gröbner basis of the ideal of relations with respect to the above order, that is, the algebras $\hat{A}$ and A are PBW algebras.*

Proof. The leading monomials of the relations (3.1) w. r. t. the above reverse lexicographical order are

$$\begin{array}{ll} e_{in}x_n \text{ for } 1 \le i < n, e_{n-1,n}x_{n-1}, & t \in 6\mathbb{Z}, \\ x_i x_j \text{ for } n \ge i > j \ge 1, & t \in 6\mathbb{Z}+1, \\ x_1 e_{1i} \text{ for } n \ge i > 1, x_2 e_{12}, & t \in 6\mathbb{Z}+2, \\ e_{1i} x_1 \text{ for } n \ge i > 1, e_{12} x_2, & t \in 6\mathbb{Z}+3, \\ x_i x_j \text{ for } 1 \le i < j \le n, & t \in 6\mathbb{Z}+4, \\ x_n e_{in} \text{ for } 1 \le i < n, x_{n-1} e_{n-1,n}, & t \in 6\mathbb{Z}+5, \end{array} \tag{3.2}$$

Let B be the Z-algebra defined by the same generators as A and the above monomial relations. We have $\dim B_{ij} = \dim A_{ij}$ for $j = i, j = i+1, j = i+2$ and

$\dim B_{ij} \geq \dim A_{ij}$ for $j \geq i+3$. By [PP, Prop. 10.1 of Ch. 4], to show that A is PBW it is sufficient to check the equalities $\dim B_{i,i+3} = \dim A_{i,i+3}$ for all i.

Because of the symmetry of the monomial relations above, we have $\dim B_{i,i+3} = \dim B_{i+3s,i+3s+3}$ for all $s \in \mathbb{Z}$. Since the algebra A is 3-periodic, it is enough to show that the equalities $\dim B_{i,i+3} = \dim A_{i,i+3}$ hold for $i = 0, 1, 2$. Here $\dim A_{i,i+3} = \dim \hat{A}_{i \bmod 3, i+3}$ is the coefficient of t^3 in the $(i \bmod 3, i \bmod 3)$-th entry of the Hilbert series given in Proposition 3.1, that is,

$$\dim \hat{A}_{0,3} = \dim \hat{A}_{1,4} = 1 - 5/4\, n^2 + 1/4\, n^4$$

or

$$\dim \hat{A}_{2,5} = 1 + 1/2\, n^4 - 1/2\, n^3 - 2\, n^2.$$

To find $\dim B_{t,t+3}$, let us calculate the nonzero paths of length 3 in the algebra B. The integers i, j, k, l below belong to the interval $[1, \dots, n]$. We have

$$\dim B_{0,3}$$

$$= \mathrm{Card}\{e_{ij}x_k x_l | i < j, k \leq l\} - \mathrm{Card}\{e_{in}x_n x_l | i < n\} - \mathrm{Card}\{e_{n-1,n}x_{n-1}x_l | l \leq n-1\}$$

$$= \frac{n(n-1)}{2}\frac{n(n+1)}{2} - n(n-1) - (n-1) = \frac{n^4}{4} - \frac{5n^2}{4} + 1,$$

$$\dim B_{1,4} = \dim B_{0,3} = \frac{n^4}{4} - \frac{5n^2}{4} + 1 \text{ by symmetry}$$

and

$$\dim B_{2,5} = \mathrm{Card}\{x_i e_{jk} x_l | j < k\} - \mathrm{Card}\{x_n e_{in} x_l | i < n\} - \mathrm{Card}\{x_{n-1} e_{n-1,n} x_l\}$$

$$- \mathrm{Card}\{x_i e_{1k} x_1 | k > 1\} - \mathrm{Card}\{x_i e_{12} x_2\} + \mathrm{Card}\{x_n e_{1n} x_1\}$$

$$= n^2 \frac{n(n-1)}{2} - \frac{n(n-1)}{2} - n - \frac{n(n-1)}{2} - n + 1 = \frac{n^4}{2} - \frac{n^3}{2} - 2n^2 + 1.$$

We obtain the equality $\dim B_{t,t+3} = \dim \hat{A}_{t,t+3}$ for each $t = 0, 1, 2$, thus, the algebra A is PBW. □

Remark 3.3. There is another proof of Proposition 3.2 based on the Diamond Lemma and the Buchberger criterion for Gröbner bases.

Remark 3.4. Note that while the algebra A is 3-periodic, we have shown only that A is PBW as a 6-periodic algebra. We do not know whether A is PBW as a 3-periodic algebra, that is, whether $\hat{A}$ is PBW or not.

3.3. **Bounded processing.** Given an algebra R defined by a set of generators X and a Gröbner basis of relations G (given an admissible order of monomials on X), one can identify each element of the algebra with a linear combinations of the words on X which are normal (=irreducible) with respect to G. Recall that the multiplication of normal words after this identification is defined as follows. We may assume that the Gröbner basis G is reduced, that is, each its element $g \in G$ has the form $g = \hat{g} - \bar{g}$, where $\hat{g}$ is its leading monomial and $-\bar{g}$ is a linear combination of lower monomials. Given two normal words u and v, one applies (if possible) to the concatenation uv a *reduction* by some element of the Gröbner $g \in G$, that is, one replaces a subword $\hat{g}$ in uv by the noncommutative polynomial $(-\bar{g})$. In the resulted noncommutative polynomial, one applies to all its nonzero terms additional reductions, etc. After a finite number of steps, all nonzero terms of the resulted noncommutative polynomial $u * v$ became irreducible. By the definition of Gröbner

basis, the linear combination of the normal words $u * v$ is defined uniquely and is identified with the product of u and v in the algebra R.

