

Hyperconifold Transitions, Mirror Symmetry, and String Theory

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Abstract

Multiply-connected Calabi-Yau threefolds are of particular interest for both string theorists and mathematicians. Recently it was pointed out that one of the generic degenerations of these spaces (occurring at codimension one in moduli space) is an isolated singularity which is a finite cyclic quotient of the conifold; these were called hyperconifolds. It was also shown that if the order of the quotient group is even, such singular varieties have projective crepant resolutions, which are therefore smooth Calabi-Yau manifolds. The resulting topological transitions were called hyperconifold transitions, and change the fundamental group as well as the Hodge numbers. Here Batyrev's construction of Calabi-Yau hypersurfaces in toric fourfolds is used to demonstrate that certain compact examples containing the remaining hyperconifolds — the \mathbb{Z}_3 and \mathbb{Z}_5 cases — also have Calabi-Yau resolutions. The mirrors of the resulting transitions are studied and it is found, surprisingly, that they are ordinary conifold transitions. These are the first examples of conifold transitions with mirrors which are more exotic extremal transitions. The new hyperconifold transitions are also used to construct a small number of new Calabi-Yau manifolds, with small Hodge numbers and fundamental group \mathbb{Z}_3 or \mathbb{Z}_5 . Finally, it is demonstrated that a hyperconifold is a physically sensible background in Type IIB string theory. In analogy to the conifold case, non-perturbative dynamics smooth the physical moduli space, such that hyperconifold transitions correspond to non-singular processes in the full theory.

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1 Introduction and discussion

This paper is a follow-up to [1], in which a class of threefold singularities and associated topological transitions were studied. These are isolated Calabi-Yau threefold singularities which are quotients of the conifold by a finite cyclic group \mathbb{Z}_N ; such a singularity was named a \mathbb{Z}_N -hyperconifold. They occur naturally in singular Calabi-Yau varieties which are limits of families of smooth multiply-connected spaces, when the generically-free group action on the covering space develops a fixed point.

It was shown in [1] that any projective variety with a \mathbb{Z}_{2M} -hyperconifold singularity has a projective crepant resolution, establishing the existence of hyperconifold *transitions* between smooth compact Calabi-Yau threefolds. The analysis was not sufficient to demonstrate the existence of the remaining cases — the \mathbb{Z}_3 - and \mathbb{Z}_5 -hyperconifold transitions — as the local resolution process did not guarantee that the resolved manifold was projective (and hence Kähler). Like the more familiar conifold transitions, hyperconifold transitions change the Hodge numbers; for a \mathbb{Z}_N -hyperconifold transition, the change is

$$\delta(h^{1,1}, h^{2,1})_{\mathbb{Z}_N} = (N - 1, -1) . \tag{1}$$

A novel feature is that the fundamental group can also change.

The present work has several objectives. We work mainly within the class of Calabi-Yau hypersurfaces in toric fourfolds, first described systematically by Batyrev [2] and then enumerated by Kreuzer and Skarke [3]. The formalism is reviewed in Section 2,

and then used in Section 3 to demonstrate that \mathbb{Z}_3 - and \mathbb{Z}_5 -hyperconifold transitions do connect compact Calabi-Yau manifolds. Perhaps more interestingly, it can also be used to study the mirror processes to these transitions, which turn out to be ordinary conifold transitions. They therefore provide a counter-example to an old conjecture of Morrison [4] that the mirror of a conifold transition is another conifold transition. The examples herein show that, while this is a very tempting conjecture, it is not true in general. They also motivate a modest conjecture, that the mirror process to any \mathbb{Z}_N -hyperconifold transition is a conifold transition in which the intermediate variety has N nodes. It is probably possible to use the local techniques of [5, 6] to prove this [7].

The mirror conifold transitions have another interesting feature. Batyrev and Kreuzer showed that within the class of Calabi-Yau hypersurfaces in toric fourfolds, mirror symmetry exchanges the fundamental group (which in these cases can only be $\mathbb{Z}_2, \mathbb{Z}_3$ or \mathbb{Z}_5) with the Brauer group, which is the torsion part of $H^3(X, \mathbb{Z})$ [8]. Since the hyperconifold transitions studied here destroy the fundamental group, their mirror conifold transitions should destroy the Brauer group. This is not a new phenomenon (see for example [9]), but here mirror symmetry gives a clear reason for it to occur.

Once we know that hyperconifold transitions exist, we can use them to try to construct new Calabi-Yau manifolds. This was mentioned in [1], but no explicit examples were given. In Section 3.1.3 and Section 3.2.1, we use the new results of this paper to construct some previously unknown Calabi-Yau manifolds via \mathbb{Z}_3 - and \mathbb{Z}_5 -hyperconifold transitions.

If two Calabi-Yau manifolds are mathematically connected by a topological transition, we might ask whether the corresponding physical theories, obtained by compactifying string theory on these spaces, are also smoothly connected. It is shown in Section 4 that the physical moduli space, at least in Type IIB string theory, is perfectly smooth through a point corresponding to a hyperconifold transition. The story is very similar to that of a conifold transition, worked out in [10]. The results of [1] and the present paper therefore have significant implications for the connectedness of the moduli space of Calabi-Yau threefolds, and the associated string vacua. Soon after Reid suggested the idea that all threefolds with $c_1 = 0$ may be connected by conifold transitions [11], this was shown to be true for almost all known Calabi-Yau examples [12, 13]. But conifold transitions cannot change the fundamental group, so this cannot be the whole story. Hyperconifold transitions then fill an important gap, since they still involve relatively mild singularities, but can change the fundamental group as well as the Hodge numbers. Whether conifold and hyperconifold transitions between them can connect all Calabi-Yau threefolds is an interesting open question.

