

# Necessary Optimality Conditions for Optimal Control Problems with Equilibrium Constraints

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**Abstract.** This paper introduces and studies the optimal control problem with equilibrium constraints (OCPEC). The OCPEC is an optimal control problem with a mixed state and control equilibrium constraint formulated as a complementarity constraint and it can be seen as a dynamic mathematical program with equilibrium constraints (MPEC). It provides a powerful modeling paradigm for many practical problems such as bilevel optimal control problems and dynamic principal-agent problems. In this paper, we propose Fritz John type weak, Clarke, Mordukhovich and strong stationarities for the OCPEC. Moreover, we give some sufficient conditions to ensure that the local minimizers of the OCPEC are Fritz John type weakly stationary, Mordukhovich stationary and strongly stationary respectively.

**Key Words.** Optimal control problem with equilibrium constraints, necessary optimality condition, weak stationarity, Mordukhovich stationarity, strong stationarity.

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# 1 Introduction

We are given a time interval  $[t_0, t_1] \subseteq \mathbb{R}$ , a multifunction  $U$  mapping  $[t_0, t_1]$  to nonempty subsets of  $\mathbb{R}^m$ , and a dynamic function  $\phi : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ . A control or control function  $u(\cdot)$  is a measurable function on  $[t_0, t_1]$  with values in a control constraint set  $U(t)$  for almost every  $t \in [t_0, t_1]$ . The state or state trajectory, corresponding to a given control  $u(\cdot)$ , refers to a solution  $x(\cdot)$  of the following controlled differential equation:

$$\dot{x}(t) = \phi(t, x(t), u(t)) \text{ almost everywhere (a.e.) } t \in [t_0, t_1], \quad (1.1)$$

$$(x(t_0), x(t_1)) \in E, \quad (1.2)$$

where  $E$  is a given closed subset of  $\mathbb{R}^n \times \mathbb{R}^n$  and  $\dot{x}(t)$  is the vector of first-order derivative of the state  $x(\cdot)$  at time  $t$ . The differential equation (1.1) linking the state  $x(\cdot)$  and the control  $u(\cdot)$  is referred to as the state equation. In optimal control, one looks for a state and control pair  $(x(\cdot), u(\cdot))$  satisfying the state equation (1.1) and the boundary condition (1.2) so as to minimize an objective function  $J(x(\cdot), u(\cdot))$ . In practice, there are generally extra constraints to be satisfied by the state and control pair. Such constraints are called mixed state and control constraints (mixed constraints for short).

Pang and Stewart [29] recently introduced the class of controlled differential variational inequality (DVI) problem :

$$\begin{aligned} \dot{x}(t) &= \phi(t, x(t), u(t)) \quad \text{a.e. } t \in [t_0, t_1], \\ (x(t_0), x(t_1)) &\in E, \\ u(t) &\in K, \langle u' - u(t), \Upsilon(t, x(t), u(t)) \rangle \geq 0 \quad \forall u' \in K \quad \text{a.e. } t \in [t_0, t_1], \end{aligned}$$

where  $\Upsilon : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  is a vector-valued function and  $K$  is a closed convex subset in  $\mathbb{R}^m$ . The DVI provides a powerful modeling paradigm for many practical problems such as differential Nash equilibrium games ([4, 29]), multi-rigid-body dynamics with frictional contacts ([37]), and hybrid engineering systems ([18]). In the case where  $K = \mathbb{R}_+^m$ , the DVI becomes the controlled differential complementarity problem (DCP)

$$\begin{aligned} \dot{x}(t) &= \phi(t, x(t), u(t)) \quad \text{a.e. } t \in [t_0, t_1], \\ (x(t_0), x(t_1)) &\in E, \\ 0 &\leq u(t) \perp \Upsilon(t, x(t), u(t)) \geq 0 \quad \text{a.e. } t \in [t_0, t_1], \end{aligned} \quad (1.3)$$

where  $a \perp b$  means that vector  $a$  is perpendicular to vector  $b$ . In the case where  $K$  can be expressed as a set of solutions satisfying some inequality constraints such as

$$K = \{u \in \mathbb{R}^m : g(u) \leq 0\}$$

where  $g(\cdot)$  is a convex vector-valued function, when  $g(\cdot)$  is linear or Slater's constraint quali-

fication holds, the DVI can also be represented as a DCP,

$$\begin{aligned}
\dot{x}(t) &= \phi(t, x(t), u(t)) && \text{a.e. } t \in [t_0, t_1], \\
(x(t_0), x(t_1)) &\in E, \\
-\Upsilon(t, x(t), u(t)) + \nabla g(u(t))\lambda(t) &= 0 && \text{a.e. } t \in [t_0, t_1], \\
0 \leq \lambda(t) \perp -g(u(t)) \geq 0 &&& \text{a.e. } t \in [t_0, t_1],
\end{aligned} \tag{1.4}$$

where  $\nabla g$  denotes the transposed Jacobian of  $g$  and  $\lambda(t)$  is the multiplier associated with the inequality constraint  $g(u(t)) \leq 0$ .

