LOCALLY RICH COMPACT SETS

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ABSTRACT. We construct a compact metric space that has any other compact metric space as a tangent, with respect to the Gromov-Hausdorff distance, at all points. Furthermore, we give examples of compact sets in the Euclidean unit cube, that have almost any other compact set of the cube as a tangent at all points or just in a dense subset. Here the "almost all compact sets" means that the tangent collection contains a contracted image of any compact set of the cube and that the contraction ratios are uniformly bounded. In the Euclidean space, the distance of sub-sets is measured by the Hausdorff distance. Also the geometric properties and dimensions of such spaces and sets are studied.

1. INTRODUCTION

Tangent measures and sets give information of the local structure of a Radon measure or a compact set. These tangent objects often have more regular structure than the original object in question. For example, a tangent set of a totally disconnected self-affine carpet can contain intervals, see [4]. However, this is not always the case. O'Neil [28] constructed a Radon measure μ on $\mathbb{R}^d$ that has any other Radon measure of $\mathbb{R}^d$ as a tangent measure at μ almost all points. Furthermore, in his PhD thesis [27], he showed that this is a typical property of Radon measures. This was later re-proved by Sahlsten [29] by using a different method. Buczolich studied the (micro-)tangent sets of continuous functions and obtained results in a very similar fashion. He proved that the graph of typical continuous function on [0,1] has a graph of any other continuous function (on [-1,1] for which $f(0) = 0$) as a tangent set at $(x, f(x))$ for Lebesgue almost all $x \in (0,1)$, see [8, Theorem 5]. This work was later extended to the local maximum and minimum points of f by Buczolich and Ráti [9]. The dynamics of the "zooming in" operation or "scenery flow" has also been studied. For example, [5, 6, 7] study the scenery flow of Cantor and Julia sets. Furstenberg [16] studied the distribution of tangent measures in euclidean unit cube and this idea has been further developed in the study of typical tangent measure distributions, see [21, 22].

We study compact metric spaces and compact sub-sets of euclidean spaces, and their tangent properties. In our results, there are no measures involved. We call a compact metric space *locally rich*, if it has a "large" collection of tangent spaces at all points. A precise definition will be given in Definition 1.2. Our main result is that there exist locally rich spaces. We also show that locally richness is a typical property. A property P of points $x \in X$ is satisfied for *typical* $x \in X$ if the complement of set

$$\{x \in X : x \text{ satisfies } P\}$$

is of first category. A subset of a topological space X is of first category, if it is a countable union of sets whose closure in X has empty interior; otherwise it is called of second category.

The distance of compact sets K and F in a metric space (X, d_X) can be measured by the Hausdorff distance, defined by

$$d_H(K, F) = \inf\{\varepsilon : F \subset K^\varepsilon \text{ and } K \subset F^\varepsilon\},$$

where $K^\varepsilon = \{x \in X : \text{dist}(x, K) < \varepsilon\}$. One could highlight the ambient space by denoting d_H^X, but in this work the space in which distances are measured is always clear from the context, so we only write d_H. We use the standard notations $\dim_H, \underline{\dim}_M, \overline{\dim}_M$, and $\dim_A$ to denote the Hausdorff, lower Minkowski, upper Minkowski and Assouad dimension, respectively. For basic properties of these dimensions, we refer to [10, 25] and especially [24] for the Assouad dimension.

The paper is organized as follows. In Section 1, we present the basic definitions. In Sections 2 and 3 we study locally rich sets in metric spaces and Euclidean spaces respectively. In Section 4, we construct an infinitely generated self-similar set and consider its tangent sets. In Section 5, we study the geometric properties of locally rich sets. To finish the paper, we consider other ways to zoom sets.

1.1. **Tangents of a metric space.** The distance of different metric spaces is measured by the Gromov-Hausdorff distance. For references, see [1, 17, 19] or [20]. The idea is that the distance of metric spaces is measured by the infimum of the Hausdorff distances of their isometric images in l^∞, the space of bounded real valued sequences. This is possible by the Fréchet embedding theorem[1] [20, Theorem 3.1.11]. When only considering compact spaces, we could instead use embedding to $C([0, 1])$, the space of continuous functions on $[0, 1]$, by a theorem of Banach [2, Théorème 9, p. 185]. This space has the advantage of being separable. However, l^∞ is enough for us, since we do not need the structure of the ambient space. The Gromov-Hausdorff distance of metric spaces X and Y is defined as

$$d_{GH}(X, Y) = \inf_{i,j} d_H(i(X), j(Y)),$$

where the Hausdorff distance is considered in l^∞ and the infimum is taken over all isometric embeddings of X and Y to l^∞. This distance does not make a difference between isometric spaces and can, on the other hand, give the value ∞ for unbounded spaces. Therefore it is reasonable to consider only the isometry classes of non-empty compact metric spaces. A rather standard notation for this collection is $\mathcal{M}_C$. The following proposition about $(\mathcal{M}_C, d_{GH})$ is the base of our work on metric spaces.

Proposition 1.1. *The distance d_{GH} is a metric on $\mathcal{M}_C$. Furthermore, $(\mathcal{M}_C, d_{GH})$ is complete and separable, and the countable dense set can be chosen to be the set of finite spaces, where all distances are rational.*

The fact that $(\mathcal{M}_C, d_{GH})$ is a metric space, is proved in [1, Proposition 4.5.2], completeness in [1, Theorem 4.5.6], and separability in [19, Theorem 2.4]. Since $\mathcal{M}_C$ is complete, Baire's theorem tells that it is of second category, and we may talk about typicality.

Let (X, d_X) be a compact metric space and $x \in X$. We define the zooming map $T_{r,x}$ for all $x \in X$ and $r > 0$ by

$$T_{x,r}(X, d_X) = (B(x, r), r^{-1}d|_{B(x,r)}),$$

where $B(x, r)$ is the closed ball in the original metric of X. The metric is often clear from the context and then we may write $T_{x,r}(X)$ for short. We say that a compact metric space

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[1]This theorem is sometimes represented as a consequence of the Kuratowski embedding theorem. We learned from [20, Notes to chapter 3] that actually Fréchet [15] proved his theorem already in 1909, while Kuratowski's paper [23] appeared in 1935.

Y is a tangent of X at x if there exists a sequence $r_i \searrow 0$ so that

$$T_{x,r_i}(X) \xrightarrow{d_{GH}} Y$$

as $i \to \infty$. We denote the collection of all tangent spaces of X at x by $\mathrm{Tan}(X,x)$. Observe that $\mathrm{Tan}(X,x)$ is a closed set in $\mathcal{M}_C$. From the definition of $T_{x,r}$ it is clear that $\mathrm{diam}\, T_{x,r}(X) \leq 2$ for any X. On the other hand, this could be modified to any other constant by adding a constant scaling factor in front of r^{-1} in the definition of $T_{x,r}$. Therefore it is reasonable to focus on some class of compact metric spaces with a uniformly bounded diameter. Since our interest is in the geometry of the spaces and not so much on the diameter, we just choose to focus on the spaces of diameter at most one. Let us denote $\mathcal{M}_C(1) = \{K \in \mathcal{M}_C : \mathrm{diam}\, K \leq 1\}$. This sub-set is closed, and so it is also complete. It again follows, from the well known theorem of Baire, that $\mathcal{M}_C(1)$ is of second category.

Definition 1.2. A metric space $X \in \mathcal{M}_C(1)$ is called *locally rich* if $\mathrm{Tan}(X,x) = \mathcal{M}_C(1)$ for all $x \in X$.

1.2. **Tangent sets in Euclidean spaces.** When considering compact subsets of Euclidean space, we could of course measure distances with the Gromov-Hausdorff distance. Instead of working in l^∞, a more natural way would be to modify d_{GH} so that the infimum is taken over isometries of the Euclidean space. This is sometimes called the Euclidean Gromov-Hausdorff distance, and denoted by d_{EH}. It is obvious that $d_{GH} \leq d_{EH} \leq d_H$, so results concerning tangents obtained for d_H are also valid for d_{GH} and d_{EH}. Our aim is to obtain tangent results with respect to d_H, and when this is not possible we allow a small scaling and translation, to obtain the desired tangent results. With this in mind, we give following notations.

Let $Q = [-1,1]^d$ and $\mathcal{K}$ be the space of non-empty compact sub-sets of Q. Let $\mathcal{K}$ be endowed with the Hausdorff metric. It is fairly easy to show that $(\mathcal{K}, d_H)$ is a compact metric space. For a proof in a more general case, see [12, 2.10.21]. For sub-collections $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{K}$, we write $\mathcal{A} \approx \mathcal{B}$, if there exists $\lambda_0 > 0$, so that for every $B \in \mathcal{B}$, there exists a set $A \in \mathcal{A}$, a real number $\lambda_0 < \lambda < \lambda_0^{-1}$, a vector $a \in A$ and a vector $b \in B$, so that

$$\lambda(A - a) = B - b.$$

Note that this relation is an equivalence relation. Define mappings $\hat{T}_{x,t} \colon \mathbb{R}^d \to \mathbb{R}^d$, $\hat{T}_{x,t}(y) = \frac{y-x}{t}$ and $T_{x,t} \colon \mathcal{K} \to \mathcal{K}$,

$$T_{x,t}(E) = \hat{T}_{x,t}(E) \cap Q$$

for all $x \in Q$ and $t > 0$. We say that $F \in \mathcal{K}$ is a tangent set of $E \in \mathcal{K}$ at $x \in E$ if there exists a sequence $t_n \searrow 0$ so that

$$T_{x,t_n}(E) \xrightarrow{d_H} F.$$

Since there is no danger of misunderstanding, we use same notation $\mathrm{Tan}(E,x)$ to denote the collection of all tangent sets of a set E at x as we did for the collection of all tangent spaces of a metric space X at x. Due to compactness of $\mathcal{K}$, this collection is never empty.

Definition 1.3. A compact set $K \in \mathcal{K}$ is called *locally rich at* x if $\mathrm{Tan}(K,x) \approx \mathcal{K}$. It is called *locally rich* if $\mathrm{Tan}(K,x) \approx \mathcal{K}$ for all $x \in K$.

