

# Finite-size effect for four-loop Konishi of the $\beta$ -deformed $\mathcal{N} = 4$ SYM

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## Abstract

We propose that certain twists of the  $su(2|2)$   $S$ -matrix elements describe the  $\beta$ -deformation of  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory. We compute the perturbative four-loop anomalous dimension of the Konishi operator of the deformed gauge theory from the Lüscher formula based on these twisted  $S$ -matrix elements. The result agrees exactly with the perturbative gauge theory computations.

# 1 Introduction

The AdS/CFT correspondence [1] is a duality between the  $\mathcal{N} = 4$  super Yang-Mills (SYM) theory and type IIB string theory on  $AdS_5 \times S^5$ . Integrability in the planar limit [2] has played a crucial role in clarifying the correspondence, which relates strong and weak couplings.

Asymptotic Bethe ansatz equations (BAEs) for all orders and sectors, which have provided a computational tool, were conjectured in [3]. Another important approach was to construct an exact  $S$ -matrix based on the integrability [4]. The  $S$ -matrix for fundamental excitations was determined from the  $su(2|2)$  symmetry algebra in [5, 6] up to an overall scalar factor which was fixed in [7, 8] using crossing symmetry [9]. The conjectured BAEs can be derived by diagonalizing the Bethe-Yang matrix [5, 10]. These asymptotic BAEs, however, have a fundamental limitation since they are valid only when the size of the spin chain becomes infinite. It has been pointed out in [11] that the asymptotic BAEs should fail because of wrapping interactions which arise when the order of perturbation theory goes beyond the size of the spin chain.

The deviations from the asymptotic BAEs have been quantitatively studied in the weak-coupling limit using generalized Lüscher corrections and compared with perturbative SYM computations [12] in a series of papers [13, 14]. The results show that the Lüscher corrections agree exactly with the deviations which is another triumph for the exact  $S$ -matrix of the  $\mathcal{N} = 4$  SYM.

Our main interest in this paper is to generalize the wrapping correction analysis to the  $\beta$ -deformed SYM theory. The deformed SYM theory is obtained by replacing the original  $\mathcal{N} = 4$  superpotential for the chiral superfields by:

$$W = ih \operatorname{tr}(e^{i\pi\beta} \phi\psi Z - e^{-i\pi\beta} \phi Z\psi). \quad (1)$$

The deformation breaks the supersymmetry down to  $\mathcal{N} = 1$  but still maintains the conformal invariance in the planar limit to all perturbative orders [15, 16], since the deformation becomes exactly marginal for real  $\beta$  if

$$h\bar{h} = g_{\text{YM}}^2, \quad (2)$$

where  $g_{\text{YM}}$  is the Yang-Mills coupling constant.

Under the AdS/CFT duality, it is believed that this  $\beta$ -deformed SYM theory is related to a string theory with the Lunin-Maldacena background [17]. When the deformation parameter

is real, the string theory on this deformed background maintains the classical integrability [18, 19], and has identical excitations such as giant magnons, whose finite-size effects have been obtained by transforming the  $AdS_5 \times S^5$  background under some T-duality [19]. Perturbative integrability for the deformed SYM was studied in [20, 21, 22]. An important development in the deformed SYM theory was the perturbative computation in [23, 24] of anomalous dimensions for the one and two magnon states in the  $su(2)$  sector up to four loops. These hints of the integrability of the deformed SYM theory can be established firmly if one can construct an exact factorizable  $S$ -matrix which is consistent with the wrapping corrections.

In this note we propose that certain twists of the  $su(2|2)$   $S$ -matrix elements describe the  $\beta$ -deformed SYM theory. We show that, via the Lüscher formula, these twisted amplitudes lead to the correct wrapping corrections for the  $su(2)$  Konishi operator.