Before moving on, it may be helpful to illustrate the hyperconifold phenomenon by considering a simple non-compact example. Let the group \mathbb{Z}_2 act on \mathbb{C}^4 as follows:

$$(y_1, y_2, y_3, y_4) \rightarrow (-y_1, -y_2, -y_3, -y_4) .$$

Then suppose we have a hypersurface \tilde{X} given by a polynomial equation $f = 0$. If we

want \tilde{X} to be invariant under \mathbb{Z}_2 , and its quotient $X = \tilde{X}/\mathbb{Z}_2$ to be Calabi-Yau, then the polynomial f must be invariant. As such, it can be written (perhaps after a change of coordinates) as

$$f = \alpha_0 + y_1 y_4 - y_2 y_3 + \mathcal{O}(y^3), \quad (2)$$

since no invariant linear terms exist. For $\alpha_0 \neq 0$, \tilde{X} is smooth, and does not contain the origin, so the quotient X is also smooth, with fundamental group isomorphic to \mathbb{Z}_2 . However, if we take the limiting case $\alpha_0 = 0$, we see that \tilde{X} then contains the origin, and that this point is a node, or conifold singularity. The corresponding singularity on X is therefore a \mathbb{Z}_2 quotient of the conifold. Locally, it looks like the vector bundle $\mathcal{O}(-2, -2) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$, with the zero section projected to a point. Blowing up the singular point gives a crepant resolution of the singularity by restoring this zero section. For more details, including the toric data for this and the other hyperconifold singularities, see [1].

Our main interest here is in compact Calabi-Yau threefolds, and transitions between them. Most known multiply-connected Calabi-Yau threefolds are obtained as free quotients of complete intersections in products of projective spaces. A few examples were discovered long ago [14, 15, 16], and recently a more systematic search has been performed, leading to a complete enumeration of the manifolds which can be constructed this way [17, 18, 19]. A smaller number of examples occur as hypersurfaces in toric fourfolds [8], or as free non-toric quotients of such hypersurfaces [20], which is a largely unexplored class.² The cyclic fundamental groups which are known to occur are those of order $N = 2, 3, 4, 5, 6, 8, 10, 12$. In all cases, there is an action of \mathbb{Z}_N on the ambient space which has fixed points, and these are missed by a generic member of the family of embedded Calabi-Yau threefolds. If such a threefold is deformed until it does contain a fixed point, the quotient variety develops a hyperconifold singularity.³

2 Toric geometry and the Batyrev construction

Here we will briefly review Batyrev's construction of Calabi-Yau hypersurfaces in toric varieties [2]. This will serve mainly to establish notation, as several conventions have been used in the literature. We will specialise to the case of Calabi-Yau threefolds in toric fourfolds.

Let N be a lattice, $N \cong \mathbb{Z}^4$, and M its dual lattice. It is convenient to choose a basis for N , with corresponding dual basis for M , so we can use coordinates. Points of N correspond to one-parameter subgroups of the algebraic torus $\mathbb{T}^4 = (\mathbb{C}^*)^4$ via the map

$$N \ni (n_1, n_2, n_3, n_4) \mapsto \{(\lambda^{n_1}, \lambda^{n_2}, \lambda^{n_3}, \lambda^{n_4}) \mid \lambda \in \mathbb{C}^*\},$$

²There are also certain exceptional cases, such as the quotients of the Horrocks-Mumford quintic [21] and the Gross-Popescu manifolds [22, 23], but these are not discussed here.

³It is possible for worse singularities to occur instead, because the quadratic terms in the analogue of Equation (2) may always be degenerate. This does not seem to happen in products of projective spaces.

while points of M correspond to monomials on \mathbb{T}^4 considered as an algebraic variety, via the map

$$M \ni (m_1, m_2, m_3, m_4) \mapsto t_1^{m_1} t_2^{m_2} t_3^{m_3} t_4^{m_4} . \quad (3)$$

We will denote by χ^m the monomial associated to $m \in M$. The two lattices are naturally embedded in the vector spaces $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, respectively.

Batyrev's construction begins with a polytope Δ in $M_{\mathbb{R}}$, which satisfies the following conditions:

- The vertices of Δ are lattice points *i.e.* they lie on $M \subset M_{\mathbb{R}}$.
- The faces of Δ lie on hyperplanes of the form

$$H_n = \{m \in M_{\mathbb{R}} \mid \langle m, n \rangle \geq -1\}$$

where $n \in N$ is a primitive lattice vector.⁴ Note that this implies that Δ contains the origin as its unique interior point.

Such a Δ is called reflexive. We also define the dual polytope $\Delta^* \subset N_{\mathbb{R}}$ by

$$\Delta^* = \{n \in N_{\mathbb{R}} \mid \langle m, n \rangle \geq -1 \ \forall m \in \Delta\} .$$

By taking cones over the faces of Δ^* , we get the fan for a toric variety which we will denote by \mathbb{P}_{Δ} (the notation reflects the fact that every variety constructed this way is projective). It is a simple fact that Δ^* is also reflexive.

We need one final definition. Given a Laurent polynomial $f = \sum_{m \in M} c_m \chi^m$, its Newton polytope is the convex hull in $M_{\mathbb{R}}$ of those points for which $c_m \neq 0$. We will be interested in those f which have Δ as their Newton polytope. The vanishing of such an f gives an affine sub-variety of \mathbb{T}^d , and the closure of this inside \mathbb{P}_{Δ} is a Calabi-Yau variety.

Since both Δ and Δ^* are reflexive, we can reverse their roles in the above construction. The two families of Calabi-Yau hypersurfaces are then mirror to each other.

2.1 Homogeneous coordinates

It is very convenient to use homogeneous coordinates for the ambient toric space, as introduced by Cox in [24]. Let Σ be a fan for a toric variety Z . Then we can construct Z from Σ as follows.

Suppose Σ contains d one-dimensional cones, which are rays, and let v_{ρ} be the first lattice vector on the ρ 'th ray. We associate with it a complex coordinate z_{ρ} . Together, these are coordinates on \mathbb{C}^d , and will be our homogeneous coordinates for Z . As in the construction of ordinary projective space, our first step is to delete a certain subset of \mathbb{C}^d . In short, we *keep* the set where $z_{\rho_1}, \dots, z_{\rho_k}$ vanish simultaneously if and only if the vectors $v_{\rho_1}, \dots, v_{\rho_k}$ span a cone in Σ . We then impose a number of equivalence

⁴A lattice vector is called primitive if it is the first lattice point on a ray.

relations on the resulting space, one for each linear relation satisfied by the vectors, as follows

$$\sum_{\rho} a_{\rho} v_{\rho} = 0 \Rightarrow (z_1, z_2, \dots, z_d) \sim (\lambda^{a_1} z_1, \lambda^{a_2} z_2, \dots, \lambda^{a_d} z_d) \quad \forall \lambda \in \mathbb{C}^* .$$

There can be further, discrete identifications, which will be important for us, but we will postpone their discussion for now.

