A Long-run Risks Model with Long- and Short-run Volatilities: Explaining Predictability and Volatility Risk Premium

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In this paper, we extend the long-run risks model of Bansal and Yaron (BY, 2004) to allow both a long- and a short-run volatility component in consumption growth, long-run risks, and dividend growth. Our two volatility model better captures macroeconomic volatility than a single volatility model, and can reconcile simultaneously the large negative market variance risk premium, differing predictability in excess returns, consumption, dividends, and stock market volatility, all of which are difficult to explain previously by the BY model.

The long-run risks model of Bansal and Yaron (henceforth BY, 2004) is perhaps currently the most viable asset pricing model which successfully explains simultaneously the equity risk premium puzzle, the risk-free rate puzzle, the high level of market volatility, and many other stylized facts about the stock market, consumption and dividend-price ratio. Consequently, it is not surprising that it has attracted a lot of attention, with important subsequent studies by Bansal, Khatchatrian, and Yaron (2005), Bansal, Kiku, and Yaron (2007a, b), Chen, Collin-Dufresne, and Goldstein (2008), Constantinides and Ghosh (2008), Drechsler and Yaron (2008), Eraker (2008), Hansen, Heaton, and Li (2008), Pakos (2008), Avramov and Hore (2009), Bansal, Dittmar, and Kiku (2009), and Beeler and Campbell (2009), among others.

However, there remain three major problems with the model. First, the BY model implies a much stronger predictability of consumption growth by dividend-price ratio than observed in the data, as pointed out by Bansal, Kiku and Yaron (henceforth BKY, 2007a). Second, the model implies a variance risk premium that is too small to match that of the options market, as shown by Drechsler and Yaron (2008). Third, the model asserts a much stronger predictive power of dividend-price ratio on future stock return volatility than is found in the data, as demonstrated by Beeler and Campbell (2009). The existing solutions to the first two problems remain piecemeal and incomplete, and to the third one is unknown. For example, BKY provides an improved version of the model that deals with the first problem, but their approach does not address the second. While Drechsler and Yaron (2008) provide a jump process solution to the second problem, their approach does not solve the first. A simple and important question is: Can the BY model be extended in such a way to overcome all the three problems simultaneously?

In this paper, we provide a new extension of the BY and BKY models by introducing both a long-run and a short-run volatility component into the consumption and dividend processes. Bansal and Shaliastovich (2008a) seem the first to use a two-factor volatility model in the context of long-run risks model, based on the empirical evidence of Stock and Watson (2002) and Kandel and Stambaugh (1990). While Stock and Watson highlight a long-run decline in aggregate uncertainty, Kandel and Stambaugh emphasize shorter run volatility which mean-reverts more quickly after rising during recessions. Bansal and Shaliastovich argue that the two volatility components view is consistent with the evidence on macroeconomic volatility, and they show that separating short- and long-run consumption volatility helps their model to match the expectation hypothesis violations in bond and foreign exchange markets. In contrast to their study, we focus here on extending the BY and BKY models in a more general way to resolve the aforementioned three problems.

What role do the two volatility components play in our new model? First, we show theoretically that the implied stock return process must also have the same two volatility factors. This is consistent with recent stock volatility studies by Alizadeh, Brandt, and Diebold (2002), Chernov, Gallant, Ghysels and Tauchen (2003), Chacko and Viceira (2003), Christoffersen, Jacobs, Ornthanalai and Wang (2008), and Lu and Zhu (2009), among others, that is, there are two volatility factors in the stock market. As a result, the new model can be easily calibrated to explain the large negative variance risk premium, resolving one of the three problems facing the BY model. Second, we show that one-factor volatility models, such as those of the BY and BKY, cannot all different degrees of predictability on volatility by using excess returns, consumption and dividends, because the predictive regression slopes are theoretically almost identical. Hence, to explain the empirical fact of differing predictability in volatility, at least two volatility components have to be present in the model. The new two volatility model exactly meets this demand, and hence it resolves another one of the aforementioned three problems.

For the last of the three problems, BKY show that a more persistent volatility process, and covariances among dividends, consumption, and long-run risks are important in resolving it. We retain the persistence in our long-run volatility, and the covariance structures in our new model, and hence we can also retain the desired properties of both the BY and BKY models. As a result, with the addition of one more volatility factor, our new model can resolve all of the three problems completely and simultaneously.

From a methodology perspective, our new model is a parsimonious one. This is because the two volatility components are necessary to match the variance risk premium and to explain the volatility predictability. On the technical aspects, we cast our extension of the BY and BKY models in continuous-time. This allows us not only to avoid the occasional negative volatility problem in the discrete-time counterparts, but also to obtain approximate analytical solutions for many functions of economic interest, such as derivative prices, measures of volatility and the slope coefficients of various predictive regressions.

In their comprehensive analysis of the BY and BKY models, Beeler and Campbell (2009) provide profound insights and raise a number of important challenges. Nevertheless, it seems that our extension of the BY and BKY is capable of meeting all the known and major challenges in the equity market associated with excess stock returns, consumption and dividend-price ratios. The remaining challenge is how well the model performs in the fixed income markets. While this issue will be discussed in detail later in the paper, it may be useful to point out at the outset that this challenge is also the one that all existing asset pricing models face. The literature has not yet provided a model that prices assets well in *both* the equity and fixed income markets. For example, the well-known Fama and French (1993) three-factor model explains the cross-sectional differences in equity expected returns, but has little to say about the rich dynamics of the fixed income markets. On the other hand, the well-known affine models (see, e.g., Dai and Singleton, 2003) can explain the term-structure well with level, slope and curvature factors, but they are not useful in explaining the cross-sectional equity expected returns. While Beeler and Campbell's (2009) emphasis on jointly modelling the equity and fixed income markets is well placed and points out the direction for perhaps the next most important breakthrough to be made in asset pricing, we believe that the long-run risks models are still important, because they provide valuable insights on how to explain well the equity markets, which is the first step for explaining well jointly the equity and fixed income markets.

The rest of the paper is organized as follows. Section I provides a short review of the BY model. Section II provides the extension with both the long-run and short-run volatility components, and solves various functions of interest approximate analytically. Section III calibrates the model and examines its implications and ability to explain the major problems facing the BY and BKY models. Section IV discusses the remaining challenges and future research. Section V concludes.

I. A Short Review of BY

In this section, we provide a short review of the BY model, which will be useful for understanding our extension and its comparison with other models.

The BY model assumes a representative investor who has Epstein-Zin-Weil preferences (Epstein and Zin 1989, Weil 1989), and makes his optimal portfolio decision under the following discrete-time processes for consumption and dividends:

$$
\begin{aligned}
\log(C_{t+1}/C_t) &= \mu + X_t + \sigma_t \eta_{t+1} \\
X_{t+1} &= \alpha X_t + \varphi_x \sigma_t e_{t+1} \\
\sigma_{t+1}^2 &= \sigma^2 + \kappa(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1} \\
\log(D_{t+1}/D_t) &= \mu_d + \varphi X_t + \varphi_d \sigma_t u_{t+1},
\end{aligned} \tag{1}
$$

where X_t is the long-run risk that affects both consumption and dividend growth, $\log(C_{t+1}/C_t)$ and $\log(D_{t+1}/D_t)$, and is persistent with autoregression (AR) coefficient α and volatility $\varphi_x \sigma_t$; the variance process σ_t^2 is also a time-varying AR process with coefficient κ; $\eta_{t+1}, e_{t+1}, w_{t+1}$ and u_{t+1} are independent shocks drawn from the standard normal distribution.

The intuition behind the model is that X_t captures the long-run growth prospects of the economy. Shocks in both long-run X_t and short-run η_{t+1} drive the consumption growth and asset prices. The fear for adverse long-run growth requires high risk premium to compensate. Along with the long- and short-run shocks in dividend growth, asset prices can be very volatile. As a result, the BY model can successfully explain, among other stylized facts of the equity market, the equity risk premium, the risk-free rate, the volatility of the market return, and the predictability of the price-dividend ratio on the stock market returns.

However, there are some strong implications from the original BY model that are inconsistent with market data. Empirically, the price-dividend ratio has little power in predicting consumption growth, but the model implies that it predicts consumption growth and excess stock returns equally well. BKY address this issue by increasing the persistence of the volatility to make it a more important factor, but the importance of long-run consumption risk factor is reduced, so is the importance of the predictability on consumption growth. As a result, as demonstrated later, the increase in the persistence of the volatility entails a too small (virtually zero) variance risk premium that is inconsistent with data. Drechsler and Yaron (2008) allow jumps in the volatility process that is capable of explaining the large negative variance risk premium, but their approach does not apply to explain either the state predictability problem above or the volatility predictability problem below. Since multi-factor stochastic volatility models are generally superior in capturing volatility risk premium and the volatility term structure dynamics (see, e.g., Christoffersen, Jacobs, Ornthanalai and Wang, 2008, and Lu and Zhu, 2009), we adopt this framework in this paper. As it turns out, this approach not only resolves the previous two problems, but also helps to address the volatility predictability problem pointed out by Beeler and Campbell (2009). Overall, despite the huge success of the BY and BKY models elsewhere, their extensions are called for to resolve the variance risk premium problem and the various predictability problems.

II. The New Long-Run Risks Model

In this section, we first motivate our dynamic processes for the state variables of the new long-run risks model, and then solve the model in terms of the state variables. Subsequently, we provide approximate analytical solutions to functions of interest: the consumption-wealth ratio, market prices of risks, price-dividend ratio, and the market return volatility.