## 2 Finite-size effect of the $\beta$ -deformed SYM

### 2.1 Asymptotic Bethe ansatz equation

The asymptotic BAEs have been conjectured for the  $\beta$ -deformed SYM theory in [20, 22]. The BAEs become quite simple for the  $su(2)$  sector which is relevant to the Konishi operator. There are two “impurities” which carry two Bethe roots  $u_1, u_2$  in the spin chain of length  $L = 4$  which satisfy

$$e^{ip_1 L} = e^{2\pi i \beta L} e^{2i\theta(p_1, p_2)} \frac{u_1 - u_2 + i}{u_1 - u_2 - i}, \quad e^{ip_2 L} = e^{2\pi i \beta L} e^{-2i\theta(p_1, p_2)} \frac{u_2 - u_1 + i}{u_2 - u_1 - i}. \quad (3)$$

The momenta  $p_i$  are related to the Bethe roots by

$$u_i = \frac{1}{2} \cot \frac{p_i}{2} \sqrt{1 + 16g^2 \sin^2 \frac{p_i}{2}}, \quad (4)$$

where

$$g^2 = \frac{g_{\text{YM}}^2 N}{16\pi^2}. \quad (5)$$

The energy for a magnon with momentum  $p_i$  is given by

$$E(p_i) = \sqrt{1 + 16g^2 \sin^2 \frac{p_i}{2}}. \quad (6)$$

One can deduce the momentum conservation relation from the BAEs

$$p_1 + p_2 = 4\pi\beta. \quad (7)$$

We will define shifted momentum variables by

$$\tilde{p}_i \equiv p_i - 2\pi\beta, \quad \tilde{p}_1 + \tilde{p}_2 = 0. \quad (8)$$

The BAE and the energy are expressed simply in terms of  $\tilde{p} = \tilde{p}_1$  as follows:

$$e^{4i\tilde{p}} = e^{2i\theta(\tilde{p}+2\pi\beta, -\tilde{p}+2\pi\beta)} \left( \frac{u(\tilde{p} + 2\pi\beta) - u(-\tilde{p} + 2\pi\beta) + i}{u(\tilde{p} + 2\pi\beta) - u(-\tilde{p} + 2\pi\beta) - i} \right) \quad (9)$$

$$E_{\text{total}}(\beta) = E(\tilde{p} + 2\pi\beta) + E(-\tilde{p} + 2\pi\beta). \quad (10)$$

We can solve the BAE perturbatively in  $g^2$  by expanding both  $\tilde{p}$  and  $E$

$$\tilde{p} = \tilde{p}_0 + g^2\tilde{p}^{(1)} + g^4\tilde{p}^{(2)} + g^6\tilde{p}^{(3)} + \dots \quad (11)$$

$$E_{\text{total}}(\beta) = 2 + g^2E_1(\beta) + g^4E_2(\beta) + g^6E_3(\beta) + g^8E_4(\beta) + \dots \quad (12)$$

The leading order BAE is

$$e^{4i\tilde{p}_0} = \frac{\cot(\frac{\tilde{p}_0}{2} - \pi\beta) + \cot(\frac{\tilde{p}_0}{2} + \pi\beta) + 2i}{\cot(\frac{\tilde{p}_0}{2} - \pi\beta) + \cot(\frac{\tilde{p}_0}{2} + \pi\beta) - 2i} \quad (13)$$

with a solution

$$\cos \tilde{p}_0 = \frac{1 \mp 3\Delta}{4 \cos(2\pi\beta)}, \quad (14)$$

where we defined  $\Delta$  by [23]

$$\Delta \equiv \frac{\sqrt{5 + 4 \cos(4\pi\beta)}}{3}. \quad (15)$$

From now on we focus on the “-” sign only keeping in mind that the other solution can be obtained by changing the sign of  $\Delta$ . The energy in the leading order becomes

$$E_1(\beta) = 8 \sin^2 \left( \frac{\tilde{p}_0}{2} - \pi\beta \right) + 8 \sin^2 \left( \frac{\tilde{p}_0}{2} + \pi\beta \right) = 6(1 + \Delta). \quad (16)$$

Higher order BAEs and their solutions can be obtained iteratively along with the energy corrections as follows:

$$E_2 = -\frac{3}{\Delta} - 15 - 21\Delta - 9\Delta^2 \quad (17)$$

$$E_3 = -\frac{3}{4\Delta^3} + \frac{153}{4\Delta} + 114 + \frac{495}{4}\Delta + 54\Delta^2 + \frac{27}{4}\Delta^3 \quad (18)$$

$$E_4 = \frac{3(1 + \Delta)^4(-1 - 2\Delta + 49\Delta^2 + 84\Delta^3 - 1359\Delta^4 - 5562\Delta^5 - 2673\Delta^6 + 1944\Delta^7)}{8\Delta^5(1 + 3\Delta)^2} + \left( -\frac{9}{\Delta} + 27 + 54\Delta - 90\Delta^2 - 189\Delta^3 - 81\Delta^4 \right) \zeta(3), \quad (19)$$

where the term proportional to  $\zeta(3)$  originates from the dressing phase [8].