1.3. **Symbol space notations.** The symbol space notations will often be used in our constructions. For the reader's convenience, we summarize it here. Let $\Sigma = \prod_{i=1}^\infty X_i$, where X_i are fixed arbitrary sets. An element ω of Σ is a mapping $\omega \colon \mathbb{N} \to \bigcup_{i=1}^\infty X_i$ so that $\omega(n) \in X_n$ for all $n \in \mathbb{N}$. One can also think of $\omega \in \Sigma$ as a sequence $(\omega(1), \omega(2) \ldots)$. For any $n \in \mathbb{N}$ we denote $\Sigma_n = \prod_{i=1}^n X_i$. When $\omega \in \Sigma_n$ we denote $|\omega| = n$ and call

this n the length of ω. If $\omega \in \Sigma$ then the length of ω is infinite. If the length of ω is greater than n, then let $\omega|_n$ denote the restriction of ω to length n. That is $\omega|_n \in \Sigma_n$ and $\omega|_n(m) = \omega(m)$, for all $m \leq n$. When $\omega \neq \omega'$, we let $\omega \wedge \omega'$ denote the symbol $\omega|_N$, where $N = \max_{n \in \mathbb{N}}\{n : \omega|_n = \omega'|_n\}$. When the length of ω is n, we denote $[\omega] = \{\omega' \in \Sigma : \omega = \omega'|_n\}$, and call it a cylinder. Note that the cylinder consists of sequences of the form

$$\omega\omega' = (\omega(1), \omega(2), \ldots, \omega(n), \omega'(1), \omega'(2), \ldots)$$

for ω' of infinite length.

2. Locally rich metric spaces

Our main result is that there exists a locally rich metric space. In Theorem 2.2 we construct an example of such a space. We also study the dimensional properties of this space. Finally, Theorem 2.4 shows that typically compact metric spaces are locally rich.

By Proposition 1.1, we can choose a dense sequence $(\gamma(n))_{n=1}^\infty$ in $\mathcal{M}_C(1)$ so that each element is a finite space with rational distances. Set $\delta_n = \min\{d_{\gamma(n)}(x,y)\}$ and fix a sequence $r_n \searrow 0$ so that $r_1 = 1$ and $r_n < \delta_{n-1}$ for all $n \geq 2$. Let $\mathbf{r}(n) = \prod_{i=1}^n r_i$.

Heuristically, we want to scale the space $\gamma(2)$ and put it into a small neighborhood of each point of $\gamma(1)$ so that when zooming into any point of $\gamma(1)$ with an appropriate scale, one would approximately see $\gamma(2)$. Next we would scale $\gamma(3)$ into a smaller neighborhood of each point of the previous space so that an appropriate zoom gives a better approximation of $\gamma(3)$. It is possible to continue this way, but the lack of an ambient space, other than l^∞, forces us to define new metric spaces at each step. On the other hand, metric spaces are just points and distances, so it is possible to just give the distances, say in a matrix, in these finite cases. Finally one could wish that this sequence converges to a compact metric space, and that the limit space is locally rich. However, we do not actually do such a construction, since the simplest way is to just give the final compact metric space and then prove that is has the desired properties.

Set $\Sigma = \prod_{n=1}^\infty \gamma(n)$. For $\omega, \omega' \in \Sigma$, define

$$d_\Sigma(\omega, \omega') = \max_{n \in \mathbb{N}} \mathbf{r}(n) d_{\gamma(n)}(\omega(n), \omega'(n)).$$

The structure of Σ is illustrated in Figure 1. Recall that $r_{n+1} < d_{\gamma(n)}(x,y) \leq 1$ for all $x, y \in \gamma(n)$. It is straightforward to check that (Σ, d_Σ) is a metric space. Since $\gamma(n) \in \mathcal{M}_C(1)$ for all $n \in \mathbb{N}$, we have $\mathrm{diam}(\Sigma) = \mathrm{diam}(\gamma(1)) \leq 1$. Note also that since $r_{n+1} < \delta_n$, the definition of the metric is equivalent to $d_\Sigma(\omega, \omega') = \mathbf{r}(n) d_{\gamma(n)}(\omega(n), \omega'(n))$, where $n = |\omega \wedge \omega'| + 1$. Also, it is clear that (Σ, d_Σ) is compact. It is even possible to construct a converging sub-sequence by hand from a given sequence, since each element of the product is finite. On the other hand, one could argue that the metric d_Σ gives the standard product topology and therefore (Σ, d_Σ) is compact by Tihonov's theorem. Now all that is left is to show that (Σ, d_Σ) is locally rich. This is done in Theorem 2.2, but first we give an approximation lemma.

Lemma 2.1. *For all $\omega \in \Sigma$, we have $d_{GH}(T_{\omega, \mathbf{r}(n)}(\Sigma), \gamma(n)) \leq r_{n+1}$.*

Proof. For any $\omega \in \Sigma$ the space $T_{\omega, \mathbf{r}(n)}(\Sigma)$ equals to $[\omega|_{n-1}]$ with the metric $\mathbf{r}(n)^{-1}d_\Sigma$ (restricted to $[\omega|_{n-1}]$). Thus, the set

$$\{\omega|_{n-1}x\alpha : x \in \gamma(n)\} \subset T_{\omega, \mathbf{r}(n)}(\Sigma)$$

is isometric to $\gamma(n)$, for any $\alpha \in \prod_{i=n+1}^\infty \gamma(i)$. For fixed α, the isometry is given by $x \mapsto \omega|_{n-1}x\alpha$, since

$$d_{T_{\omega, \mathbf{r}(n)}(\Sigma)}(\omega|_{n-1}x\alpha, \omega|_{n-1}y\alpha) = \mathbf{r}(n)^{-1}d_\Sigma(\omega|_{n-1}x\alpha, \omega|_{n-1}y\alpha) = d_{\gamma(n)}(x,y).$$
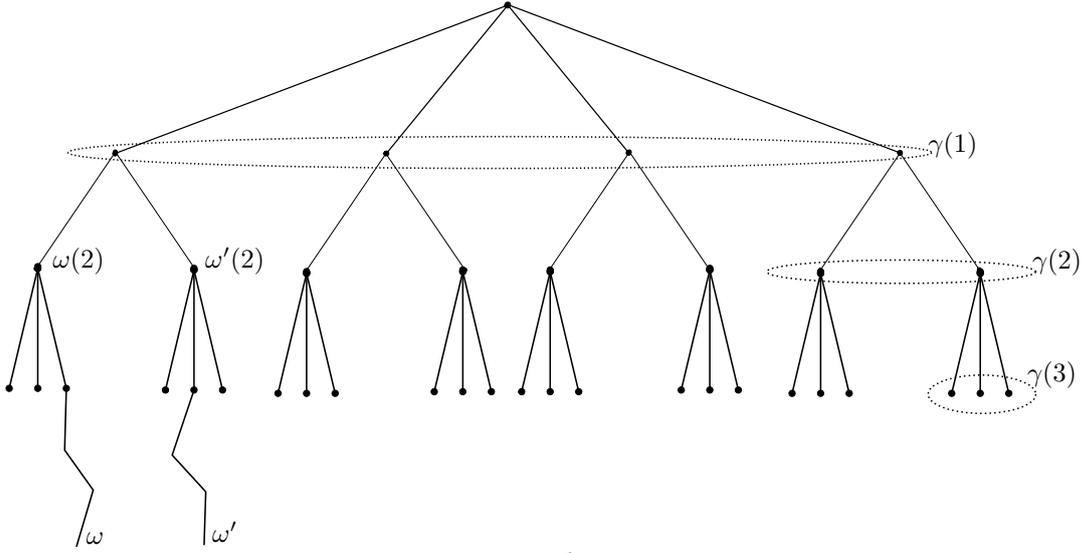
FIGURE 1. The space Σ is a tree, where the number of offspring in each vertex at level n is $\#\gamma(n+1)$. Here the distance between ω and ω' equals to $\mathbf{r}(2)d_{\gamma(2)}(\omega(2),\omega'(2))$.

Letting i denote this isometry in question, we clearly have

$$d_H\big(T_{\omega,\mathbf{r}(n)}(\Sigma), i(\gamma(n))\big) \leq r_{n+1}$$

and so $d_{GH}(T_{\omega,\mathbf{r}(n)}(\Sigma), \gamma(n)) \leq r_{n+1}$. □

Theorem 2.2. *For all $\omega \in \Sigma$, we have $\mathrm{Tan}(\Sigma,\omega) = \mathcal{M}_C(1)$.*

Proof. Let $K \in \mathcal{M}_C(1)$ and $\omega \in \Sigma$. For any ε, we find an integer n, in fact infinitely many n, so that $d_{GH}(K,\gamma(n)) \leq \varepsilon$ and $r_{n+1} \leq \varepsilon$. By Lemma 2.1, we have

$$d_{GH}(T_{\omega,\mathbf{r}(n)}(\Sigma), K) \leq d_{GH}(T_{\omega,\mathbf{r}(n)}(\Sigma), \gamma(n)) + d_{GH}(K,\gamma(n)) \leq 2\varepsilon.$$

Thus K is a tangent set of Σ at ω. Thus we complete the proof by the arbitrary choice of K and ω. □

Remark 2.3. In order to tangent out whole $\mathcal{M}_C$ from Σ, instead of just $\mathcal{M}_C(1)$, we could modify the definition of a tangent space by allowing a finite scaling of the metric at each step. This is similar to the definition of a tangent measure. On the other hand, we could concentrate on the similarity classes of metric spaces. Meaning that two spaces are equivalent if they are the same up to isometry and scaling of the metric with a constant.

In the next theorem, we show that a typical compact metric space is locally rich. It is interesting that this somewhat weird and irregular property is, in terms of category, very common, but this seems to be the nature of (Baire) typical objects.

