A note on LU decomposition of the Discrete Fourier Transform matrix

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July 2, 2015

Abstract

We describe some properties of the lower triangular Toeplitz matrix T_q with coefficients $t_{i,j} = 1/(q;q)_{i-j}$, where $(z;q)_k$ is the q-Pochhammer symbol. We identify explicitly the inverse of T_q and show that both this matrix and its transpose appear in LU decomposition of the Vandermonde matrix V_q having coefficients $v_{i,j} = q^{ij}$. When q is the n-th root of unity, our result gives an explicit LU decomposition of the Discrete Fourier Transform matrix.

Keywords: discrete Fourier transform, Vandermonde matrix, LU decomposition, Toeplitz matrix, q-Binomial Theorem, q-Pochhammer symbol
2010 Mathematics Subject Classification : Primary 15A23, Secondary 15B05

1 Introduction and main results

The Discrete Fourier Transform (DFT) matrix $F = \{f_{k,l}\}_{0\le k,l\le n-1}$ is defined as

$$f_{k,l} = e^{-2\pi \mathrm{i} kl/n}, \quad 0 \le k, l \le n-1. \tag{1}$$

There exist several explicit matrix factorizations of F, which have found important applications in developing Fast Fourier Transform algorithms (see [2]). Our goal in this note is to prove an explicit LU decomposition of the DFT matrix (and of a more general Vandermonde matrix V_q, defined below) and to study the structure of the lower-triangular Toeplitz matrix appearing in this decomposition.

In what follows, we assume that $n \in \mathbb{N}$ and $q \in \mathbb{C}$. We define the q-Pochhammer symbol

$$(z;q)_n := (1-z)(1-zq)\cdots(1-zq^{n-1}), \quad n \ge 1, \tag{2}$$

and $(z;q)_0 := 1$. The following matrices of size $n \times n$ will be considered in this paper:

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Research supported by the Natural Sciences and Engineering Research Council of Canada.

(i) A lower-triangular Toeplitz matrix $T_q = \{t_{i,j}\}_{0\le i,j\le n-1}$ defined by $t_{i,j} = 1/(q;q)_{i-j}$ if $i \ge j$:

$$T_q := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{(q;q)_1} & 1 & 0 & 0 & \dots \\ \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & 0 & \dots \\ \frac{1}{(q;q)_3} & \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(ii) A Vandermonde matrix $V_q = \{q^{ij}\}_{0\le i,j\le n-1}$:

$$V_q := \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & q & q^2 & q^3 & \dots \\ 1 & q^2 & q^4 & q^6 & \dots \\ 1 & q^3 & q^6 & q^9 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(iii) A diagonal matrix D_q having coefficients $\{q^i\}_{0\le i\le n-1}$ on the main diagonal:

$$D_q := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & q & 0 & 0 & \dots \\ 0 & 0 & q^2 & 0 & \dots \\ 0 & 0 & 0 & q^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(iv) A diagonal matrix P_q having coefficients $\{(q;q)_i\}_{0\le i\le n-1}$ on the main diagonal:

$$P_q := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & (q;q)_1 & 0 & 0 & \dots \\ 0 & 0 & (q;q)_2 & 0 & \dots \\ 0 & 0 & 0 & (q;q)_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(v) A "shift" matrix S:

$$S := \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We will also denote by I the $n \times n$ identity matrix. Note that the matrices T_q and $T_{q^{-1}}$ are well-defined for all $q \in \mathbb{C} \setminus \mathcal{A}_n$, where the set $\mathcal{A}_n$ is given by

$$\mathcal{A}_n := \{q \in \mathbb{C} : q^j = 1 \text{ for some } j = 1, 2, \dots, n-1\}.$$

In the next proposition we summarize some properties of the Toeplitz matrix T_q.

Proposition 1. *Assume that* $q \in \mathbb{C} \setminus \mathcal{A}_n$.

(i) $T_q T_{q^{-1}} = (I - S)^{-1}$.