Motivated by the studies for the DVI, we consider a class of controlled differential complementarity system where in addition to the state equation (1.1) and the boundary condition (1.2), the state and control pair  $(x(\cdot), u(\cdot))$  satisfies some mixed equality and inequality constraints, as well as a mixed equilibrium system formulated as a complementarity system:

$$0 \leq G(t, x(t), u(t)) \perp H(t, x(t), u(t)) \geq 0 \quad \text{a.e. } t \in [t_0, t_1], \tag{1.5}$$

where  $G, H : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^l$ . If  $G_i(t, x(t), u(t)) = H_i(t, x(t), u(t)) = 0$ , then we say that the index  $i$  is degenerate. It is obvious that such a system includes the DCPs (1.3) and (1.4) as special cases. Correspondingly, it is natural to determine what is the “best” control (or the “best” state and control pair) satisfying such a system to achieve some given objective. A simple example is to find the best control from such a system so that the final state  $x(t_1)$  will reach some prescribed target from an given initial state  $x(t_0)$ . In this paper, we introduce a class of optimal control problems with equilibrium constraints (OCPEC) in which one looks for a state and control pair  $(x(\cdot), u(\cdot))$  from such a system so as to minimize an objective function  $J(x(\cdot), u(\cdot))$ . Mathematically, the OCPEC considered in this paper is of the form

$$\begin{aligned}
\text{(OCPEC)} \quad \min \quad & J(x(\cdot), u(\cdot)) \\
\text{s.t.} \quad & \dot{x}(t) = \phi(t, x(t), u(t)) \quad \text{a.e. } t \in [t_0, t_1], \\
& g(t, x(t), u(t)) \leq 0, \quad h(t, x(t), u(t)) = 0 \quad \text{a.e. } t \in [t_0, t_1],
\end{aligned} \tag{1.6}$$

$$\begin{aligned}
& 0 \leq G(t, x(t), u(t)) \perp H(t, x(t), u(t)) \geq 0 \quad \text{a.e. } t \in [t_0, t_1], \\
& u(t) \in U(t) \quad \text{a.e. } t \in [t_0, t_1], \\
& (x(t_0), x(t_1)) \in E,
\end{aligned} \tag{1.7}$$

where  $g : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{l_1}$  and  $h : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{l_2}$ .

The OCPEC can be considered as a dynamic version of the mathematical program with equilibrium constraints (MPEC) which has been an active area of research in recent years (see, e.g., the monographs [26, 28]). The OCPEC provides a powerful modeling paradigm for many practical problems such as the dynamic optimization of chemical processes with changes in the number of equilibrium phases [31]. A large part of source problems of the OCPEC comes from bilevel optimal control problems (see, e.g, [1, 15, 16, 45, 46]), Stackelberg differential

games (see, e.g., [17,43]), and dynamic principal-agent problems (see, e.g., [25,34]) when there exist inequality constraints in the lower-level problem. For those problems, if the lower-level problem, which is a parametric optimal control problem, is replaced by the Pontryagin's maximum principle (see [30,38]) which is the well-known first-order necessary optimality condition for optimal control problems, then an OCPEC results; see, e.g., [15, Section 6.1].

It is desirable to know whether there exists an optimal control before solving the OCPEC. The Filippov's existence theorem for Mayer problems which is due to Filippov [12] (see, e.g., [3, Theorem 9.2.i]) requires the convexity of the velocity set  $\phi(t, x, \mathcal{U}(t, x))$  where

$$\mathcal{U}(t, x) := \{u \in U(t) : g(t, x, u) \leq 0, h(t, x, u) = 0, 0 \leq G(t, x, u) \perp H(t, x, u) \geq 0\}.$$

The velocity set is in general nonconvex due to the existence of the complementarity constraint. Hence the classical existence theorem may not be applicable and one may need to look for new ways to prove the existence of optimal controls for the OCPEC or use the existence theorem in the relaxed control setting (see [39,50]). We leave these challenging questions for future research.

In this paper, we assume that an optimal control exists for the OCPEC and focus on deriving its necessary optimality conditions. To the best of our knowledge, there is no such result in the literature so far. Although deriving necessary optimality conditions for optimal control problems with mixed constraints is a highly challenging problem, some progresses have been made; see, e.g., [6–10,20,21,24]. Unfortunately, none of these results are applicable to the OCPEC and its reformulations. The constraint (1.5) is obviously equivalent to that for almost every  $t \in [t_0, t_1]$ ,

$$G(t, x(t), u(t)) \geq 0, H(t, x(t), u(t)) \geq 0, G(t, x(t), u(t))^\top H(t, x(t), u(t)) \leq 0, \quad (1.8)$$

where  $^\top$  denotes the transpose, which is clearly a system of inequalities. However, all the inequalities in (1.8) never hold strictly at the same time. This means that the Mangasarian-Fromovitz constraint qualification (MFCQ) is violated at any point satisfying (1.8). The classical necessary optimality conditions for optimal control problems with mixed equality and inequality constraints generally require the linear independence constraint qualification (LICQ) (see, e.g., [20]) or the Mangasarian-Fromovitz condition (MFC) (see, e.g., [7]) over some neighborhood of the local minimizer. But both LICQ and MFC are stronger than the MFCQ. Thus the classical necessary optimality conditions for optimal control problems with equality and inequality constraints cannot be applied to the OCPEC with the complementarity constraint (1.5) reformulated as inequality constraints (1.8). In the MPEC literature, by using the so-called ‘‘piecewise programming’’ approach (see, e.g., [26,44]), the feasible region of an MPEC is locally reformulated as a union of finitely many pieces where each piece is expressed as a system of equality and inequality constraints, and then it can be shown that the strong (S-) stationarity holds under the so-called MPEC LICQ. It is obvious that such an approach fails for the dynamic complementarity system (1.5). A well-known technique to