A. The Model and Solution

Our model is an extension of the BY and BKY models and is cast in the continuous-time framework. In this subsection, we first provide a generalized version of model (1) with long- and short-run volatilities, and then solve the value function of the representative investor's optimization problem, which lays the foundation for solutions to the other functions of interest.

Parallel to the discrete-time model (1), we consider the following model for the consumption and dividends processes and their related variables,

$$\begin{aligned}
\frac{dC_t}{C_t} &= (\mu + X_t)dt + \sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)}dZ_{1t} \\
dX_t &= -\alpha X_t dt + \varphi_x \sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)}dZ_{2t} \\
\frac{dD_t}{D_t} &= (\mu_d + \varphi X_t)dt + \varphi_d \sqrt{V_{1t}\delta_d + V_{2t}(1-\delta_d)}dB_t + \sigma_{dc}\sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)}dZ_{1t} \\
&\quad + \sigma_{dx}\sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)}dZ_{2t} + \sigma_{dv}\sqrt{V_{1t}}dw_{1t} \\
dV_{1t} &= \kappa_1(\bar{V}_1 - V_{1t})dt + \sigma_1\sqrt{V_{1t}}dw_{1t} \\
dV_{2t} &= \kappa_2(\bar{V}_2 - V_{2t})dt + \sigma_2\sqrt{V_{2t}}dw_{2t}, \qquad 0 < \kappa_1 < \kappa_2,
\end{aligned} \tag{2}$$

where $dZ_{1t}, dZ_{2t}, dB_t, dw_{1t}$ and dw_{2t} are independent shocks drawn from the Brownian motion. When $\delta_x = 1, \delta_c = 1, \delta_d = 1$, and $\sigma_{dc} = \sigma_{dx} = \sigma_{dv} = 0$, the above model reduces to the continuous-time limit of the BY model. When $\delta_x = 1, \delta_c = 1, \delta_d = 1$, and $\sigma_{dc} = 0$, it becomes the continuous-time limit of the BKY model. Therefore, the above model is a continuous-time extension of both the BY and BKY models that nests the original ones as special cases.

The key feature of the new model is that the consumption growth has a variance level of $V_{1t}\delta_c + V_{2t}(1-\delta_c)$, a convex combination of the long- and short-run variances V_{1t} and V_{2t}. This convex combination not only decomposes the total variance into two plausible components, but also allows for easy solutions below. The same variance decomposition, adjusted for the leverage factor φ_x, is also applied to the long-run risk X_t. The dividend growth process is treated similarly, except that it allows for various covariations with C_t and X_t, as in the BKY model. Finally, the long- and short-run variances follow two independent standard square-root Heston (1993) processes.

Bansal and Shaliastovich (2008a) appear the first to use a two-factor volatility model in the context of long-run risks models. However, their model is a special case of our model with $\delta_x = 0, \delta_c = 1$, and is a specification on the consumption growth progress only, and they focus on the bond and foreign exchange markets in a model with inflation dynamics. In contrast, we focus here on extending the classical BY model to explain the various predictability patterns and the variance risk premium. While there are ample studies in the volatility term structure literature about the two-factor volatility model (as mentioned in the introduction), Bansal and Shaliastovich (2008b) provide an intriguing motivation from an economic standpoint. They introduce uncertainty about measures of future consumption growth, and show that the additional volatility component can arise through learning from signals with time-varying precision.

To solve the equilibrium prices and other quantities of interest, following the BY model, we also use the Epstein-Zin-Weil preference, but in continuous-time. Based on Duffie and Epstein (1992), we can define the intertemporal value function recursively by

$$J_t = E_t[\int_t^T f(C_s, J_s)ds]. \tag{3}$$

Then the representative investor's objective is to choose consumption and security holdings to

$$\max_{\{C_s\}} E_0[\int_0^T f(C_s, J_s)ds], \tag{4}$$

where $f(C,J)$ is a *normalized aggregator* related to current consumption C_t and continuation value function J_t, and is given by

$$f(C,J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J[(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}})^{1-\frac{1}{\psi}} - 1], \tag{5}$$

with β the rate of time preference, $\gamma > 0$ the relative risk aversion, and $\psi > 0$ the elasticity of intertemporal substitution (IES). If we set $\psi = 1/\gamma$ in (5), as shown by Duffie and Epstein (1992), we obtain the standard additive expected utility of constant relative risk aversion (CRRA).

Following Duffie and Epstein (1992), we have the pricing kernel

$$\pi_t = \exp\left[\int_0^t f_J(C_s, J_s)ds\right] f_C(C_t, J_t) \tag{6}$$

with $J = J(c, x, v_1, v_2)$, a solution of

$$\begin{aligned} f(c,J) + cf_c(\mu + c) + \frac{1}{2}[\delta_c v_1 + (1-\delta_c)v_2]c^2 J_{cc} + J_c \cdot (-\alpha c) + \frac{1}{2}[\delta_x v_1 + (1-\delta_x)v_2]J_{xx} \\ + J_{v_1} \cdot \kappa_1(\bar{V}_1 - v_1) + \frac{1}{2}\sigma_1^2 v_1 J_{v_1 v_1} + J_{v_2} \cdot \kappa_2(\bar{V}_2 - v_2) + \frac{1}{2}\sigma_2^2 v_2 J_{v_2 v_2} = 0. \end{aligned} \tag{7}$$

Let g_1 be the long-term mean of the consumption-wealth ratio,[1]

$$g_1 = \overline{\left(\frac{C_t}{W_t}\right)} = \overline{\exp(c_t - \omega_t)}, \tag{8}$$

where the lowercase variables are the log variables. With the standard log-linear approximation which Campbell (1993) develops in discrete time, and Chacko and Viceira (2005) use first in continuous time, we have

$$\frac{C_t}{W_t} = \exp(c_t - \omega_t) \approx g_1 - g_1 \log g_1 + g_1 \log(C_t/W_t). \tag{9}$$

Conjecturing a solution for J of the following form,

$$J(W_t, X_t, V_{1t}, V_{2t}) = \exp(A_0 + A_1 X_t + A_2 V_{1t} + A_3 V_{2t})\frac{W_t^{1-\gamma}}{1-\gamma}, \tag{10}$$

we can solve the Bellman equation (7) to obtain (see Appendix A.1)

$$\begin{aligned} A_0 &= \frac{1}{g_1\psi}[\theta\xi + (1-\gamma)\mu + \kappa_1\bar{V}_1\psi A_2 + \kappa_2\bar{V}_2\psi A_3] \\ A_1 &= \frac{1-\gamma}{(g_1+\alpha)\psi} \\ A_2 &= \frac{-b_1 - \sqrt{b_1^2 - 4a_1c_1}}{2a_1} \\ A_3 &= \frac{-b_2 - \sqrt{b_2^2 - 4a_2c_2}}{2a_2} \end{aligned} \tag{11}$$

[1] It can be solved endogenously once the model parameters are known. Appendix A.3 provides the details.

with

$$a_1 = \tfrac{1}{2}\sigma_1^2\psi^2, \quad b_1 = -(g_1+\kappa_1)\psi, \quad c_1 = -\tfrac{1}{2}\gamma(1-\gamma)\delta_c + \tfrac{1}{2}\varphi_x^2\delta_x\tfrac{(1-\gamma)^2}{(g_1+\alpha)^2} \tag{12}$$

$$a_2 = \tfrac{1}{2}\sigma_2^2\psi^2, \quad b_2 = -(g_1+\kappa_2)\psi, \quad c_2 = -\tfrac{1}{2}\gamma(1-\gamma)\delta_c(1-\delta_c) + \tfrac{1}{2}\varphi_x^2(1-\delta_x)\tfrac{(1-\gamma)^2}{(g_1+\alpha)^2}. \tag{13}$$

The solution is approximate in general and exact when $\psi = 1$. Armed with this solution, we are ready to solve for various functions of interest, which are also approximate in general and exact when $\psi = 1$.[2]

B. Consumption-wealth Ratio

Consider first the consumption-wealth ratio. Based on (A6) and its proof, we immediately have

$$\frac{C_t}{W_t} = \beta^{\psi}\exp\{(A_{0a} + A_{1a}X_t + A_{2a}V_{1t} + A_{3a}V_{2t})\}, \tag{14}$$

where $A_{ia} = A_i\frac{1-\psi}{1-\gamma}$ for $i = 0,1,2,3$. The ratio is loglinear in the state variables, and has similar functional form as in the BY model. In particular,

$$A_{1a} = -\frac{1-\frac{1}{\psi}}{g_1+\alpha}, \tag{15}$$

which is exactly the same as the continuous analogue of BY's $-A_1$.[3] Hence, the same interpretation applies that, when $\psi < 1$, the income effect dominates, and when $\psi > 1$, the substitution effect dominates. In addition, the consumption-wealth ratio is more sensitive to the expected growth rate when the persistence of expected growth shocks, measured by $1/\alpha$, increases.

However, there are now two volatilities in Equation (14). This is expected since they are in the basic dynamics equations. Due to their symmetric formulations entered into the consumption and dividends dynamics, the two volatility components impact on the ratio in the same way, with proportional coefficients A_{2a} and A_{3a} depending on the volatility parameters via similar functional forms. When $\psi > 1$, both A_{2a} and A_{3a} are positive. The same intuition of the BY about volatility also holds here. For example, a rise in either of the volatilities will make consumption more volatile, which lowers asset valuations and increases the risk premia on all assets. In addition, an increase

[2] As found in similar approximations elsewhere as well as our own studies, the approximate solution is accurate around commonly calibrated parameters.