Theorem 2.4. *A typical point of $(\mathcal{M}_C(1), d_{GH})$ is a locally rich space.*

Proof. "Similarly" as before, we set $\gamma(n,k) = \gamma(n) \times \gamma(k)$ and endow this space with a metric

$$d_{n,k}(\alpha,\beta) = \max\{d_{\gamma(n)}(\alpha(1),\beta(1)), r_{n+1}d_{\gamma(k)}(\alpha(2),\beta(2))\},$$

for $\alpha = (\alpha(1),\alpha(2))$ and $\beta = (\beta(1),\beta(2)) \in \gamma(n,k)$. Recall that $r_{n+1} < \delta_n$. Fix a sequence $\varepsilon_n \searrow 0$ so that $r_{n+1} + 2\varepsilon_n < \delta_n$ and denote

$$(2.1) \qquad\qquad \mathcal{G}(n,k) := U_{d_{GH}}(\gamma(n,k), r_{n+1}\epsilon_n),$$

an open ball in $\mathcal{M}_C(1)$ with center $\gamma(n,k)$ and radius $r_{n+1}\epsilon_n$. It is not hard to see that $\bigcup_{n=N}^{\infty} \mathcal{G}(n,k)$ is open and dense subset of $\mathcal{M}_C(1)$ for any $N \in \mathbb{N}$. It is open, since it is a union of open balls. The denseness follows, since the sequence $\gamma(n)$ is dense and $r_n \searrow 0$. Let $\mathcal{G}(k) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \mathcal{G}(n,k)$, then if $X \in \mathcal{G}(k)$ there are infinitely many $n \in \mathbb{N}$ so that zooming to $x \in X$ with scale $r_{n+1} + 2r_{n+1}\varepsilon_n$ gives an $r_{n+1}\epsilon_n/(r_{n+1} + 2r_{n+1}\varepsilon_n)$-approximation of the metric space $(\gamma(k), r_{n+1}(r_{n+1} + 2r_{n+1}\epsilon_n)^{-1}d_{\gamma(k)})$. Since

$$\frac{r_{n+1}\epsilon_n}{r_{n+1} + 2r_{n+1}\varepsilon_n} \to 0 \qquad \text{and} \qquad \frac{r_{n+1}}{r_{n+1} + 2r_{n+1}\epsilon_n} \to 1$$

as $n \to \infty$, one can tangent out $\gamma(k)$ at all points of X. Thus from every element of $\bigcap_{k=1}^{\infty} \mathcal{G}(k)$ one can tangent out any $\gamma(k)$ at all points. By the observation in the introduction, $\mathrm{Tan}(X,x)$ is closed so any $X \in \bigcap_{k=1}^{\infty} \mathcal{G}(k)$ is locally rich.

In the end we notice that the complement of $\bigcap_{k=1}^{\infty} \mathcal{G}(k)$ is of first category since $\mathcal{G}(n,k)$ are open and dense sets in a complete metric space $\mathcal{M}_C(1)$. $\square$

2.1. **Properties of Σ.** A metric space X is called *locally doubling* if there exits a constant $N(x) \in \mathbb{N}$, so that every ball $B(x,r) \subset X$ can be covered by at most $N(x)$ balls of radius $r/2$, where $N(x)$ does not depend on r. If one can choose a uniform N for all $x \in X$, then X is called *doubling*. As one might guess, (Σ, d_Σ) is not locally doubling anywhere. Intuitively the reason is that in small scales, Σ contains approximations of spaces with larger and larger doubling constants.

Remark 2.5. A locally rich space is not locally doubling anywhere, and thus has Assouad dimension equal to ∞.

Proof. The latter claim follows by from the first one, see [19, Proposition 1.15]. Assume on the contrary that a locally rich space (X, d_X) is locally doubling at x with constant N. Let K be a metric space of $N+1$ elements with all distances equal to 1. Since $K \in \mathrm{Tan}(X,x)$, we find sequence $t_n \searrow 0$, for which $\#T_{x,t_n}(X) \geq N+1$ and a set $A \subset T_{x,t_n}(X)$ with $\#A = N+1$ and $d_{T_{x,t_n}(X)}(x,y) > 3/4$ for all $x,y \in A$. Therefore, it takes at least $N+1$ balls of radius $2^{-1}t_n$ to cover the ball $B(x,t_n)$. $\square$

Remark 2.6. From Remark 2.5 one can see that $\dim_A \Sigma = \infty$ regardless of the choice of the sequence (r_n) in the construction. It turns out that the other dimensions depend heavily on (r_n). For example, if $r_n \leq (\#\gamma(n))^{-n}$, we have that

$$\frac{\log N(\Sigma, \mathbf{r}(n))}{-\log \mathbf{r}(n)} \leq \frac{\log \prod_{i=1}^{n} \#\gamma(i)}{\log \prod_{i=1}^{n} (\#\gamma(i))^i} \to 0,$$

where $N(\Sigma, \mathbf{r}(n))$ denotes the minimal number of balls with radius $\mathbf{r}(n)$ needed to cover Σ. This implies that $\underline{\dim}_M \Sigma = 0$.

We do not know how to estimate the Hausdorff dimension of Σ from below. One way could be to consider the lower local dimension of a Borel measure on Σ. For definitions and properties of local dimensions of measures, see [11]. Let μ be the unique measure satisfying $\mu[\omega|_n] = \prod_{i=1}^{n} p_i(\omega(i))$ where $p_i(x)$ are positive real numbers for each $x \in X_i$ satisfying $\sum_{x \in X_i} p_i(x) = 1$ for all $i \in \mathbb{N}$. Choosing $\mathbf{r}(n+1) < r \leq \mathbf{r}(n)$ gives that

$$\frac{\log \mu B(\omega, r)}{\log r} \geq \frac{\log \mu B(\omega, \mathbf{r}(n))}{\log \mathbf{r}(n+1)} = \frac{\log \mu[\omega|_n]}{\log \mathbf{r}(n+1)} = \frac{\log \prod_{i=1}^{n} p_i(\omega(i))}{\log \mathbf{r}(n+1)}.$$

Choosing the weights $p_n(x)$ and the sequence (r_n) fixes the estimate for the lower local dimension of μ for all ω and therefore also for $\dim_H \Sigma$. However, it is not clear how to choose (r_n) large enough. The conditions are that $r_n \searrow 0$ and $r_n < \delta_{n-1}$, the minimum of distances between points of $\gamma(n-1)$ and it takes a lot more detailed construction to get

a grip of δ_n. On the other hand in Theorem 5.3, we show that a set E in euclidean space, might have positive Hausdorff dimension, even if one can tangent out $\{0\}$ from each $x \in E$. Therefore we conjecture that a locally rich space can have positive Hausdorff dimension.

3. Locally rich sets in Euclidean spaces

In this section, we show that there exists locally rich sets in Q. Theorem 3.2 gives an example of such a space, and in Theorem 3.6 we show that this is a typical property of compact sets. The construction is of course similar to the construction of the locally rich metric space, but the geometry of $\mathbb{R}^d$ and the definition of d_H give some technical differences. For example, there needs to be enough space between the construction pieces. Also, in any compact set K, there are always points x for which $T_{x,r}(K)$ is at least half empty.

Let us fix the notations

$$\mathcal{K}_0 = \{K \subset Q : K \text{ compact and } 0 \in K\}$$

$$\mathcal{S} = \{\bigcup_{i=0}^{N-1} x_i : x_i \in \mathbb{Q}^d \cap (-1,1)^d, x_0 = 0, N < \infty\}.$$

Clearly $\mathcal{S}$ is countable and dense in $\mathcal{K}_0$. Choose a sequence $(\gamma_i)_{i=1}^{\infty} \subset \mathcal{S}$, so that each element of $\mathcal{S}$ occurs infinitely many times. Note that each γ_i is finite. Let

(3.1) $$\delta_n = \min\{|x-y| : x \in \gamma_n, y \in \gamma_n \cup \partial Q \text{ and } x \neq y\}.$$

Fix a sequence (r_n) with $r_1 = 1$, $r_i \searrow 0$ and let $\mathtt{r}(n) = \prod_{i=1}^{n} r_i$.

Theorem 3.1. *There exists a compact set X with the property that $\mathrm{Tan}(X,x) = \mathcal{K}_0$ for all x in a dense subset of X.*

Notice that for any compact set $E \in \mathcal{K}$, we have $\mathrm{Tan}(E,x) \neq \mathcal{K}_0$ at the "boundary" point of E. Here the "boundary" of E is the set $E \cap \partial Q'$ where Q' is the smallest cube that contain E and $\partial Q'$ is the boundary of Q'. Since every element of $\mathrm{Tan}(X,x)$ contains the original point zero, the key in obtaining Theorem 3.2 is that we do not try to get $\mathrm{Tan}(X,x) = \mathcal{K}$ but only $\mathrm{Tan}(X,x) \approx \mathcal{K}$.

Theorem 3.2. *There exists a compact set X with the property that $\mathrm{Tan}(X,x) \approx \mathcal{K}$ for all $x \in X$.*

Theorem 3.1 is similar to [28, Theorem 3], but does not follow directly from that, since generally the convergence of measures does not imply the convergence of the supports of the measures. Not even in the case when, the support of the limit measure is finite, see Example 3.4. We may nevertheless use the construction of [28], we just need to check the convergence of compact sets instead of the weak convergence of measures. Theorem 3.2 generalizes Theorem 3.1. It is rather surprising that this tangent property holds for all points, since it not possible for measures, see [29, Proposition 5.1].

Let us now construct the compact set in question in Theorems 3.1 and 3.2. Unlike in the construction of a locally rich metric space, we require that

(3.2) $$8\sqrt{d}r_{n+1} \leq \delta_n,$$

to guarantee that we have enough empty space in between the screens that we are focusing on. Here d is the dimension of the ambient space. When $\mathtt{i} \in \Sigma = \prod_{i=1}^{\infty} \gamma_i$, let $\mathtt{i}(n)$ denote the n:th coordinate of $\mathtt{i}$. Define the projection $\pi \colon \Sigma \to Q$ by setting

$$\{\pi\mathtt{i}\} = \bigcap_{n=1}^{\infty} \left(\mathtt{r}(n+1)Q + \sum_{k=1}^{n} \mathtt{r}(k)\mathtt{i}(k) \right).$$

Finally, let μ be a Bernoulli measure on Σ (e.g. $\mu[\mathtt{i}] = \prod_{n=1}^{|\mathtt{i}|}(\#\gamma(n))^{-1}$). For each $E \in \mathcal{S}$ fix a sequence $e(n)$ so that $\gamma_{e(n)} = E$ and define

$$V_E = \{\mathtt{i} \in \Sigma : \mathtt{i}(e(n)) = 0 \text{ infinitely often}\}.$$

A Borel-Cantelli type argument [28, Lemma 4] shows that $\mu(\bigcap_{E \in \mathcal{S}} V_E) = 1$. The set $\pi\Sigma$ is compact since Σ is compact and π is continuous. The assumption (3.2) ensures that π is one to one.