derive a necessary optimality condition for an MPEC called the Clarke (C-) stationarity is to use the following equivalent reformulation of the complementarity constraint:

$$\min\{G, H\} = 0$$

where “min” denotes the componentwise minimum, to replace the complementarity system  $0 \leq G \perp H \geq 0$  (see, e.g., [35, 48]). This technique, however, is also not applicable to the OCPEC since such an approach leads to an optimal control problem with a *nonsmooth* mixed equality constraint for which there does not exist any applicable necessary optimality condition in the control literature. Yet another equivalent reformulation of the complementarity constraint is  $(G, H) \in \mathcal{C}^l$  where

$$\mathcal{C}^l := \{(a, b) \in \mathbb{R}^l \times \mathbb{R}^l : 0 \leq a \perp b \geq 0\}, \quad (1.9)$$

is the complementarity cone. It is known that this reformulation is useful to obtain a necessary optimality condition in the form of Mordukhovich (M-) stationarity in the MPEC literature; see, e.g., [48]. Using this reformulation, the OCPEC can be equivalently reformulated as

$$\begin{aligned} (P_s) \quad & \min && J(x(\cdot), u(\cdot)) \\ & \text{s.t.} && \dot{x}(t) = \phi(t, x(t), u(t)) \quad \text{a.e. } t \in [t_0, t_1], \\ & && (x(t), u(t)) \in S(t) \quad \text{a.e. } t \in [t_0, t_1], \\ & && (x(t_0), x(t_1)) \in E, \end{aligned} \quad (1.10)$$

with

$$S(t) := \left\{ (x, u) \in \mathbb{R}^n \times U(t) : \begin{array}{l} g(t, x, u) \leq 0, h(t, x, u) = 0 \\ (G(t, x, u), H(t, x, u)) \in \mathcal{C}^l \end{array} \right\}. \quad (1.11)$$

An optimal control problem in the form of  $(P_s)$  with an abstract mixed constraint  $S(t)$  was recently studied by Clarke and De Pinho [7]. In this paper we first derive a slightly sharper necessary optimality condition for  $(P_s)$  than Clarke and De Pinho [7, Theorem 2.1] and then apply it to the problem with  $S(t)$  defined as in (1.11). We hope that we would get the M-stationarity as in the MPEC literature. Unfortunately for the OCPEC, no sign information on the multipliers associated with the degenerate indices can be derived and consequently, we can only obtain a weak stationarity. In order to get more sign information on the multipliers associated with the degenerate indices, we further apply the Weierstrass condition to obtain the second set of multipliers. A counter example shows that in general these two sets of multipliers may be different in measure. However, under the MPEC LICQ, since the multipliers corresponding to the weak stationarity are unique, these two sets of multipliers coincide almost everywhere and we obtain the S-stationarity with one set of multipliers.

We organize our paper as follows. In Section 2 we review some basic concepts in nonsmooth analysis and derive a sharper necessary optimality condition for problem  $(P_s)$ . In Section 3 we derive necessary optimality conditions for the OCPEC. Section 4 illustrates our results using a simple example.

## 2 Preliminary and preliminary results

Throughout this paper,  $\|\cdot\|$  denotes the Euclidean norm and  $\mathcal{B}_\delta(x) := \{y : \|y - x\| < \delta\}$  the open ball centered at  $x$  with positive radius  $\delta$ . Given a subset  $\Omega \subseteq \mathbb{R}^n$ , the boundary, the closure, the convex hull, and the closed convex hull are denoted by  $\text{bd } \Omega$ ,  $\text{cl } \Omega$ ,  $\text{co } \Omega$ , and  $\text{clco } \Omega$ , respectively. Moreover,  $\text{dist}_\Omega(x)$  denotes the Euclidean distance from  $x$  to  $\Omega$ . For any  $a, b \in \mathbb{R}^n$ , we denote by  $\langle a, b \rangle$  the inner product of vector  $a$  and vector  $b$ . Given a mapping  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $\nabla\psi$  stands for the transposed Jacobian of  $\psi$ . The Minkowski sum of a singleton  $\{a\}$  and an arbitrary set  $A$  is denoted by  $a + A$ .

### 2.1 Background in variational analysis

We now review some basic concepts and results in variational analysis that will be used later on; see, e.g., [5, 27, 33] for more details. Given a subset  $\Omega \subseteq \mathbb{R}^n$  and  $x \in \text{cl } \Omega$ , the proximal normal cone to  $\Omega$  at  $x$  is defined by

$$\mathcal{N}_\Omega^P(x) := \{v \in \mathbb{R}^n : \exists \sigma \geq 0 \text{ s.t. } \langle v, y - x \rangle \leq \sigma \|y - x\|^2 \quad \forall y \in \Omega\},$$

the limiting normal cone to  $\Omega$  at  $x$  is defined by

$$\mathcal{N}_\Omega^L(x) := \{v \in \mathbb{R}^n : \exists x^k \rightarrow x, \exists v^k \rightarrow v \text{ with } v^k \in \mathcal{N}_\Omega^P(x^k) \quad \forall k\},$$

and the Clarke normal cone to  $\Omega$  at  $x$  is defined by

$$\mathcal{N}_\Omega^C(x) := \text{clco } \mathcal{N}_\Omega^L(x),$$

which also holds true even if the space is not finite dimensional but is an Asplund space [27]. For a multifunction  $\Xi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ , its graph and domain are defined, respectively, by

$$\text{gph } \Xi := \{(x, u) : u \in \Xi(x)\} \quad \text{and} \quad \text{dom } \Xi := \{x : \Xi(x) \neq \emptyset\}.$$

Both the limiting normal cone and the Clarke normal cone are closed in the sense that their graphs are closed. It is obvious that the following inclusions hold

$$\mathcal{N}_\Omega^P(x) \subseteq \mathcal{N}_\Omega^L(x) \subseteq \mathcal{N}_\Omega^C(x).$$

The following expression for the limiting normal cone of the complementarity cone  $\mathcal{C}^l$  is well-known (see, e.g., [47, Proposition 3.7]) and will be used in Section 3.

