# Optimal Control of Underactuated Nonholonomic Mechanical Systems

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***Abstract*— In this paper we use an affine connection formulation to study an optimal control problem for a class of nonholonomic, under-actuated mechanical systems. In particular, we aim at minimizing the norm-squared of the control input to move the system from an initial to a terminal state. We consider systems evolving on general manifolds. The class of nonholonomic systems we study in this paper includes, in particular, wheeled-type vehicles, which are important for many robotic locomotion systems. The two special aspects of this optimal control problem are the nonholonomic constraints and under-actuation. Nonholonomic constraints restrict the evolution of the system to a distribution on the manifold. The nonholonomic connection is used to express the constrained equations of motion. Furthermore, it is used to take variations of the cost functional. Many robotic systems are under-actuated since control inputs are usually applied through the robot's internal configuration space only. While we do not consider symmetries with respect to group actions in this paper, the fact that the system is under-actuated is taken into account in our problem formulation. This allows one to compute reaction forces due to any inputs applied in directions orthogonal to the constraint distribution. We illustrate our ideas by considering a simple example on a three-dimensional manifold.**

## I. INTRODUCTION

In this paper we use the theory of affine connections to study force minimizing optimal control problems for a large class of nonholonomic under-actuated mechanical systems. Mechanical systems considered in this paper may be nonlinear and evolving on algebraic (for holonomically constrained systems) and/or abstract manifolds such Lie groups (in particular, the group of rigid body motions in three dimensional space, $\mathsf{SE}(3)$, and its subgroups). The class of nonholonomic systems we study in this paper includes, in particular, any wheeled-type vehicle, such as robots on wheels and or tracks. The fact that most of these robotic systems apply torques and forces internal to the system, which makes these system move in an undulatory fashion (see [1] and references therein for more on undulatory locomotion), without the application of any external forces, makes the system under-actuated. In fact, control inputs that are applied through the shape space in the absence of any control authority through the group space (that is, the *fiber*.) Hence, including under-actuated systems in our study is crucial in covering a wide range of robotic applications.

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Nonholonomic mechanical control systems have a long and complex history which is described in, for example, [2], [3] (in particular, Chapter 5) and [4]. Of much interest in the present work are the recent developments that utilize a geometric approach [5], [3] and, in particular, the theory of affine connections [6], [4]. These methods offer a coordinate-free differential approach to mechanics and control that avoids many of the issues that arise in classical mechanics such singularity and change of coordinates, complexity of notation and the lack of a geometric picture. For more on differential-geometric mechanics and its use in the context of dynamics and control, we refer the reader to [7], [3], [4]. For the treatment of under-actuated systems using affine connections, we refer the reader to [8].

Aside from [9], [10], previous results usually treat kinematic systems that usually aim at minimizing energy. In this paper the cost function is the square of the norm of the total applied control. We treat second order (i.e., dynamic) nonholonomic systems and allow for under-actuation. As will be seen in this paper, the set of necessary optimality conditions are coordinate-free and generic for a large class of nonholonomic mechanical systems. Given problem-specific data, one can specialize the result to the specific problem at hand. This process can be automated using symbolic manipulation packages such as Mathematica® and toolboxes such as those introduced in [11][1].

While most of the systems appearing in robotics naturally posses symmetries with respect to a group action, which leads to the reduced equations of motion for the system, in this work we provide a framework for treating nonholonomic systems in the context of optimal control using the theory of affine connections, regardless of the presence of any symmetries. In the case where symmetries do exist, one can usually do more by utilizing the structure of the equations of motion as done in [9], [10]. In [9], [10], however, the authors use the momentum equation form of the reduced equations of motion [5]. The problem of optimally controlling systems with symmetry will be treated in a future paper. In particular, we are interested in understanding how results based on an affine connection approach and Lagrange's multiplier method relate to results based on the momentum equation form that appear in [9], [10]. For more on systems with symmetry we refer the reader to, for example, [1], [5], [7], [12], [13] and references therein.

[1] These packages are available online for which a reference is provided in [11].

The paper is arranged as follows. In Section II, we briefly describe how nonholonomic mechanical systems are treated using the theory of affine connections. We also state the relationship of this approach to the Lagrange-d'Alembert equations of motion for nonholonomic systems. In Section III, we introduce the optimal control problem and derive the necessary optimality conditions using the theory of affine connections. In Section IV, we use the vertical coin (equivalently, the inline or ice skate) as a simple example to illustrate how to perform the computations. Finally, in Section V, we summarize our results and describe areas of current and future research.

## II. Review of Affine Differential Geometry and Nonholonomic Systems

### A. Riemannian Manifolds and Affine Connections

In this section we give brief definitions of the various objects from affine connection theory that are essential to this paper. For more complete studies, we refer the reader to the mathematically-oriented text [14] or the more mechanically-oriented text [4].

Let $\mathsf{Q}$ be a smooth ($\mathcal{C}^\infty$) *Riemannian manifold* with the *Riemannian metric* defined by $g_{\mathbf{q}} : \mathsf{T_q Q} \times \mathsf{T_q Q} \to \mathbb{R}$ at some point $\mathbf{q} \in \mathsf{Q}$, where $\mathsf{TQ} = \cup_{\mathbf{q}} \mathsf{T_q Q}$ is the *tangent bundle* of all tangent space $\mathsf{T_q Q}$ at all points $\mathbf{q} \in \mathsf{Q}$. Thus the length of a tangent vector $\mathbf{v_q} \in \mathsf{T_q Q}$ is given by $\sqrt{g_{\mathbf{q}}(\mathbf{v_q}, \mathbf{v_q})}$.

A *Riemannian connection* on $\mathsf{Q}$, denoted $\nabla$, is a mapping that assigns to any two smooth vector fields $\mathbf{X}$ and $\mathbf{Y}$ on $\mathsf{Q}$ a new vector field, $\nabla_{\mathbf{X}}\mathbf{Y}$. For the properties of $\nabla$, we refer the reader to [14], [15], [3]. The operator $\nabla_{\mathbf{X}}$, which assigns to every vector field $\mathbf{Y}$ the vector field $\nabla_{\mathbf{X}}\mathbf{Y}$, is called the *covariant derivative of* $\mathbf{Y}$ *with respect to* $\mathbf{X}$.

The *Lie bracket* of the vector fields $\mathbf{X}$ and $\mathbf{Y}$ will be denoted by $[\mathbf{X}, \mathbf{Y}]$ and is defined by the identity: $[\mathbf{X}, \mathbf{Y}]f = \mathbf{X}(\mathbf{Y}f) - \mathbf{Y}(\mathbf{X}f)$, for all $\mathcal{C}^\infty$ functions $f : \mathsf{Q} \to \mathbb{R}$. Given vector fields $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ on $\mathsf{Q}$, define the vector field $\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}$ by the identity

$$\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}. \qquad (1)$$

$\mathcal{R}$ is trilinear in $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ and is a tensor of type $(1, 3)$, which is called the *curvature tensor* of $\mathsf{Q}$.

Finally, we will employ the musical isomorphism $\sharp_g : \mathsf{T^*Q} \to \mathsf{TQ}$ (called the "sharp") and its inverse $\flat_g : \mathsf{TQ} \to \mathsf{T^*Q}$ (the "flat") associated with the metric $g$ and defined by the relation $\mathsf{T^*Q} \ni \mathbf{Y}^\flat(\mathbf{X}) = g(\mathbf{Y}, \mathbf{X})$, for all $\mathbf{X} \in \mathsf{TQ}$. The sharp is induced from the definition of the flat.