Lemma 3.3. *If* $\mathtt{i} \in \bigcap_{E \in \mathcal{S}} V_E$*, then* $\mathcal{S} \subset \operatorname{Tan}(\pi\Sigma, \pi\mathtt{i})$*.*

Proof. Let $E \in \mathcal{S}$ and $\mathtt{i} \in \bigcap_{E \in \mathcal{S}} V_E$. Let $k(n)$ be a sub-sequence of $e(n)$ so that $\mathtt{i}(k(n)) = 0$ for all $n \in \mathbb{N}$. Since $k(n) = 0$ we have that $|\pi\mathtt{i} - \pi\mathtt{i}|_{k(j)}\mathtt{j}| \leq r_{k(n)+1}$, where $\mathtt{j} = (0, 0 \ldots)$. Since E is finite and $E \subset (-1, 1)^d$ we have that for large n that

$$d_H(T_{\pi\mathtt{i}, \mathtt{r}_{k(n)}}(\pi\Sigma), E) \leq 2r_{k(n)+1}.$$

This proves the claim, since $r_n \to 0$ as $n \to \infty$. $\qquad\square$

Proof of Theorem 3.1. Set $X = \pi\Sigma$ and $m = \pi\mu$. It is obvious that $\operatorname{Tan}(X, x) \subset \mathcal{K}_0$ for all $x \in X$. By combining Lemma 3.3 and the fact that tangent collections are closed, we get that $\operatorname{Tan}(X, x) \supset \mathcal{K}_0$ for m almost all x. It is clear that any set A with $m(A) = 1$ is dense in X. $\qquad\square$

Proof of Theorem 3.2. Set $X = \pi\Sigma$. Let $E \in \mathcal{K}$ and $x = \pi\mathtt{i}$. Fix a sequence $k(n)$ so that $\gamma_{k(n)} \to E$ in the Hausdorff metric. Denote $\mathtt{i}(n) = i_n$. Since the origin is in every $\gamma(n)$, we can define $\mathtt{j}_n = (i_1, \ldots, i_{n-1}, 0, 0, \ldots)$. Consider the point $T_{\pi\mathtt{j}_{k(n)}, \mathtt{r}_{k(n)}}(x) =: x_n$, see Figure 2. This point is in Q so the sequence (x_n) has a converging sub-sequence. Let $j(n)$ be so that $x_{k(j(n))}$ converges to $\overline{x} \in Q$. By the choice of $k(n)$,

$$T_{\pi\mathtt{j}_{k(j(n))}, \mathtt{r}_{k(j(n))}}(X) \xrightarrow{d_H} E.$$

By this and the assumption (3.2), we have that

$$T_{\pi\mathtt{i}, 2\mathtt{r}_{k(j(n))}}(X) \xrightarrow{d_H} aE + b,$$

where $a = \frac{1}{2}$ and $b \in Q$ is so that $aE + b \subset Q$. $\qquad\square$

Example 3.4. Let δ_x denote the Dirac measure at x. Fix $x \in Q^d$ and let Let $\mu_i = (N_i)^{-1}(\delta_x + \delta_0 + \sum_{k=1}^{N_i-2} \delta_{x(i,k)})$, where $x(i, k)$ are arbitrary points with $\|x(i, k)\| \leq i^{-1}$ and $x(i, k) \neq x(i, l)$ for all $k \neq l$. If $N_i \to \infty$, then it is clear that $\mu_i \to \delta_0$ but $\operatorname{spt}\mu_i \to \{0\} \cup \{x\}$.

Remark 3.5. As in remark 2.6, we can modify this construction, namely by modifying the sequence r_i, so that the set $\pi\Sigma$ has Hausdorff dimension zero. By a similar technique we can also adjust the construction of O'Neil, such that $\dim_H \mu = 0$, where μ is the measure of [28, Theorem 3]. Here $\dim_H \mu$ is the Hausdorff dimension of μ as defined in [11, (10.8)]. This is sometimes called the lower Hausdorff dimension of μ.

Theorem 3.6. *A typical compact set of* $\mathcal{K}$ *is locally rich.*

For each $n \in \mathbb{N}$, decompose the cube Q into 3nd disjoint $2 \cdot 3^{-n}$-adic subcubes (note that $Q = [-1, 1]^d$). Let $\mathcal{D}_n$ be the union of these 3nd subcubes. Define $C_n := \cup_{C \in \mathcal{D}_n}\{x_c\}$ where x_c is the center point of the sub-cube C. Recall the definition of γ_n and δ_n at the beginning of this section. Let $A_n \subset C_n$ and define $A_{n,k} := \cup_{a \in A_n}(a + 3^{-(n+1)}\gamma_k)$. It is easy to see that $A_{n,k} \in \mathcal{K}$. Let

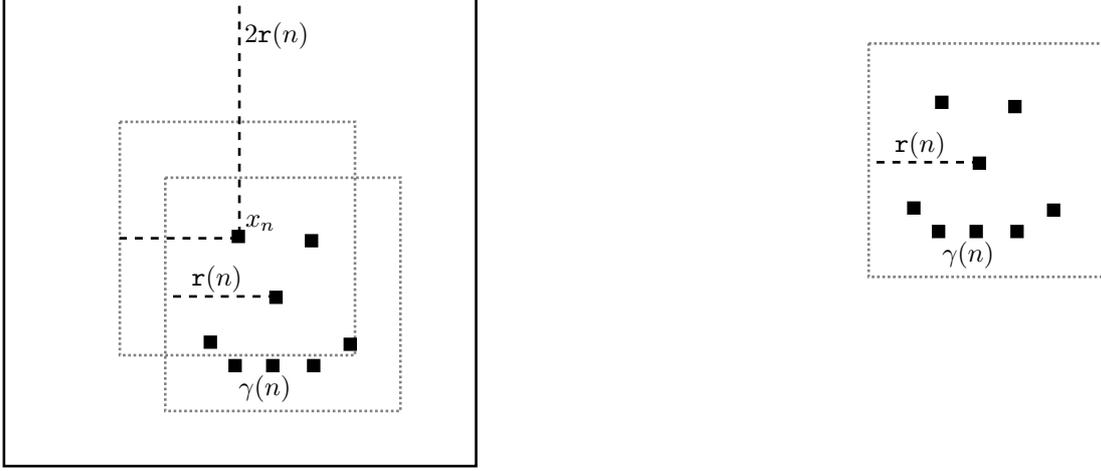$$(3.3) \qquad \mathcal{G}(n, k) := \bigcup_{A_n \subset C_n} U_{\mathcal{K}}(A_{n,k}, 3^{-2n}\delta_k),$$

FIGURE 2. The point x_n is not in the middle of the screen, so one must use a doubled screen to see the set $\gamma(n)$. The assumption (3.2) ensures that there is enough space to do so.

where $U_{\mathcal{K}}(A_{n,k}, 3^{-2n}\delta_k)$ is the open ball of $\mathcal{K}$ with center $A_{n,k}$ and radius $3^{-2n}\delta_k$.

Lemma 3.7. *Let $E \in \mathcal{G}(n,k)$, then for each $x \in E$, there exist vector $b(x) \in Q$, such that*

$$(3.4) \qquad d_H(T_{x, 2 \cdot 3^{-(n+1)}}(E), \frac{1}{2}\gamma_k + b(x)) < 3^{-n}\delta_k.$$

Proof. Since $E \in \mathcal{G}(n,k)$, so there exist $A \subset C_n$, such that $E \in U_{\mathcal{K}}(A, 3^{-2(n+1)}\delta_k)$. For each $x \in E$, there exist $a \in A$ such that $Q(a, 3^{-(n+1)}) \subset Q(x, 2 \cdot 3^{-(n+1)}) \subset Q(x, 3^{-n})$. Thus there is $b(x) \in Q$, such that $T_{x, 2 \cdot 3^{-(n+1)}}(Q(a, 3^{-(n+1)})) = \frac{1}{2}Q + b(x) \subset Q$. By the structure of $\mathcal{G}(n,k)$, we see that $E \cap Q(x, 3^{-(n+1)}2) = E \cap Q(a, 3^{-(n+1)})$. Thus by the elementary geometry we arrive at the estimate. $\square$

Proof of Theorem 3.6. For any k and N, the set $\bigcup_{n=N}^{\infty} \mathcal{G}(n,k)$ is open. It is also dense, since any compact set can be approximated by $2 \cdot 3^{nd}$-adic cubes. Let

$$\mathcal{G}_k := \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \mathcal{G}(n,k),$$

which is a countable intersection of dense open sets. Thus the complement of $\mathcal{G}_k$ is of first category. Furthermore, let $E \in \mathcal{G}_k$, then $E \in \mathcal{G}(n,k)$ happens infinitely often. By the Lemma 3.7. Thus for each $x \in E$, we can find vector $c(x) \in Q$ such that $\{\frac{1}{2}\gamma_k + c(x)\} \in \text{Tan}(E, x)$.
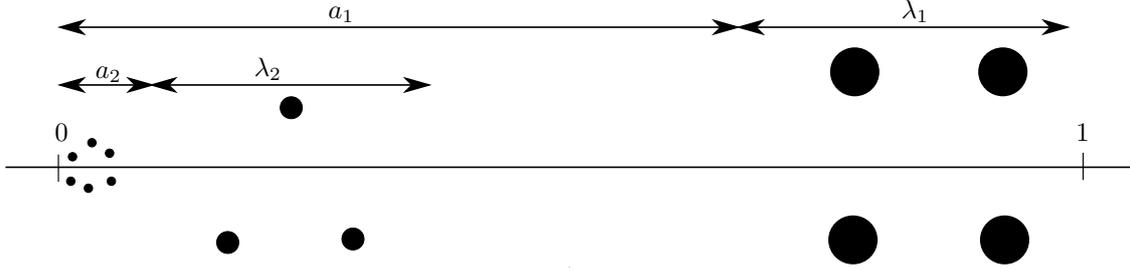
By the above discussion, we have that the complement of $\bigcap_{k=1}^{\infty} \mathcal{G}_k$ is of first category and for each $E \in \bigcap_{k=1}^{\infty} \mathcal{G}_k$, we have $\text{Tan}(E, x) \approx \mathcal{K}$ for all $x \in E$. It means that every element of $\bigcap_{k=1}^{\infty} \mathcal{G}_k$ is locally a rich set, thus we have finished the proof. $\square$

Remark 3.8. Define $\mathcal{G}(n,k)$ by

$$(3.5) \qquad \widetilde{\mathcal{G}}(n,k) := \bigcup_{A \subset C_n} U_{\mathcal{K}}(A, (\#\gamma_k)^{-n} 3^{-n^2}),$$

where $(\#\gamma_k)$ means the cardinality of γ_k. Let

$$\widetilde{\mathcal{G}} := \bigcap_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \widetilde{\mathcal{G}}(n,k).$$

FIGURE 3. First level approximation of the set C_∞

By the same argument as above, we have that the complement of $\widetilde{\mathcal{G}}$ is of first category. Notice that for every $E \in \widetilde{\mathcal{G}}(n,k)$, we can find at most $(\#\gamma_k)3^n$ balls with radius $(\#\gamma_k)^{-n}3^{-n^2}$ so that they cover E. It implies that every element E of $\widetilde{\mathcal{G}}$ satisfies $\underline{\dim}_M E = 0$. Thus a typical compact set of $\mathcal{K}$ has zero lower box-counting dimension. The result that typically a compact has zero lower box-counting dimension, was proved earlier in [13], by a different method.