## 2.2 Perturbative gauge theory results

We summarize here the perturbative computation of anomalous dimensions of the  $su(2)$  Konishi operators  $\text{Tr}(XZXZ)$  and  $\text{Tr}(ZZXX)$  of the  $\beta$ -deformed SYM up to four loops [23]. One of the two eigenvalues of the dilatation operator is given by

$$\begin{aligned} \gamma &= 4 + g^2\gamma_1 + g^4\gamma_2 + g^6\gamma_3 + g^8\gamma_4 + \dots \\ \gamma_1 &= 6(1 + \Delta) \end{aligned} \tag{20}$$

$$\gamma_2 = -\frac{3}{\Delta} - 15 - 21\Delta - 9\Delta^2 \tag{21}$$

$$\gamma_3 = -\frac{3}{4\Delta^3} + \frac{153}{4\Delta} + 114 + \frac{495}{4}\Delta + 54\Delta^2 + \frac{27}{4}\Delta^3 \tag{22}$$

$$\begin{aligned} \gamma_4 &= -\frac{3}{8\Delta^5} + \frac{33}{2\Delta^3} - \frac{1701}{4\Delta} - 1230 - \frac{2427}{2}\Delta - 180\Delta^2 + 162\Delta^4 + \frac{2997}{8}\Delta^3 \\ &+ \left( -\frac{9}{\Delta} + 297 + 702\Delta + 234\Delta^2 - 405\Delta^3 - 243\Delta^4 \right) \zeta(3) - 360(1 + \Delta)^2 \zeta(5), \end{aligned} \tag{23}$$

and the other eigenvalue is obtained by changing the sign of  $\Delta$ .

Up to  $g^6$  order these results match exactly with the asymptotic BAE results. One can compare (20) with (16), (21) with (17), and (22) with (18). At wrapping order  $g^8$  there is a discrepancy between (23) and (19),

$$\Delta E_{\text{wrapping}} = g^8 \left[ -54(1 + \Delta)^3(-5 + 3\Delta)\zeta(3) - 360(1 + \Delta)^2\zeta(5) + \frac{81(1 - 3\Delta)^2(1 + \Delta)^4}{(1 + 3\Delta)^2} \right]. \tag{24}$$

We will explain this difference at the leading wrapping order by a finite-size effect based on the Lüscher formula.

## 2.3 Lüscher formula

The Lüscher formula for multi-particle states for a theory with non-diagonal  $S$ -matrix has been proposed and checked in [13]. For the case of the Konishi operator, the wrapping correction is given by

$$\Delta E_{\text{wrapping}} = - \sum_{\ell=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \left( \frac{z^-}{z^+} \right)^L \sum_{j,j'} (-1)^{F_{(jj')}} [\mathcal{S}^{(\ell 1)}(z^\pm, x_1^\pm) \mathcal{S}^{(\ell 1)}(z^\pm, x_2^\pm)]_{(jj')(11)}^{(jj')(11)}, \tag{25}$$

where the rapidities of physical particles are given by

$$x_i^\pm = \frac{1}{4g} \left( \cot \frac{p_i}{2} \pm i \right) \left( 1 + \sqrt{1 + 16g^2 \sin^2 \frac{p_i}{2}} \right) \tag{26}$$

and that of the mirror  $\ell$ -particle bound states is

$$z^\pm = \frac{q + i\ell}{4g} \left( \sqrt{1 + \frac{16g^2}{\ell^2 + q^2}} \pm 1 \right). \quad (27)$$

$F_{(jj')} = F_j + F_{j'}$  is given by

$$F_j = \begin{cases} 0 & \text{if } j = 1, \dots, 2\ell \\ 1 & \text{if } j = 2\ell + 1, \dots, 4\ell. \end{cases} \quad (28)$$

As explained in [13], we need to choose  $L = 4$  because we are including the ‘string frame’ phase factors into the effective length.