In the cases of interest, Σ consists of cones over (some triangulation of) the faces of a reflexive polytope Δ^* . Calabi-Yau hypersurfaces can now be defined by the vanishing of homogeneous polynomials, which are obtained from points of M via a homogeneous version of Equation (3):

$$M \ni m \mapsto \prod_{\rho} z_{\rho}^{\langle m, v_{\rho} \rangle + 1} . \quad (4)$$

3 Transitions between toric hypersurfaces and their quotients

In this section we will turn to examples of hyperconifold transitions between Calabi-Yau hypersurfaces in toric fourfolds. The required analysis of reflexive polytopes was greatly assisted by the software package PALP [25].

3.1 The \mathbb{Z}_3 quotient of the bicubic

The family of ‘bicubic’ manifolds $X^{2,83}$ are hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$, cut out by a single polynomial of bidegree (3, 3). Products of projective spaces are toric varieties, so we can use Batyrev’s formalism for Calabi-Yau hypersurfaces in toric fourfolds [2].

If we take homogeneous coordinates (z_0, z_1, z_2) on the first \mathbb{P}^2 and (z_3, z_4, z_5) on the second \mathbb{P}^2 , then we can take the corresponding vectors in $N \cong \mathbb{Z}^4$ to be

$$\begin{array}{rcccccc} & z_0 & z_1 & z_2 & z_3 & z_4 & z_5 \\ \hline \mathbb{P}^2 \times \mathbb{P}^2 : & 1 & 0 & -1 & 0 & 0 & 0 \\ & 0 & 1 & -1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 0 & -1 \\ & 0 & 0 & 0 & 0 & 1 & -1 \end{array} .$$

It is easy to see that the linear relations between these vectors induce the two expected rescalings of the coordinates. The convex hull of these six points is a reflexive polytope Δ^* .

Using Equation (4), we can write down the monomial corresponding to a point of the dual lattice M in the present case:

$$(m_1, m_2, m_3, m_4) \mapsto z_0^{1+m_1} z_1^{1+m_2} z_2^{1-m_1-m_2} z_3^{1+m_3} z_4^{1+m_4} z_5^{1-m_3-m_4} \equiv \chi^m . \quad (5)$$

It is easy enough to check that the polytope Δ , dual to Δ^* above, corresponds exactly to bicubic monomials under this map.

We now define an action of \mathbb{Z}_3 , generated by

$$g_3 : z_i \rightarrow \zeta^i z_i , \tag{6}$$

where $\zeta = \exp(2\pi i/3)$. The resulting orbifold $(\mathbb{P}^2 \times \mathbb{P}^2)/\mathbb{Z}_3$ is also toric, and we obtain its fan simply by sub-dividing the lattice N . It is instructive to carry this out explicitly. Polynomials on the quotient are exactly those polynomials on the covering space which are invariant under the \mathbb{Z}_3 action. Under this action, we see from (5) that

$$\chi^m \rightarrow \zeta^{m_1 - m_2 + m_3 - m_4} \chi^m ,$$

so the sub-lattice $M' \subset M$ corresponding to \mathbb{Z}_3 -invariants is determined by the condition $m_1 - m_2 + m_3 - m_4 \equiv 0 \pmod{3}$. The polytope Δ is also reflexive with respect to M' , and so determines a family of Calabi-Yau hypersurfaces in the quotient.

A short algebraic exercise determines a basis for the corresponding dual lattice $N' \subset N_{\mathbb{R}}$, which is a refinement of the lattice N :

$$N' = \left\langle (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), \left(-\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) \right\rangle .$$

We can re-express the generators of our fan in terms of this basis:

	z_0	z_1	z_2	z_3	z_4	z_5
$(\mathbb{P}^2 \times \mathbb{P}^2)/\mathbb{Z}_3 :$	1	0	-1	0	1	-1
	0	1	-1	0	-1	1
	0	0	0	1	1	-2
	0	0	0	0	3	-3

Generic hypersurfaces determined by Δ miss the orbifold points, and therefore give a family of smooth, multiply connected Calabi-Yau threefolds, with Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 29)$. This is well known; see for example [26]. We are now interested in specialising to the case where the Calabi-Yau hypersurface intersects one of these singularities, and therefore has a \mathbb{Z}_3 -hyperconifold singularity.

Let us focus on the fixed point $z_1 = z_2 = z_4 = z_5 = 0$. There is a single monomial which is non-zero at this point: $z_0^3 z_3^3$. This corresponds to the point $(2, -1, 2, -2) \in M'$, which is in fact a vertex of Δ . So we obtain a hyperconifold singularity precisely when the coefficient of this monomial vanishes, and in this case the polynomial f has a reduced Newton polytope $\widehat{\Delta}$, obtained as the convex hull of the lattice points in Δ , minus the vertex above.

So it is now clear how the resolution process proceeds: the new dual polytope, $\widehat{\Delta}^*$, is larger than Δ^* , and the extra vertices correspond to exceptional divisors resolving the orbifold singularity in the ambient space. The results of [2] imply that this resolves

the \mathbb{Z}_3 -hyperconifold singularity as well, but we will check this explicitly below. We see that in this case, and indeed all those in the present paper, the hyperconifold transition is a link in the web of toric hypersurfaces described by Kreuzer and Skarke [3].

In [1], the local toric structure of the \mathbb{Z}_3 -hyperconifold singularity was described, and the corresponding toric diagram is reproduced in Figure 1, along with those for the two distinct local crepant resolutions. The resolution we have just implicitly constructed must correspond to one of these. It will turn out to be the first, but to see this we will have to go into more detail.

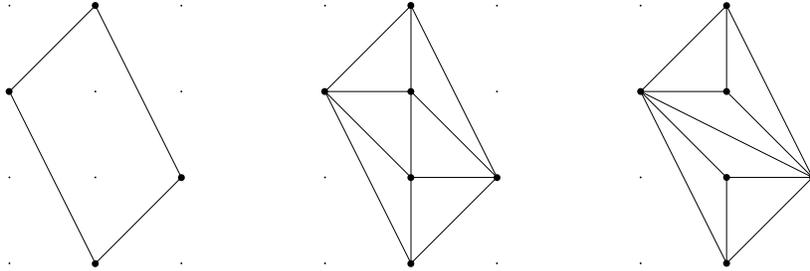


Figure 1: The toric diagram for the \mathbb{Z}_3 -hyperconifold, and its two crepant resolutions. The first resolution is the one which occurs in the example of this section.