4. LOCALLY RICH INFINITELY GENERATED SELF-SIMILAR SET

Here we consider the sub-space $\mathcal{K}_0^+$ of $\mathcal{K}$, where

$$\mathcal{K}_0^+ = \{K \in \mathcal{K} : 0 \in K \text{ and } x_1 \geq 0 \text{ for all } x \in K\}.$$

It is clear that $\mathcal{K}_0^+ \approx \mathcal{K}$ and $\mathcal{K}_0^+$ is also a separable metric metric space. As in the previous constructions, we choose a countable dense set $\{\gamma_n\}_{n=1}^\infty$ in $\mathcal{K}_0^+$. Moreover we require that each set γ_n is a finite union points and $0 \in \gamma_n$. Fix sequences $a_n \searrow 0$ and $\lambda_n \searrow 0$ with the following properties

(1) $a_1 + \lambda_1 \leq 1$
(2) $a_{n+1} + \lambda_{n+1} < \frac{a_n}{n}$ for all $n \in \mathbb{N}$
(3) $\frac{a_n}{\lambda_n} \to 0$ as $n \to \infty$

For example, one could choose $a_n = 2^{-n^2}$ and $\lambda_n = na_n$ for all $n \in \mathbb{N}$. By setting $\widetilde{\gamma}_n = \vec{a}_n + \lambda_n\gamma_n$, where $\vec{a}_n = (a_n, 0, 0, \ldots)$, we have $\widetilde{\gamma}_n \subset Q$ for all $n \in \mathbb{N}$ and

$$C_0 = \{0\} \cup \bigcup_{n=1}^\infty \widetilde{\gamma}_n.$$

The set C_0 is made of the elements of $(\gamma_n)_{n=1}^\infty$ scaled with λ_n and positioned so that the leftmost point of the n:th piece on the x_1-axis is a_n. The condition (1) ensures that the construction stays in Q and (2) gives the separation of the pieces. Obviously zero is the only clustering point of C_0, so C_0 is compact. The next theorem shows that the tangent space of C_0 at zero is large, as might be expected.

Theorem 4.1. *We have* $\mathrm{Tan}(C_0, 0) = \mathcal{K}_0^+$.

Proof. It is obvious that $\mathrm{Tan}(C_0, 0) \subset \mathcal{K}_0^+$. For any $K \in \mathcal{K}_0^+$, there exist a sub-sequence $\{\gamma_{n(k)}\}_{k=1}^\infty$, such that $\gamma_{n(k)} \xrightarrow{d_H} K$ as $k \to \infty$. By choosing real numbers $t_{n(k)} = a_{n(k)} + \lambda_{n(k)}$, we get that $T_{0, t_{n(k)}}(C_0) = A_{n(k)} \cup B_{n(k)}$, where

$$A_{n(k)} = \frac{1}{t_{n(k)}}\left(\{0\} \cup \bigcup_{m=n(k)+1}^\infty \widetilde{\gamma}_m\right) \quad \text{and} \quad B_{n(k)} = \frac{1}{t_{n(k)}}(\vec{a}_{n(k)} + \lambda_{n(k)}\gamma_{n(k)}).$$

By the triangle inequality, we have

$$d_H(A_{n(k)} \cup B_{n(k)}, K) \leq d_H(A_{n(k)} \cup B_{n(k)}, B_{n(k)}) + d_H(B_{n(k)}, K) \leq \frac{a_{n(k)}}{t_{n(k)}} + d_H(B_{n(k)}, K).$$

Thus by the choice of the sequences (a_n) and (λ_n), we have $T_{0,t_{n(k)}}(C_0) \xrightarrow{d_H} K$. Thus the result follows. $\qquad\square$

4.1. **Infinitely generated self-similar set.** Since each $\widetilde{\gamma}_n$ is a finite set of points we may consider a countable set of similitude contractions, so that each point of $\widetilde{\gamma}_n$ is the image of zero under one of these contractions. More precisely, denote $\widetilde{\delta}_n = (8\sqrt{d})^{-1} \min\{|x-y| : x, y \in \gamma_n, x \neq y\}$ and set

$$f_{n,m}(x) = \lambda_n \widetilde{\delta}_n \varepsilon_n x + \xi_{n,m} \tag{4.1}$$

for $n \geq 1, n \in \mathbb{N}$ and $1 \leq m \leq \#\gamma_n$, where $\varepsilon_1 \leq 1/2$ and $\varepsilon_n \searrow 0$ and the translations $\xi_{n,m}$ are fixed so that the union of $f_{n,m}(0)$ over m equals $\widetilde{\gamma}_n$. We add the mapping $f_{0,1} \equiv 0$. Note that for fixed n, the mappings $f_{n,m}$ have the same contraction ratio $r_{(n,m)}$. Usually, we denote this number by r_n for short ($r_{(0,1)} = 0$).

Now we may set $C_1 = \bigcup_{n=0}^{\infty} \bigcup_{m=1}^{\#\gamma_n} f_{n,m}(Q)$, and then recursively set

$$C_k = \bigcup_{n=0}^{\infty} \bigcup_{m=1}^{\#\gamma_n} f_{n,m}(C_{k-1})$$

for all $k \geq 2$. Now the sets C_k are compact and $C_k \subset C_{k-1}$ for all $k \geq 2$. Therefore we may set

$$C_{\infty} = \bigcap_{k=1}^{\infty} C_k.$$

Obviously, C_{∞} is compact. The sequence (ε_n) ensures that $\mathrm{Tan}(C_1, 0) = \mathrm{Tan}(C_0, 0)$ and thus $\mathrm{Tan}(K, 0) = \mathrm{Tan}(C_0, 0)$ for any $C_0 \subset K \subset C_1$. Especially this applies to C_{∞}.

By setting

$$I = \{(n,m) : n \geq 1, 1 \leq m \leq \#\gamma_n\}, \tag{4.2}$$

and the elements of $I^{\mathbb{N}}$ and $I^* := \bigcup_{n=1}^{\infty} I^n$ are called symbols. We denote the length of $\mathtt{i} \in I^*$ by $|\mathtt{i}|$. The length of symbols in $I^{\mathbb{N}}$ is infinite. If $n \leq |\mathtt{i}|$, we denote $\mathtt{i}|_n = (i_1, \ldots, i_n)$. We can consider the infinitely generated self-similar IFS where the mappings $f_{n,m}$ are from the construction of C_{∞}. Define the projection mapping $\pi \colon I^{\mathbb{N}} \to Q$ by

$$\{\pi(\mathtt{i})\} = \bigcap_{n=1}^{\infty} f_{\mathtt{i}|_n}(Q),$$

where $f_{\mathtt{i}|_n} = f_{i_1} \circ \cdots \circ f_{i_n}$. Similarly, we denote $r_{\mathtt{i}|_n} = r_{i_1} \cdots r_{i_n}$. It is clear that the limit set $\bigcup_{\mathtt{i} \in I^{\mathbb{N}}} \{\pi(\mathtt{i})\} =: F$ is a subset of C_{∞}. Since C_{∞} is compact, we also have that $\overline{F} \subset C_{\infty}$. On the other hand, let $x \in S$ where

$$S = \{x \in C_{\infty} : x = f_{\mathtt{i}}(0) \text{ for some } \mathtt{i} \in I^*\}, \tag{4.3}$$

and let $\mathtt{j} \in I^{\mathbb{N}}$ be arbitrary. It is clear from the construction of the mappings that $\pi((n,1)\mathtt{j}) \to 0$ as $n \to \infty$. Thus we have $\pi(\mathtt{i}(n,1)\mathtt{j}) \to x$ and so $S \subset \overline{F}$. Since S is dense in C_{∞} we also have that $C_{\infty} \subset \overline{F}$. Now we have that the set C_{∞} is the closure of a limit set of an infinitely generated self-similar IFS. This gives us tools for estimating the dimensions of C_{∞}. By [26, Corollary 3.17], we know that

$$\dim_{\mathrm{H}} F = \inf\{s > 0 : \sum_{i \in I} r_i^s < 1\}.$$

This gives that $\dim_{\mathrm{H}} F > 0$ by the reason that $\sum_{i \in I} r_i^s > 1$ whence s is small. Since $F \subset C_\infty$, so we have $\dim_{\mathrm{H}} C_\infty > 0$. Also, [26, Theorem 3.1] gives that $\dim_{\mathrm{P}} F = \dim_{\mathrm{P}} C_\infty = \overline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} C_\infty$.

4.2. **Tangent properties of C_∞.** Here we investigate the tangent sets of C_∞. Theorem 4.2 shows that the tangent space of C_∞ is large in a dense subset. For finitely generated self-similar sets, the tangent sets should are similar to the set itself [3, Theorem and Remarks]. For infinitely generated sets we have a similar result for "finitely generated points", meaning points of the form $\pi \mathtt{i}$ with $\mathtt{i} \in I_N^{\mathbb{N}} = \{(n, m_n) \in I : n \leq N\}^{\mathbb{N}}$, see Theorem 4.3. In Remark 4.4 and Theorem 4.5, we consider the "infinitely generated points" and especially the sequences that grow fast enough.