The total  $S$ -matrix is a tensor product of two twisted  $su(2|2)$   $S$ -matrices,

$$\mathcal{S}^{(\ell 1)}(z^\pm, x_i^\pm) = S_{\text{scalar}}^{(\ell 1)}(q, u_i) \left[ \mathcal{S}_{\text{matrix}}^{(\ell 1)}(q, u_i) \otimes \mathcal{S}_{\text{matrix}}^{(\ell 1)}(q, u_i) \right], \quad (29)$$

and the  $S$ -matrix part in the Lüscher formula can be rewritten as

$$\begin{aligned} & \sum_{j, j'} (-1)^{F_{(jj')}} \left[ \mathcal{S}^{(\ell 1)}(q, u_1) \mathcal{S}^{(\ell 1)}(q, u_2) \right]_{(jj')(11)}^{(jj')(11)} \\ &= S_{\text{scalar}}^{(\ell 1)}(q, u_1) S_{\text{scalar}}^{(\ell 1)}(q, u_2) \left( \sum_j (-1)^{F_j} \left[ \mathcal{S}_{\text{matrix}}^{(\ell 1)}(q, u_1) \mathcal{S}_{\text{matrix}}^{(\ell 1)}(q, u_2) \right]_{j1}^{j1} \right)^2. \end{aligned} \quad (30)$$

For the  $su(2)$  Konishi operator it is enough to consider scattering amplitudes between  $\ell$ -particle bound states and  $A_1$  which are diagonal. We propose that these matrix elements are given explicitly by

$$\mathcal{S}^{(\ell 1)j1}_{j1} = \begin{cases} S^{(\ell 1)j1}_{j1} = a_5^5, & j = 1, \dots, \ell + 1, \\ S^{(\ell 1)j1}_{j1} = 2a_8^8, & j = \ell + 2, \dots, 2\ell \end{cases} \quad (31)$$

for the bosonic states and

$$\mathcal{S}^{(\ell 1)j1}_{j1} = \begin{cases} e^{i\pi\beta} S^{(\ell 1)j1}_{j1} = e^{i\pi\beta} a_9^9, & j = 2\ell + 1, \dots, 3\ell, \\ e^{-i\pi\beta} S^{(\ell 1)j1}_{j1} = e^{-i\pi\beta} \frac{a_9^9 + a_3^3}{2}, & j = 3\ell + 1, \dots, 4\ell \end{cases} \quad (32)$$

for the fermionic states. Explicit expressions for the undeformed matrix elements  $a_j^j$  are given in [13, 14].

Let us emphasize that contrary to [13] we calculate here the Lüscher correction of the  $su(2)$  representative of the Konishi operator. In the  $\beta$ -deformed theory the two representatives

have different anomalous dimensions and the direct comparison is available only in the  $su(2)$  case. This calculation is also new in the undeformed case where the Lüscher correction for only the  $sl(2)$  representative has been calculated so far. There are calculations in the  $su(2)$  sector based on the  $Y$ -system [25] but we are not able to use this approach since the deformed-TBA equations are not available.

Since the exponential factor becomes

$$\left(\frac{z^-}{z^+}\right)^4 = \frac{256g^8}{(q^2 + \ell^2)^4} + \dots, \quad (33)$$

we may consider only a leading term in each expression. The scalar part of the  $S$ -matrix is the same as the undeformed case,

$$S_{\text{scalar}}^{(\ell 1)}(q, u) = \frac{[q - i(\ell - 1) - 2u]}{[q + i(\ell - 1) - 2u]} \frac{(2u + i)^2}{[(q - 2u)^2 + (\ell + 1)^2]}. \quad (34)$$

The sum in the matrix part is nontrivial. Taking the  $g \rightarrow 0$  limits on the  $S$ -matrix elements given in Eqs.(31) and (32), we obtain at leading order

$$\mathcal{S}_{\text{matrix}}^{(\ell 1)}(q, u)_{j1}^{j1} \approx \begin{cases} \frac{2u - q - i(\ell - 1)}{2u + i}, & j = 1, \dots, \ell + 1, \\ \frac{(2u + i)}{(2u - q)^2 + (\ell + 1)^2}, & j = \ell + 2, \dots, 2\ell, \\ e^{i\pi\beta} \frac{2u - q - i(\ell + 1)}{\sqrt{4u^2 + 1}}, & j = 2\ell + 1, \dots, 3\ell, \\ e^{-i\pi\beta} \frac{(2u - i)(\ell^2 + (2u - q + i)^2)}{(2u + i)(2u - q + i(\ell - 1))\sqrt{4u^2 + 1}}, & j = 3\ell + 1, \dots, 4\ell. \end{cases} \quad (35)$$