(ii) For $m \in \mathbb{N}$ *we have*

$$T_q D_{q^{-m}} T_{q^{-1}} D_{q^m} = I + \sum_{j=1}^{m-1} \frac{(q^{1-m}; q)_j}{(q)_j} S^j. \tag{3}$$

Remark 1. Note that the matrix $H := (I - S)^{-1}$ – which appears in item (i) – is a lower triangular Toeplitz matrix having coefficients $h_{i,j} = 1$ if $i \geq j$ and $h_{i,j} = 0$ otherwise. Also, the results of item (i) and item (ii) with $m = 1$ provide an explicit expression for the inverse of the matrix T_q:

$$(T_q)^{-1} = T_{q^{-1}}(I - S) = D_{q^{-1}} T_{q^{-1}} D_q.$$

Finally, we would like to point out that the matrix in the right-hand side of (3) is a lower-triangular Toeplitz matrix, having m non-zero diagonals. This matrix has coefficient 1 on the main diagonal and the coefficient $(q^{1-m}; q)_j/(q; q)_j$ on the sub-diagonal number j, for $1 \leq j \leq m - 1$.

The following theorem is our main result in this note. Here T'_q denotes the transpose of the matrix T_q.

Theorem 1. *Assume that* $q \in \mathbb{C} \setminus \mathcal{A}_n$. *Then*

$$V_q = P_q T_q D_{q^{-1}} (P_{q^{-1}})^{-1} T'_q P_q. \tag{4}$$

By multiplying the matrices in (4) and using the identity

$$(1/q; 1/q)_j = (-1)^j q^{-j(j+1)/2} (q; q)_j, \tag{5}$$

we arrive at the following corollary, which provides an explicit LU decomposition of the Vandermonde matrix V_q.

Corollary 1. *We have* $V_q = L_q \times U_q$, *where* $L_q = \{l_{i,j}\}_{0 \leq i,j \leq n-1}$, $U_q = \{u_{i,j}\}_{0 \leq i,j \leq n-1}$ *and*

$$l_{i,j} = \frac{(q; q)_i}{(q; q)_j (q; q)_{i-j}} \ \text{ if } \ i \geq j \ \text{ and } \ l_{i,j} = 0 \text{ otherwise}, \tag{6}$$

$$u_{i,j} = (-1)^i q^{i(i-1)/2} \frac{(q; q)_j}{(q; q)_{j-i}} \ \text{ if } \ i \leq j \ \text{ and } \ u_{i,j} = 0 \text{ otherwise}. \tag{7}$$

One important application of the above result is obtained by setting $q = e^{-2\pi \mathrm{i}/n}$: this gives an explicit LU decomposition of the DFT matrix F defined by (1).

We hope that the our factorization result (4) will find future applications in developing new Fast Fourier Tranform methods. This result reduces the problem of computing the matrix-vector product $V_q \times \mathbf{x}$ (for a vector $\mathbf{x} \in \mathbb{C}^n$) to potentially simpler problems of computing products $T_q \times \mathbf{x}$ and $T'_q \times \mathbf{x}$. As we have demonstrated in Proposition 1 (see also formulas (13) and (14) below), the Toeplitz matrix T_q has a lot of structure which may lead to fast algorithms for computing matrix-vector products $T_q \times \mathbf{x}$.

2 Proofs

The proofs are based on the q-Binomial Theorem (see [1][Theorem 10.2.1]), which states that

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{j\ge 0} \frac{(a;q)_j}{(q;q)_j} z^j, \quad |q|<1, \ |z|<1. \tag{8}$$

Here $(z;q)_\infty := \prod_{l\ge 0}(1-zq^l)$ and the product converges for all $z\in\mathbb{C}$ and $|q|<1$. We also record the two corollaries of the q-Binomial Theorem which will be needed later: for $|q|<1$

$$\frac{1}{(z;q)_\infty} = \sum_{j\ge 0} \frac{z^j}{(q;q)_j}, \quad |z|<1, \tag{9}$$

$$(z;q)_\infty = \sum_{j\ge 0} \frac{(-1)^j q^{j(j-1)/2}}{(q;q)_j} z^j, \quad z\in\mathbb{C}. \tag{10}$$

Proof of Proposition 1: Let us prove the result in item (i). The Toeplitz matrix T_q can be written as

$$T_q = I + \sum_{j\ge 1} \frac{S^j}{(q;q)_j}. \tag{11}$$

The above formula is easy to derive, given the fact that S^j is a zero matrix for $j\ge n$ while for $1\le j\le n-1$ it is a matrix whose coefficients have value 1 on the sub-diagonal number j and zeros everywhere else. Note that the series in (11) terminates at $j=n-1$ and we do not need to worry about convergence issues.