Theorem 4.2. *Let S be as in (4.3). Then S is dense in C_∞ and $\mathrm{Tan}(C_\infty, x) = \mathcal{K}_0^+$ for all $x \in S$.*

Proof. The first claim is trivial. For the second one, fix $x = f_{\mathtt{i}}(0)$. Now $T_{0,t}(C_\infty) = T_{f_{\mathtt{i}}(0), r_{\mathtt{i}} t}(C_\infty)$ for all $0 < t < 1$ and thus we have $\mathrm{Tan}(C_\infty, x) = \mathrm{Tan}(C_0, 0) = \mathcal{K}_0^+$ by Theorem 4.1. $\square$

Theorem 4.3. *If $x = \pi \mathtt{i}$, where $\mathtt{i} \in \bigcup_{n=1}^\infty I_n^{\mathbb{N}}$, then we have $\mathrm{Tan}(C_\infty, x) \subset \{Q \cap (\alpha C_\infty + \beta)\}_{\alpha \geq 1, \beta \in [-\alpha, \alpha]}$.*

Proof. Let $\mathtt{i} \in I_N^{\mathbb{N}}$ and Let $T_{x,t_n}(C_\infty) \xrightarrow{d_H} K$. Denote the smallest contraction ratio of the mappings $f_{n,m_n}, n = 1, \dots N$ by $\underline{c}(N)$. For all $t > 0$ let $k(t)$ be so that $\mathrm{diam}([\pi \mathtt{i}|_{k(t)}]) \leq t < \mathrm{diam}(\pi[\mathtt{i}|_{k(t)-1}])$. By our construction and choice of ε_n, we have that

$$\mathrm{diam}(\pi[\mathtt{i}|_n]) \leq \mathrm{dist}(\pi[\mathtt{i}|_n], \pi[\mathtt{j}|_n])$$

for all $\mathtt{i}, \mathtt{j} \in I^{\mathbb{N}}$ and $n \in \mathbb{N}$. Thus the screen $[x - t, x + t]^d$ intersects only one cylinder of level $k(t) - 1$. Since $\mathrm{diam}(\pi[\mathtt{i}|_{k(t)}]) \leq t$ we now have, by self-similarity, that $T_{x,t_n}(C_\infty) = Q \cap (\alpha_n C_\infty + \beta_n)$, where $1 \leq \alpha_n \leq \underline{c}(N)^{-1}$ and $\beta_n \in [-\alpha_n, \alpha_n]^d$.

Let $(j(n))_{n=1}^\infty$ be a sequence so that both $\alpha_{j(n)}$ and $\beta_{j(n)}$ converge. Call the limits α and β respectively. Now it is clear that

$$T_{x,t_{j(n)}}(C_\infty) = Q \cap (\alpha_{j(n)} C_\infty + \beta_{j(n)}) \xrightarrow{d_H} Q \cap (\alpha C_\infty + \beta)$$

Since $T_{x,t_n}(C_\infty)$ converges to K, the limit is the same for all sub-sequences and thus we have proved the claim. $\square$

Remark 4.4. It is a bit technical to say exactly which collections of α and β are needed to get $\mathrm{Tan}(C_\infty, x) = \{Q \cap (\alpha C_\infty + \beta)\}_{\alpha, \beta}$. If $x = f_{\mathtt{i}}(0)$ and $|\mathtt{i}| = N$, then $[x - t, x + t]^d$ intersects infinitely many cylinders of level N for all $t > 0$ and the above proof does not work. Also, if $\mathtt{i} \in I^{\mathbb{N}} \setminus \bigcup_{n=1}^\infty I_n^{\mathbb{N}}$ then the sequence α_n is not bounded in general and so the above proof does not work. It turns out that the infinitely generated points divide into further sub-groups according to the "index growth rate".

Theorem 4.5. *If $x = \pi \mathtt{i}$, where $\mathtt{i} \in I^{\mathbb{N}} \setminus \bigcup_{n=1}^\infty I_n^{\mathbb{N}}$ and $i_n \geq n$ for all n large enough, then $\mathrm{Tan}(C_\infty, x) = \mathcal{K}_0^+$.*

Proof. For $K \in \mathcal{K}$ we find a sequence $k(n)$ so that setting $t_n = a_{k(n)} + \lambda_{k(n)}$ gives

$$T_{0,t_n}(C_\infty) \xrightarrow{d_H} \frac{1}{2} K - x_l(\frac{1}{2} K) =: K'.$$

For large n we have $i_n \geq n$. This implies

$$|f_{\mathtt{i}|_n}(0) - \pi \mathtt{i}| = r_{\mathtt{i}|_n} |0 - \pi \sigma^n \mathtt{i}| \leq r_{\mathtt{i}|_n} (a_{n+1} + \lambda_{n+1}) \leq r_{\mathtt{i}|_n} a_n.$$

Now we have

$$d_H(T_{\pi\mathtt{i},r_{\mathtt{i}|_{k(n)}}t_n}(C_\infty), K') \le d_H(T_{\pi\mathtt{i},r_{\mathtt{i}|_{k(n)}}t_n}(C_\infty), T_{f_{\mathtt{i}|_{k(n)}}(0),r_{\mathtt{i}|_{k(n)}}t_n}(C_\infty))$$

$$+ d_H(T_{f_{\mathtt{i}|_{k(n)}}(0),r_{\mathtt{i}|_{k(n)}}t_n}(C_\infty), K')$$

$$\le \frac{r_{\mathtt{i}|_{k(n)}}a_{k(n)}}{r_{\mathtt{i}|_{k(n)}}t_n} + d_H(T_{0,t_n}(C_\infty), K')$$

$$\le \frac{a_{k(n)}}{a_{k(n)} + \lambda_{k(n)}} + d_H(T_{0,t_n}(C_\infty), K') \to 0$$

as $n \to \infty$. $\qquad\qquad\square$

Remark 4.6. The assumption $i_n \ge n$ for large n is used to verify $|0 - \pi\sigma^n\mathtt{i}| \le a_{n+1} + \lambda_{n+1}$. Otherwise we can not get convergence for all sub-sequences $k(n)$.

5. GEOMETRY AND DIMENSION

In this section, we give a closer study to the geometric properties of locally rich sets in Euclidean spaces. We also study different dimensions of such sets.

5.1. **Tangent sets and dimension.** We show that using the construction of C_∞, one can obtain a set of any given Hausdorff dimension that is locally rich in a dense subset. We begin by constructing a variant of C_∞ so that it will have Hausdorff dimension zero. We start with the mappings $\{f_{n,m_n}\}$ that were used to create the set C_1. As has been showed, iterating these mappings gives a set with Hausdorff dimension strictly greater than zero. To make the Hausdorff dimension smaller, we only need to scale the mappings at each level. To this end, fix the sequence $(\alpha_k)_{k=1}^\infty$ with $\alpha_k = 2^{-k}$, choose the sequence (ε_n) in the construction of the mappings f_{n,m_n} so that $r_n \le (2\#\gamma_n)^{-n}$, and for any $t > 0$ let $k(t) \in \mathbb{N}$ be such that $1 - tk(t) < -1$. For any $t > 0$, we now have

$$(5.1) \qquad \sum_{\mathtt{i} \in I} r_\mathtt{i}^t = \sum_{i=1}^\infty \#\gamma_i(r_i)^t$$

$$\le \sum_{i=1}^\infty 2^{-it}(\#\gamma_i)^{1-it}$$

$$\le c'(t) + \sum_{i=k(t)}^\infty 2^{-it}$$

$$= c(t) < \infty.$$

Recall the notations from (4.1) and (4.2). We define the collections of mappings $\mathcal{F}_k$ by setting $\mathcal{F}_k = \{\widetilde{f}_{n,m_n} = \alpha_k\lambda_n\widetilde{\delta}_n\varepsilon_n x + \xi_{n,m_n}\}_{(n,m_n)\in I}$. Set $K_0 = Q$. Next set K_1 to be the union of images of K_0 under the mappings of $\mathcal{F}_1$ and generally

$$K_n = \{0\} \cup \bigcup_{k=1}^n \bigcup_{g_k \in \mathcal{F}_k} g_1 \circ \cdots \circ g_k \circ \cdots \circ g_n(K_0).$$

Again, each K_n is compact since zero is the only accumulation point of K_1, so we get a non empty compact set $K_\infty = \bigcap_{n=0}^\infty K_n$. Note that K_∞ is not a limit set of self similar IFS since we scaled the mappings at each level k by a factor α_k, and $\alpha_k \to 0$. By our choice of sequence (α_k) and (ϵ_k), and applying Theorem 4.1, we have that K_∞ has the same tangent spaces as C_∞ at the points of $\mathcal{S}$ defined in (4.3). It remains to show that $\dim_\mathrm{H} K_\infty = 0$, as we claimed.

Theorem 5.1. *The set K_∞ has Hausdorff dimension zero.*

Proof. It suffices to show that $\mathcal{H}^t(K_\infty) < 0$ for all $t > 0$. So fix $t > 0$ and consider the covering of K_∞ by K_n. Let $\delta(n) = \max\{r_{\mathtt{i}} : \mathtt{i} \in I^n\}$. Now using (5.1) gives

$$
\begin{aligned}
\mathcal{H}^t_{2\delta(n)}(K_\infty) &\leq \sum_{\mathtt{i} \in I^n} (\alpha_{\mathtt{i}} r_{\mathtt{i}})^t \\
&= (\prod_{i=1}^n \alpha_i)^t (\sum_{\mathtt{i} \in I} r_{\mathtt{i}}^t)^n \\
&= 2^{-\frac{1}{2}(n-1)(n)t}(c(t))^n \\
&= 2^{-\frac{1}{2}(n-1)(n)t + n\log_2 c(t)} \to 0.
\end{aligned}
$$

$\square$

Corollary 5.2. *For every $s \in [0, d]$, there exist compact set X with Hausdorff dimension s, and countable dense subset $E \subset X$, such that $\mathrm{Tan}(X, x) = \mathcal{K}_0^+$ for all $x \in E$.*

Proof. It is well known that for every $s \in [0, d]$, there exist compact $F \in \mathcal{K}_0^+$ with the Hausdorff dimension s and without interior points, see [25, Section 4.12]. We apply the Whitney's decomposition for the open set $\mathbb{R}^d \setminus F$. There exists cubes $\{Q_n\}$ such that $\mathbb{R}^d \setminus F = \bigcup_{n=1}^\infty Q_n$, the interior of Q_i and interior of Q_j are disjoint for $i \neq j$, and

$$
\mathrm{diam}(Q_n) \leq d(Q_n, F) \leq 4\,\mathrm{diam}(Q_n)
$$

for all $n \in \mathbb{N}$. Denote the center point of Q_n by x_n and set $C = \bigcup_{n=1}^\infty \{x_n\}$. Choose a large closed ball B, such that it contains F, and let $E = B \cap C$. It is clear that $F \subset \overline{E}$. By setting

$$
X = F \cup \bigcup_{x \in C} (x + \frac{1}{5}\,\mathrm{diam}(Q_x) K_\infty).
$$

we have that X and E satisfy the given conditions. $\square$

Like we mentioned in the end of Section 2, even the fact that one can tangent out $\{0\}$ from all points of a set E, does not give any information on the Hausdorff dimension of E. See Theorem 5.3 below.