[3] Note that Bansal and Yaron (2004) use the ratio of wealth to consumption, but we use the ratio of consumption to wealth. Hence our A_{ia}'s have the opposite sign of theirs. The same applies to the price-dividend ratio below.

in the persistence of volatility shocks, that is, a decrease in either κ_1 or κ_2, will magnify the effects of volatility shocks on valuation ratios, since the investor would perceive changes in economic uncertainty as being long lasting.

C. Risk-free Rate and Market Prices of Risks

Recall that the pricing kernel is given by Equation (6). Based on the definition for f in Equation (5), we have

$$\begin{aligned} f_J &= \xi_1 - g_1(A_1X_t + A_2V_{1t} + A_3V_{2t})\frac{1-\gamma\psi}{1-\gamma} \\ f_C &= \beta^{\psi\gamma}\exp\left[(B + A_1X_t + A_2V_{1t} + A_3V_{2t})\frac{1-\gamma\psi}{1-\gamma}\right]C_t^{-\gamma}, \end{aligned}$$

where

$$\xi_1 = (\theta-1)\xi - \beta - g_1\frac{1-\gamma\psi}{1-\gamma}A_0. \tag{16}$$

Applying Ito's Lemma to π_t in Equation (6), we have

$$\frac{d\pi_t}{\pi_t} = -(r_f dt + \lambda_1 dZ_{1t} + \lambda_2 dZ_{2t} + \lambda_3 dw_{1t} + \lambda_4 dw_{2t}), \tag{17}$$

where the risk-free rate r_f and the market prices of risks, $\lambda_i, i = 1,2,3,4$, are given below.

First, the risk-free rate is

$$r_f = -(r_0 + r_1X_t + r_2V_{1t} + r_3V_{2t}), \tag{18}$$

where

$$\begin{aligned} r_0 &= \xi_1 + (\kappa_1A_2\bar{V}_1 + \kappa_2A_3\bar{V}_2)\frac{1-\gamma\psi}{1-\gamma} - \gamma\mu \\ r_1 &= -\frac{1}{\psi} \\ r_2 &= -(g_1+\kappa_1)A_2\frac{1-\gamma\psi}{1-\gamma} + \frac{1}{2}\left(\frac{1-\gamma\psi}{1-\gamma}\right)^2(A_1^2\varphi_x^2\delta_x + A_2^2\sigma_1^2) + \frac{1}{2}\gamma(\gamma+1)\delta_c \\ r_3 &= -(g_1+\kappa_2)A_3\frac{1-\gamma\psi}{1-\gamma} + \frac{1}{2}\left(\frac{1-\gamma\psi}{1-\gamma}\right)^2[A_1^2\varphi_x^2(1-\delta_x) + A_3^2\sigma_2^2] + \frac{1}{2}\gamma(\gamma+1)(1-\delta_c). \end{aligned} \tag{19}$$

Note that $r_1 < 0$, implies that the risk-free rate increases with increasing expectation of growth. As usual, we get $\partial X/\partial r_f = \psi$, i.e., a one percent increase in asset return will induce ψ percent increase in consumption, hence the interpretation of ψ as the elasticity of the intertemporal substitution.

Further, when $\gamma > 1/\psi$, because A_2 and A_3 are all positive, both r_2 and r_3 are positive, implying that the risk-free rate decreases as the consumption uncertainty increases.

The market prices of risks are

$$\begin{aligned} \lambda_1 &= \gamma\sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)} \\ \lambda_2 &= -\frac{1-\gamma\psi}{1-\gamma}A_1\varphi_x\sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)} \\ \lambda_3 &= -\frac{1-\gamma\psi}{1-\gamma}A_2\sigma_1\sqrt{V_{1t}} \\ \lambda_4 &= -\frac{1-\gamma\psi}{1-\gamma}A_3\sigma_2\sqrt{V_{2t}}. \end{aligned} \tag{20}$$

As usual, the magnitude of the risk aversion relative to the reciprocal of the IES determines whether agents prefer early or late resolution of uncertainty regarding consumption path. When agents prefer early resolution of uncertainty, that is, when $\gamma > 1/\psi$, the market price for long run risk is positive. In addition, because A_1, A_2 and A_3 are all increasing in magnitude as the persistence parameters, $1/\alpha$, $1/\kappa_1$ and $1/\kappa_2$ increase, consequently, risk premia increase in magnitude as persistent parameters increase.

Moreover, the market prices of risks for variances are negative when $\gamma > 1/\psi$. This is consistent with the empirical evidence that the variance risk premium is negative. Intuitively, the negative sign of the variance risk premium indicates that investors regard increases in market volatility as unfavorable shocks to the investment opportunity. However, in contrast to the BY and BKY models, the market variance risk premium is determined by both the long- and short-run volatilities. As will be clear later, because of the rich dynamics of these two components, the associated parameters can be chosen in such a way to explain the market variance risk premium, while the previous models do not allow for such flexibility.

Finally, notice that when $\gamma = 1/\psi$, we obtain the results for power utility. In this case, all the risk premia other than that of the instantaneous consumption growth become zero, and hence it will not be possible for the Breeden (1979) CCAPM model to match the market risk premium.

D. Price-dividend Ratio

With the market prices of risks given in Equation (20), we can solve the price-dividend ratio as follows:

$$\frac{D_t}{P_t} = \exp\{(A_{0m} + A_{1m}X_t + A_{2m}V_{1t} + A_{3m}V_{2t})\}, \tag{21}$$

where

$$A_{1m} = -\frac{\varphi - \frac{1}{\psi}}{g_{1m} + \alpha} \tag{22}$$

and the other A_{im}'s are given in Appendix A.2, with g_{1m}, similar to g_1, as the long-term mean of D_t/P_t. Note that when $\varphi > 1$, $|A_{1m}| > |A_{1a}|$. Consequently, news about expected growth rate leads to a larger reaction in the price of the dividend claim than in the price of the consumption claim. In addition, when φ is big enough, A_{2m} and A_{3m} are positive, resulting in the well-known leverage effect, i.e., shocks to return is negatively correlated with shocks to variance process.

E. The Market Volatility

Applying Ito's Lemma to

$$P_t = D_t \exp\{-(A_{0m} + A_{1m}X_t + A_{2m}V_{1t} + A_{3m}V_{2t})\},$$

we obtain

$$\begin{aligned}
\frac{dP_t}{P_t} &= (\mu_d - (A_{2m}\kappa_1\bar{V}_1 + A_{3m}\kappa_2\bar{V}_2) + (\varphi + \alpha A_{1m})X_t + A_{2m}\kappa_1 V_{1t} + A_{3m}\kappa_2 V_{2t})dt \\
&\quad + \varphi_d\sqrt{V_{1t}\delta_d + V_{2t}(1-\delta_d)}dB_t + \sigma_{dc}\sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)}dZ_{1t} \\
&\quad + \sigma_{dx}\sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)}dZ_{2t} + \sigma_{dv}\sqrt{V_{1t}}dw_{1t} \\
&\quad - A_{1m}\varphi_x\sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)}dZ_{2t} \\
&\quad - A_{2m}\sigma_1\sqrt{V_{1t}}dw_{1t} - A_{3m}\sigma_2\sqrt{V_{2m}}dw_{2t} \\
&= [\mu_d - (A_{2m}\kappa_1\bar{V}_1 + A_{3m}\kappa_2\bar{V}_2) + (\varphi + \alpha A_{1m})X_t + A_{2m}\kappa_1 V_{1t} + A_{3m}\kappa_2 V_{2t}]dt + \sqrt{V_t}dZ_t,
\end{aligned}$$

where dZ_t is another Brownian motion, and hence the variance of the price process is

$$V_t = c_1 V_{1t} + c_2 V_{2t} \tag{23}$$

with

$$c_1 = \phi_d^2\delta_d + \sigma_{dc}^2\delta_c + (\sigma_{dx} - A_{1m}\varphi_x)^2\delta_x + (\sigma_{dv} - A_{2m}\sigma_1)^2 \tag{24}$$

$$c_2 = \phi_d^2(1-\delta_d) + \sigma_{dc}^2(1-\delta_c) + (\sigma_{dx} - A_{1m}\varphi_x)^2(1-\delta_x) + A_{3m}^2\sigma_2^2. \tag{25}$$

III. Calibration

In this section, based on the BY, BKY and Beeler and Campbell (2009), we first calibrate the model parameters and examine their moment matching properties. Then we discuss predictability of excess returns, consumption and dividends growth, as well as their volatilities. Finally, we analyze the risk premium on the market variance.

A. Calibrated Parameters and Moment Matching

To compare the results, we first translate the parameters of the BY and BKY models into their continuous-time counterparts. With the monthly estimates provided by Bansal and Yaron (2004) and Beeler and Campbell (2009), we can annualize them and match the discrete- and continuous-time processes. In this way, all the parameters of the continuous-time models corresponding to either the BY or the BKY model can be obtained. We report the results in Table I.

To retain the good and desired properties of the BY and BKY models, we set most of the common parameters of our two-factor volatility model at roughly the same values with their one-factor models, while calibrate the additional parameters to meet various requirements. This will be clear as we go along in discussing the various results below. On the preference parameters, Bansal and Yaron (2004) use both $\gamma = 7.5$ and 10. We use, however, a lower value of $\gamma = 6$. Theoretically, this is a more plausible level of risk aversion.[4] However, it will imply a lower market risk premium, which will be discussed further below. On parameters governing the consumption growth, all the parameters, μ, α and φ_x, are virtually the same across the models (the δ's are additional parameters). For the common parameters in the dividend growth process, the new model's calibrations are somewhat different, but not by much. For the common volatility parameters, we calibrate our parameters closely to the BKY model. The volatility of the BKY model is much more persistent than the BY model. Although κ_1 in our new model has almost the same value as BKY, the total volatility is not as persistent as in BKY, because there are two components in our model, and the larger short-run volatility component is not persistent at all. These parameter values in our model imply reasonable volatility components for the market returns that are consistent with the volatility literature (see, e.g., Christoffersen, Jacobs, Ornthanalai and Wang, 2008, and Lu and Zhu, 2009).