It turns out that $\widehat{\Delta}^*$, which corresponds to a space in which the orbifold singularity is resolved, is obtained from Δ^* by adding two more one-dimensional cones, which are contained in the top-dimensional cone of Δ^* corresponding to the orbifold point.⁵ We will call the corresponding new homogeneous coordinates z_6, z_7 ; our list of coordinates, and corresponding lattice points, is now

z_0	z_1	z_2	z_3	z_4	z_5	z_6	z_7
1	0	-1	0	1	-1	-1	0
0	1	-1	0	-1	1	0	0
0	0	0	1	1	-2	-1	0
0	0	0	0	3	-3	-1	1

The new relations are

$$3v_6 - v_1 - 2v_2 - v_4 - 2v_5 = 0$$

$$3v_7 - 2v_1 - v_2 - 2v_4 - v_5 = 0$$

PALP gives us the various faces of $\widehat{\Delta}^*$. There are four non-simplicial facets, which must be triangulated in order to resolve the corresponding toric fourfold. We will focus

⁵It should be noted that adding just one of the two new cones gives a polytope which is not reflexive; there is no ‘halfway house’ between Δ^* and $\widehat{\Delta}^*$.

on one such facet; the other three can be treated identically. Its vertices correspond to the homogeneous coordinates z_0, z_1, z_4, z_5, z_7 . The two-dimensional faces of this polyhedron are then

$$\begin{aligned} &\langle z_0 z_1 z_4 \rangle, \quad \langle z_0 z_1 z_5 \rangle, \quad \langle z_0 z_4 z_5 \rangle, \\ &\langle z_1 z_4 z_7 \rangle, \quad \langle z_1 z_5 z_7 \rangle, \quad \langle z_4 z_5 z_7 \rangle. \end{aligned}$$

We see that z_0 and z_7 appear thrice each, while the other coordinates each appear four times; this implies that the polyhedron looks like Figure 2. It has an obvious maximal triangulation, given by adding a new two-dimensional face $\langle z_1 z_4 z_5 \rangle$, which divides it into two minimal tetrahedra. In fact, we have no choice but to take this triangulation — we are resolving an orbifold point of $(\mathbb{P}^2 \times \mathbb{P}^2)/\mathbb{Z}_3$, in which z_1, z_4, z_5 are certainly allowed to vanish simultaneously, so this two-face was already there.

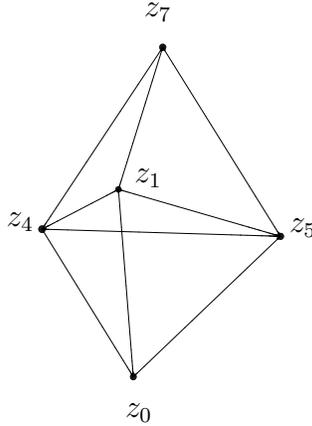


Figure 2: One of the non-simplicial faces of $\widehat{\Delta}^*$. Adding the two-simplex $\langle z_1 z_4 z_5 \rangle$ gives a maximal triangulation. Vertices are labelled by the corresponding homogeneous coordinates.

Batyrev tells us that the procedure above resolves the \mathbb{Z}_3 -hyperconifold singularity, and we would like to know to which local resolution this corresponds, where the two possibilities are shown in Figure 1. To answer this question we will examine the exceptional set of the resolution. Inspection of Figure 1, and the ‘star construction’ of toric geometry, as described in [27], tell us that in the first case, the exceptional set consists of two copies of the Hirzebruch surface \mathbb{F}_1 , intersecting along a \mathbb{P}^1 , while in the second it consists of two disjoint surfaces, each isomorphic to \mathbb{P}^2 .

The two components of the exceptional set in the case at hand are given by $z_6 = 0$ and $z_7 = 0$, respectively. Let us examine the component $z_6 = 0$ first. After the triangulation described above, $z_6 = 0$ implies that $z_0 \neq 0$ and $z_3 \neq 0$. We can therefore set $z_0 = z_3 = 1$, using the usual rescaling relations of the two \mathbb{P}^2 s. This leaves us with homogeneous coordinates z_1, z_2, z_4, z_5, z_7 for some toric threefold, and remaining identifications which are equivalent to the following:

$$(z_1, z_2, z_4, z_5, z_7) \sim (\mu z_1, \lambda z_2, \mu z_4, \lambda z_5, \mu^{-2} \lambda z_7) \quad , \quad \lambda, \mu \in \mathbb{C}^* .$$

The interpretation of this is that z_1, z_4 are homogeneous coordinates for a base \mathbb{P}^1 , while z_2, z_5, z_7 are homogeneous coordinates on the fibres of the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ (indeed, a careful inspection of the fan reveals that before taking the quotient, we must delete the sets $\{z_1 = z_4 = 0\}$ and $\{z_2 = z_5 = z_6 = 0\}$). The exceptional divisor in our Calabi-Yau hypersurface is then given by restricting the equation $f = 0$ to this threefold.

If we take all the monomials coming from $\widehat{\Delta}$ and set $z_0 = z_3 = 1$ and $z_6 = 0$, we are left with

$$z_1 z_2, z_2 z_4, z_1 z_5, z_4 z_5, z_1^3 z_7, \\ z_1^2 z_4 z_7, z_1 z_4^2 z_7, z_4^3 z_7.$$

Consider an arbitrary linear combination of these. If we set z_1, z_4 to any values, we are left with something linear in the homogeneous coordinates of the \mathbb{P}^2 fibre. So the exceptional divisor is a \mathbb{P}^1 bundle over the base \mathbb{P}^1 parametrised by z_1, z_4 , i.e. it is a Hirzebruch surface. An identical analysis holds for the component $z_7 = 0$, and it is easily checked that the two components overlap on a \mathbb{P}^1 , so the resolution realised is the first of those in Figure 1.

Finally, we ask about the topological data of the resolved Calabi-Yau. This can be calculated directly from the polytope $\widehat{\Delta}$, and the Hodge numbers turn out to be $(h^{1,1}, h^{2,1}) = (4, 28)$. So the change realised by the hyperconifold transition is $\delta(h^{1,1}, h^{2,1}) = (2, -1)$, in accord with the argument of [1] which implied Equation (1) — we imposed a single condition on the complex structure, and the resolution introduced two new divisor classes. Furthermore, the new family of manifolds $X^{4,28}$ are simply-connected, because the fundamental group was destroyed by allowing a fixed point of the \mathbb{Z}_3 action to develop; this also follows simply from Theorem 1.6 of [8].