### B. Nonholonomic Systems and the Constrained Affine Connection

In this section we introduce the affine connection viewpoint of mechanical control systems. The discussion presented here is based on the material found in [6], [4], [16]. Let $\mathsf{Q}$ be a $\mathcal{C}^\infty$ $n$-dimensional manifold with the tangent and cotangent bundles denoted by $\mathsf{TQ}$ and $\mathsf{T^*Q}$, respectively. An under-actuated constrained simple mechanical control system is given by the quadruple $(\mathsf{Q}, g, \mathcal{F}, \mathcal{D})$, where $g_{\mathbf{q}} : \mathsf{T_q Q} \times \mathsf{T_q Q} \to \mathbb{R}$ is the kinetic energy (Riemannian) metric on $\mathsf{Q}$ at $\mathbf{q} \in \mathsf{Q}$. The collection of covectors $\mathcal{F} = \{\mathbf{F}^1, \mathbf{F}^2, \ldots, \mathbf{F}^p\} \in \mathsf{T^*Q}$, $p < n$, is a set of linearly independent 1-forms on $\mathsf{Q}$ that represent the directions of the forces and torques acting on the system given by

$$\boldsymbol{\tau} = \sum_{i=1}^{p} \tau_i \mathbf{F}^i \in \mathsf{T^*Q}. \qquad (2)$$

Hence, the system is underactuated with underactuation degree $n - p$. The subspace $\mathcal{D}$ is an $(n - m)$-dimensional nonholonomic distribution on $\mathsf{Q}$, where the $m$ constraints are given by

$$\boldsymbol{\omega}^i_{\mathbf{q}}(\mathbf{v_q}) = 0, \ i = 1, \ldots, m, \qquad (3)$$

where $\boldsymbol{\omega}^i \in \mathsf{T^*Q}$ are one-forms on $\mathsf{Q}$. In this paper we only consider systems that evolve in a potential-free environment. The presence of a potential does not introduce additional theoretical challenges to our treatment and, hence, we omit it here for the sake of simplicity. In a future archival version of this work, we will include potentials to study a wider range of nonholonomic systems.

For a simple mechanical control system without a potential the Lagrangian $L : \mathsf{T_q Q} \to \mathbb{R}$ is given by

$$L(\mathbf{q}, \mathbf{v}) = \frac{1}{2} g(\mathbf{v}, \mathbf{v}). \qquad (4)$$

The Lagrange d'Alembert principle then gives the following equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = \sum_{i=1}^{k} \lambda^j \boldsymbol{\omega}^j_{\mathbf{q}} + \sum_{i=1}^{m} \tau_i \mathbf{F}^i, \qquad (5)$$

where $\lambda^j$ are Lagrange multipliers such that $\boldsymbol{\lambda} = \sum_{j=1}^{k} \lambda^j \boldsymbol{\omega}^j$ represents reaction forces. The system of equations (5) is equivalently written using the affine connection as

$$\begin{aligned} \nabla_{\mathbf{v}(t)}\mathbf{v}(t) &= \boldsymbol{\lambda}(t)^{\sharp_g} + \mathbf{u}(t) \\ \dot{\mathbf{q}}(t) &= \mathbf{v}(t) \\ \mathbf{v}(t) &\in \mathcal{D}_{\mathbf{q}(t)}, \end{aligned} \qquad (6)$$

where $\nabla$ is the Levi-Civita connection compatible with the metric $g$, $\boldsymbol{\lambda}(t)$ is a section of $\mathcal{D}^\perp$ (the $g$-orthogonal complement of $\mathcal{D}$) and

$$\mathbf{u}(t) = \boldsymbol{\tau}^{\sharp_g}(t) = \sum_{i=1}^{p} \tau_i(t) \left(\mathbf{F}^i\right)^{\sharp_g} = \sum_{i=1}^{p} \tau_i(t) \mathbf{Y}_i, \qquad (7)$$

where $\mathbf{Y}_i = \left(\mathbf{F}^i\right)^{\sharp_g}$ are the corresponding input vector fields.

If we define $\mathcal{P} : \mathsf{TQ} \to \mathcal{D} \subseteq \mathsf{TQ}$ and $\mathcal{Q} : \mathsf{TQ} \to \mathcal{D}^\perp \subseteq \mathsf{TQ}$ to be the complementary $g$-orthogonal projectors, then the equations (6) are equivalently written as

$$\begin{aligned} \bar{\nabla}_{\mathbf{v}(t)}\mathbf{v}(t) &= \mathcal{P}(\mathbf{u}(t)) \\ \dot{\mathbf{q}}(t) &= \mathbf{v}(t), \end{aligned} \qquad (8)$$

where now we only require that the initial velocity be $\mathbf{v}(0) \in \mathcal{D}$ to ensure that the flow remains on the constrained disctribution. The connection $\bar{\nabla}$ is called the *nonholonomic affine connection* and is given by

$$\begin{aligned}\bar{\nabla}_{\mathbf{X}}\mathbf{Y} &= \nabla_{\mathbf{X}}\mathbf{Y} + (\nabla_{\mathbf{X}}\mathcal{Q})(\mathbf{Y}) \\ &= \mathcal{P}(\nabla_{\mathbf{X}}\mathbf{Y}) + \nabla_{\mathbf{X}}(\mathcal{Q}(\mathbf{Y})),\end{aligned} \tag{9}$$

for all $\mathbf{X}, \mathbf{Y} \in \mathsf{TQ}$. Note that $\bar{\nabla}_{\mathbf{X}}\mathbf{Y} \in \mathcal{D}$ for all $\mathbf{Y} \in \mathcal{D}$ and $\mathbf{X} \in \mathsf{TQ}$ [4], [6]. The constrained connection also appears in [17]. We now give further properties of the nonholonomic connection $\bar{\nabla}$, in particular, how they operate on functions and one-forms.

**Lemma II.1.** $\bar{\nabla}_{\mathbf{X}} f = \nabla_{\mathbf{X}} f$ for all $f \in \mathcal{C}^{\infty}(\mathsf{Q})$.

**Proof.** This is obvious since for any affine connection $\tilde{\nabla}$ we have $\tilde{\nabla}_{\mathbf{X}} f = \mathcal{L}_{\mathbf{X}} f = \mathbf{X}(f)$, the Lie derivative of $f$ with respect to the vector field $\mathbf{X}$. This is true since the Lie derivative $\mathcal{L}$ is independent of the choice of $\tilde{\nabla}$. ■

**Lemma II.2.** For all $\boldsymbol{\lambda} \in \mathsf{T}^*\mathsf{Q}$ we have

$$\bar{\nabla}_{\mathbf{X}}\boldsymbol{\lambda} = \nabla_{\mathbf{X}}\boldsymbol{\lambda} - (\nabla_{\mathbf{X}}\mathcal{Q})^*(\boldsymbol{\lambda}),$$

for all $\mathbf{X} \in \mathsf{TQ}$, where $*$ denotes the adjoint of a map. Note here that $\mathcal{Q}$ (and $\mathcal{P}$) is a $(1,1)$ tensor and so is $\bar{\nabla}_{\mathbf{X}}\mathcal{Q}$ and its adjoint $\bar{\nabla}_{\mathbf{X}}\mathcal{Q}$.