**Proposition 2.1** *For any  $(a, b) \in \mathcal{C}^l$  where  $\mathcal{C}^l$  is defined in (1.9),*

$$\mathcal{N}_{\mathcal{C}^l}^L(a, b) = \left\{ (d^1, d^2) : \begin{array}{ll} d_i^1 = 0 & \text{if } a_i > 0, \quad d_i^2 = 0 \quad \text{if } b_i > 0 \\ d_i^1 < 0, d_i^2 < 0 \text{ or } d_i^1 d_i^2 = 0 & \text{if } a_i = b_i = 0 \end{array} \right\}.$$

Given a lower semicontinuous function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and a point  $x$  with  $\varphi(x)$  finite, the limiting subdifferential of  $\varphi$  at  $x$  is defined by

$$\partial^L \varphi(x) := \left\{ v \in \mathbb{R}^n \mid \exists x^k \rightarrow x, v^k \rightarrow v \text{ with } \lim_{y \rightarrow x^k} \frac{f(y) - f(x^k) - \langle v^k, y - x^k \rangle}{\|y - x^k\|} \geq 0 \forall k \right\}.$$

If  $\varphi$  is Lipschitz continuous near  $x$ , then the Clarke subdifferential of  $\varphi$  at  $x$  can be defined by

$$\partial^C \varphi(x) := \text{clco } \partial^L \varphi(x),$$

which also holds true even if the space is not finite dimensional but is an Asplund space [27]. Both the limiting subdifferential and Clarke subdifferential are also closed in the sense that their graphs are closed.

Given a point  $(x, u) \in \text{clgph} \Xi$  for multifunction  $\Xi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ , the coderivative  $D^* \Xi(x, u) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  of  $\Xi$  at  $(x, u)$  is defined by

$$D^* \Xi(x, u)(y) := \{v \in \mathbb{R}^n : (v, -y) \in \mathcal{N}_{\text{gph} \Xi}^L(x, u)\}.$$

The symbol  $D^* \Xi(x)$  is used when  $\Xi$  is single-valued at  $x$  and  $u = \Xi(x)$ . In particular, if  $\Xi(\cdot)$  is single-valued and Lipschitz continuous near  $x$ , then

$$D^* \Xi(x)(y) = \partial^L \langle y, \Xi \rangle(x) \quad \forall y \in \mathbb{R}^m.$$

## 2.2 Local error bound condition and constraint qualifications

In this subsection, we consider the following constrained system:

$$\Omega := \{z \in \mathcal{D} : g(z) \leq 0, h(z) = 0, (G(z), H(z)) \in \mathcal{C}^l\}, \quad (2.1)$$

where  $\mathcal{D}$  is a closed subset in  $\mathbb{R}^d$ , and  $g : \mathbb{R}^d \rightarrow \mathbb{R}^{l_1}, h : \mathbb{R}^d \rightarrow \mathbb{R}^{l_2}, G, H : \mathbb{R}^d \rightarrow \mathbb{R}^l$  are all strictly differentiable. We say that the local error bound condition holds (for the constraint system representing the set  $\Omega$  as in (2.1)) at  $\bar{z} \in \Omega$  if there exist  $\tau > 0$  and  $\delta > 0$  such that

$$\text{dist}_\Omega(z) \leq \tau(\|g(z)_+\| + \|h(z)\| + \text{dist}_\mathcal{D}(G(z), H(z))) \quad \forall z \in \mathcal{B}_\delta(\bar{z}) \cap \mathcal{D}.$$

It is well-known that the local error bound condition at  $\bar{z}$  is equivalent to the calmness of the perturbed constrained system

$$\Omega(y) := \{z \in \mathcal{D} : g(z) + y \leq 0, h(z) + y = 0, 0 \leq G(z) + y \perp H(z) + y \leq 0\} \quad (2.2)$$

at  $(0, \bar{z})$  (see, e.g., [19]). The local error bound condition is a very weak condition and there exist many sufficient conditions for it to hold; see, e.g., [14, 19, 40–42, 49]. The following constraint qualifications are such sufficient conditions.

**Definition 2.1 (MPEC constraint qualifications)** Let  $\bar{z} \in \Omega$  where  $\Omega$  is defined in (2.1). When  $\mathcal{D} = \mathbb{R}^d$ , we say that the MPEC LICQ holds at  $\bar{z}$  if the family of gradients

$$\{\nabla g_i(\bar{z}) : i \in I_g(\bar{z})\} \cup \{\nabla h_i(\bar{z}) : i = 1, \dots, l_2\} \cup \{\nabla G_i(\bar{z}) : i \in I_G(\bar{z})\} \cup \{\nabla H_i(\bar{z}) : i \in I_H(\bar{z})\}$$

is linearly independent, where  $I_\psi(\bar{z}) := \{i : \psi_i(\bar{z}) = 0\}$  for a vector-valued function  $\psi$ .