3.1.1 The mirror transition

Batyrev's construction allows us to easily identify the mirror of a Calabi-Yau hypersurface in a toric variety: we simply exchange the roles of the polytopes Δ and Δ^* . This will allow us to identify the process which is mirror to the above transition; on general grounds, it will be a projection from $X^{29,2} \subset \mathbb{P}_{\Delta^*}$ to a singular member of $X^{28,4} \subset \mathbb{P}_{\widehat{\Delta}^*}$, followed by a smoothing. Surprisingly, we will see that this turns out to be an ordinary conifold transition, as discussed in Section 1.

We obtain $\widehat{\Delta}$ from Δ by removing a single vertex, corresponding to blowing down a divisor.⁶ Four other points, which were interior to higher-dimensional faces of Δ , become vertices of $\widehat{\Delta}$. We will use w instead of z for the homogeneous coordinates in this section, to avoid confusion, and order them so that these four are w_1, w_2, w_3, w_4 ; the corresponding points are the vertices of a two-face. In any maximal triangulation

⁶The divisor, which is a threefold, may be blown down to a curve or a surface, depending on the chosen triangulation of $\widehat{\Delta}$. We are interested in maximal triangulations, in which case the divisor is blown down to a surface, as we will see.

of $\widehat{\Delta}$, this must be divided into two triangles, by adding to the fan either $\langle w_1 w_2 \rangle$ or $\langle w_3 w_4 \rangle$. We will consider the first option. Then adding the new point to pass to Δ corresponds to blowing up along the toric surface S given by $w_1 = w_2 = 0$. We see this by noting that, if we call the new coordinate w_0 , the associated vectors satisfy $u_0 - u_1 - u_2 = 0$.

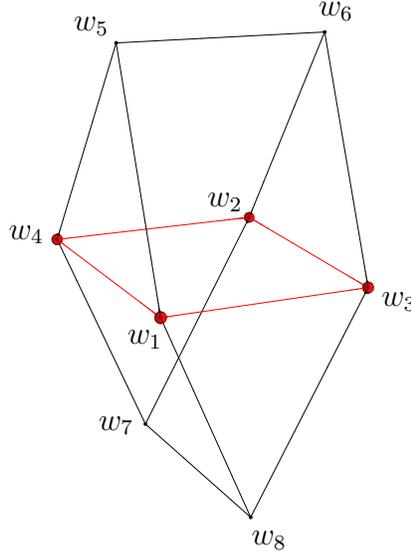


Figure 3: Part of the boundary of $\widehat{\Delta}$, with the vertices labelled by the corresponding coordinates. The red two-face can be triangulated by adding the one-face $\langle w_1 w_2 \rangle$. This one-face is then bisected by the new ray which is added to pass to Δ .

Now consider the singular Calabi-Yau varieties, which are mirror to the hyperconifold from the last section. These spaces are singular members of $X^{28,4}$, given by setting to zero the coefficients of the two monomials coming from the points v_6, v_7 of $\widehat{\Delta}^*$, which do not belong to Δ^* . It is a quick check that these two monomials are the only ones which are not identically zero on the surface $w_1 = w_2 = 0$, so the singular varieties contain the surface S from above, and are resolved when it is blown up.

In the ambient toric space, the exceptional divisor is a \mathbb{P}^1 bundle over S , since S is codimension two. We want to calculate the exceptional set in the resolved Calabi-Yau manifolds $X^{29,2}$. After the introduction of w_0 , the coordinates w_1 and w_2 become homogeneous coordinates on the \mathbb{P}^1 fibres. Inspecting Figure 3, we see that on the exceptional divisor, given by $w_0 = 0$, we must have $w_i \neq 0 \forall i > 8$, so we can use the toric scaling relations to set $w_i = 1 \forall i > 8$. Then, setting $w_0 = 0$, the most general polynomial defining our Calabi-Yau hypersurface is

$$w_1(\alpha_1 w_5^3 w_4 + \alpha_2 w_8^3 w_3) + w_2(\alpha_3 w_7^3 w_4 + \alpha_4 w_8^3 w_3) = 0 .$$

3.1.3 New manifolds from the $\mathbb{Z}_3 \times \mathbb{Z}'_3$ quotient

All the manifolds discussed above are hypersurfaces in toric fourfolds, and therefore already appear in the Kreuzer-Skarke list; we merely showed that they are connected by hyperconifold transitions. Here we turn to an example of how new Calabi-Yau manifolds can be constructed by considering hyperconifold transitions from known ones.

A smooth sub-family of bicubics actually admit a free action by $\mathbb{Z}_3 \times \mathbb{Z}'_3$, giving a smooth quotient family with Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 11)$. The first \mathbb{Z}_3 still acts as in Equation (6), but the second does not act torically, instead permuting the homogeneous coordinates of each \mathbb{P}^2 ,

$$g'_3 : z_0 \rightarrow z_1 \rightarrow z_2 \rightarrow z_0, \quad z_3 \rightarrow z_4 \rightarrow z_5 \rightarrow z_3.$$

The quotient manifolds $X^{2,11} = X^{2,83} / \mathbb{Z}_3 \times \mathbb{Z}'_3$ are therefore not toric hypersurfaces, and we will see that we can generate genuinely new manifolds by hyperconifold transitions from them.

The action of g'_3 on the ambient space permutes the nine fixed points of g_3 , which therefore fall into three orbits of three. So now, when we ask for a fixed point of g_3 to develop on the covering space $X^{2,83}$, three will in fact develop, and these are identified by the action of g'_3 . Taking the quotient by just \mathbb{Z}_3 and simultaneously resolving the three singularities will realise the transition from $X^{2,29}$ to $X^{8,26}$, i.e. the first three links in the chain of last section, in one step. We can restrict the Kähler form such that the exceptional divisors over each point have the same volume, and in this way $X^{8,26}$ inherits a free action of \mathbb{Z}'_3 , by which we can quotient. We have therefore in fact described a hyperconifold transition from $X^{2,11} = X^{2,83} / \mathbb{Z}_3 \times \mathbb{Z}'_3$ to a new manifold $X^{4,10} = X^{8,26} / \mathbb{Z}'_3$. As before, we can now perform the same process for the remaining \mathbb{Z}'_3 -orbits of fixed points, of which there are two.