**Proof.** Given our knowledge of how $\bar{\nabla}$ acts on vector fields (equation (9)) and Lemma II.1, we have

$$\begin{aligned}\bar{\nabla}_{\mathbf{X}}\boldsymbol{\lambda}(\mathbf{Z}) &= \bar{\nabla}_{\mathbf{X}}(\boldsymbol{\lambda}(\mathbf{Z})) - \boldsymbol{\lambda}\left(\bar{\nabla}_{\mathbf{X}}\mathbf{Z}\right) \\ &= \nabla_{\mathbf{X}}(\boldsymbol{\lambda}(\mathbf{Z})) - \boldsymbol{\lambda}(\nabla_{\mathbf{X}}\mathbf{Z} + (\nabla_{\mathbf{X}}\mathcal{Q})(\mathbf{Z})) \\ &= (\nabla_{\mathbf{X}}\boldsymbol{\lambda})(\mathbf{Z}) + \boldsymbol{\lambda}(\nabla_{\mathbf{X}}\mathbf{Z}) - \boldsymbol{\lambda}(\nabla_{\mathbf{X}}\mathbf{Z}) \\ &\quad -\boldsymbol{\lambda}((\nabla_{\mathbf{X}}\mathcal{Q})(\mathbf{Z})) \\ &= (\nabla_{\mathbf{X}}\boldsymbol{\lambda})(\mathbf{Z}) - \left((\nabla_{\mathbf{X}}\mathcal{Q})^*(\boldsymbol{\lambda})\right)(\mathbf{Z})\end{aligned}$$

for all vector fields $\mathbf{X}, \mathbf{Z} \in \mathsf{TQ}$. For the first equality we used $\bar{\nabla}_{\mathbf{X}}(\boldsymbol{\lambda}(\mathbf{Z})) = \left(\bar{\nabla}_{\mathbf{X}}\boldsymbol{\lambda}\right)(\mathbf{Z}) + \boldsymbol{\lambda}\left(\bar{\nabla}_{\mathbf{X}}\mathbf{Z}\right)$. ■

Finally, recall the definition of the curvature tensor $\mathcal{R}$, which arises naturally in higher order optimal control problems, associated with an affine connection $\nabla$ given by equation (1). Associated with the nonholonomic affine connection is the nonholonomic curvature tensor, denoted $\bar{\mathcal{R}}$, that also satisfies equation (1) but with $\bar{\nabla}$ replacing $\nabla$ everywhere. We have the following observation for the nonholonomic curvature tensor $\bar{\mathcal{R}}$.

**Lemma II.3.** The nonholonomic curvature tensor satisfies

$$\begin{aligned}\bar{\mathcal{R}}(\mathbf{X},\mathbf{Y})\mathbf{Z} &= \mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} \\ &\quad + \left(\left(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \nabla_{[\mathbf{X},\mathbf{Y}]}\right)\mathcal{Q}\right)(\mathbf{Z}) \\ &\quad + (\nabla_{\mathbf{X}}\mathcal{Q})\left[(\nabla_{\mathbf{Y}}\mathcal{Q})(\mathbf{Z})\right] - (\nabla_{\mathbf{Y}}\mathcal{Q})\left[(\nabla_{\mathbf{X}}\mathcal{Q})(\mathbf{Z})\right]\end{aligned}$$

for all $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathsf{TQ}$.

**Proof.** The proof comes from the definition in equation (1) of the nonholonomic curvature tensor in terms of the nonholonomic connection. One then uses the definition of the nonholonomic connection in equation (9) to substitute $\bar{\nabla}$ in the equation for $\bar{\mathcal{R}}$ in (1). The rest of the proof is straightforward algebraic operations. ■

## III. Optimal Control of a Nonholonomic System

In this section we introduce the optimal control problem and derive the necessary optimality conditions. In this paper, we use Lagrange's multiplier method for constrained problems in the calculus of variations. We only investigate normal extremals, which is a reasonable assumption for simple mechanical control systems that occur in engineering. We also note that, while the system is controlled through the shape space only and is, hence, inherently, under-actuated, *a basic assumption is that the system is controllable.*

**Problem III.1.** Minimize

$$\mathcal{J}(\boldsymbol{\tau}) = \int_0^T \frac{1}{2} g^E(\mathbf{u},\mathbf{u})\,\mathrm{d}t \tag{10}$$

subject to the dynamics given in equation (8) and some initial and terminal conditions $\mathbf{q}(0)$ and $\mathbf{q}(T)$, respectively. The $(0,2)$ tensor $g^E$ is the standard identity metric on $\mathbb{R}^n$. The state $\mathbf{q}(T)$ is assumed to be reachable by the system from $\mathbf{q}(0)$.

Note that we want to minimize the norm of $\mathbf{u}$ as opposed to the norm of its projection $\mathcal{P}(\mathbf{u})$. In other words, generally, the above formulation does not attempt to minimize the constrained applied torques. Intuitively, one forecasts that no control forces and torques should be applied in directions that violate the constraints since these will be squandered by creating only more reaction forces (that maintain the constraints) with no net useful motion, or by violating the constraints altogether. For example, for the *rolling* vertical coin [5], if excessive torque is applied in the rolling direction, the rolling constraint may be violated. Moreover, the application of any side forces will not contribute to the net motion of the system due to the strict no-side-slip (the "knife edge") constraint. As will be shown below, it turns out that the control will be constrained to lie in $\mathcal{D}$ as expected, hence, not allowing for violation of the constraints or the application of unnecessary control.

In undulatory locomotion, which is of main interest in this work, by definition, we usually require that the control be applied through the shape (internal configuration) space only. Hence, we need to impose the constraint that the generalized control vector field in the group directions be zero. We do this as follows. Let $\tilde{\mathbf{F}}_i$, $i = 1,\ldots,n-p$, form an orthogonal set of co-vector fields complement to the co-vector fields $\mathbf{F}_i$, $i = 1,\ldots,p$. Then, define a Lagrange multiplier one-form $\boldsymbol{\xi} = \sum_{i=1}^{n-p} \xi_i \tilde{\mathbf{F}}_i$ such that

$$\boldsymbol{\xi}(\mathbf{u}) = 0. \tag{11}$$

We only treat the general case when $\mathsf{Q}$ is an arbitrary manifold and specialize the result to trivial fiber bundles in future publications. By trivial fiber bundles we mean manifolds of the form $\mathsf{Q} = \mathsf{G} \times \mathsf{S}$, where $\mathsf{G}$ is a Lie group that represents the fiber, or overall configuration of the system, and $\mathsf{S}$ is the shape space, or internal configuration, of the system. We begin by forming the appended cost functional

$$\mathcal{J} = \int_0^T \frac{1}{2} g^E(\mathbf{u},\mathbf{u}) + \boldsymbol{\xi}(\mathbf{u}) + \boldsymbol{\mu}\left(\dot{\mathbf{q}} - \mathbf{v}\right) + \boldsymbol{\eta}\left(\bar{\nabla}_{\mathbf{v}}\mathbf{v} - \mathcal{P}\left(\mathbf{u}\right)\right) \mathrm{d}t, \tag{12}$$

where $\boldsymbol{\mu}, \boldsymbol{\eta} \in \mathcal{D}^*\mathsf{Q}$ are Lagrange multipliers.

**Remark.** Let $\boldsymbol{\omega}^j$, $j = m+1, \ldots, n$, be the set of one-forms that span $\mathcal{D}^*$. This set of one forms along with the one-forms $\omega^i$, $i = 1, \ldots, m$, form a basis for $\mathsf{T}^*\mathsf{Q}$. In the above, we view $\boldsymbol{\mu} \in \mathcal{D}^*$, the co-tangent constraint distribution spanned by the one forms $\boldsymbol{\omega}^j$ for , as a one-form on $\mathsf{Q}$ such that $\boldsymbol{\mu} : \mathcal{D} \to \mathbb{R}$. On the other hand, it is important to note that the derivatives of the velocity vector field is general not going to be in the constraint distribution $\mathcal{D}$. Recall that if $\mathbf{v} \in \mathcal{D}$, then $\bar{\nabla}_{\mathbf{X}}\mathbf{v} \in \mathcal{D}$ for all $\mathbf{X} \in \mathsf{TQ}$. Hence, the argument of $\boldsymbol{\eta}$ is always in $\mathcal{D}$ and we view $\mathcal{D}^* \ni \boldsymbol{\eta} : \mathcal{D} \to \mathbb{R}$. Moreover, observe that $\mathcal{P}^*\boldsymbol{\omega} = \boldsymbol{\omega}$ for all $\boldsymbol{\omega} \in \mathcal{D}^*$, where $\mathcal{P}^* : \mathcal{D}^* \subseteq \mathsf{T}^*\mathsf{Q} \to \mathsf{T}^*\mathsf{Q}$ is the adjoint of the map $\mathcal{P}$. •