We say that the MPEC linear condition holds if functions  $g(\cdot), h(\cdot), G(\cdot), H(\cdot)$  are linear and  $\mathcal{D}$  is a union of finitely many polyhedral sets.

We say that the MPEC quasi-normality holds at  $\bar{z}$  if there is no nonzero vector  $(\lambda, v, \mu, \nu)$  such that

- $0 \in \nabla g(\bar{z})\lambda + \nabla h(\bar{z})v - \nabla G(\bar{z})\mu - \nabla H(\bar{z})\nu + \mathcal{N}_{\mathcal{D}}^L(\bar{z})$ ,
- $\lambda \geq 0$ ,  $\lambda_i = 0$   $i \notin I_g(\bar{z})$ ,  $\mu_i = 0$   $i \notin I_G(\bar{z})$ ,  $\nu_i = 0$   $i \notin I_H(\bar{z})$ ,  $\mu_i > 0, \nu_i > 0$  or  $\mu_i \nu_i = 0$   $i \in I_G(\bar{z}) \cap I_H(\bar{z})$ ,
- there exists a sequence  $\{z^k\} \subseteq \mathcal{D}$  converging to  $\bar{z}$  such that for each  $k$ ,

$$\begin{aligned} \lambda_i > 0 &\implies g_i(z^k) > 0, & v_i \neq 0 &\implies v_i h_i(z^k) > 0, \\ \mu_i \neq 0 &\implies \mu_i G_i(z^k) < 0, & \nu_i \neq 0 &\implies \nu_i H_i(z^k) < 0. \end{aligned}$$

It should be noted that the MPEC quasi-normality is a weak condition which holds automatically if the MPEC linear condition holds with  $\mathcal{D} = \mathbb{R}^d$  and is also implied by the MPEC LICQ.

**Proposition 2.2** *The local error bound condition holds at  $\bar{z} \in \Omega$  if the MPEC linear condition holds or the MPEC quasi-normality holds at  $\bar{z}$ .*

**Proof.** If the MPEC linear condition holds, then it is easy to see that the perturbed constrained system  $\Omega(y)$  defined in (2.2) is a polyhedral multifunction and hence the local error bound condition holds [32]. Moreover, it follows from [14, Theorem 5.2] that the MPEC quasi-normality at  $\bar{z}$  implies the local error bound condition at  $\bar{z}$ . ■

### 2.3 Optimal control problems with a set constraint

In this subsection, we consider the optimal control problem  $(P_s)$  where

$$J(x(\cdot), u(\cdot)) := \int_{t_0}^{t_1} F(t, x(t), u(t)) dt + f(x(t_0), x(t_1)).$$

Here  $F : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  and  $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ . The basic hypotheses on the problem data, in force throughout this subsection, are the following:  $F(\cdot), \phi(\cdot)$  are  $\mathcal{L} \times \mathcal{B}$  measurable,

$S(\cdot)$  is  $\mathcal{L}$  measurable, and  $f(\cdot)$  is locally Lipschitz continuous, where  $\mathcal{L} \times \mathcal{B}$  denotes the  $\sigma$ -algebra of subsets of appropriate spaces generated by product sets  $M \times N$  where  $M$  is a Lebesgue ( $\mathcal{L}$ ) measurable subset in  $\mathbb{R}$  and  $N$  is a Borel ( $\mathcal{B}$ ) measurable subset in  $\mathbb{R}^n \times \mathbb{R}^m$ .

We refer to any absolutely continuous function as an arc. An admissible pair for  $(P_s)$  is a pair of functions  $(x(\cdot), u(\cdot))$  on  $[t_0, t_1]$  for which  $u(\cdot)$  is a control and  $x(\cdot)$  is an arc that satisfies all the constraints in  $(P_s)$ . Given a measurable radius function  $R : [t_0, t_1] \rightarrow (0, \infty]$ , as in [7] we say that an admissible pair  $(x^*(\cdot), u^*(\cdot))$  is a local minimizer of radius  $R(\cdot)$  for  $(P_s)$  if there exists  $\epsilon > 0$  such that for every pair  $(x(\cdot), u(\cdot))$  admissible for  $(P_s)$  which also satisfies

$$\|x(t) - x^*(t)\| \leq \epsilon, \|u(t) - u^*(t)\| \leq R(t) \text{ a.e. } t \in [t_0, t_1], \int_{t_0}^{t_1} \|\dot{x}(t) - \dot{x}^*(t)\| dt \leq \epsilon,$$

we have  $J(x^*(\cdot), u^*(\cdot)) \leq J(x(\cdot), u(\cdot))$ . Note that the so-called  $W^{1,1}$  local minimizer in the optimal control literature is the case where the radius function  $R(\cdot)$  is identically  $+\infty$ .

Let  $(x^*(\cdot), u^*(\cdot))$  be a local minimizer of radius  $R(\cdot)$  for  $(P_s)$ . For every given  $t \in [t_0, t_1]$ , a radius function  $R(\cdot)$ , and a positive constant  $\epsilon$ , we define a neighborhood of the point  $(x^*(t), u^*(t))$  as follows:

$$S_*^{\epsilon, R}(t) := \{(x, u) \in S(t) : \|x - x^*(t)\| \leq \epsilon, \|u - u^*(t)\| \leq R(t)\}. \quad (2.3)$$

Other than the basic hypotheses on the problem data, we also assume that the following assumptions hold for  $(P_s)$ .