In summary, we obtain a short chain of hyperconifold transitions,

$$X^{2,11} \rightsquigarrow X^{4,10} \rightsquigarrow X^{6,9} \rightsquigarrow X^{8,8},$$

where the last three spaces all have fundamental group \mathbb{Z}_3 , being free quotients by g'_3 of $X^{8,26}$, $X^{14,23}$, $X^{20,20}$ respectively. The families $X^{4,10}$ and $X^{8,8}$ are certainly new manifolds, since no existing manifolds have the same Hodge numbers and fundamental group. $X^{6,9}$, on the other hand, could well be the same as Yau's famous three-generation manifold [28, 29]. This suspicion is strengthened by the fact that their covering spaces have the same Hodge numbers.

3.2 The \mathbb{Z}_5 quotient of the quintic

A smooth quintic hypersurface in \mathbb{P}^4 is a Calabi-Yau manifold, with Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 101)$. If we take homogeneous coordinates z_i , $i = 0, \dots, 4$, an action

of \mathbb{Z}_5 on \mathbb{P}^4 can be defined by

$$g_5 : z_i \rightarrow \zeta^i z_i ,$$

where $\zeta = \exp(2\pi i/5)$. It is well known that generic quintic polynomials invariant under this action determine smooth Calabi-Yau manifolds without fixed points. The resulting family of smooth quotients are $X^{1,21}$. We can perform an analysis almost identical to that in Section 3.1 to show that there is a hyperconifold transition to a simply-connected family $X^{5,20}$; here we only sketch the details.

Imposing a single condition on the complex structure of $X^{1,21}$, we can arrange for the covering space to contain one of the fixed points of the \mathbb{Z}_5 action, say $(1, 0, 0, 0, 0)$, and this gives rise to a \mathbb{Z}_5 -hyperconifold in $X^{1,21}$. As in the example of Section 3.1, the singular point is also a fixed point of the torus action on the ambient space, and when we resolve it we get another toric fourfold. The resolution introduces four new coordinates in this case, which we label z_5, z_6, z_7, z_8 , such that the homogeneous coordinates and corresponding vectors are:

z_0	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
1	0	0	-4	3	-2	0	-1	1
0	1	0	-3	2	-1	0	0	1
0	0	1	-2	1	-1	0	0	1
0	0	0	5	-5	2	-1	1	-2

The proper transforms of the singular Calabi-Yau varieties are smooth Calabi-Yau manifolds $X^{5,20}$ in this new ambient space. The Hodge numbers follow from the general formula (1), and are again confirmed by PALP.

The toric diagram for the \mathbb{Z}_5 -hyperconifold singularity is shown in Figure 5, and it is easy to see that there are several possible crepant resolutions. The topology of the one realised by the resolution constructed here is more difficult to find than in the analogous problem of Section 3.1, and has not been investigated.

The mirror to the transition above can be found by the same method as that in Section 3.1, and the story is very similar. The mirror varieties to the singular members of $X^{1,21}$ are singular members of $X^{20,5}$, all of which contain a toric surface S' , and generically have five nodes which lie on this surface. Blowing up along S' resolves the nodes, leading to smooth manifolds $X^{21,1}$, and this resolution is mirror to the deformation of the hyperconifold. Similarly, the nodes can be smoothed by passing to a general member of $X^{20,5}$, which is mirror to the resolution of the hyperconifold. So the mirror to this \mathbb{Z}_5 -hyperconifold transition is a conifold transition in which the intermediate variety has five nodes, consistent with our conjecture of Section 1. Note that $X^{20,5}$ has no torsion in its cohomology, but $X^{21,1}$ has Brauer group \mathbb{Z}_5 .

3.2.1 More transitions, and another new manifold

Again there are further hyperconifold transitions possible; this time there are five orbifold points in the original ambient space $\mathbb{P}^4/\mathbb{Z}_5$, and we get the following chain of

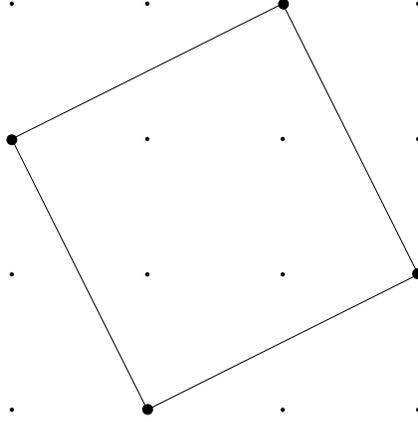


Figure 5: The toric diagram for the \mathbb{Z}_5 -hyperconifold.

transitions:

$$X^{1,21} \rightsquigarrow X^{5,20} \rightsquigarrow X^{9,19} \rightsquigarrow X^{13,18} \rightsquigarrow X^{17,17} \rightsquigarrow X^{21,16} .$$

The first manifold has fundamental group \mathbb{Z}_5 , while the other five all have torsion-free (co)homology.

In analogy with the bicubic case, we can now consider the action of a second group \mathbb{Z}'_5 , which acts by permuting the homogeneous coordinates of \mathbb{P}^4 . It is well known that there is a family of smooth hypersurfaces invariant under $\mathbb{Z}_5 \times \mathbb{Z}'_5$, giving rise to the quotient $X^{1,5} = X^{1,101} / \mathbb{Z}_5 \times \mathbb{Z}'_5$.

Now all five fixed points of \mathbb{Z}_5 are identified by the \mathbb{Z}'_5 action, so if we look for hyperconifold transitions from $X^{1,5}$, we get just one,

$$X^{1,5} \rightsquigarrow X^{5,4} ,$$

where the new manifold has fundamental group \mathbb{Z}_5 , and is a free quotient of $X^{21,16}$ from above. Once again, we have found a brand new manifold, in fact the first one known with Hodge numbers $(h^{1,1}, h^{2,1}) = (5, 4)$.

It is clear that there are many new manifolds, some with quite small Hodge numbers, waiting to be found via hyperconifold transitions from known spaces. No systematic approach to this has been attempted.