[4]Technically, this smaller value is useful to ensure the Heston volatility models are well behaved.

The moments of the asset pricing variables computed from monthly data from February 1947 to March 2007, as well as those implied by all the three models are presented in Table II. Those of the data and of the BY and BKY models are based on Beeler and Campbell (2009) with $\gamma = 10$, and those of our new model are evaluated by the analytical formulas of the previous section with the calibrated parameters in Table I. Note that the moments of our new model are largely in agreement with the data and with the BY and BKY models, except two notable differences. First, the market risk premium is 3.58%, lower than about 6% from the data, and lower than about 6.6% from both the BY and BKY models. As mentioned earlier, this is due to our choice of a lower $\gamma = 6$. The question is whether 3.58% is a reasonable market risk premium level. According to Jagannathan, McGrattan and Scherbina (2001), and Fama and French (2002), the market risk premium is declining, and the former article even states that "the premium averaged about 7 percentage points during 1926–70 and only about 0.7 of a percentage point after that." In addition, Donaldson, Kamstra and Kramer (2009) find, using various estimation approaches, that the risk premium is 3.5% within 50 basis points. Hence, a 3.58% risk premium from our model is reasonable. The second exception is that the volatility of the risk-free rate in our model is higher than those of the BY and BKY models, but this is in fact desired as it matches the data better. Overall, the new factor model matches the moments of the data nicely, and does so as well as the BY and BKY models.

Finally, it may be useful to comment on the parsimony of our proposed model. The original BY model is widely regarded as parsimonious for explaining many stylized facts. Our extension builds upon the original model by adding six new key parameters to account for many necessary aspects of the new model. δ_x, δ_c, and δ_d are useful to measure the relative mixture of the two volatility components, and σ_{dc}, σ_{dx}, and σ_{dv} capture the covariances among dividends, consumption, and long-run risk. Since σ_{dx}, and σ_{dv} are already in BKY, there are essentially 4 new key parameters. As it turns out, all of them are important in describing the data. There are in addition a few parameters of the second volatility component which are necessary based on the variance term structure literature (e.g., Egloff, Leippold, and Wu, 2009, and Lu and Zhu, 2009). Overall, considering the huge differences in what the BY or BKY models imply and what the data speak about the additional stylized facts on the equity market, it is necessary for us to add those new parameters to make our model work. In short, in terms of what it has to do, our model appears a very parsimonious one.

B. Predictability of Excess Returns, Consumption and Dividends

To examine predictability of the variables, we, following Beeler and Campbell (2009), consider the following three K-period regressions

$$(r_{t+j} - r_{f,t+j}) + \cdots + (r_{t+j+K} - r_{f,t+j+K}) = \alpha_{jK} + \beta(p_t - d_t) + \epsilon_{jKt} \tag{26}$$

$$\Delta c_{t+j} + \cdots + \Delta c_{t+j+K} = \alpha_{jK} + \beta(p_t - d_t) + \epsilon_{jKt} \tag{27}$$

and

$$\Delta d_{t+j} + \cdots + \Delta d_{t+j+K} = \alpha_{jK} + \beta(p_t - d_t) + \epsilon_{jKt}, \tag{28}$$

where r and r_f are the market return and risk-free rate, respectively; and c_t and d_t are logarithms of consumption and dividends. To explain the observed regression patterns, our idea is to derive the regression slope coefficients as functions of the model parameters that can be chosen in such a way to make the model implied regression slope coefficients match closely with those of the data.

We provide the formulas for $K = 1$ only for notational simplicity, while the general case is a straightforward extension. As $K = 1$, the regressors of the above three regressions all have the same functional form of

$$dY_t = [a_0 + a_1 X_t + a_2 V_{1t} + a_3 V_{2t}]dt + \sqrt{b_1 V_{1t} + b_2 V_{2t}} dZ_t, \tag{29}$$

where dY_t corresponds to $d\ln P_t - r_f dt$, $d\ln C_t$ and $d\ln D_t$, respectively, with

$$a_1 = \varphi + \alpha A_{1m} + r_1 + g_{1m} A_{1m},\ a_2 = A_{2m}\kappa_1 - \frac{b_1}{2} + r_2 + g_{1m} A_{2m},\ a_3 = A_{3m}\kappa_2 - \frac{b_2}{2} + r_3 + g_{1m} A_{3m}$$

in the first case,

$$a_1 = 1,\ a_2 = -\frac{\delta_c}{2},\ a_3 = -\frac{1-\delta_c}{2}$$

in the second case, and

$$a_1 = \varphi,\ a_2 = -\frac{\varphi_d^2\delta_d + \sigma_{dc}^2\delta_c + \sigma_{dv}^2 + \sigma_{dx}^2\delta_x}{2}, a_3 = -\frac{\varphi_d^2(1-\delta_d) + \sigma_{dc}^2(1-\delta_c) + \sigma_{dx}^2(1-\delta_x)}{2}$$

in the last one. The regression slope coefficient is then analytically obtained as

$$b = \frac{\mathrm{Cov}(\Delta_\tau y, p - d)}{\mathrm{Var}(p - d)}, \tag{30}$$

where

$$\begin{aligned}
\mathrm{Cov}(\Delta_\tau y, p-d) &= -\Big[a_1 A_{1m}\frac{\sigma_x^2}{2\alpha^2}(1-e^{-\alpha\tau}) + a_2 A_{2m}\frac{\sigma_1^2\bar{V}_1}{2\kappa_1^2}(1-e^{-\kappa_1\tau}) \\
&\quad + a_3 A_{3m}\frac{\sigma_2^2\bar{V}_2}{2\kappa_2^2}(1-e^{-\kappa_2\tau})\Big] \qquad (31)\\
\mathrm{Var}(p-d) &= A_{1m}^2\frac{\sigma_x^2}{2\alpha} + A_{2m}^2\frac{\sigma_1^2\bar{V}_1}{2\kappa_1} + A_{3m}^2\frac{\sigma_2^2\bar{V}_2}{2\kappa_2} \qquad (32)
\end{aligned}$$

with

$$\sigma_x^2 = \varphi_x^2[\bar{V}_1\delta_x + \bar{V}_2(1-\delta_x)]. \qquad (33)$$

The proof is given in Appendix A.4.

Table III reports the calibrated results, which are in the last column of the table. The first three columns are the regression betas from the data, the BY and the BKY models, all of which are taken from Beeler and Campbell (2009).[5] On the predictability of excess stock returns by the price-dividend ratio, the data have betas ranging from −0.059 to −0.421 as horizon increases from 1 year to 5 years, but the BY model does not match this with betas from −0.007 to −0.039. In contrast, the BKY does the matching well with betas from −0.078 to −0.368. In line with the BKY results, our new model performs as well with betas from −0.074 to −0.351.

Opposite to its low predictability on excess stock returns, the BY model forecasts consumption growth with betas from 0.114 to 0.338, which are substantially higher than those from data, which are lower than 0.01 in absolute values. The BY calibration also has too much power in predicting dividend growth. In contrast, the BKY model and our new one match the data fairly well on both fronts. Overall, our model inherits much of the good properties of the BKY model, and so it performs as well in explaining the predictability of excess returns, consumption and dividends.

However, the BKY calibration inevitably increases the predictability of volatility because of the increased importance of the volatility in the price-dividend ratio. In addition, as we show later, as long as there is a single volatility component, the predictive regression slope coefficients of log volatility for the three variables (excess return, consumption growth, dividend growth) should be nearly the same. This suggests that adding jumps without additional state variables, as in Drechsler and Yaron (2008), is unlikely to match the volatility predictability. Therefore, we must go beyond

[5] We do not report R^2 for brevity, since the R^2s must match once the second moments and betas are matched.

the BKY model to understand the differing predictability of volatility, as well as the large negative variance premium, which are addressed in the next two subsections.

C. Predictability of Volatility: Excess Returns, Consumption and Dividends

Following Beeler and Campbell (2009), we use the realized volatility measure. There are two steps in computing this measure. First, we run an AR(1) regression of each variable of interest y_{t+1},

$$y_{t+1} = b_0 + b_1 y_t + u_{t+1}, \tag{34}$$

where y_{t+1} is the excess return or consumption growth or dividend growth. Second, the K-period realized volatility is defined as the sum of the absolute values of the residuals,

$$\text{Vol}_{t,t+K-1} = \sum_{k=0}^{K-1} |u_{t+k}| \tag{35}$$

over K periods with K, as before, the horizon of interest.

Then, the predictability of volatility is examined from the regression of the log of K-period realized volatility on the log price-dividend ratio,

$$\ln[\text{Vol}_{t+1,t+K}] = \alpha + \beta(p_t - d_t) + \xi_t. \tag{36}$$

To match the volatility predictability of the data, we need to solve the continuous-time version of the regression slope.