**Assumption 2.1** (i) *There exist measurable functions  $k_x^\phi(\cdot), k_x^F(\cdot), k_u^\phi(\cdot), k_u^F(\cdot)$  such that for almost every  $t \in [t_0, t_1]$  and for every  $(x^1, u^1), (x^2, u^2) \in S_*^{\epsilon, R}(t)$ , we have*

$$\begin{aligned} \|\phi(t, x^1, u^1) - \phi(t, x^2, u^2)\| &\leq k_x^\phi(t)\|x^1 - x^2\| + k_u^\phi(t)\|u^1 - u^2\|, \\ |F(t, x^1, u^1) - F(t, x^2, u^2)| &\leq k_x^F(t)\|x^1 - x^2\| + k_u^F(t)\|u^1 - u^2\|. \end{aligned}$$

(ii) *There exists a positive measurable function  $k_S(\cdot)$  such that for almost every  $t \in [t_0, t_1]$ , the bounded slope condition holds, that is,*

$$(x, u) \in S_*^{\epsilon, R}(t), (\alpha, \beta) \in \mathcal{N}_{S(t)}^P(x, u) \implies \|\alpha\| \leq k_S(t)\|\beta\|. \quad (2.4)$$

(iii) *The functions  $k_x^\phi(\cdot), k_x^F(\cdot), k_S(\cdot)[k_u^\phi(\cdot) + k_u^F(\cdot)]$  are integrable and there exists a positive number  $\eta > 0$  such that  $R(t) \geq \eta k_S(t)$  for almost every  $t \in [t_0, t_1]$ .*

Assumption 2.1(i) can be considered as a Lipschitz condition in variable  $(x, u)$  with measurable Lipschitz constants. The condition is automatically satisfied with time independent Lipschitz constants when  $u^*(\cdot)$  is bounded over  $[t_0, t_1]$ , the radius function  $R(\cdot)$  is a positive constant function, and functions  $F(\cdot), \phi(\cdot)$  are locally Lipschitz continuous in variable  $(t, x, u)$ .

Assumption 2.1(ii) is a key condition proposed in [7] to derive the necessary optimality conditions. We will investigate some sufficient conditions for such an assumption to hold in our problem setting in Section 3.

In the following, we give a slightly sharper result than Clarke and De Pinho [7, Theorem 2.1] for  $(P_s)$ , which will be used in obtaining our main results. The conclusion is slightly different from the original theorem [7, Theorem 2.1] in that the Weierstrass condition holds only on the open ball  $\{u : \|u - u^*(t)\| < R(t)\}$ . In fact, it was shown in [2] by a counter example that the Weierstrass condition may not hold on the closed ball  $\{u : \|u - u^*(t)\| \leq R(t)\}$  even for a standard optimal control problem without mixed constraints.

**Theorem 2.1** *Let  $(x^*(\cdot), u^*(\cdot))$  be a local minimizer of radius  $R(\cdot)$  for  $(P_s)$  and Assumption 2.1 hold. Then there exist a number  $\lambda_0 \in \{0, 1\}$  and an arc  $p(\cdot)$  such that*

(a) *the nontriviality condition holds:  $(\lambda_0, p(t)) \neq 0 \quad \forall t \in [t_0, t_1]$ ;*

(b) *the transversality condition holds:*

$$(p(t_0), -p(t_1)) \in \lambda_0 \partial^L f(x^*(t_0), x^*(t_1)) + \mathcal{N}_E^L(x^*(t_0), x^*(t_1));$$

(c) *the Euler inclusion holds: For almost every  $t \in [t_0, t_1]$ ,*

$$\begin{aligned} (\dot{p}(t), 0) \in \text{co}\{(w, 0) : (w, 0) \in \partial^L \{ \langle -p(t), \phi(t, \cdot, \cdot) \rangle + \lambda_0 F(t, \cdot, \cdot) \}(x^*(t), u^*(t)) \\ + \mathcal{N}_{S(t)}^L(x^*(t), u^*(t))\}; \end{aligned} \quad (2.5)$$

(d) *the Weierstrass condition holds: For almost every  $t \in [t_0, t_1]$ ,*

$$\begin{aligned} (x^*(t), u) \in S(t), \|u - u^*(t)\| < R(t) \implies \\ \langle p(t), \phi(t, x^*(t), u) \rangle - \lambda_0 F(t, x^*(t), u) \leq \langle p(t), \phi(t, x^*(t), u^*(t)) \rangle - \lambda_0 F(t, x^*(t), u^*(t)). \end{aligned}$$

**Proof.** It follows from [7, Theorem 2.1] that the results (a), (b) and (d) hold. Then it is sufficient to show that the Euler inclusion (c) holds. Note that (2.5) is slightly sharper than the corresponding one in [7, Theorem 2.1]. In fact, in the proof of [7, Theorem 2.1], for the case where  $F \equiv 0$  we can get (2.5) easily by making use of equation (6) and the second equation from the end on Page 4521 in [7]. The general case in which a nonzero  $F$  is present is reducible to the already treated one by augmentation as explained at the end of the proof of [7, Theorem 2.1]. ■