Inserting these into Eq.(25) and integrating using the residue at  $q = i\ell$ , we get <sup>1</sup>

$$\Delta E_{\text{wrapping}} = \sum_{\ell=1}^{\infty} \left[ \frac{f_1(u_1, u_2)}{\ell^5} + \frac{f_2(u_1, u_2)}{\ell^3} + f_3(u_1, u_2, \ell) \right] \quad (36)$$

where<sup>2</sup>

$$f_1(u_1, u_2) = -\frac{2560(1 + 2u_1^2 + 2u_2^2)^2}{(4u_1^2 + 1)^2(4u_2^2 + 1)^2} \quad (37)$$

and

$$\begin{aligned} f_2 &= \frac{\text{num}}{(4u_1^2 + 1)^4(4u_2^2 + 1)^4} \quad (38) \\ \text{num} &= 2048(-1 + 5u_1^2 + 48u_1^4 + 96u_1^6 - 2u_1u_2 - 16u_1^3u_2 - 32u_1^5u_2 + 5u_2^2 + 224u_1^2u_2^2 \\ &+ 1024u_1^4u_2^2 + 1536u_1^6u_2^2 + 768u_1^8u_2^2 - 16u_1u_2^3 - 128u_1^3u_2^3 + 64u_1^5 \\ &- 256u_1^5u_2^3 + 48u_2^4 + 1024u_1^2u_2^4 + 3200u_1^4u_2^4 + 2560u_1^6u_2^4 - 32u_1u_2^5 \\ &- 256u_1^3u_2^5 - 512u_1^5u_2^5 + 96u_2^6 + 1536u_1^2u_2^6 + 2560u_1^4u_2^6 + 64u_2^8 + 768u_1^2u_2^8). \quad (39) \end{aligned}$$

<sup>1</sup>Summing up the contributions of the other residues gives a vanishing result.

<sup>2</sup> We have expressed the deformation parameter  $\beta$  in terms of  $u_1, u_2$  using Eq.(7) for simplicity.

The  $f_3$  is too complicated to write (1775 terms in the numerator).

The BAE roots  $u_1, u_2$  at the leading order can be used to evaluate the wrapping correction,

$$u_1 = \frac{1}{2} \cot \left( \frac{\tilde{p}_0}{2} + \pi\beta \right), \quad u_2 = \frac{1}{2} \cot \left( -\frac{\tilde{p}_0}{2} + \pi\beta \right). \quad (40)$$

Using  $\tilde{p}_0$  in Eq.(14), we obtain

$$u_1 = \frac{(1 - 3\Delta)^2}{2\sqrt{-1 + 9\Delta^2} \left( 3\sqrt{1 - \Delta^2} + 2\sqrt{1 + \frac{2}{1+3\Delta}} \right)}, \quad (41)$$

$$u_2 = \frac{(1 - 3\Delta)^2}{2\sqrt{-1 + 9\Delta^2} \left( 3\sqrt{1 - \Delta^2} - 2\sqrt{1 + \frac{2}{1+3\Delta}} \right)}. \quad (42)$$

Inserting these exact expressions into Eq.(36) and summing up the infinite terms exactly, we obtain

$$\Delta E_{\text{wrapping}} = g^8 \left[ -54(1 + \Delta)^3(-5 + 3\Delta)\zeta(3) - 360(1 + \Delta)^2\zeta(5) + \frac{81(1 - 3\Delta)^2(1 + \Delta)^4}{(1 + 3\Delta)^2} \right] \quad (43)$$

which matches exactly with Eq.(24). The wrapping correction for the second anomalous dimension can be computed from the same Lüscher formula by simply replacing  $\Delta$  with  $-\Delta$ .

The proposed  $S$ -matrix elements can be constructed from a Drinfeld twist of the  $su(2|2)$   $S$ -matrix [26] which we will report in a separate publication [27]. Furthermore this  $S$ -matrix can lead to the conjectured asymptotic BAEs via nested BAE analysis. We hope this will lead to studying the  $\beta$ -deformed SYM theory in a more rigorous way.