3.3 Calabi-Yau hypersurfaces in $\mathbb{P}^4_{(2,1,1,1,1)}$

It is clear from the previous sections that hyperconifolds do not only occur in the moduli space of multiply-connected Calabi-Yau manifolds. More generally, they can occur in families of varieties which are complete intersections in an ambient space with orbifold singularities, where a generic member of the family does not intersect the singularities. As an example, we will consider Calabi-Yau hypersurfaces in the weighted projective space $\mathbb{P}^4_{(2,1,1,1,1)}$.

If the homogeneous coordinates $(z_0, z_1, z_2, z_3, z_4)$ are assigned weights $(2, 1, 1, 1, 1)$, then the resulting weighted projective space has a \mathbb{Z}_2 orbifold singularity at the point $(1, 0, 0, 0, 0)$. This can be seen by considering the affine patch $z_0 \neq 0$, and noticing that there are two choices of rescaling parameter which set $z_0 \rightarrow 1$; they are $\pm \frac{1}{\sqrt{z_0}}$. The corresponding local coordinates are therefore subject to the identification $(y_1, y_2, y_3, y_4) \sim (-y_1, -y_2, -y_3, -y_4)$.

The family of Calabi-Yau hypersurfaces in this space are cut out by degree six (weighted) homogeneous polynomials. A generic such polynomial can, by a $GL(4, \mathbb{C})$ transformation on the last four coordinates, be put in the form

$$f = \alpha_0 z_0^3 + z_0^2(z_1 z_4 - z_2 z_3) + \dots$$

The corresponding smooth hypersurfaces are simply-connected, by Theorem 1.6 of [8], and the Hodge numbers are $(h^{1,1}, h^{2,1}) = (1, 103)$.

In the local coordinates near the orbifold point, f is just

$$f = \alpha_0 + (y_1 y_4 - y_2 y_3) + \dots,$$

so that over the distinguished locus in moduli space given by $\alpha_0 = 0$, the family develops a \mathbb{Z}_2 -hyperconifold singularity. As mentioned earlier, this singularity can be resolved by blowing up the orbifold point in the ambient space, taking us to a new family of smooth simply-connected Calabi-Yau threefolds, with Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 102)$. This is easily confirmed by use of the toric formalism.

This is an example of a hyperconifold transition between two simply-connected families, which furthermore does not belong to a series of such transitions starting with a multiply-connected manifold (if so, the Hodge number $h^{1,1}$ would have to be larger).

It should also be mentioned that hyperconifolds are not the only possibility in analogous situations. In some cases, a singularity will arise which is a quotient of a hypersurface singularity more severe than a node. The reader can see an example of this by considering Calabi-Yau hypersurfaces in $\mathbb{P}_{(4,1,1,1,1)}^4$.

4 Hyperconifolds in Type IIB string theory

Singular Calabi-Yau varieties are particularly interesting in the context of string compactification where, contrary to intuition, they often give rise to a consistent physical theory. It has been known since the pioneering work of [30, 31] that orbifold singularities can be understood in the context of string perturbation theory, whereas conifold singularities represent singularities of the worldsheet theory. However, in non-perturbative Type IIB string theory, the conifold singularity is resolved by the effects of light D-brane states [32]. Furthermore, when it is mathematically possible to carry out a conifold *transition* to a new Calabi-Yau manifold, this manifests in the physics

as a new branch of the low-energy moduli space [10]. It was suggested in [1] that a similar story should hold for hyperconifolds, and we will now show that this is indeed the case. The following argument is closely modelled on that of [10, 32], and also uses the insights of [33] about D-branes wrapped on multiply-connected cycles. Much of what follows is well known, but is included in order to give a relatively self-contained account.

4.1 The conifold

For expository reasons, we will consider the case where a hyperconifold singularity arises in a space $X = \tilde{X}/\mathbb{Z}_N$, so that we can first consider the conifold singularity which occurs on the covering space \tilde{X} .

On a Calabi-Yau threefold \tilde{X} , the homology group $H_3(\tilde{X}, \mathbb{Z})$ has a symplectic basis (with respect to the intersection form, which is necessarily symplectic) represented by three-cycles $\{A^I, B_I\}_{I=1, \dots, h^{2,1}(\tilde{X})+1}$. The complex structure moduli space of \tilde{X} admits complex homogeneous coordinates Z^I , and holomorphic ‘functions’ F_I defined in terms of the holomorphic three-form Ω by [34]

$$Z^I = \int_{A^I} \Omega, \quad F_I = \int_{B_I} \Omega.$$

The moduli space metric is Kähler, with Kähler potential

$$K = -\log \left[i \left(\bar{Z}^I F_I - Z^I \bar{F}_I \right) \right]. \quad (7)$$

The low-energy dynamics of the complex structure moduli fields is that of a non-linear sigma model with metric following from this potential.

There are harmonic three-forms $\{\alpha_I, \beta^I\}_{I=1, \dots, h^{2,1}(\tilde{X})+1}$ on \tilde{X} which are dual to the above cycles, and also related to each other by the Hodge star operator, $\beta^I = *\alpha_I$. The IIB theory contains a four-form potential $C^{(4)}$, with self-dual five-form field strength⁸ $F^{(5)} = *F^{(5)}$. Upon compactification, this gives rise to a number of massless $U(1)$ gauge fields, one for each of the harmonic three-forms, via a Kaluza-Klein reduction:

$$C^{(4)} = \sum_I \left(C^I \wedge \alpha_I + \tilde{C}_I \wedge \beta^I \right) + \dots. \quad (8)$$

The C_I and \tilde{C}^I are massless four-dimensional vector fields, and the self-duality constraint on $F^{(5)}$ implies the usual four-dimensional electric-magnetic duality relation $d\tilde{C}_I = *dC^I$. These vector fields pair up with the moduli fields Z^I to give the bosonic content (C^I, Z^I) of $h^{2,1}(\tilde{X}) + 1$ $\mathcal{N} = 2$ vector multiplets, corresponding to the gauge group $U(1)^{h^{2,1}(\tilde{X})+1}$.

⁸In a background where all antisymmetric tensor fields are set to zero, as we consider here, we have simply $F^{(5)} = dC^{(4)}$.