Taking variations of equation (12) we obtain

$$\boldsymbol{\delta}\mathcal{J} = \int_0^T g^E\left(\nabla_{\mathbf{W}}\mathbf{u}, \mathbf{u} + \boldsymbol{\xi}^{\sharp_{g^E}}\right) + \boldsymbol{\mu}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{W} - \nabla_{\mathbf{W}}\mathbf{v}\right) + \boldsymbol{\eta}\left(\nabla_{\mathbf{W}}\bar{\nabla}_{\mathbf{v}}\mathbf{v} - \left(\nabla_{\mathbf{W}}\mathcal{P}\right)(\mathbf{u}) - \mathcal{P}\left(\nabla_{\mathbf{W}}\left(\mathbf{u}\right)\right)\right) \mathrm{d}t,$$

where $\nabla_{\mathbf{W}}$ is the covariant derivative with respect to the variation vector field $\mathbf{W} \in \mathsf{TQ}$ given by

$$\mathbf{W} = \left.\frac{\partial \mathbf{q}(t,\epsilon)}{\partial \epsilon}\right|_{\epsilon=0}, \tag{13}$$

with $\mathbf{q}(t,\epsilon)$ being the one-parameter variation of the optimal curve $\mathbf{q}(t)$. In the above expression, we used the fact that $\tilde{\nabla}_{\mathbf{X}}\boldsymbol{\lambda}\left(\mathbf{Z}\right) = \left(\tilde{\nabla}_{\mathbf{X}}\boldsymbol{\lambda}\right)(\mathbf{Z}) + \boldsymbol{\lambda}\left(\tilde{\nabla}_{\mathbf{X}}\mathbf{Z}\right)$, for any affine connection $\tilde{\nabla}$, vector fields $\mathbf{X}$ and $\mathbf{Z}$ and any co-vector field $\boldsymbol{\lambda}$ (see page 78 in [18]). When $\boldsymbol{\lambda}$ is a Lagrange multiplier, say $\boldsymbol{\mu}$ or $\boldsymbol{\eta}$, the terms $\left(\nabla_{\mathbf{W}}\boldsymbol{\mu}\right)(\dot{\mathbf{q}} - \mathbf{v})$ and $\left(\nabla_{\mathbf{W}}\boldsymbol{\eta}\right)\left(\bar{\nabla}_{\mathbf{v}}\mathbf{v} - \mathcal{P}\left(\mathbf{u}\right)\right)$ give us the equations of motion (8) later when we set $\delta\mathcal{J}$ equal to zero. Hence, as usually done in optimal control theory, these two terms can be omitted without affecting the rest of the derivation.

We now study the term $\nabla_{\mathbf{W}}\bar{\nabla}_{\mathbf{v}}\mathbf{v}$. Using the definition of the constrained connection in equation (9) we get

$$\nabla_{\mathbf{W}}\bar{\nabla}_{\mathbf{v}}\mathbf{v} = \nabla_{\mathbf{W}}\left[\mathcal{P}\left(\nabla_{\mathbf{v}}\mathbf{v}\right) + \nabla_{\mathbf{v}}\left(\mathcal{Q}\left(\mathbf{v}\right)\right)\right].$$

However, since the dynamics given by equation (8) guarantee that $\mathbf{v} \in \mathcal{D}$ (assuming $\mathbf{v}(0) \in \mathcal{D}$), we have $\mathcal{Q}(\mathbf{v}) = 0$. Therefore, we obtain

$$\nabla_{\mathbf{W}}\bar{\nabla}_{\mathbf{v}}\mathbf{v} = \left(\nabla_{\mathbf{W}}\mathcal{P}\right)\left(\nabla_{\mathbf{v}}\mathbf{v}\right) + \mathcal{P}\left(\nabla_{\mathbf{W}}\nabla_{\mathbf{v}}\mathbf{v}\right).$$

Recall that the curvature tensor, $\mathcal{R}$, associated with the unconstrained connection $\nabla$, satisfies the identity given by equation (1). Since $\mathbf{W}$ is arbitrary, and hence independent of $\mathbf{q}$, then $[\mathbf{W}, \mathbf{v}] = 0$ such that [19]

$$\nabla_{\mathbf{W}}\nabla_{\mathbf{v}}\mathbf{v} = \nabla_{\mathbf{v}}\nabla_{\mathbf{W}}\mathbf{v} + \mathcal{R}\left(\mathbf{W}, \mathbf{v}\right)\mathbf{v}.$$

Finally, we conclude that

$$\nabla_{\mathbf{W}}\bar{\nabla}_{\mathbf{v}}\mathbf{v} = \left(\nabla_{\mathbf{W}}\mathcal{P}\right)\left(\nabla_{\mathbf{v}}\mathbf{v}\right) + \mathcal{P}\left(\nabla_{\mathbf{v}}\nabla_{\mathbf{W}}\mathbf{v} + \mathcal{R}\left(\mathbf{W}, \mathbf{v}\right)\mathbf{v}\right).$$

Using this identity, we obtain

$$\boldsymbol{\delta}\mathcal{J} = \int_0^T g^E\left(\bar{\nabla}_{\mathbf{W}}\mathbf{u}, \mathbf{u} + \boldsymbol{\xi}^{\sharp_{g^E}}\right) + \boldsymbol{\mu}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{W} - \nabla_{\mathbf{W}}\mathbf{v}\right) + \boldsymbol{\eta}\Big(\mathcal{P}\left(\nabla_{\mathbf{v}}\nabla_{\mathbf{W}}\mathbf{v} + \mathcal{R}\left(\mathbf{W}, \mathbf{v}\right)\mathbf{v}\right) + \left(\nabla_{\mathbf{W}}\mathcal{P}\right)\left(\nabla_{\mathbf{v}}\mathbf{v} - \mathbf{u}\right) - \mathcal{P}\left(\nabla_{\mathbf{W}}\left(\mathbf{u}\right)\right)\Big)\mathrm{d}t.$$

For the two first terms in the argument of $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$, we integrate by parts and use the fact that $\mathbf{W}(0) = \mathbf{W}(T) = \nabla_{\mathbf{W}}\mathbf{v}(0) = \nabla_{\mathbf{W}}\mathbf{v}(T) = 0$ to obtain

$$\int_0^T \boldsymbol{\mu}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{W}\right)\mathrm{d}t = -\int_0^T \left(\nabla_{\mathbf{v}}\boldsymbol{\mu}\right)(\mathbf{W})\,\mathrm{d}t \tag{14}$$

$$\int_0^T \boldsymbol{\eta}\left(\mathcal{P}\left(\nabla_{\mathbf{v}}\nabla_{\mathbf{W}}\mathbf{v}\right)\right)\mathrm{d}t = -\int_0^T \nabla_{\mathbf{v}}\left(\mathcal{P}^*\left(\boldsymbol{\eta}\right)\right)\left(\nabla_{\mathbf{W}}\mathbf{v}\right)\mathrm{d}t = -\int_0^T \nabla_{\mathbf{v}}\boldsymbol{\eta}\left(\nabla_{\mathbf{W}}\mathbf{v}\right)\mathrm{d}t,$$

where we made use of the fact that $\mathcal{P}^*\boldsymbol{\eta} = \boldsymbol{\eta}$ and that $\boldsymbol{\eta} \in \mathcal{D}^*$.