**Remark 2.1** *Theorem 2.1 is a Fritz John (FJ) type necessary optimality condition. In the case where  $\lambda_0 = 0$ , no information on the objective functions can be derived from the necessary optimality condition and it becomes useless. Hence the case where  $\lambda_0 = 1$  is desirable. It*

follows from Theorem 2.1 that if there is no nonzero abnormal multiplier, i.e., the following implication holds:

$$\begin{cases} (p(t_0), -p(t_1)) \in \mathcal{N}_E^L(x^*(t_0), x^*(t_1)), \\ (\dot{p}(t), 0) \in \text{co} \{ (w, 0) : (w, 0) \in \partial^L \langle -p(t), \phi(t, \cdot, \cdot) \rangle (x^*(t), u^*(t)) \\ \quad + \mathcal{N}_{S(t)}^L(x^*(t), u^*(t)) \} \quad \text{a.e. } t \in [t_0, t_1], \\ (x^*(t), u) \in S(t), \|u - u^*(t)\| < R(t) \\ \implies \langle p(t), \phi(t, x^*(t), u) \rangle \leq \langle p(t), \phi(t, x^*(t), u^*(t)) \rangle \quad \text{a.e. } t \in [t_0, t_1], \\ \implies p(t) = 0 \text{ for some } t \in [t_0, t_1], \end{cases}$$

then the conclusion of Theorem 2.1 holds with  $\lambda_0 = 1$ . Such a condition is automatically satisfied in the case of free (right or left) end point, that is,  $E = E_0 \times \mathbb{R}^n$  or  $E = \mathbb{R}^n \times E_1$  with closed subsets  $E_0, E_1$  in  $\mathbb{R}^n$ , since it follows from result (b) in Theorem 2.1 that  $p(t_1) = 0$  or  $p(t_0) = 0$ . Throughout the paper, all the derived necessary optimality conditions are FJ type conditions. The desired case where  $\lambda_0 = 1$  can be obtained provided that there is no nonzero abnormal multiplier, which is always true if the end point is free.

### 3 Necessary optimality conditions for OCPECs

In this section, we develop necessary optimality conditions for the OCPEC under the following basic hypotheses.

**Assumption 3.1 (Basic assumption)**  $F : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  and  $\phi : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  are  $\mathcal{L} \times \mathcal{B}$  measurable,  $g : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^l$ ,  $h : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{l_2}$ , and  $G, H : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^l$  are  $\mathcal{L}$  measurable in variable  $t$  and strict differentiable in variable  $(x, u)$ ,  $U : [t_0, t_1] \rightrightarrows \mathbb{R}^m$  is a  $\mathcal{L}$  measurable multifunction,  $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is locally Lipschitz continuous, and  $E$  is a closed subset in  $\mathbb{R}^n \times \mathbb{R}^n$ .

In fact, we can extend our results easily to the case where the mappings  $g(\cdot), h(\cdot), G(\cdot), H(\cdot)$  are only Lipschitz continuous in variable  $(x, u)$  and strictly differentiable at  $(x^*(t), u^*(t))$ . But for simplicity of the exposition, we just assume that they are strictly differentiable in variable  $(x, u)$  as in Assumption 3.1.

Given an admissible pair  $(x(\cdot), u(\cdot))$  and any given  $t \in [t_0, t_1]$ , we denote the index sets by

$$\begin{aligned} I_t^-(x, u) &:= \{i : g_i(t, x(t), u(t)) < 0\}; \\ I_t^0(x, u) &:= \{i : g_i(t, x(t), u(t)) = 0\}; \\ \mathcal{I}_t^{+0}(x, u) &:= \{i : G_i(t, x(t), u(t)) > 0 = H_i(t, x(t), u(t))\}; \\ \mathcal{I}_t^{00}(x, u) &:= \{i : G_i(t, x(t), u(t)) = 0 = H_i(t, x(t), u(t))\}; \\ \mathcal{I}_t^{0+}(x, u) &:= \{i : G_i(t, x(t), u(t)) = 0 < H_i(t, x(t), u(t))\}. \end{aligned}$$

Moreover, we denote

$$\Psi(t, x, u; \lambda^g, \lambda^h, \lambda^G, \lambda^H) := g(t, x, u)^\top \lambda^g + h(t, x, u)^\top \lambda^h - G(t, x, u)^\top \lambda^G - H(t, x, u)^\top \lambda^H.$$

**Theorem 3.1** *Let  $(x^*(\cdot), u^*(\cdot))$  be a local minimizer of radius  $R(\cdot)$  for the OCPEC and Assumptions 2.1 and 3.1 hold. If for almost every  $t \in [t_0, t_1]$ , the local error bound condition for the system representing  $S(t)$  as in (1.11) holds at  $(x^*(t), u^*(t))$ , then there exist a number  $\lambda_0 \in \{0, 1\}$ , an arc  $p(\cdot)$ , and measurable functions  $\lambda^g(\cdot), \lambda^h(\cdot), \lambda^G(\cdot), \lambda^H(\cdot)$  such that the following conditions hold:*

- (i) *the nontriviality condition  $(\lambda_0, p(t)) \neq 0 \quad \forall t \in [t_0, t_1]$ ;*
- (ii) *the transversality condition*

$$(p(t_0), -p(t_1)) \in \lambda_0 \partial^L f(x^*(t_0), x^*(t_1)) + \mathcal{N}_E^L(x^*(t_0), x^*(t_1));$$

- (iii) *the Euler adjoint inclusion: For almost every  $t \in [t_0, t_1]$ ,*

$$\begin{aligned} (\dot{p}(t), 0) &\in \partial^C \{ \langle -p(t), \phi(t, \cdot, \cdot) \rangle + \lambda_0 F(t, \cdot, \cdot) \} (x^*(t), u^*(t)) + \{0\} \times \mathcal{N}_{U(t)}^C(u^*(t)) \\ &\quad + \nabla_{x,u} \Psi(t, x^*(t), u^*(t); \lambda^g(t), \lambda^h(t), \lambda^G(t), \lambda^H(t)), \\ \lambda^g(t) &\geq 0, \quad \lambda_i^g(t) = 0 \quad i \in I_t^-(x^*, u^*), \\ \lambda_i^G(t) &= 0 \quad i \in \mathcal{I}_t^{+0}(x^*, u^*), \quad \lambda_i^H(t) = 0 \quad i \in \mathcal{I}_t^{0+}(x^*, u^*); \end{aligned}$$