Theorem 5.3. *For any $s \in [0, d]$, there is a compact set so that it has Hausdorff dimension s and one can tangent out $\{0\}$ at all its points.*

For the proof, we briefly give the construction of homogeneous Cantor sets. For more details, see e.g. [25, Chapter 4], [10, Chapter 4] or [14]. Let $\{m_k\}_{k=1}^\infty \subset \mathbb{N}$ with $m_k \geq 2$ for all $k \in \mathbb{N}$ and $\{\lambda_k\}_{k=1}^\infty \subset (0, 1)$ with the property $m_k\lambda_k < 1$ for all $k \in \mathbb{N}$. Let E_0 be the unit interval $[0, 1]$. For interval $[0, 1]$ the 1 th level intervals $I_1, \cdots, I_{m_1}$ contained in $[0, 1]$ are of equal length λ_1 and equally spaced with the left-hand ends of I_1 and $[0, 1]$ coinciding, and the right-hand ends of I_{m_1} and $[0, 1]$ coinciding. Given E_k, a collection of $\prod_{i=1}^k m_i$ disjoint interval with equal length $\prod_{i=1}^k \lambda_i$. For each k th level interval I, the $(k+1)$ th level intervals $I_1, ..., I_{m_{k+1}}$ contained in I are of equal length $\prod_{i=1}^{k+1} \lambda_i$ and equally spaced, with the left-hand ends of I_1 and I coinciding, and the right-hand ends of I_{m+1} and I coinciding. We define the limit set of this construction by

$$
E(\{m_k\}, \{\lambda_k\}) = \bigcap_{k=1}^\infty E_k.
$$

For the Hausdorff dimension of $E(\{m_k\}, \{\lambda_k\})$, we have that

$$(5.2) \qquad \dim_H E(\{m_k\}, \{\lambda_k\}) = \liminf_{k \to \infty} \frac{\log \prod_{i=1}^{k} m_i}{-\log \prod_{i=1}^{k} \lambda_i},$$

see [14, Theorem 2].

Proof of Theorem 5.3. For the case $s = 0$, we chose any finite union of points as our E. Let us continue in one dimension. Let $0 < s < 1$, $m_k = k$ and $\lambda_k = k^{-\frac{1}{s}}$ for $k \in \mathbb{N}$. By (5.2), we have that $\dim_H E(\{m_k\}, \{\lambda_k\}) = s$. By a geometrical observation, for large k we have

$$(5.3) \qquad \frac{\prod_{i=1}^{k} \lambda_i}{\mathrm{dist}(I, J)} \leq \frac{k - 1}{k^{\frac{1}{s}} - k}$$

for all different k th interval I and J. Since $0 < s < 1$, we have that the right hand side of (5.3) goes to zero as $k \to \infty$. This implies that $\{0\} \in \mathrm{Tan}(E, x)$ for all $x \in E := E(\{m_k\}, \{\lambda_k\})$.

Next, let $E = \prod_{i=1}^{d} E_i$, where $E_i = E(\{m_k\}, \{\lambda_k\})$ for all i. For any point $x \in E$, there are $x_i \in E_i$, such that $x = (x_1, \cdots, x_d)$. Let $\epsilon > 0$. By the structure of $E(\{m_k\}, \{\lambda_k\})$, there is positive t_n, such that $d_H(T_{x_i, t_n}(E_i), \{0\}) < \epsilon$ for all i. Thus $d_H(T_{x, t_n}(E), \{0\}) < \epsilon \sqrt{d}$. By the arbitrary choice of ϵ, we have that $\{0\} \in \mathrm{Tan}(E, x)$ for all $x \in E$. By a general product formula [10, Product formula 7.2], the number sd is a lower bound for $\dim_H E$. By using the natural covering of E, we get

$$\mathcal{H}^{ds}(E) \leq \liminf_{n \to \infty} \sum_{i=1}^{(\prod_{k=1}^{n} m_k)^d} |\sqrt{d} \prod_{k=1}^{n} \lambda_k|^{ds}$$

$$= d^{\frac{ds}{2}} \liminf_{n \to \infty} \left(\prod_{k=1}^{n} m_k \right)^d |\prod_{k=1}^{n} \lambda_k|^{ds}$$

$$= d^{\frac{ds}{2}} \left(\liminf_{n \to \infty} n! ((n!)^{-\frac{1}{s}})^s \right)^d$$

$$= d^{\frac{ds}{2}},$$

which gives the upper bound $\dim_H E \leq sd$.

For the case $s = d$, let $0 < s_1 < s_2 \cdots$ be an increasing sequence numbers with $s_n \nearrow d$. By the above argument, we have that for each s_n, there is a set E_n with $\dim_H E_n = s_n$ and $\{0\}$ is a tangent set at all points of each E_n. Let

$$E = \{0\} \cup \bigcup_{n=1}^{\infty} (2^{-n^2} + n 2^{-n^2} E_n).$$

We have that $\dim_H E = \sup_{n \geq 1} \{\dim_H E_n\} = d$. For the point zero, applying the same argument as in Theorem 4.1, gives that $\{0\} \in \mathrm{Tan}(E, 0)$. For other points $x \in E$, this follows by our choice of E_n, and the sequences 2^{-n^2} and $n 2^{-n^2}$. Thus we have completed the proof. $\qquad \square$

Example 5.4. Let $E \in \mathcal{K}$ and x be a Lebesgue density point of E, then $\mathrm{Tan}(E, x) = Q$.

Proof. Denote $r_E = \sup\{r : B(x, r) \subset Q \setminus E\}$, by elementary geometry we see that $d_H(E, Q) = r_E$. Denote by $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}^d$. It's not hard to see that $\frac{\mathcal{L}(E \cap B(x, r))}{\alpha(d) r^d} \to 1$ if and only if $\frac{\mathcal{L}(E \cap Q(x, r))}{(2r)^d} \to 1$ where $\alpha(d)$ is the measure of the unit ball and $Q(x, r)$ means the cube with center x and side-length $2r$.

Let $t_n \searrow 0$. Set $E_n := T_{x,t_n}(E)$, we have $\mathcal{L}(E_n) = \mathcal{L}(E \cap Q(x, t_n))t_n^{-d}$. If x is the density point of E, then $\mathcal{L}(E_n) \to 2^d = \mathcal{L}(Q)$. It means that $r_{E_n} \to 0$. By the above result, we conclude that $E_n \xrightarrow{d_H} Q$. By the arbitrary choice of t_n, we have completed the proof. $\square$

Notice that the converse is not true in the above example. For example, let $E = \{0\} \cup \{\pm\frac{1}{n}\}_{n \geq 1}$, then $\text{Tan}(E, 0) = [-1, 1]$.

Example 5.5. Let $E \in \mathcal{K}$ with $0 < \mathcal{H}^s(E) < \infty$. If $\{0\} \in \text{Tan}(E, x)$ for $\mathcal{H}^s$ almost all $x \in E$, then $\underline{D}^s(E, x) = 0$ for $\mathcal{H}^s$ almost all $x \in E$. Here $\underline{D}^s(E, x)$ and $\overline{D}^s(E, x)$ mean the lower and upper s-densities of E at x respectively, see [25, Chapter 6].

Since $\{0\} \in \text{Tan}(E, x)$, there exist a sequence $r_n \searrow 0$, such that $d_H(T_{x,r_n}(E, x), \{0\}) \to 0$. Thus for any $\epsilon > 0$, there exist N, such that $d_H(T_{x,r_n}(E, x), \{0\}) < \epsilon$ for all $n \geq N$. It means that $(B(x, r_n) \setminus B(x, \epsilon r_n)) \cap E = \emptyset$, then

$$\frac{\mathcal{H}^s(E \cap B(x, r_n))}{2^{-s} r_n^s} = \frac{\mathcal{H}^s(E \cap B(x, \epsilon r_n))}{2^{-s} (\epsilon r_n)^s} \epsilon^s.$$

Let $n \to \infty$, then $\underline{D}^s(E, x) \leq \epsilon^s \overline{D}^s(E, x)$, where $\underline{D}^s(E, x)$ and $\overline{D}^s(E, x)$ are lower density and upper density of E at point x respectively. Since $\overline{D}^s(E, x) \leq 1$ for $\mathcal{H}^s$ a.e. $x \in E$, the results follows.

5.2. **Porosity of sets.** A subset E of $\mathbb{R}^d$ is called porous, if there exist $0 < \alpha < 1$, such that for each $x \in \mathbb{R}^d$, and $r > 0$, the ball $B(x, r)$ contains a open ball $U(y, \alpha r)$ that does not meet E (, $E \cap U(y, \alpha r) = \emptyset$). In this case we also call E the (α)-porous. The concept of porosity is closely related to the Assouad dimension. The connection is the following: A subset E of $\mathbb{R}^d$ is porous if and only if $\dim_A E < d$. For details, we refer to [24, Theorem 5.2].

Proposition 5.6. *Let E be (α)- porous, then every element of $\text{Tan}(E, x)$ is also (α)-porous for all $x \in E$.*

Proof. Let $x \in E$ and $F \in \text{Tan}(E, x)$, then there exist a sequence $t_n \searrow 0$, such that $d_H(T_{x,t_n}(E), F) \to 0$. Denote $E_n := T_{x,t_n}(E), n \in \mathbb{N}$. Since $d_H(E_n, F) \to 0$, so for any $\epsilon \in (0, \alpha r)$ there exist N, such that

(5.4) $$d_H(E_N, F) < \epsilon.$$

By the porosity of E and a simple geometric observation, we have that E_N is (α)- porous. Let $y \in \mathbb{R}^d$ and $r > 0$, then we find $z_N \in \mathbb{R}^d$ so that $U(z_N, \alpha r) \subset B(y, r)$ and $U(z_N, \alpha r) \cap E_N = \emptyset$. Together with (5.4) we have

$$U(z_N, \alpha r - \epsilon) \cap F = \emptyset.$$

Let $\epsilon \to 0$, then the open ball $U(z_N, \alpha r) \cap F = \emptyset$. By the arbitrary choice of the ball $B(y, r)$ and $x \in E$, we have completed the proof. $\square$

By applying the Proposition 5.6, we know that locally rich set can't be porous set, and thus it has Assouad dimension d (Here we apply the fact that: A subset E of $\mathbb{R}^d$ is porous if and only if $\dim_A E < d$ again). We put this result as the following corollary.