Next, we refer the reader to the properties of a curvature tensor $\tilde{\mathcal{R}}$ defined in terms of the connection $\tilde{\nabla}$ and a metric $\tilde{g}$ found in [20], Proposition 2.5 on page 91. From these properties, one can show that the curvature satisfies $\tilde{g}\left(\tilde{\mathcal{R}}\left(\mathbf{W}, \mathbf{v}\right)\mathbf{v}, \mathbf{X}\right) = \tilde{g}\left(\tilde{\mathcal{R}}\left(\mathbf{X}, \mathbf{v}\right)\mathbf{v}, \mathbf{W}\right)$, where $\tilde{\mathcal{R}}$ is the curvature tensor based on a connection $\tilde{\nabla}$ that is compatible with $\tilde{g}$ (this is shown in [15], for example). Going through the proof of Proposition 2.5 in [20], one can see that all the derivation can be generalized to an arbitrary metric $\hat{g}$ and not only to $\tilde{g}$. Hence, we have $\hat{g}\left(\tilde{\mathcal{R}}\left(\mathbf{W}, \mathbf{v}\right)\mathbf{v}, \mathbf{X}\right) = \hat{g}\left(\tilde{\mathcal{R}}\left(\mathbf{X}, \mathbf{v}\right)\mathbf{v}, \mathbf{W}\right)$, for any positive definite metric $\hat{g}$ on $\mathsf{Q}$. In the context of our problem, this gives

$$\boldsymbol{\eta}\left(\mathcal{P}\left(\mathcal{R}\left(\mathbf{W}, \mathbf{v}\right)\mathbf{v}\right)\right) = g^E\left(\left(\mathcal{P}^*\boldsymbol{\eta}\right)^{\sharp_{g^E}}, \mathcal{R}\left(\mathbf{W}, \mathbf{v}\right)\mathbf{v}\right) = g^E\left(\mathbf{W}, \mathcal{R}\left(\boldsymbol{\eta}^{\sharp_{g^E}}, \mathbf{v}\right)\mathbf{v}\right). \tag{15}$$

Moreover, note that

$$\boldsymbol{\eta}\left(\mathcal{P}\left(\nabla_{\mathbf{W}}\mathbf{u}\right)\right) = \left(\mathcal{P}^*\boldsymbol{\eta}\right)\left(\nabla_{\mathbf{W}}\mathbf{u}\right) = g^E\left(\boldsymbol{\eta}^{\sharp_{g^E}}, \nabla_{\mathbf{W}}\mathbf{u}\right). \tag{16}$$

Finally, recall that $\mathcal{P}$ is a $(1,1)$ tensor. Hence $\nabla_{\mathbf{W}}\mathcal{P}$ is also a $(1,1)$ tensor (for this check out any book on differential geometry, though it is explicitly stated in [6]). So $\left(\nabla_{\mathbf{W}}\mathcal{P}\right)^* : \mathsf{T}^*\mathsf{Q} \to \mathsf{T}^*\mathsf{Q}$ is the dual of $\nabla_{\mathbf{W}}\mathcal{P}$. With this observation we have

$$\boldsymbol{\eta}\left(\nabla_{\mathbf{W}}\mathcal{P}\left(\nabla_{\mathbf{v}}\mathbf{v} - \mathbf{u}\right)\right) = \left(\left(\nabla_{\mathbf{W}}\mathcal{P}\right)^*\boldsymbol{\eta}\right)\left(\boldsymbol{\lambda}^{\sharp_{g^E}}\right) = g^E\left(\left(\left(\nabla_{\mathbf{W}}\mathcal{P}\right)^*\boldsymbol{\eta}\right)^{\sharp_{g^E}}, \boldsymbol{\lambda}^{\sharp_{g^E}}\right) \tag{17}$$

where we recall from equation (6) that $\boldsymbol{\lambda}$ is the net reaction generalized force co-vector field.

Using equations (14)-(17), we find that

$$\begin{aligned}\boldsymbol{\delta}\mathcal{J} &= \int_0^T g^E\left(\nabla_{\mathbf{W}}\mathbf{u}, \mathbf{u}+\boldsymbol{\xi}^{\sharp_{g^E}}-\boldsymbol{\eta}^{\sharp_{g^E}}\right)\\ &\quad -\nabla_{\mathbf{v}}\boldsymbol{\mu}\left(\mathbf{W}\right)+\left[\mathcal{R}\left(\boldsymbol{\eta}^{\sharp_{g^E}}, \mathbf{v}\right)\mathbf{v}\right]^{\flat_{g^E}}\left(\mathbf{W}\right)\\ &\quad +\left(\left(\nabla_{\mathbf{W}}\mathcal{P}\right)^*\boldsymbol{\eta}\right)\left(\boldsymbol{\lambda}^{\sharp_{g^E}}\right)\\ &\quad +g^E\left(\nabla_{\mathbf{W}}\mathbf{v}, -\boldsymbol{\mu}^{\sharp_{g^E}}-\left(\nabla_{\mathbf{v}}\boldsymbol{\eta}\right)^{\sharp_{g^E}}\right)\mathrm{d}t.\end{aligned}$$

Two important points need to be emphasized. First, the term $\nabla_{\mathbf{W}}\mathcal{P}$ involves variations in the configuration variables only since $\mathcal{P}$ is an operator that depends only on the configuration variables. That is, while $\mathcal{P}$ acts on the velocity vector field $\mathbf{v}_{\mathbf{q}}$ at the point $\mathbf{q}$, $\mathcal{P}$ as a projector depends solely on the point $\mathbf{q}$. Thus, this term depends on $\mathbf{W}$ only.

The second observation we wish to make is that $\nabla_{\mathbf{W}}\mathbf{u}$ or $\nabla_{\mathbf{W}}\mathbf{v}$ each can be separated into two terms. For example, for $\nabla_{\mathbf{W}}\mathbf{u}$, the first variation term involves only variations in the control components $\tau_i$, $i = 1, \ldots, m$, while the second will involve configuration variations $\mathbf{W}$ only coming through variations of the basis vectors, denoted by $\mathbf{Y}_i$. This should be realized in order to completely (and rigorously) separate variations in configuration, velocity and control variables. For more on this, see for example the discussion in Section II in [21] and the definitions of the operators $B$ and $\delta$ therein. It turns out that if we simply ignore this separation step and treat $\nabla_{\mathbf{W}}\mathbf{u}$ and $\nabla_{\mathbf{W}}\mathbf{v}$ as terms that involve variations in control and velocity variables only, respectively, and no variations in the configuration variables, then we end up with exactly the same result. We emphasize that the separation step is the correct rigorous mathematical approach, while ignoring it is not rigorous albeit reduces the number of steps to obtain the *same correct* result.

From the above discussion, we realize that since $\mathbf{W}$, $\nabla_{\mathbf{W}}\mathbf{u}$ and $\nabla_{\mathbf{W}}\mathbf{v}$ are independent variations and since for a normal extremal we must have $\boldsymbol{\delta}\mathcal{J} = 0$, we conclude that along the optimal trajectory we must have

$$\begin{aligned}\mathbf{u}+\boldsymbol{\xi}^{\sharp_{g^E}} &= \boldsymbol{\eta}^{\sharp_{g^E}}\\ \nabla_{\mathbf{v}}\boldsymbol{\mu} &= \left[\mathcal{R}\left(\boldsymbol{\eta}^{\sharp_{g^E}}, \mathbf{v}\right)\mathbf{v}\right]^{\flat_{g^E}}+\boldsymbol{\eta}\left(\left(\nabla\mathcal{P}\right)\boldsymbol{\lambda}^{\sharp_{g^E}}\right)\\ \nabla_{\mathbf{v}}\boldsymbol{\eta} &= -\boldsymbol{\mu},\end{aligned} \tag{18}$$

where $\flat_{g^E}$ and $\sharp_{g^E}$ are the musical isomorphisms with respect to the standard metric $g^E$.