Now suppose we approach a point in complex structure moduli space where \tilde{X} develops a conifold singularity. At the conifold point, a particular three-sphere vanishes, and we will assume that this is the cycle A^1 . We chose our basis of harmonic three-forms so that only α_1 has a non-zero integral over this cycle,

$$\int_{A^1} \alpha_1 = 1 .$$

A D3-brane couples electrically to $C^{(4)}$, so the action for such a brane which is wrapped around A^1 and follows a worldline γ in the four non-compact dimensions contains the term

$$I_{D3} \supset \int_{A^1 \times \gamma} C^{(4)} = \int_{A^1} \alpha_1 \int_{\gamma} C_1 = \int_{\gamma} C_1 .$$

In the four-dimensional theory, these states therefore manifest as a hypermultiplet carrying unit electric charge under the $U(1)$ corresponding to the gauge field C_1 . The mass of this hypermultiplet saturates a BPS bound coming from the $\mathcal{N} = 2$ supersymmetry algebra [32],

$$M_{D3} \propto |Z^1| = \left| \int_{A^1} \Omega \right| \rightarrow 0 .$$

At the conifold point, then, this hypermultiplet becomes massless, and so should be included in the low-energy theory. If instead it is integrated out, it exactly reproduces the classical singularity of the moduli space, via a divergent one-loop contribution to F_1 [35],

$$F_1 \sim \text{const.} + \frac{1}{2\pi i} Z^1 \log Z^1 . \quad (9)$$

If this is substituted into Equation (7), it is easily seen that the moduli space metric becomes singular at $Z^1 = 0$. However, this is now seen to be merely an artifact of integrating out massless states.

The above is a telegraphic account of Strominger's description of conifold singularities in type IIB string theory. Now we will ask what happens when the conifold singularity of \tilde{X} lies over a \mathbb{Z}_N -hyperconifold on $X = \tilde{X}/\mathbb{Z}_N$.

4.2 Hyperconifolds and their resolutions

First, we observe that the moduli space of X is just a subspace of that of \tilde{X} , and inherits its Kähler geometry. Since by assumption the cycle A^1 is mapped to itself by the \mathbb{Z}_N action, Z^1 is a good coordinate on this subspace, and we get exactly the same singularity implied by Equation (9). If the hyperconifold singularity is to make physical sense, we must find states on X which become massless at $Z^1 = 0$ and again reproduce Equation (9) if integrated out.

Such states are easy to identify. We now have a vanishing cycle A^1/\mathbb{Z}_N , which again can be wrapped by a D3-brane. But the worldvolume theory of such a brane contains a $U(1)$ gauge field, so now that the worldvolume has fundamental group \mathbb{Z}_N , its vacuum

becomes N -fold degenerate, corresponding to the N choices of discrete Wilson line [33]. So instead of a single massless hypermultiplet, the theory on the quotient space X contains N such hypermultiplets.⁹

One might expect that these extra states lead to conflict with Equation (9), since each hypermultiplet will give the same contribution to F_1 . But this is a little too hasty. Equation (9) comes about from a one-loop calculation, so the contribution of each hypermultiplet is proportional to the square of its charge, and we need to check whether this changes when passing from \tilde{X} to X . When we perform the Kaluza-Klein expansion of $C^{(4)}$ in Equation (8), the normalisation of the resulting kinetic terms for the C^I depends on the normalisation of the α_I , which is¹⁰

$$\int_{\tilde{X}} \alpha_I \wedge * \alpha_I = 1 .$$

The same condition should hold on X , but now we are integrating over only $1/N$ times the volume. The harmonic forms in which we expand $C^{(4)}$ on X should therefore be $\alpha'_I = \sqrt{N} \alpha_I$ (where I now ranges over only those values for which α_I is invariant under the group). As such, the charge of a D3-brane wrapped on A^1/\mathbb{Z}_N is

$$\int_{A^1/\mathbb{Z}_N} \alpha'_I = \frac{1}{N} \int_{A^1} \sqrt{N} \alpha_I = \frac{1}{\sqrt{N}} .$$

There are N such hypermultiplets, so when integrated out they give

$$F_1 \sim \text{const.} + N \times \left(\frac{1}{\sqrt{N}} \right)^2 \frac{1}{2\pi i} Z^1 \log Z^1 ,$$

which agrees with Equation (9). We conclude that hyperconifold singularities are smoothed by the presence of massless D-brane states, just like the familiar case of the conifold.

In this paper, and in [1], it has been shown that hyperconifolds can be resolved to pass to a new Calabi-Yau manifold. Since we now know that the singularity itself is physically innocuous, we should expect that the theory develops a new branch of moduli space corresponding to its resolution. This is true, and the process is completely analogous to the conifold case, discussed in [10].

First recall that each hypermultiplet contains two complex scalars, each charged under the $U(1)$ gauge group, so at the hyperconifold point the theory develops $4N$

⁹We might also wonder about massless states coming from winding modes of strings which attain zero length on the hyperconifold. See [33] for a nice explanation of why these need not be considered separately.

¹⁰We have normalised the α_I by the condition $\int_{A^I} \alpha_I = \delta_I^J$. Since β^I is Hodge-dual to α_I and Poincaré dual to A^I , we automatically get

$$\int_{\tilde{X}} \alpha_I \wedge * \alpha_I = \int_{\tilde{X}} \alpha_I \wedge \beta^I = \int_{A^I} \alpha_I = 1 .$$

new massless scalar degrees of freedom, transforming non-trivially under the $U(1)$. We now argue that some of these are flat directions, corresponding to the the new Kähler parameters of the resolution.

The $\mathcal{N} = 2$ vector multiplet of interest contains one real and one complex auxiliary scalar, which in $\mathcal{N} = 1$ language are respectively the D -term associated with the vector C^1 , and the F -term associated with the complex modulus Z^1 . At the hyperconifold point $Z^1 = 0$, the vacuum conditions become just $D = F = 0$. These auxiliary fields are functions of the scalar components of the hypermultiplets charged under C^1 , so we get three real conditions on these scalars. There is also a one-parameter group of gauge rotations, which removes another degree of freedom. So we do indeed get a new $4N - 4 = 4(N - 1)$ -dimensional branch of moduli space, parametrised by $N - 1$ hypermultiplets coming from the new massless states. Giving vacuum expectation values to these fields Higgses the $U(1)$ and gives mass to both C^1 and Z^1 . In this way it corresponds to moving into the moduli space of the resolution of the hyperconifold; the new hypermultiplets are identified with the new Kähler parameters, and the fact that Z^1 becomes massive corresponds to the loss of a single complex structure parameter.

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