There are several imminent questions. While our analysis successfully generated the four-loop Lüscher corrections of the  $su(2)$  Konishi operator, we have not succeeded in the one impurity system, whose perturbative results are given in [23, 24], and for which Lüscher corrections have been calculated for  $\beta = 1/2$  and generic values of  $L$  in [28]. Another challenge is to derive the asymptotic BAEs based on our  $S$ -matrices which will eventually lead to the thermodynamic Bethe ansatz, thereby generalizing the TBA equations in [29]. We hope to address these issues in the near future.

*Note Added:* In the recent paper [30], our result for the four-loop Lüscher correction of the  $su(2)$  Konishi operator is confirmed, and also results for a single impurity are obtained using a (non-symmetric) deformation of the  $Y$ -system.



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## References

- [1] J. M. Maldacena, “The large  $N$  limit of superconformal field theories and supergravity”, Adv. Theor. Math. Phys. **2**, 231 (1998) [arXiv:hep-th/9711200];  
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B **428**, 105 (1998) [hep-th/9802109];  
E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].
- [2] J. A. Minahan and K. Zarembo, “The Bethe-ansatz for  $\mathcal{N} = 4$  super Yang-Mills,” JHEP **0303**, 013 (2003) [hep-th/0212208].
- [3] N. Beisert and M. Staudacher, “Long-range  $PSU(2, 2|4)$  Bethe ansaetze for gauge theory and strings,” Nucl. Phys. B **727**, 1 (2005) [hep-th/0504190].
- [4] M. Staudacher, “The factorized  $S$ -matrix of CFT/AdS,” JHEP **0505**, 054 (2005) [arXiv:hep-th/0412188].
- [5] N. Beisert, “The  $su(2|2)$  dynamic  $S$ -matrix,” Adv. Theor. Math. Phys. **12**, 945 (2008) [arXiv:hep-th/0511082];  
N. Beisert, “The Analytic Bethe Ansatz for a Chain with Centrally Extended  $su(2|2)$  Symmetry,” J. Stat. Mech. **0701** P017 (2007) [arXiv:nlin/0610017].
- [6] G. Arutyunov, S. Frolov and M. Zamaklar, “The Zamolodchikov-Faddeev algebra for  $AdS_5 \times S^5$  superstring,” JHEP **0704**, 002 (2007) [arXiv:hep-th/0612229].
- [7] N. Beisert, R. Hernandez and E. Lopez, “A crossing-symmetric phase for  $AdS(5) \times S^{*5}$  strings,” JHEP **0611**, 070 (2006) [arXiv:hep-th/0609044].

- [8] N. Beisert, B. Eden and M. Staudacher, “Transcendentality and crossing,” J. Stat. Mech. **0701**, P021 (2007) [arXiv:hep-th/0610251].
- [9] R. A. Janik, “The  $AdS_5 \times S^5$  superstring worldsheet  $S$ -matrix and crossing symmetry,” Phys. Rev. D **73**, 086006 (2006) [arXiv:hep-th/0603038].
- [10] M.J. Martins and C.S. Melo, “The Bethe ansatz approach for factorizable centrally extended  $S$ -matrices,” Nucl. Phys. **B785**, 246 (2007) [arXiv:hep-th/0703086].
- [11] J. Ambjorn, R. A. Janik and C. Kristjansen, “Wrapping interactions and a new source of corrections to the spin-chain / string duality,” Nucl. Phys. B **736**, 288 (2006) [arXiv:hep-th/0510171].
- [12] F. Fiamberti, A. Santambrogio, C. Sieg, and D. Zanon, “Anomalous dimension with wrapping at four loops in  $\mathcal{N} = 4$  SYM ,” Nucl. Phys. **B805**, 231 (2008) [hep-th/0806.2095].
- [13] Z. Bajnok and R. A. Janik, “Four-loop perturbative Konishi from strings and finite size effects for multiparticle states,” Nucl. Phys. B **807**, 625 (2009) [arXiv:0807.0399].
- [14] Z. Bajnok, R. A. Janik, and T. Lukowski, “Four loop twist two, BFKL, wrapping and strings,” Nucl. Phys. B **816**, 376 (2009) [arXiv:0811.4448].
- [15] R. G. Leigh and M. J. Strassler, “Exactly marginal operators and duality in four-dimensional  $\mathcal{N} = 1$  supersymmetric gauge theory,” Nucl. Phys. **B447**, 95 (1995) [arXiv:hep-th/9503121].
- [16] A. Mauri, S. Penati, A. Santambrogio, and D. Zanon, “Exact results in planar  $\mathcal{N} = 1$  superconformal Yang-Mills theory,” JHEP **11**, 024 (2005) [arXiv:hep-th/0507282].
- [17] O. Lunin and J. M. Maldacena, “Deforming field theories with  $U(1) \times U(1)$  global symmetry and their gravity duals,” JHEP **05**, 033 (2005) [hep-th/0502086].
- [18] S. Frolov, “Lax pair for strings in Lunin-Maldacena background,” JHEP **05**, 069 (2005) [hep-th/0503201].
- [19] D. V. Bykov and S. Frolov, “Giant magnons in TsT-transformed  $AdS_5 \times S^5$ ,” JHEP **07**, 071 (2008) [arXiv:hep-th/0805.1070].
- [20] S. A. Frolov, R. Roiban, and A. A. Tseytlin, “Gauge - string duality for superconformal deformations of  $\mathcal{N} = 4$  super Yang-Mills theory,” JHEP **07**, 045 (2005) [hep-th/0503192].