**Remark. (The Unconstrained, Fully Actuated Problem)** In the unconstrained case $\mathcal{P} = \mathcal{I} : \mathsf{TQ} \to \mathsf{TQ}$ is simply the identity map on $\mathsf{TQ}$ for all $\mathbf{q} \in \mathsf{Q}$. Similarly for $\mathcal{P}^*$, $\mathcal{P}^* = \mathcal{I}^* : \mathsf{T}^*\mathsf{Q} \to \mathsf{T}^*\mathsf{Q}$ is the identity map in $\mathsf{T}^*\mathsf{Q}$ for all $\mathbf{q} \in \mathsf{Q}$. $\mathcal{Q}$ will have a null space $\mathcal{N}\left(\mathcal{Q}_{\mathbf{q}}\right) = \mathsf{T}_{\mathbf{q}}\mathsf{Q}$ for all $\mathbf{q} \in \mathsf{Q}$. In the fully actuated case we have $\boldsymbol{\xi} = 0$. Therefore, in the unconstrained, fully-actuated case the necessary conditions (18) reduce to

$$\begin{aligned}\mathbf{u} &= \boldsymbol{\eta}^{\sharp_{g^E}}\\ \nabla_{\mathbf{v}}\boldsymbol{\mu} &= \left[\mathcal{R}\left(\boldsymbol{\eta}^{\sharp_{g^E}}, \mathbf{v}\right)\mathbf{v}\right]^{\flat_{g^E}}\\ \nabla_{\mathbf{v}}\boldsymbol{\eta} &= -\boldsymbol{\mu}.\end{aligned}$$

These are precisely the result obtained in, say, [21]. •

## IV. Example: The Vertical Coin

### A. Constrained Equations of Motion

We now give an example to illustrate the above approach and compare the result to traditional methods. In this section we study optimal control of the vertical coin (i.e., it can not fall sideways). The system is shown in Figure 1. The mass of the coin is $m$ and its mass moment of inertia about the vertical axis is $J$. The position of the point of contact between the coin and the plane is denoted by $(q_1, q_2)$ while its heading direction is denoted by $q_3$ as shown in the figure. The configuration $\mathbf{q}$ is then given by $\mathbf{q} = (q_1, q_2, q_3)$ and the configuration space is simply $\mathsf{SE}(2)$. The control input is denoted by $u_1$ for the force applied to the center of mass of the coin and $u_2$ for the torque applied about the vertical axis.

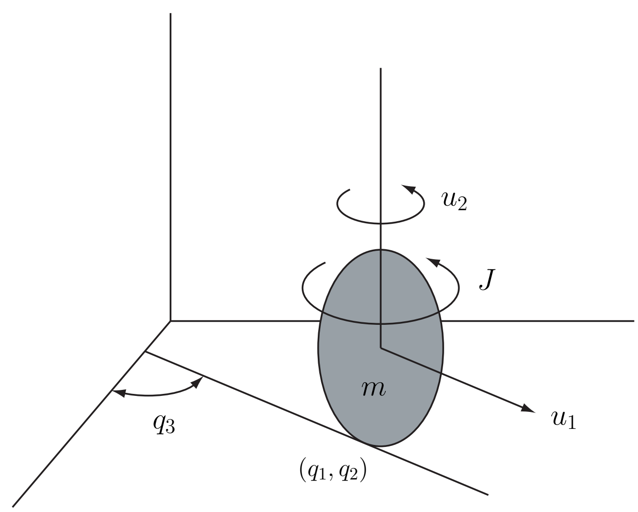


Fig. 1. The vertical coin.

The constraint we have is that the coin can not slip sideways (i.e., it satisfies a "knife-edge" constraint). This constraint is symbolically expressed in differential form by equation (3) with $m = 1$ and

$$\boldsymbol{\omega}^1_{\mathbf{q}} = \sin q_3 \mathrm{d}q_1 - \cos q_3 \mathrm{d}q_2. \tag{19}$$

Hence, the constraint distribution is given by the span of the vector fields

$$\begin{aligned}\mathbf{X}_1(\mathbf{q}) &= \frac{\partial}{\partial q_3}\\ \mathbf{X}_2(\mathbf{q}) &= \cos q_3 \frac{\partial}{\partial q_1} + \sin q_3 \frac{\partial}{\partial q_2}.\end{aligned} \tag{20}$$

The lagrangian for the vertical coin is given by

$$L(\mathbf{q},\dot{\mathbf{q}}) = g(\dot{\mathbf{q}},\dot{\mathbf{q}}) = \frac{1}{2}m\left(\dot{q}_1^2+\dot{q}_2^2\right)+\frac{1}{2}J\dot{q}_3^2$$

and, hence, the components of the metric $g$ are given by

$$g_{11} = g_{22} = m,\ g_{33} = J \tag{21}$$

where all other components are zero.

We denote the unconstrained connection by $\nabla$. Since the metric is coordinate independent, the unconstrained Christoffel symbols $\Gamma^i_{jk}$ are all zero. The curvature on $\mathsf{SE}(2)$ is identically zero. (For why curvature is zero on $\mathsf{SE}(2)$, see [22].) For the constrained system, one can check that the Christoffel symbols corresponding to the constrained connection $\bar{\nabla}$ are given by

$$\bar{\Gamma}^1_{13} = -\bar{\Gamma}^2_{23} = \sin(2q_3),\ \bar{\Gamma}^1_{23} = \bar{\Gamma}^2_{13} = -\cos(2q_3), \tag{22}$$

where all other Christoffel symbols are zero.

The external generalized force is given by

$$\boldsymbol{\tau} = u_1\mathbf{F}^1 + u_2\mathbf{F}^2, \tag{23}$$

where one can check that the inputs have the directions

$$\mathbf{F}^1 = \cos q_3 \mathrm{d}q_1 + \cos q_3 \mathrm{d}q_2,\ \mathbf{F}^2 = \mathrm{d}q_3. \tag{24}$$

The vector fields $\mathbf{Y}_i = \left(\mathbf{F}^i\right)^{\sharp_g}$ are then given by

$$\begin{aligned}\mathbf{Y}_1 &= \frac{\cos q_3}{m}\frac{\partial}{\partial q_1} + \frac{\sin q_3}{m}\frac{\partial}{\partial q_2}\\ \mathbf{Y}_2 &= \frac{1}{J}\frac{\partial}{\partial q_3}\end{aligned}$$

such that the input is given by

$$\mathbf{u} = \frac{u_1\cos q_3}{m}\frac{\partial}{\partial q_1} + \frac{u_1\sin q_3}{m}\frac{\partial}{\partial q_2} + \frac{u_2}{J}\frac{\partial}{\partial q_3}. \tag{25}$$

The projection map $\mathcal{P} : \mathsf{TQ} \to \mathcal{D}$ is a $(1,1)$ tensor whose elements are computed as follows. Let $\mathbf{Z} \in \mathsf{TQ}$ be an arbitrary vector field. Then $\mathcal{P}(\mathbf{Z}) = \sum_{i,j} C^{ij} g(\mathbf{Z},\mathbf{X}_i)\mathbf{X}_j$, where $C^{ij}$ is the inverse of the matrix $C_{ij} = g(\mathbf{X}_i,\mathbf{X}_j)$, $i,j = 1,2$ [16]. If we have $\mathcal{P} = \mathcal{P}^i_j \mathrm{d}q_i \otimes \frac{\partial}{\partial q_j}$, one can check that the components of $\mathcal{P}$ are given by

$$\begin{aligned}\mathcal{P}^1_1 &= \cos^2 q_3,\ \mathcal{P}^1_2 = \mathcal{P}^2_1 = \cos q_3 \sin q_3,\\ \mathcal{P}^2_2 &= \sin^2 q_3,\ \mathcal{P}^3_3 = 1,\end{aligned} \tag{26}$$

where all other components are zero. Hence, we have

$$\mathcal{P}(\mathbf{u}) = \frac{u_1\cos q_3}{m}\frac{\partial}{\partial q_1} + \frac{u_1\sin q_3}{m}\frac{\partial}{\partial q_2} + \frac{u_2}{J}\frac{\partial}{\partial q_3} = \mathbf{u}.$$

One could have anticipated this result since $\mathbf{u}$ is applied in directions lying inside the constraint distribution $\mathcal{D}$. This gives the right hand side of the nonholonomic equation motion given in equation (8).