Corollary 5.7. *Let $E \in \mathcal{K}$ be a locally rich set, then $\dim_A E = d$.*

Definition 5.8. Let $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$. The local porosity of E at x at distance r is

$$por(E, x, r) = \sup\{\alpha \geq 0 : B(y, \alpha r) \subset B(x, r) \setminus E \text{ for some } y \in \mathbb{R}^d\}.$$

The upper and lower porosities of E at x are defined as

$$\overline{por}(E,x) = \limsup_{r \to 0} por(E,x,r) \text{ and } \underline{por}(E,x) = \liminf_{r \to 0} por(E,x,r),$$

respectively. By the definition, we have

$$0 \leq \underline{por}(E,x) \leq \overline{por}(E,x) \leq \frac{1}{2},$$

for any $x \in E$. Note that if E is a (α)- porous set of $\mathbb{R}^d$, then $por(E,x,r) \geq \alpha$ for any $x \in \mathbb{R}^d$ and $r > 0$. Thus $\inf_{x \in E} \underline{por}(E,x) \geq \alpha$. We conclude that if E is porous of $\mathbb{R}^d$, then $\inf_{x \in E} \underline{por}(E,x) > 0$. The following simple example shows that the converse is not true.

Example 5.9. Let $E = \{0\} \cup \{\frac{1}{n}\}_{n=1}^{\infty}$ be a subset of $\mathbb{R}$, then

$$\underline{por}(E,x) = \overline{por}(E,x) = \frac{1}{2} \text{ for all } x \in E,$$

and E is not porous in $\mathbb{R}$.

Proposition 5.10. *Let $E \in \mathcal{K}$. If $\{0\} \in \mathrm{Tan}(E,x)$, then $\overline{por}(E,x) = \frac{1}{2}$.*

Proof. There exist $t_n \searrow 0$, such that $d_H(T_{x,t_n}(E), F) \to 0$ as $n \to \infty$. For any $\epsilon \in (0,1)$, there is N such that for any $n \geq N$, we have

$$(5.5) \qquad d_H(T_{x,t_n}(E), 0) < \epsilon.$$

Thus there is an open ball $U(z, \frac{1-\epsilon}{2}) \subset B(0,1)$ and $U(z, \frac{1-\epsilon}{2}) \cap T_{x,t_n}(E) = \emptyset$. We also have $T_{x,t_n}^{-1}(U(z, \frac{1-\epsilon}{2})) \cap E = \emptyset$. It follows that $\overline{por}(E,x,t_n) \geq \frac{1-\epsilon}{2}$. Let $\epsilon \to 0$, then $\overline{por}(E,x) \geq \frac{1}{2}$. We complete the proof, since the nature upper bound of $\overline{por}(E,x)$ is $\frac{1}{2}$. $\square$

As an implication of Theorem 3.6 and Proposition 5.10, we have the following corollary.

Corollary 5.11. *Typically a compact set of $\mathcal{K}$ has upper porosity $\frac{1}{2}$ at all its points.*

A Borel regular measure μ on metric space X is said to be *doubling* if there is a constant $C \geq 1$ such that for any $x \in X$ and $0 < r < \infty$ we have

$$(5.6) \qquad 0 < \mu(B(x,2r) \leq C\mu(B(x,r)) < \infty.$$

In this case we also say that μ is C-doubling. We denote by $\mathcal{D}(X)$ the collection of non-trivial doubling measures on X. Let $E \subset X$, we say that E is *thin* if E has zero measure for all doubling measures on X. It is well known that porous set are thin, but the converse is not true, for more details see [30]. In the following, we only consider $\mathcal{D}([0,1]^d)$.

It's not hard to see that by applying the density theorem and Proposition 5.10 we have the following fact. For the convenience of reader, we show the details here.

Proposition 5.12. *Let $E \in \mathcal{K}$ and $\{0\} \in \mathrm{Tan}(E,x)$ for all $x \in E$, then E is thin.*

Proof. Let $x \in E$. By proposition 5.10, we have $\overline{por}(E,x) = \frac{1}{2}$. There exist a sequence (t_n), $t_n \searrow 0$, such that $\lim_{n \to \infty} por(E,x,t_n) \to \frac{1}{2}$. Thus for every $\epsilon > 0$, there exists an integer N, such that for all $n \geq N$ we have $E \cap (B(x,t_n) \setminus B(x,\epsilon t_n)) = \emptyset$.

Let $\mu \in \mathcal{D}(Q)$. By the above calculation, it is easy to see that for all $n \geq N$,

$$(5.7) \qquad \frac{\mu(E \cap B(x,t_n))}{\mu(B(x,t_n))} \leq \frac{\mu(B(x,\epsilon t_n))}{\mu(B(x,t_n))} \leq C\epsilon^s,$$

where positive constants C and s only depend on the measure μ, see [18, Chapter 13]. By the above estimate we have that $\liminf_{r \to 0} \dfrac{\mu(E \cap B(x,r))}{\mu(B(x,r))} = 0$ for all $x \in E$. Thus we have $\mu(E) = 0$ by applying the density point theorem for Radon measures, see [25, Corollary 2.14]. $\square$

By applying Proposition 5.12, we have the following corollary.

Corollary 5.13. *Let $E \in \mathcal{K}$ be a locally rich set, then E is thin.*

Corollary 5.14. *By Theorem 3.6 and Corollaries 5.7 and 5.13, we conclude that a typical set of $\mathcal{K}$ has Assouad dimension d and is thin.*

Remark 5.15. Corollary 5.14 follows implicitly from [13] in a different manner. In [13], they proved that a typical set $E \in \mathcal{K}$ has $\underline{\dim}_{\mathrm{M}}(E) = 0$ and $\overline{\dim}_{\mathrm{M}}(E) = d$. By the well known results that $\dim_H(E) \leq \underline{\dim}_{\mathrm{M}}(E) \leq \overline{\dim}_{\mathrm{M}}(E) \leq \dim_A(E)$, we have the dimension part of our claim. For the thin part, we recall the following results: A subset of uniformly perfect metric space with Hausdorff dimension zero is thin [18, Chapter 13].

By Theorem 3.6 and Remark 5.15, we know that typically a compact set $X \subset \mathbb{R}^d$ is locally rich and has $\overline{\dim}_M X = d$. We end this section with the following question.

Question 5.16. If $X \subset \mathbb{R}^d$ is locally rich, is it true that $\overline{\dim}_M X = d$?

6. GLOBAL STRUCTURE OF SETS

The converse to the local structure of sets is the global structure of sets. Here we briefly discuss, what can be the global structure of an unbounded closed set in Euclidean space. In other words, what can one see when zooming out? To be formal, we show the following definitions. We say that $F \in \mathcal{K}$ is a photograph of a closed $E \subset \mathbb{R}^d$ at $x \in E$ if there exists a sequence $t_n \nearrow \infty$ so that

$$T_{x,t_n}(E) \xrightarrow{d_H} F.$$

Denote by $P(E, x)$ all the photograph of E at x. We say that E is *globally rich* at $x \in E$ if $P(E, x) \approx \mathcal{K}$. If this happens for all $x \in E$ then we just say that E is *globally rich*.

Recall the subspace $\mathcal{K}_0^+$ of $\mathcal{K}$ and the choice of $(\gamma_i)_{i=1}^\infty$ at the beginning of Section 3. Fix sequences $a_n \nearrow \infty$ and $\lambda_n \nearrow \infty$ with the following properties

  (1) $a_n + \lambda_n + 1 \leq a_{n+1}$ for all $n \in \mathbb{N}$
  (2) $\frac{a_n}{a_n + \lambda_n} \leq \frac{1}{n}$ as $n \to \infty$

For example, one could choose $a_n = 2^{n^2}$ and $\lambda_n = na_n$ for all $n \in \mathbb{N}$. We set $\widetilde{\gamma}_n = \vec{a}_n + \lambda_n \gamma_n$, where $\vec{a}_n = (a_n, 0, 0, \dots)$ and let

$$A := \bigcup_{n=1}^\infty \widetilde{\gamma}_n.$$

The set A is made of the elements of $(\gamma_n)_{n=1}^\infty$ scaled with λ_n and positioned so that the leftmost point of the n th piece on the x_1-axis is a_n. The condition (2) gives the separation of the pieces. Obviously A is closed set of $\mathbb{R}^d$. By applying the same argument as in Theorem 4.1 to this zooming out case, we have the following result. We omit the proof here, but note the result is the same for all points, since any two points get arbitrarily close when looking from far enough. This is a major difference to the zooming in case.

Theorem 6.1. *We have $P(A, x) = \mathcal{K}_0^+$ for all $x \in A$.*

Note that the set A of Theorem 6.1 is a countable set of (different) points. By our earlier results we know that there exist locally rich sets. Now we are going to put the copy of a locally rich set of suitable size at every point of A, thus we can have one set such that it is not only locally rich, but also globally rich.

Corollary 6.2. *There exists a closed set $E \subset \mathbb{R}^d$ such that $P(E, x) \approx \mathcal{K}$ and $Tan(E, x) \approx \mathcal{K}$ for all $x \in E$.*

Proof. Let A be the set of Theorem 6.1 and $R \subset Q, 0 \in R$ be a locally rich set with diameter less than one. Let

$$E = \bigcup_{n=1}^{\infty} \bigcup_{x \in \widetilde{\gamma}_n} (x + 2^{-n}\delta_n R),$$

where δ_n is from (3.1). It's follows directly that E is a closed set. For any $x \in E$, by the choice of 2^{-n} we still have that $P(E,x) = \mathcal{K}_0^+$. By the definition of $(\widetilde{\gamma}_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$, we have that $(a + 2^{-m}\delta_m R) \cap (b + 2^{-n}\delta_n R) = \emptyset$ for $a \in \widetilde{\gamma}_m, b \in \widetilde{\gamma}_n$ with $a \neq b$. Thus by the local richness of R we have $\mathrm{Tan}(E,x) \approx \mathcal{K}$. We complete the proof by the simple fact that $\mathcal{K}_0^+ \approx \mathcal{K}$. $\square$

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