- [21] D. Berenstein and S. A. Cherkis, “Deformations of  $\mathcal{N} = 4$  SYM and integrable spin chain models,” Nucl. Phys. **B702**, 49 (2004) [hep-th/0405215].
- [22] N. Beisert and R. Roiban, “Beauty and the twist: The Bethe ansatz for twisted  $\mathcal{N} = 4$  SYM,” JHEP **08**, 039 (2005) [hep-th/0505187].
- [23] F. Fiamberti, A. Santambrogio, C. Sieg, and D. Zanon, “Finite-size effects in the superconformal  $\beta$ -deformed  $\mathcal{N} = 4$  SYM,” JHEP **08**, 057 (2008) [hep-th/0806.2103].
- [24] F. Fiamberti, A. Santambrogio, C. Sieg, and D. Zanon, “Single impurity operators at critical wrapping order in the beta-deformed  $\mathcal{N} = 4$  SYM,” JHEP **08**, 034 (2009) [hep-th/0811.4594].
- [25] N. Gromov, V. Kazakov, and P. Vieira, “Integrability for the Full Spectrum of Planar AdS/CFT,” Phys. Rev. Lett. **103**, 131601 (2009) [arXiv:hep-th/0901.3753].
- [26] N. Reshetikhin, “Multiparameter quantum groups and twisted quasitriangular Hopf algebras,” Lett. Math. Phys. **20**, 331 (1990); I. Roditi, “Integrable Systems and Quantum Groups,” Braz. J. Phys. **30**, 357 (2000).
- [27] C. Ahn, Z. Bajnok, D. Bombardelli, and R. I. Nepomechie, in preparation.
- [28] J. Gunnesson, “Wrapping in maximally supersymmetric and marginally deformed  $\mathcal{N} = 4$  Yang-Mills,” JHEP **0904**, 130 (2009) [arXiv:0902.1427];  
M. Beccaria and G. F. De Angelis, “On the wrapping correction to single magnon energy in twisted  $\mathcal{N} = 4$  SYM,” Int. J. Mod. Phys. A **24**, 5803 (2009) [arXiv:0903.0778].
- [29] D. Bombardelli, D. Fioravanti and R. Tateo, “Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal,” J. Phys. A **42**, 375401 (2009) [arXiv:0902.3930];  
N. Gromov, V. Kazakov, A. Kozak and P. Vieira, “Exact Spectrum of Anomalous Dimensions of Planar  $\mathcal{N} = 4$  Supersymmetric Yang-Mills Theory: TBA and excited states,” Lett. Math. Phys. **91**, 265 (2010) [arXiv:0902.4458];  
G. Arutyunov and S. Frolov, “Thermodynamic Bethe Ansatz for the  $AdS_5 \times S^5$  Mirror Model,” JHEP **05** 068 (2009) [arXiv:0903.0141].
- [30] N. Gromov and F. Levkovich-Maslyuk, “Y-system and  $\beta$ -deformed  $\mathcal{N} = 4$  Super-Yang-Mills,” [arXiv:1006.5438].