**Remark.** If the force $u_1$ applied to the center of mass was restricted to be along the $x$-direction, $\mathcal{P}(\mathbf{u})$ will not be equal to $\mathbf{u}$. In this case, $\mathbf{u}$ will be projected down to the constraint distribution by creating a reaction force perpendicular to the direction of motion to prevent any motion that violates the knife-edge constraint. •

Finally recall that $\bar{\nabla}_{\mathbf{v}}\mathbf{v}$ is given in coordinates by

$$\begin{aligned}\bar{\nabla}_{\mathbf{v}}\mathbf{v} &= \sum_{i,j,k=1}^{3}\left(\ddot{q}_i + \bar{\Gamma}^i_{jk}\dot{q}_j\dot{q}_k\right)\frac{\partial}{\partial q_i}\\ &= \left[\ddot{q}_1 + \dot{q}_3\left(\dot{q}_1\sin(2q_3) - \dot{q}_2\cos(2q_3)\right)\right]\frac{\partial}{\partial q_1}\\ &\quad + \left[\ddot{q}_2 + \dot{q}_3\left(-\dot{q}_1\cos(2q_3) - \dot{q}_2\sin(2q_3)\right)\right]\frac{\partial}{\partial q_2}\\ &\quad + \ddot{q}_3\frac{\partial}{\partial q_3},\end{aligned}$$

where we have used the constrained connection coefficients given in equation (22). This gives the left hand side of equation (8). Hence, the equations of motion are given by

$$\begin{aligned}\ddot{q}_1 &= \left(-\dot{q}_1\sin(2q_3) + \dot{q}_2\cos(2q_3)\right)\dot{q}_3 + \frac{\cos q_3 u_1}{m}\\ \ddot{q}_2 &= \left(\dot{q}_1\cos(2q_3) + \dot{q}_2\sin(2q_3)\right)\dot{q}_3 + \frac{\sin q_3 u_1}{m}\\ \ddot{q}_3 &= \frac{u_2}{J},\end{aligned} \tag{27}$$

which simplify to

$$\begin{aligned}\ddot{q}_1 &= -\dot{q}_2\dot{q}_3 + \frac{\cos q_3 u_1}{m}\\ \ddot{q}_2 &= \dot{q}_1\dot{q}_3 + \frac{\sin q_3 u_1}{m}\\ \ddot{q}_3 &= \frac{u_2}{J}\end{aligned} \tag{28}$$

after using the constraints (19).

### *B. Optimality Conditions*

We now apply equations (18) for the vertical coin. First, we need to find a basis for $\mathcal{D}^*$, which we take to be

$$\begin{aligned}\boldsymbol{\omega}^2_{\mathbf{q}} &= \cos q_3 \mathrm{d}q_1 + \sin q_3 \mathrm{d}q_2\\ \boldsymbol{\omega}^3_{\mathbf{q}} &= \mathrm{d}q_3.\end{aligned} \tag{29}$$

Once can check that these are orthogonal to $\boldsymbol{\omega}^1_{\mathbf{q}}$. Hence, the lagrange multipliers are given by

$$\begin{aligned}\boldsymbol{\mu} &= \mu^2\boldsymbol{\omega}^2_{\mathbf{q}} + \mu^3\boldsymbol{\omega}^3_{\mathbf{q}}\\ &= \mu^2\cos q_3\mathrm{d}q_1 + \mu^2\sin q_3\mathrm{d}q_2 + \mu^3\mathrm{d}q_3\end{aligned} \tag{30}$$

and

$$\begin{aligned}\boldsymbol{\eta} &= \eta^2\boldsymbol{\omega}^2_{\mathbf{q}} + \eta^3\boldsymbol{\omega}^3_{\mathbf{q}}\\ &= \eta^2\cos q_3\mathrm{d}q_1 + \eta^2\sin q_3\mathrm{d}q_2 + \eta^3\mathrm{d}q_3.\end{aligned} \tag{31}$$

The under-actuated direction is given by $\tilde{\mathbf{F}}^1 = -\sin q_3 \mathrm{d}q_1 + \cos q_3 \mathrm{d}q_2$ such that $\boldsymbol{\xi} = \xi_1\tilde{\mathbf{F}}^1$ and

$$\boldsymbol{\xi}^{\sharp_{g^E}} = \frac{-\sin q_3\xi_1}{m}\frac{\partial}{\partial q_1} + \frac{\cos q_3\xi_1}{m}\frac{\partial}{\partial q_2}.$$

The first of equations (18) then gives

$$\begin{aligned}u_1 &= m\eta^2\\ u_2 &= J\eta^3\\ \xi_1 &= 0.\end{aligned} \tag{32}$$

In fact, $\xi_1$ is nothing but the generalized reaction force created by the knife edge constraint in reaction to any forces applied normal to the constraint.

We now obtain the differential equation for $\boldsymbol{\mu}$. The curvature tensor on $\mathsf{Q}$ is identically zero in this case. Moreover, since there are no external forces and torques other than the control inputs, which are applied in $\mathcal{D}$, then the constraint reaction forces $\boldsymbol{\lambda}$ are identically zero in our example. Generally the reaction forces $\boldsymbol{\lambda}$ won;t be zero. Finally, $\nabla$ has identically zero Christofel symbols. Thus, $\nabla_{\mathbf{v}}$ becomes a simple time derivative. Hence, the equation for $\boldsymbol{\mu}$ simply gives

$$\begin{aligned}\dot{\mu}^2 \cos q_3 &= \mu^2 \sin q_3 \dot{q}_3\\ \dot{\mu}^2 \sin q_3 &= -\mu^2 \cos q_3 \dot{q}_3\\ \dot{\mu}^3 &= 0.\end{aligned}$$

Multiplying the first equation by $\cos q_3$ and the second by $\sin q_3$ and summing we obtain

$$\begin{aligned}\dot{\mu}^2 &= 0\\ \dot{\mu}^3 &= 0.\end{aligned} \tag{33}$$

This gives the $\boldsymbol{\mu}$ differential equation.

Finally, we compute $\nabla_{\mathbf{v}}\boldsymbol{\eta} = -\boldsymbol{\mu}$. Hence, $\boldsymbol{\eta}$ satisfies the following differential equation equations

$$\begin{aligned}\dot{\eta}^2 \cos q_3 &= -\mu^2 \cos q_3 + \eta^2 \dot{q}_3 \sin q_3\\ \dot{\eta}^2 \sin q_3 &= -\mu^2 \sin - \eta^2 \dot{q}_3 \cos q_3\\ \dot{\eta}^3 &= -\mu^3.\end{aligned}$$

Multiplying the first equation by $\cos q_3$ and the second by $\sin q_3$ and adding both expressions we finally obtain

$$\begin{aligned}\dot{\eta}^2 &= -\mu^2\\ \dot{\eta}^3 &= -\mu^3.\end{aligned} \tag{34}$$

We see that the necessary conditions for this example are particularly simple. This is due to the fact that the vertical coin has a manifold $\mathsf{SE}(2)$ that is isomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$, which is a differentially flat space. This flatness and the positive definiteness and convexity of the cost functional (10) render the problem convex, with simple linear necessary conditions that can be solved analytically for the global optimal solution. The necessary conditions are solved to get

$$\begin{aligned}\mu^2(t) &= \mu_0^2\\ \mu^3(t) &= \mu_0^3\\ \eta^2(t) &= -\mu_0^2 t + \eta_0^2\\ \eta^3(t) &= -\mu_0^3 + \eta_0^3,\end{aligned} \tag{35}$$

where $\mu_0^2, \mu_0^3, \eta_0^2, \eta_0^3$ are initial conditions determined from the boundary conditions on the state of the system $\mathbf{q}$ and $\mathbf{v}$. These expressions may then be used to compute the optimal control law from equation (32).