First, we note that the innovation process u_{t+1} in the AR(1) process (34) is a discrete-time version of dZ_t in equation (29). Thus, in the continuous limit, the discrete realized volatility defined in (35) can be written as

$$\text{Vol}_{t,t+\tau} = \int_0^\tau \sqrt{V_t}|dZ_t| = \frac{2}{\sqrt{2\pi}} \int_0^\tau \sqrt{V_t} dt, \tag{37}$$

where we have used the relation

$$|dZ_t| = \frac{2}{\sqrt{2\pi}} dt$$

for a standard Brownian Motion Z_t. Then, a key step in approximating the log of τ period realized volatility in Equation (37) is to use the following approximate equality,

$$\ln \int_0^\tau \sqrt{V_t}|dZ_t| \approx Const + \frac{1}{2\tau} \int_0^\tau \frac{V_t}{\bar{V}} dt, \tag{38}$$

where $\bar{V}$ is the unconditional mean of V_t (see Appendix A.5). Then, we can obtain approximately the volatility regression coefficient as

$$\beta = \frac{\mathrm{Cov}(\Delta_\tau y, p-d)}{\mathrm{Var}(p-d)}, \tag{39}$$

where

$$\mathrm{Cov}(\Delta_\tau y, p-d) = -\left[b_1 A_{2m} \frac{\sigma_1^2 \bar{V}_1}{2\kappa_1^2}(1-e^{-\kappa_1 \tau}) + b_2 A_{3m} \frac{\sigma_2^2 \bar{V}_2}{2\kappa_2^2}(1-e^{-\kappa_2 \tau}) \right], \tag{40}$$

$\mathrm{Var}(p-d)$ is the same as given by equation (32), and b_1 and b_2 are given by

$$b_1 = c_1, \;\; b_2 = c_2$$

in the case of excess return volatility,

$$b_1 = \delta_c, \;\; b_2 = 1 - \delta_c$$

in the case of consumption growth volatility, and

$$b_1 = \varphi_d^2 \delta_d + \sigma_{dc}^2 \delta_c + \sigma_{dx}^2 \delta_x + \sigma_{dv}^2, \quad b_2 = \varphi_d^2(1-\delta_d) + \sigma_{dc}^2(1-\delta_c) + \sigma_{dx}^2(1-\delta_x)$$

in the final case of dividend growth volatility.

The approximate equality (38) is not only important for obtaining the volatility slope regression coefficients, but also important for understanding why the one-factor volatility models of the BY and BKY cannot explain the cross-sectional predictability in volatility. If the variance process V_t has only one factor, the volatilities of excess return, consumption and dividends all must be a linear function of this factor. Because the term

$$\frac{1}{2\tau}\int_0^\tau \frac{V_t}{\bar{V}} dt$$

in Equation (38) is invariant to the same scaling on both V_t and $\bar{V}$, it follows that the approximated regressors in equation (38) will be the same for any of the three volatility variables. Hence, the volatility regression slope coefficients for all of the three volatility variables must be roughly the same across the variables in either the BY or the BKY models. This is indeed the case, as shown empirically by Beeler and Campbell (2009) in their comprehensive Monte Carlo studies, which are also presented in Table IV. For example, for one year horizon, the regression slope coefficients β

for volatilities of excess return, consumption and dividends are $-0.123, -0.128$, and -0.146 in BY model, and $-1.315, -1.420$, and -1.483 in BKY model, both of which show little differences across the three variables. This pattern holds for other time horizons too, as predicted by our analysis above.

However, the volatility regression slope coefficients based on data, taken from Beeler and Campbell (2009), show substantial differences across the three variables. For example, as reported in Table IV which are based on Beeler and Campbell (2009) except the last column, β is only -0.081 for excess return volatility, but a much sizeable value of -0.530 for dividend volatility, when a one-year period is considered. Therefore, in order to match the empirical evidence on volatility predictability, it is necessary to incorporate at least one more volatility factor in the BY or BKY models.

Indeed, in our proposed two-factor model with calibrated parameters reported in Table I, the volatility regression slope coefficients match remarkably well with those from the data. With varying time periods for excess return volatility, while the data imply decreasing beta values of $-0.081, -0.59$ and -0.017, so does the model with $-0.133, -0.064$ and -0.050. For the consumption volatility, the betas are $-0.481, -0.491$ and -0.564 with no apparent patterns of either increasing or decreasing, the model provides $-0.494, -0.446$ and -0.431. Finally, the match in dividends volatility is also almost perfect. In short, additional volatility factor is a key to explain the differences in volatility predictability across excess returns, consumption and dividends. As it turns out, that is also fundamentally important in explaining the market variance premium, as discussed next.

D. Variance risk premium

First, it is easy to see that, in our model, the risk premia associated with V_{1t} and V_{2t} are

$$\begin{aligned} \lambda_3\sigma_1\sqrt{V_{1t}} &= -\nu_1 V_{1t} \\ \lambda_4\sigma_2\sqrt{V_{2t}} &= -\nu_2 V_{2t}, \end{aligned} \tag{41}$$

where

$$\begin{aligned} \nu_1 &= \frac{1-\gamma\psi}{1-\gamma}A_2\sigma_1^2 \\ \nu_2 &= \frac{1-\gamma\psi}{1-\gamma}A_3\sigma_2^2. \end{aligned} \tag{42}$$

The risk-neutral processes for V_{1t} and V_{2t} are

$$\begin{aligned} dV_{1t} &= \kappa_1^Q(\frac{\kappa_1}{\kappa_1^Q}\bar{V}_1 - V_{1t})dt + \sigma_1\sqrt{V_{1t}}dw_{1t}^Q \\ dV_{2t} &= \kappa_2^Q(\frac{\kappa_2}{\kappa_2^Q}\bar{V}_2 - V_{2t})dt + \sigma_2\sqrt{V_{2t}}dw_{2t}^Q, \end{aligned} \tag{43}$$

where

$$\begin{aligned} \kappa_1^Q &= \kappa_1 - \nu_1 \\ \kappa_2^Q &= \kappa_2 - \nu_2. \end{aligned}$$

Then we can show that we need the following constraints

$$\nu_1 < \kappa_1 \quad \text{and} \quad \nu_2 < \kappa_2 \tag{44}$$

to make the risk-neutral processes stationary. Otherwise, they will blow up in finite time.

Now, it is of interest to see why the one-factor volatility models of the BY and the BKY cannot explain the market variance risk premium. Their κ's are kept small because of high persistence of consumption growth volatility. For example, with a value of $\kappa = 0.0139$ of the BKY parameter calibration, the monthly variance risk premium in absolute value will be bounded from above by $0.0139 \times 0.04/12 = 0.463$ if $V_t = 0.20^2 = 0.04$ for market price variance,[6] which is much smaller than that from data, as reported in Table V. However, with an additional volatility component that is less persistent, the new model can produce the desired variance risk premium while keeping most of the moments of the model intact. The parameters for the physical dynamics of the volatility processes, provided in Table I, are consistent with empirical findings of variance term structure literature (e.g., Egloff, Leippold, and Wu 2009, and Lu and Zhu 2009). We next calibrate the short term variance risk premium (VRP) to market data. In doing so, we actually calibrate the whole term structure of VRP because the long term variance risk premium is close to zero, as demonstrated above in the model, as well as empirically by Egloff, Leippold, and Wu (2009), and Lu and Zhu (2009), among others.

Drechsler and Yaron (2008) seem to be the first to explain the negatively large market variance risk premium by introducing a jump process into the volatility of the long-run risks model, while

[6]The variance premium is multiplied by 10000. We use this scaling and the monthly values following Drechsler and Yaron (2008).

stochastic volatility and the long-run risks general equilibrium model are analyzed by Bansal, Gallant and Tauchen (2007), Eraker (2008), and Eraker and Shaliastovich (2008), among others, and there is a large volatility literature that focuses on using reduced form models. Following Drechsler and Yaron (2008), we measure the market variance risk premium from data as the difference between the squared VIX index and realized variance that captures attitudes toward uncertainty in the equity market. Theoretically, the implied market variance risk premium is computed in three steps. First, we compute the squared VIX, or more generally, variance swap rate VS_t with maturity τ_0, which is a linear function of the instantaneous variance,

$$VS_t = \sum_{i=1}^{2}(A_i^Q + B_i^Q V_{it}), \tag{45}$$

where A_i^Q and B_i^Q $(i = 1, 2)$ are constants given by

$$A_i^Q = \frac{\kappa_i \bar{V}_i}{\kappa_i^Q}\left[1 - \frac{1 - e^{-\kappa_i^Q \tau_0}}{\kappa_i^Q \tau_0}\right], \qquad B_i^Q = \frac{1 - e^{-\kappa_i^Q \tau_0}}{\kappa_i^Q \tau_0}.$$

Second, we evaluate the realized variance over time period τ_0 as

$$RV_t = \sum_{i=1}^{2}(A_i^P + B_i^P V_{it}), \tag{46}$$

where A_i^P and B_i^P $(i = 1, 2)$ are constants given by

$$A_i^P = \bar{V}_i\left[1 - \frac{1 - e^{-\kappa_i \tau_0}}{\kappa_i \tau_0}\right], \qquad B_i^P = \frac{1 - e^{-\kappa_i \tau_0}}{\kappa_i \tau_0}. \tag{47}$$

Finally, the difference between realized variance (46) and variance swap rate (45) is the model implied variance risk premium (VRP),

$$VRP = \sum_{i=1}^{2}[(A_i^P - A_i^Q) + (B_i^P - B_i^Q)V_{it}]. \tag{48}$$

Table V reports the results, which are measured monthly and multiplied by 10000. The second and third columns, based on the data and the Drechsler and Yaron (2008) model, are taken from their paper directly. The rest are results for the BY, the BKY and our new model, which are computed based on Equation (48) with the calibrated parameters. While the data show a VRP of the level of -11.27, both the BY and BKY imply a levels of only -0.005 and -0.010, too small (in absolute value) to explain anywhere close to the true range. In contrast, both Drechsler and

Yaron (2008) and our new model imply −7.57 and −6.04, respectively, which are much closer to the empirical mean level. In addition, the standard deviation of the VRP is also matched well by the latter two models. However, the Drechsler and Yaron (2008) model has only one state variable. As evident from our earlier analysis, it is very difficult for this model to explain the volatility predictability cross-sectionally. In addition, studies in the volatility literature (see, e.g., Christoffersen, Jacobs, Ornthanalai and Wang, 2008, and Lu and Zhu, 2009) show that the two volatility components model is generally preferred in explaining the the large negative variance risk premium and variance term structure. Hence, our extension of the BY and BKY models seems to offer a promising route for future applications and further extension of the long-run risks models.