Similarly, from (5) and (11) we obtain

$$T_{q^{-1}} = I + \sum_{j\ge 1} \frac{(-1)^j q^{j(j-1)/2}}{(q;q)_j} (qS)^j. \tag{12}$$

Assume that $|q|<1$. Then formulas (9) and (11) give us

$$T_q = [(S;q)_\infty]^{-1} = (1-S)^{-1}\times(1-qS)^{-1}\times(1-q^2S)^{-1}\times\cdots. \tag{13}$$

Similarly, formulas (10) and (12) give us

$$T_{q^{-1}} = (qS;q)_\infty = (1-qS)\times(1-q^2S)\times(1-q^3S)\times\cdots. \tag{14}$$

Combining the above two identities we obtain the result in item (i) for $|q|<1$. The general case $q\in\mathbb{C}\backslash\mathcal{A}_n$ follows by analytical continuation.

The proof of the result in item (ii) uses the same ideas. Again, assume that $|q|<1$. From (5) we check that $D_{q^{-m}}T_{q^{-1}}D_{q^m}$ is a Toeplitz matrix of the form

$$D_{q^{-m}}T_{q^{-1}}D_{q^m} = I + \sum_{j\ge 1} \frac{(-1)^j q^{j(j-1)/2}}{(q;q)_j}(q^{1-m}S)^j = (q^{1-m}S;q)_\infty.$$

Using the above result and formula (13) we obtain

$$T_q D_{q^{-m}}T_{q^{-1}}D_{q^m} = [(S;q)_\infty]^{-1}\times(q^{1-m}S;q)_\infty = (q^{1-m}S;q)_{m-1}.$$

The desired desired result (3) follows by applying (8) and analytical continuation in q. □

Proof of Theorem 1: Multiplying all matrices in (4) (and using (5)) we see that formula (4) is equivalent to the following identity: for $i, j \geq 0$

$$\sum_{k=0}^{\min(i,j)} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k (q;q)_{i-k} (q;q)_{j-k}} = \frac{q^{ij}}{(q;q)_i (q;q)_j}. \tag{15}$$

We will prove the above identity by writing the Taylor series of the function

$$g(u,v) := \frac{(uv;q)_\infty}{(u;q)_\infty (v;q)_\infty}, \quad |u| < 1, \ |v| < 1, \ |q| < 1,$$

in two different ways. First of all, from formula (8) we obtain

$$g(u,v) = \frac{1}{(v;q)_\infty} \times \frac{(uv;q)_\infty}{(u;q)_\infty} = \frac{1}{(v;q)_\infty} \sum_{i \geq 0} \frac{(v;q)_i}{(q;q)_i} u^i.$$

Using the fact that $(v;q)_i/(v;q)_\infty = 1/(q^i v;q)_\infty$ and expanding this expression as Taylor series in v via (9) we conclude that

$$g(u,v) = \sum_{i \geq 0} \sum_{j \geq 0} \frac{q^{ij} u^i v^j}{(q;q)_i (q;q)_j}. \tag{16}$$

On the other hand, we can obtain the series expansion of $g(u,v)$ by applying formulas (9) and (10) in the form

$$(uv;q)_\infty = \sum_{k \geq 0} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k} u^k v^k,$$

$$\frac{1}{(u;q)_\infty} = \sum_{l \geq 0} \frac{u^l}{(q;q)_l},$$

$$\frac{1}{(v;q)_\infty} = \sum_{m \geq 0} \frac{v^m}{(q;q)_m}.$$

We multiply the above three series expansions and obtain a Taylor series representation in the form

$$g(u,v) = \sum_{k \geq 0} \sum_{l \geq 0} \sum_{m \geq 0} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k (q;q)_l (q;q)_m} u^{k+l} v^{k+m}. \tag{17}$$

Comparing the coefficients in front of the term $u^i v^j$ in both formulas (16) and (17) gives us the desired result (15). □

References

[1] G. E. Andrews, R. Askey, and R. Roy. *Special functions*. The University Press, Cambridge, 1999.

[2] C. Van Loan. *Computational Frameworks for the Fast Fourier Transform*. Society for Industrial and Applied Mathematics, 1992.