In this paper, we give a simple example for the sake of transparency of the approach. A longer version of this work will include more interesting examples in the context of optimal robotic locomotion.

### C. Verification of Results Using Classical Methods

In this section we verify the necessary conditions obtained in the last section, which were the coordinate expressions of equation (18) for the vertical coin. To do so, we derive the necessary optimality conditions for the optimal control problem Problem III.1 in coordinates using equations (27) for the dynamic constraints. The cost function in Problem (III.1), in coordinates, reads

$$\mathcal{J} = \int_0^T \frac{1}{2}\left(\frac{u_1^2}{m^2} + \frac{u_2^2}{J^2}\right) \mathrm{d}t.$$

Writing equations (27) in first order form

$$\begin{aligned}\dot{q}_1 &= v_1,\ \dot{q}_2 = v_2,\ \dot{q}_3 = v_3\\ \dot{v}_1 &= -v_2 v_3 + \frac{\cos q_3 u_1}{m} =: f_1(\mathbf{q}, \mathbf{v}, \mathbf{u})\\ \dot{v}_2 &= v_1 v_3 + \frac{\sin q_3 u_1}{m} =: f_2(\mathbf{q}, \mathbf{v}, \mathbf{u})\\ \dot{v}_3 &= \frac{u_2}{J} =: f_3(\mathbf{q}, \mathbf{v}, \mathbf{u})\end{aligned} \tag{36}$$

the appended cost function is given by

$$\begin{aligned}\mathcal{J} &= \int_0^T \frac{1}{2}\left(\frac{u_1^2}{m^2} + \frac{u_2^2}{J^2}\right)\\ &\quad + \bar{\mu}^1(\dot{q}_1 - v_1) + \bar{\mu}^2(\dot{q}_2 - v_2) + \bar{\mu}^3(\dot{q}_3 - v_3)\\ &\quad + \bar{\eta}^1(\dot{v}_1 - f_1) + \bar{\eta}^2(\dot{v}_2 - f_2) + \bar{\eta}^3(\dot{v}_3 - f_3)\,\mathrm{d}t,\end{aligned}$$

where, following traditional approaches to nonlinear optimal control theory, we take $\bar{\boldsymbol{\mu}} = \bar{\mu}^1 \mathrm{d}q_1 + \bar{\mu}^2 \mathrm{d}q_2 + \bar{\mu}^3 \mathrm{d}q_3$ and $\bar{\boldsymbol{\eta}} = \bar{\eta}^1 \mathrm{d}q_1 + \bar{\eta}^2 \mathrm{d}q_2 + \bar{\eta}^3 \mathrm{d}q_3$.

Taking variations of the cost functional we get

$$\begin{aligned}\delta\mathcal{J} &= \int_0^T \delta u_1 \left(\frac{u_1}{m^2} - \bar{\eta}^1 \frac{\partial f_1}{\partial u_1} - \bar{\eta}^2 \frac{\partial f_2}{\partial u_1}\right)\\ &\quad + \delta u_2 \left(\frac{u_2}{J^2} - \bar{\eta}^3 \frac{\partial f_3}{\partial u_3}\right)\\ &\quad - \delta q_1 \dot{\bar{\mu}}^1 - \delta q_2 \dot{\bar{\mu}}^2 - \delta q_3 \left(\dot{\bar{\mu}}^3 + \bar{\eta}^1 \frac{\partial f_1}{\partial q_3} + \bar{\eta}^2 \frac{\partial f_2}{\partial q_3}\right)\\ &\quad + \delta v_1 \left(-\bar{\mu}^1 - \dot{\bar{\eta}}^1 - \bar{\eta}^2 v_3\right)\\ &\quad + \delta v_2 \left(-\bar{\mu}^2 - \dot{\bar{\eta}}^2 + \bar{\eta}^1 v_3\right)\\ &\quad + \delta v_3 \left(-\bar{\mu}^3 - \dot{\bar{\eta}}^3 + \bar{\eta}^1 v_2 - \bar{\eta}^2 v_1\right).\end{aligned} \tag{37}$$

Setting $\delta\mathcal{J} = 0$ in equation (37) and by virtue of the independence of the variations $\delta u_j,\ \delta q_i,\ \delta v_i,\ j = 1, 2,\ j =$

$1, 2, 3$, we obtain the following necessary conditions

$$\begin{aligned}
u_1 &= m\left(\bar{\eta}^1 \cos q_3 + \bar{\eta}^2 \sin q_3\right) \\
u_2 &= J\bar{\eta}^3 \\
\dot{\bar{\mu}}^1 &= 0 \\
\dot{\bar{\mu}}^2 &= 0 \\
\dot{\bar{\mu}}^3 &= \frac{1}{2}\sin(2q_3)\left[\left(\bar{\eta}^1\right)^2 - \left(\bar{\eta}^2\right)^2\right] - \bar{\eta}^1\bar{\eta}^2\cos(2q_3) \\
\dot{\bar{\eta}}^1 &= -\bar{\mu}^1 - \bar{\eta}^2 v_3 \\
\dot{\bar{\eta}}^2 &= -\bar{\mu}^2 + \bar{\eta}^1 v_3 \\
\dot{\bar{\eta}}^3 &= -\bar{\mu}^3 + \bar{\eta}^1 v_2 - \bar{\eta}^2 v_1 .
\end{aligned} \tag{38}$$

We need to show that these equations are indeed equivalent to equations (32), (33) and (34). To do that, let $\bar{\boldsymbol{\mu}}$ and $\bar{\boldsymbol{\eta}}$ be related to $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ by

$$\begin{aligned}
\bar{\mu}^1 &= \mu^2 \cos q_3,\ \bar{\eta}^1 = \eta^2 \cos q_3 \\
\bar{\mu}^2 &= \mu^2 \sin q_3,\ \bar{\eta}^2 = \eta^2 \sin q_3 \\
\bar{\mu}^3 &= \mu^3,\ \bar{\eta}^3 = \eta^3 .
\end{aligned} \tag{39}$$

One can easily check that substituting these relationships into equations (38) and after simple algebraic manipulations, we obtain equations (32), (33) and (34). It is interesting to note the particularly simple form of the necessary conditions obtained using the affine connection approach when compared to the necessary conditions (38). At first glance, equations (38) may not appear to be solvable, whereas equations (32), (33) and (34) are clearly much easier to study.

## V. Conclusion

In this paper we used the theory of affine connections to study an optimal control problem for a class of nonholonomic, under-actuated mechanical systems. The cost function is the norm-squared of the control input exerted in moving the system from an initial to a terminal state under the assumption of controllability. We gave a brief overview of some facts from Riemannian geometry and the use of the nonholonomic connection to deriving the constrained equations of motion. We formulated an optimal control problem, where we used the nonholonomic affine connection together with Lagrange's multiplier method in the calculus of variations to derive the optimal necessary conditions. We gave a simple example on a three-dimensional manifold with a single nonholonomic constraint that captures the main features of the theoretical result. Future work will focus on the treatment of nonholonomic systems with symmetry, which naturally occur in robotic locomotion [1]. In particular, we are interested in the structure of the resulting optimality conditions and the possibility of existence of closed form extremals.

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