Furthermore, it is of interest to examine the predictability of VRP. Empirically, Drechsler and Yaron (2008) compute an AR(1) coefficient of 0.54, which is largely consistent with Bollerslev and Zhou (2007). Theoretically, the VRP should be predictable due to predictability in volatility. The model implied AR(1) coefficients (see Appendix A.6 for the computation details), reported in the last row of Table V, show that our new model matches remarkably well with the data, while the BY and BKY models are too optimistic on the predictability of VRP. These results may serve as additional evidence for supporting the new model of this paper.

IV. Future Research

The long-run risks models have been developed for the equity market. In comparison with competing models, such as Campbell and Cochrane (1999, 2000), Wachter (2006), Bekaert, Engstrom, and Xing (2009), and Barro (2009), that explain the equity risk premium, the long-run risks models seem to have come a long way in explaining a number of additional stylized facts about the equity market. With the extension of this paper, this class of models appears to meet most, if not all, of the major issues in equity market. However, there are still three major challenges facing the long-run risks models.

First, a common question is that, to an econometrician, what the long-run risks really stand for. Empirically, they are latent variables to be estimated, and it is difficult to link them to known macroeconomic risks. But this, in our view, might be exactly the advantage of the long-run risks models since no known models can perform well by relying on a few commonly measured macroeconomic variables. In the real world, one variable may be the major driving force at one time, such as oil price in the 70s, while another variable at another time, and sometimes a completely new variable may emerge, such as credit default swap rates or those causing financial crisis which, by definition, are unanticipated in a rational economy. The long-run risks might capture exactly these variables over time, i.e., those differing factors that have long and lasting effects on the consumption growth. The "event risk" in Liu, Longstaff, and Pan (2003), Barro (2009), and Liu and Loewenstein (2009) might be imagined exactly as variables of this sort, or the long-run risks with high volatility. As a result, no single name of any macroeconomic variable can be given to the ever present long-run risks.

Second, it is debatable whether there exist long-run persistent fluctuations in consumption and dividend growth rates (see, e.g., the BY, Sargent, 2007, and Beeler and Campbell, 2009). Statistically, given the data size, there appears no strong empirical evidence on either side of the debate. Our view is more pragmatic. Without assuming the persistent fluctuations, the models developed so far do not seem capable of explaining as many stylized facts and as well as the BY does, and hence it will be of great interest and urgency to create such models to beat the BY. On the other hand, assuming the persistent fluctuations, there is a workable model. It seems to us that this should not be ignored or abandoned simply because of its controversial assumptions. In fact, by studying under either extreme assumptions, it only enriches our understanding about the driving forces in each type of models. The truth may be somewhere in between, and the models may be patched together in the future, or an absolutely new line of models can be created from what we learned from the shortcoming of old models.

Thirdly, and perhaps of more practical importance, is that the long-run risks models fail miserably in explaining the real bond markets. For typically calibrated models, Beeler and Campbell (2009) point out that the price of a real consol is unrealistically high and approaches infinity. Our extension of the BY model here is of no exception to this problem either. The reason is that the consumption can have too much drop under some extremely large shocks, so that the real rate can approach zero or even negative to clear the market. As a result, a real consol that guarantees a unit consumption payoff forever must have too high a price. In addition, Beeler and Campbell (2009) also point out that the risk premia on long-term real bonds relative to short-term ones can have a negative real term premium if consumption growth follows a persistent process. These insights seem to suggest that imposing certain restrictions in the domain, adding a mean-reverting component, or allowing some reflecting boundaries can further improve our model here. The novel approach of Bansal and Shaliastovich (2008a) may be useful here too. Moreover, the classic general equilibrium model of Cox, Ingersoll, and Ross (1985) appears to offer yet another route of extension to our model via introducing endogenously state variables to explain the real bond markets. Overall, while there remain many challenges, they seem important topics for future research.

V. Conclusion

One of the fundamental problems in asset pricing is to explain various stylized facts about the equity market, for which the rational and general equilibrium long-run risks model of Bansal and Yaron (2004) seems to have come a long way toward this goal. Hence, it is not surprising that there are subsequently a number of important studies along this line of research, including Bansal, Khatchatrian, and Yaron (2005), Bansal, Kiku, and Yaron (2007a, b), Chen, Collin-Dufresne, and Goldstein (2008), Constantinides and Ghosh (2008), Drechsler and Yaron (2008), Eraker (2008), Hansen, Heaton, and Li (2008), Pakos (2008), Avramov and Hore (2009), Bansal, Dittmar, and Kiku (2009), and Beeler and Campbell (2009). However, in the equity market, there are still three major problems confronting the model. It has difficulty in explaining the predictability of consumption, and the varying degrees of volatility predictability of excess stock returns, consumption and dividends. Moreover, the model fails completely in explaining the large negative market variance risk premium. While Bansal, Kiku, and Yaron (2007b) and Drechsler and Yaron (2008) address one of the problems each, there have been no studies that can resolve all the three problems simultaneously.

This paper is the first to propose such an extension of the Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2007b) models, that solves all three problems simultaneously, while retaining many of the desired properties of the original models in explaining well other stylized facts. In our new model, there are two components of volatility: the long-run and short-run volatilities. Both components enter into the dynamics of both the consumption and dividend growth rates, and both are theoretically necessary for resolving the three problems. The implied stock volatility process of our model is consistent with findings in the volatility literature, and hence the model can justify the large negative variance risk premium. The flexibility permitted by the two-factor volatility model also allows for explaining the predictability problems. Looking forward, due to wide applications of the long-run risks model of Bansal and Yaron (2004), such as in equity and currency markets, it will be of interest to see how conclusions of these applications might be altered in light of the proposed new model. Additionally, as pointed out by Beeler and Campbell (2009), it is an open challenge to extend the long-run risks models to the fixed income markets. All of these issues are important and exciting topics of future research.

Appendix A

A.1 Derivation for the A_i's

First, we rewrite the normalized aggregator f defined in Equation (5) as

$$f(C,J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J[G-1],$$

where

$$G \equiv \Big(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\Big)^{1-\frac{1}{\psi}}. \tag{A1}$$

Then, taking derivative of $f(C,J)$ with respect to J and C , we have

$$f_J = (\theta-1)\beta G - \beta\theta \tag{A2}$$

and

$$f_C = \beta\frac{G}{C}(1-\gamma)J. \tag{A3}$$

Conjecturing a solution for J of the following form,

$$J(W_t, X_t, V_{1t}, V_{2t}) = \exp(A_0 + A_1X_t + A_2V_{1t} + A_3V_{2t})\frac{W_t^{1-\gamma}}{1-\gamma}, \tag{A4}$$

and using the standard envelope condition $f_C = J_W$, we have

$$C = J_W^{-\psi}[(1-\gamma)J]^{\frac{1-\gamma\psi}{1-\gamma}}\beta^{\psi}. \tag{A5}$$

Substituting (A3) and (A4) into (A5), we obtain

$$\frac{C}{W} = \beta^{\psi}\exp\left[(A_0 + A_1X_t + A_2V_{1t} + A_3V_{2t})\frac{1-\psi}{1-\gamma}\right]. \tag{A6}$$

Further substitution of (A6) and (A4) into (A1) implies that

$$\beta G = \frac{C_t}{W_t}.$$

Applying the log-linear approximation, we obtain

$$\beta G = \frac{C_t}{W_t} \approx g_1 - g_1\log g_1 + g_1\log(\beta G). \tag{A7}$$

This implies that

$$f = \theta J(\beta G - \beta) \approx \theta J\left[g_1\frac{1-\psi}{1-\gamma}(A_0 + A_1X_t + A_2V_{1t} + A_3V_{2t}) + \xi\right], \tag{A8}$$

where $\theta = 1 - \gamma/(1-\frac{1}{\psi})$ and $\xi = g_1 - g_1\log g_1 + g_1\psi\log\beta - \beta$. Substituting (A8) to the Bellman equation, (7), and collecting the terms containing X_t, V_{1t} and V_{2t}, we have

$$\theta g_1\frac{1-\psi}{1-\gamma}A_0 + \theta\xi + (1-\gamma)\mu + \kappa_1\bar{V}_1\psi A_2 + \kappa_2\bar{V}_2\psi A_3 = 0$$

$$X: \quad \theta g_1\frac{1-\psi}{1-\gamma}A_1 + (1-\gamma) - \alpha\psi A_1 = 0$$

$$V_1: \quad \theta g_1\frac{1-\psi}{1-\gamma}A_2 - \frac{1}{2}\gamma(1-\gamma)\delta_c + \frac{1}{2}\varphi_x^2\delta_x\psi^2A_1^2 - \kappa_1\psi A_2 + \frac{1}{2}\sigma_1^2\psi^2A_2^2 = 0$$

$$V_2: \quad \theta g_1\frac{1-\psi}{1-\gamma}A_3 - \frac{1}{2}\gamma(1-\gamma)(1-\delta_c) + \frac{1}{2}\varphi_x^2(1-\delta_x)\psi^2A_1^2 - \kappa_2\psi A_3 + \frac{1}{2}\sigma_2^2\psi^2A_3^2 = 0.$$

Solving the above algebraic equations, we obtain equation (11) for A's. Q.E.D.