Thus, one can consider the calculation of the product $u * v$ as a processing of some "machine" (like a Turing machine, if G is finite) which takes a concatenation uv as an input, finds the first occurrence of a leading monomial $\hat{g}$ of some element of the Gröbner basis, replaces it by $-\bar{g}$, etc. Since the words u and v are normal, the first replaced subword $\hat{g}$ should overlap the both parts u and v of uv. In the next steps, the region of processing in each subword should overlap one of the words from some $\bar{g}$ which appeared in a previous step. An algebra R is said to be an *algebra of r-processing* [P01] for some $r > 0$ if the region of processing do not spread beyond r letters to the right from the beginning of the right part v of the initial word uv, that is, for each pair of normal words u and $v = ws$, where the word w has length at least r, we have

$$u * v = (u * w)s.$$

Note that the above definition is compatible with the standard assumptions of the Gröbner basis theory for ideals in path algebras [FFG]. In this theory, it is assumed that the above set of generators X of a path algebra kQ of a quiver Q consists of two parts, $X = V \cup E$, where V is the set of vertices and E is the set of arrows of the quiver Q. By definition, the normal words are the paths of the corresponding quiver, that is, the paths of length 0 which are the vertices and the paths of positive length which are sequences of arrows.

We see that the definition of length of a normal word in the quiver algebra is slightly different from the general definition of length as the number of letters in word used above. However, if we try to check the property of r-processing (with $r \geq 1$) for a quotient of the path algebra by an ideal I generated by linear combination of paths of positive length (quiver algebra $R = kQ/I$), then we can assume that the words u, v, and w are paths of positive length. For these words u, v, and w, the both definitions of lengths coincide, so, one can use the second one.

In particular, the path algebra kQ of any quiver Q is an algebra of 1-processing. Another example of a quiver algebra with r-processing is our algebra A.

Proposition 3.5. *The algebra A is an algebra of 3-processing with respect to the generators and the Gröbner basis introduced in Subsection 3.2.*

Proof. Let $uv = \dots e_{ab}x_ix_je_{kl}x_sx_t\dots$ be a product of two normal words u and v. Suppose that the right subword $x_sx_t\dots$ belongs to the second part v. It is sufficient to show that in each stage of the processing, the subword which begins with x_t is stable. Since in each stage of the processing the region of processing is extended by at most one letter (because all leading monomials of the elements of the Gröbner basis have length 2), it is sufficient to show that this region does not rich x_t.

In the reductions w. r. t. the elements of the Gröbner basis G from Subsection 3.2, any two-letter word c is reduced to a linear combinations of another two-letter words c' with the following properties:

(1) each letter e is replaced by some e, and each x is replaced by some x;

(2) if $c = yz$ and $c' = y'z'$ for some letters y, z, y', z', then $z \geq z'$.

In particular, we have

(3') if $c = e_{\alpha\beta}x_s$, then $c' = e_{\alpha'\beta'}x_{s'}$ with $x_{s'} \leq x_s$.

It follows from observation of the list of the leading monomials of elements of G given in (3.2) that

(4) if the monomial $x_s x_t$ is irreducuble (i. e., normal), then for each $x_{s'} \leq x_s$ the monomial $x_{s'} x_t$ is irreducible too.

Since the monomial $x_s x_t$ is a subword of the normal word v, it is irreducible. By (1), in each stage of the processing this subword will be replaced by a linear combination of the words of the form $x_{s'} x_{t'}$, where $x_{s'} \leq x_s$ by (3'). By (4), it follows that $x_{t'} = x_t$, that is, the region of processing does not reach x_t. It follows that x_t and all letters to the right of it will be stable in each stage of the processing. □

Remark 3.6. Due to the right–left symmetry in list of the leading monomials of the Gröbner basis, A is an algebra of left 3-processing as well.

Remark 3.7. Note that the sufficient condition for r–processing given in [P01, Prop. 2] does not hold for the algebra A.

3.4. **Coherence.**

Corollary 3.8. *The algebra A is right and left coherent.*

Proof. The right coherence of A follows from Proposition 3.5 and [P01, Th. 8]. The left coherence follows from symmetry. □

Note that it follows that A is coherent both in graded and non-graded sense.

Remark 3.9. The property of 3–processing implies also the following estimate for the degrees of relations of ideals [P01, Prop. 7]: If a right sided ideal I in A is generated in degreed $\leq d$ for some d, then its relations are concentrated in degrees $\leq d + 6$. It follows that A is universally coherent in terms of [P05], see also [P05, Prop. 4.10]. Similar linear estimates for the generators of the entries of the minimal projective resolution for each finitely presented A–module follow from [P05, Prop. 4.3].

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