A.2 Derivation for the A_{im}'s

Let

$$\frac{D_t}{P_t} = \exp\{(A_{0m} + A_{1m}X_t + A_{2m}V_{1t} + A_{3m}V_{2t})\}. \tag{A9}$$

A key step in the derivations is to use the following pricing relation

$$E_t\left(\frac{dP_t}{P_t}\right) + \frac{D_t}{P_t}dt = r_f dt - E_t\left[\frac{d\pi_t}{\pi_t}\frac{dP_t}{P_t}\right]. \tag{A10}$$

With similar loglinear approximation as equation (A7), we can approximate the ratio as

$$\frac{D_t}{P_t} \approx g_{0m} + g_{1m}\log\frac{D_t}{P_t} = g_{0m} + g_{1m}((A_{0m} + A_{1m}X_t + A_{2m}V_{1t} + A_{3m}V_{2t})), \tag{A11}$$

where

$$g_{0m} = g_{1m} - g_{1m}\log g_{1m}.$$

Applying Ito's lemma to (A9), we have

$$\frac{dP_t}{P_t} = \frac{dD_t}{D_t} - (A_{1m}dX_t + A_{2m}dV_{1t} + A_{3m}dV_{2t}) + \frac{1}{2}A_{1m}^2(dX_t)^2 + \frac{1}{2}A_{2m}^2(dV_{1t})^2 + \frac{1}{2}A_{3m}^2(dV_{2t})^2.$$

Hence,

$$\begin{aligned} E_t(\frac{dP_t}{P_t})/dt &= \mu_d + \varphi X_t + \alpha A_{1m}X_t - \kappa_1 A_{2m}(\bar{V}_1 - V_{1t}) - \kappa_2 A_{3m}(\bar{V}_2 - V_{2t}) \\ &+\frac{1}{2}A_{1m}^2\varphi_x^2[V_{1t}\delta_x + V_{2t}(1-\delta_x)] + \frac{1}{2}A_{2m}^2\sigma_1^2V_{1t} + \frac{1}{2}A_{3m}^2\sigma_2^2V_{2t}. \end{aligned} \tag{A12}$$

The risk premium term in equation (A10) can thus be written as

$$
\begin{aligned}
-E_t\left[\frac{d\pi_t}{\pi_t}\frac{dP_t}{P_t}\right]/dt &= \lambda_1\sigma_{dc}\sqrt{V_{1t}\delta_c+V_{2t}(1-\delta_c)}-A_{1m}\lambda_2\varphi_x\sqrt{V_{1t}\delta_x+V_{2t}(1-\delta_x)} \\
&\quad -A_{2m}\lambda_3\sigma_1\sqrt{V_{1t}}-A_{3m}\lambda_4\sigma_2\sqrt{V_{2t}}, \qquad (A13)
\end{aligned}
$$

where λ_1, λ_2, λ_3 and λ_4 are market prices of risk as defined in equation (20), and $d\pi_t/\pi_t$ is the pricing kernel as defined in (17).

Now, substituting (A11), (A12), (A13), and risk-free rate as defined in (19) into equation (A10), and then collecting terms containing X_t, we obtain

$$A_{1m} = -\frac{\varphi-\frac{1}{\psi}}{g_{1m}+\alpha}.$$

Collecting terms containing V_{1t} and V_{2t}, we obtain an equation for A_{2m},

$$a_{2m}A_{2m}^2+b_{2m}A_{2m}+c_{2m}=0$$

with

$$a_{2m}=\frac{1}{2}\sigma_1^2, \quad b_{2m}=g_{1m}+\kappa_1-\frac{1-\gamma\psi}{1-\gamma}A_2\sigma_1^2, \quad c_{2m}=(\frac{1}{2}A_{1m}^2-\frac{1-\gamma\psi}{1-\gamma}A_1A_{1m})\varphi_x^2\delta_x+r_2.$$

Solving it, we have

$$A_{2m}=\frac{-b_{2m}\pm\sqrt{b_{2m}^2-4a_{2m}c_{2m}}}{2a_{2m}}.$$

We choose the root that goes to zero when σ_1 goes to zero. This is because when σ_1, or a_{3m} goes to zero, the price sensitivity to V_1 should be zero.

Similarly, we obtain an equation for A_{3m},

$$a_{3m}A_{3m}^2+b_{3m}A_{3m}+c_{3m}=0$$

with

$$a_{3m}=\frac{1}{2}\sigma_2^2,\ b_{2m}=g_{1m}+\kappa_2-\frac{1-\gamma\psi}{1-\gamma}A_3\sigma_2^2,\ c_{3m}=(\frac{1}{2}A_{1m}^2-\frac{1-\gamma\psi}{1-\gamma}A_1A_{1m})\varphi_x^2(1-\delta_x)+r_3.$$

The solution is

$$A_{3m}=\frac{-b_{3m}\pm\sqrt{b_{3m}^2-4a_{3m}c_{3m}}}{2a_{3m}},$$

where we choose the root in a similar fashion as for A_{2m}.

Finally, collecting the constants, we obtain

$$\mu_d - \kappa_1 A_{2m}\bar{V}_1 - \kappa_2 A_{3m}\bar{V}_2 + g_{0m} + g_{1m}A_{0m} + r_0 = 0.$$

Then

$$A_{0m} = -\frac{1}{g_{1m}}\left[\mu_d - \kappa_1 A_{2m}\bar{V}_1 - \kappa_2 A_{3m}\bar{V}_2 + g_{1m} - g_{1m}\log g_{1m} + r_0\right],$$

which is the what we need for the log-linear coefficients. Q.E.D.

A.3 Solutions to g_1 and g_{1m}

Note that the derived solutions depend on the approximation constant g_1, but we can solve it endogenously. This is because, given the model parameters, we can compute the unconditional mean of consumption-wealth ratio as a function of the parameters,

$$\begin{aligned} g_1 &= E\left(\frac{C}{W}\right) = \beta^{\psi}\exp\{A_{0a}\}\exp\left\{\frac{1}{4}A_{1a}^2\varphi_x^2\frac{(\bar{V}_1\delta_x + \bar{V}_2(1-\delta_x))}{\alpha}\right\} \\ &\quad \cdot\exp\left\{-\frac{2\kappa_1\bar{V}_1}{\sigma_1^2}\log(1-\frac{A_{2a}}{2\kappa_1/\sigma_1^2})\right\}\cdot\exp\left\{-\frac{2\kappa_2\bar{V}_2}{\sigma_2^2}\log(1-\frac{A_{3a}}{2\kappa_2/\sigma_2^2})\right\}. \end{aligned} \tag{A14}$$

Now note that the A_{ia}'s on the right hand side are also a function of g_1. Plugging in, we obtain a nonlinear function in terms of g_1 alone, and hence g_1 can be solved in terms of the fundamental parameters of the model, and can be computed numerically with many available algorithms.

Similarly, we can solve g_{1m} endogenously based on dividend-price ratio is

$$\begin{aligned} g_{1m} &= E\left(\frac{D}{P}\right) = \exp\{A_{0m}\}\exp\left\{\frac{1}{4}A_{1m}^2\varphi_x^2\frac{(\bar{V}_1\delta_x + \bar{V}_2(1-\delta_x))}{\alpha}\right\} \\ &\quad \cdot\exp\left\{-\frac{2\kappa_1\bar{V}_1}{\sigma_1^2}\log(1-\frac{A_{2m}}{2\kappa_1/\sigma_1^2})\right\}\cdot\exp\left\{-\frac{2\kappa_2\bar{V}_2}{\sigma_2^2}\log(1-\frac{A_{3m}}{2\kappa_2/\sigma_2^2})\right\}. \end{aligned} \tag{A15}$$

This is easy to solve numerically. Q.E.D.

A.4 Predictability of Variables

First, we want to show (29). Note that the log price process can be written as:

$$d\log P_t = \Big[\mu_d - (A_{2m}\kappa_1\bar{V}_1 + A_{3m}\kappa_2\bar{V}_2) + (\varphi + \alpha A_{1m})X_t + (A_{2m}\kappa_1 - \frac{1}{2}c_1^2)V_{1t} \tag{A16}$$

$$+(A_{3m}\kappa_2 - \frac{1}{2}c_2^2)V_{2t}\Big]dt + \sqrt{\delta_c V_{1t} + (1-\delta_c)V_{2t}}dZ_t, \tag{A17}$$

the log consumption process can be written as:

$$d\log C_t = \left[\mu + X_t - \frac{1}{2}\delta_c V_{1t} - \frac{1}{2}(1-\delta_c)V_{2t}\right]dt + \sqrt{\delta_c V_{1t} + (1-\delta_c)V_{2t}}dZ_{1t}, \tag{A18}$$

and the log dividend process can be written as:

$$d\log D_t = \left[\mu_d + \varphi X_t - \frac{1}{2}(\varphi_d^2\delta_d + \sigma_{dc}^2\delta_c)V_{1t} - \frac{1}{2}(\varphi_d^2(1-\delta_d) + \sigma_{dc}^2(1-\delta_c))V_{2t}\right]$$

$$+\varphi_d\sqrt{V_{1t}\delta_d + V_{2t}(1-\delta_d)}dB_t + \sigma_{dc}\sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)}dZ_{1t}. \tag{A19}$$

Combining these with the riskfree rate process and their suitable differences, we obtain (29) as well as $p_t - d_t$. Then, the covariance and variance evaluations are mechanical, though tedious, and the results are given as (31) and (32). Q.E.D.

A.5 Predictability of Volatilities

First, we prove equation (38). To do so, we apply the following approximation:

$$\frac{1}{\tau}\int_0^\tau \exp(x_s)ds \approx \exp(\frac{1}{\tau}\int_0^\tau x_s ds) \tag{A20}$$

for any process x_s. This is equivalent to an approximation of arithmetic mean by geometric mean. It is a good approximation when the variation of x_t is small in magnitude both across time and probability space. This is true for our variance processes because the magnitude of the variance is generally in the order of $10^{-3} \sim 10^{-4}$, and the variation of $\ln V_t$ is within 1. Applying the above approximation to $\ln V_t$, we have

$$\frac{1}{\tau}\int_0^\tau \sqrt{V_t}dt = \frac{1}{\tau}\int_0^\tau \exp(\frac{1}{2}\ln V_t)dt \approx \exp(\frac{1}{2\tau}\int_0^\tau \ln V_t dt). \tag{A21}$$

Hence,

$$\begin{aligned}
\ln \frac{1}{\tau}\int_0^\tau \sqrt{V_t}dt &\approx \frac{1}{2\tau}\int_0^\tau \ln V_t dt \\
&= \frac{1}{2\tau}\left[\int_0^\tau \ln \bar{V} + \int_0^\tau \ln(1+\frac{V_t-\bar{V}}{\bar{V}})dt\right] \\
&\approx \frac{1}{2}\ln \bar{V} + \frac{1}{2\tau}\int_0^\tau \frac{V_t-\bar{V}}{\bar{V}}dt \\
&= Const + \frac{1}{2\tau\bar{V}}\int_0^\tau V_t dt, \qquad \text{(A22)}
\end{aligned}$$

which is equation (38).

Because of the approximation above, we can express the volatilities as an integral of $b_1V_{1s}+b_2V_{2s}$ over $(t, t+\tau)$. Plugging in these terms into the definition of the covariance, we then obtain (40). Q.E.D.

A.6 Derivation of the AR(1) Coefficient

Consider a stochastic process of the form

$$b_1V_{1t} + b_2V_{2t},$$

where b_1 and b_2 are constants. Due to independence between V_1 and V_2, the unconditional auto-covariance can be evaluated as

$$b_1^2\frac{\sigma_1^2\bar{V}_1}{2\kappa_1}\exp(-\kappa_1\tau) + b_2^2\frac{\sigma_2^2\bar{V}_2}{2\kappa_2}\exp(-\kappa_2\tau)$$

and the unconditional variance can be evaluated as

$$b_1^2\frac{\sigma_1^2\bar{V}_1}{2\kappa_1} + b_2^2\frac{\sigma_2^2\bar{V}_2}{2\kappa_2}.$$

Hence, the AR(1) coefficient can be computed easily based on above. Q.E.D.

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Table I: Long Run Risks Parameters

The table reports the parameters for the three calibrated long-run risks models: the Bansal and Yaron (BY, 2004), and Bansal, Kiku, and Yaron (BKY, 2007a), and the new model. In Panel A, γ is the risk aversion parameter, ψ the EIS parameter, β the discount rate. Other panels provide parameters governing the consumption, dividend and volatility dynamics:

$$
\begin{aligned}
\frac{dC_t}{C_t} &= (\mu + X_t)dt + \sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)}dZ_{1t} \\
dX_t &= -\alpha X_t dt + \varphi_x \sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)}dZ_{2t} \\
\frac{dD_t}{D_t} &= (\mu_d + \varphi X_t)dt + \varphi_d \sqrt{V_{1t}\delta_d + V_{2t}(1-\delta_d)}dB_t + \sigma_{dc}\sqrt{V_{1t}\delta_c + V_{2t}(1-\delta_c)}dZ_{1t} \\
&\quad + \sigma_{dx}\sqrt{V_{1t}\delta_x + V_{2t}(1-\delta_x)}dZ_{2t} + \sigma_{dv}\sqrt{V_{1t}}dw_{1t} \\
dV_{1t} &= \kappa_1(\bar{V}_1 - V_{1t})dt + \sigma_1\sqrt{V_{1t}}dw_{1t} \\
dV_{2t} &= \kappa_2(\bar{V}_2 - V_{2t})dt + \sigma_2\sqrt{V_{2t}}dw_{2t}, \quad \kappa_1 < \kappa_2,
\end{aligned}
$$

where the parameter values are annualized whenever applicable.

Preference Parameters

	γ	ψ	β
BY	7.5 – 10	1.5	2.4%
BKY	10	1.5	3%
New	6	1.5	2.4%

Consumption Growth Dynamics

	μ	α	φ_x	δ_c	δ_x
BY	0.018	0.256	0.528	1	1
BKY	0.018	0.3	0.456	1	1
New	0.018	0.256	0.58	0.64	0.81

Dividend Growth Dynamics

	μ_d	ϕ	ϕ_d	δ_d	σ_{dc}	σ_{dx}	σ_{dv}
BY	0.018	3	4.5	1	0	0	0
BKY	0.018	2.5	5.96	1	2.6	0	0
New	0.03	3.5	3.5	0	1	-4.5	1.9

Volatility Parameters

	Factor 1			Factor 2		
	$\bar{V}_1$	σ_1	κ_1	$\bar{V}_2$	σ_2	κ_2
BY	0.027^2	0.0035	0.156			
BKY	0.025^2	0.0027	0.015			
New	0.022^2	0.004	0.014	0.039^2	1.16	6.6

Table II: Long Run Risks Moments

The table reports moments of various variables of the Long-run risks models. The second column provides those computed based on monthly data from February, 1947 to March, 2007; the third column provides those based on the BY model; and the fourth those based on the BKY model. The results in these three columns are from Beeler and Campbell (2009). The last column provides the moments based on our new model.

Moment	Data (1947.2-2007.3)	BY	BKY	New
$E(\Delta c)$	1.79	1.95	1.82	1.76
$\sigma(\Delta c)$	2.16	2.92	2.96	2.96
AC1(Δc)	0.44	0.51	0.44	0.33
$E(\Delta d)$	1.02	1.66	1.85	1.21
$\sigma(\Delta d)$	10.69	11.57	16.42	18.93
AC1(Δd)	0.14	0.40	0.29	0.11
$E(r_e)$	6.2	6.62	6.58	3.58
$\sigma(r_e)$	18.34	16.88	21.35	18.33
AC1(r_e)	0.04	0.03	0.02	0.00
$E(r_f)$	0.99	2.56	0.99	2.63
$\sigma(r_f)$	4.28	1.30	1.28	4.14
AC1(r_f)	0.59	1.28	0.86	0.11
$E(p-d)$	3.31	3.00	3.04	2.75
$\sigma(p-d)$	0.46	0.16	0.26	0.42
AC1$(p-d)$	0.88	0.77	0.95	0.94

Table III: Predictability of Excess Returns, Consumption and Dividends

The table reports the slope coefficients from predictive regressions of excess returns, consumption growth and dividend growth on log price-dividend ratios based on the data, the BY, the BKY and our new models. Except the last column, the results are from Beeler and Campbell (2009).

	Data	BY	BKY	New
		Excess return		
1Y	-0.059	-0.007	-0.078	-0.074
3Y	-0.229	-0.026	-0.226	-0.214
5Y	-0.421	-0.039	-0.368	-0.351
		Consumption growth		
1Y	0.012	0.114	0.022	0.022
3Y	0.010	0.286	0.052	0.051
5Y	-0.001	0.388	0.069	0.069
		Dividend		
1Y	0.064	0.343	0.054	0.088
3Y	0.076	0.860	0.133	0.213
5Y	0.051	1.171	0.176	0.297

Table IV: Predictability of Volatility: Excess Returns, Consumption and Dividends

The table reports the slope coefficients from predictive regressions of volatilities (of excess returns, consumption growth and dividend growth) on log price-dividend ratios based on the data, the BY, the BKY and our new models. Except the last column, the results are from Beeler and Campbell (2009).

	Data	BY	BKY	New
	Excess return volatility			
1Y	-0.081	-0.123	-1.315	-0.133
3Y	-0.059	-0.115	-1.268	-0.064
5Y	-0.017	-0.113	-1.336	-0.050
	Consumption volatility			
1Y	-0.481	-0.128	-1.420	-0.496
3Y	-0.491	-0.122	-1.382	-0.446
5Y	-0.564	-0.113	-1.336	-0.431
	Dividend volatility			
1Y	-0.530	-0.146	-1.483	-0.476
3Y	-0.478	-0.144	-1.431	-0.425
5Y	-0.496	-0.123	-1.384	-0.410

Table V: Variance Risk Premium

The table reports variance risk premiums (VRP), the standard deviation (std) of VRP, and the autoregression coefficient (AR1) for the market data, the Drechsler and Yaron (2008), the BY, the BKY and our new models. The results of the first two columns are from Drechsler and Yaron (DY, 2008). All the values are monthly and multiplied by 10000.

	Data	DY	BY	BKY	New
VRP	-12.67	-7.57	-0.005	-0.010	-6.04
std	14.38	10.65	0.000	0.000	7.31
AR1	0.54	N/A	0.99	0.99	0.58