- (iv) *the Weierstrass condition for radius  $R(\cdot)$ : For almost every  $t \in [t_0, t_1]$ ,*

$$\begin{aligned} (x^*(t), u) \in S(t), \quad \|u - u^*(t)\| < R(t) &\implies \\ \langle p(t), \phi(t, x^*(t), u) \rangle - \lambda_0 F(t, x^*(t), u) &\leq \langle p(t), \phi(t, x^*(t), u^*(t)) \rangle - \lambda_0 F(t, x^*(t), u^*(t)). \end{aligned}$$

**Proof.** For simplicity in the proof we omit the equality and inequality constraints (1.6) and the control constraint (1.7). Then  $(x^*(\cdot), u^*(\cdot))$  is a local minimizer of radius  $R(\cdot)$  for  $(P_s)$  with  $S(t)$  defined as follows:

$$S(t) = \left\{ (x, u) : (G(t, x, u), H(t, x, u)) \in \mathcal{C}^l \right\}. \quad (3.1)$$

By virtue of Theorem 2.1, results (i), (ii) and (iv) in this theorem hold. It now suffices to show (iii) by Theorem 2.1(c). Since the local error bound condition holds at  $(x^*(t), u^*(t))$  and functions  $G(t, \cdot, \cdot), H(t, \cdot, \cdot)$  are strictly differentiable, it follows from Proposition 2.1, [22, Proposition 3.4] and [27, Theorem 1.90] that

$$\begin{aligned} &\mathcal{N}_{S(t)}^L(x^*(t), u^*(t)) \\ &\subseteq \left\{ -\nabla_{x,u} G(t, x^*(t), u^*(t))\beta - \nabla_{x,u} H(t, x^*(t), u^*(t))\gamma : (\beta, \gamma) \in \mathcal{M}^*(t) \right\}, \quad (3.2) \end{aligned}$$

where  $\mathcal{M}^*(t) := \mathcal{M}_1^*(t) \times \cdots \times \mathcal{M}_l^*(t)$  and for each  $i = 1, \dots, l$ ,

$$\mathcal{M}_i^*(t) := \left\{ (\beta_i, \gamma_i) : \begin{array}{ll} \beta_i = 0 & i \in \mathcal{I}_t^{+0}(x^*, u^*) \\ \gamma_i = 0 & i \in \mathcal{I}_t^{0+}(x^*, u^*) \\ \beta_i > 0, \gamma_i > 0 \text{ or } \beta_i \gamma_i = 0 & i \in \mathcal{I}_t^{00}(x^*, u^*) \end{array} \right\}. \quad (3.3)$$

Then it follows from (2.5) and (3.2) that for almost every  $t \in [t_0, t_1]$ ,

$$\begin{aligned} (\dot{p}(t), 0) &\in \partial^C \{ \langle -p(t), \phi(t, \cdot, \cdot) \rangle + \lambda_0 F(t, \cdot, \cdot) \} (x^*(t), u^*(t)) \\ &+ \text{co} \left\{ \sum_{i=1}^l [\beta_i \nabla_{x,u} G_i(t, x^*(t), u^*(t)) + \gamma_i \nabla_{x,u} H_i(t, x^*(t), u^*(t))] : (\beta, \gamma) \in \mathcal{M}^*(t) \right\} \end{aligned} \quad (3.4)$$

Then by Carathéodory's theorem for the convex hull, it follows from (3.4) that for almost every  $t \in [t_0, t_1]$ , there exist  $\alpha \in \Delta := \{ \alpha \in \mathbb{R}_+^{n+m+1} : \sum_{j=1}^{n+m+1} \alpha_j = 1 \}$  and  $\{(\beta^j, \gamma^j) \in \mathcal{M}^*(t) : j = 1, \dots, n+m+1\}$  such that

$$\psi(t, \alpha, \beta, \gamma) \in -(\dot{p}(t), 0) + \partial^C \{ \langle -p(t), \phi(t, \cdot, \cdot) \rangle + \lambda_0 F(t, \cdot, \cdot) \} (x^*(t), u^*(t)), \quad (3.5)$$

where

$$\psi(t, \alpha, \beta, \gamma) := \sum_{j=1}^{n+m+1} \sum_{i=1}^l [\alpha_j \beta_i^j \nabla_{x,u} G_i(t, x^*(t), u^*(t)) + \alpha_j \gamma_i^j \nabla_{x,u} H_i(t, x^*(t), u^*(t))]$$

is a Carathéodory mapping since it is continuous in  $(\alpha, \beta, \gamma)$  and measurable in  $t$  by virtue of [33, Theorem 14.13]. By [33, Theorem 14.56 and Exercise 14.12], the multifunction

$$\partial^C \{ \langle -p(t), \phi(t, \cdot, \cdot) \rangle + \lambda_0 F(t, \cdot, \cdot) \} (x^*(t), u^*(t))$$

is measurable in  $t$ . Hence

$$-(\dot{p}(t), 0) + \partial^C \{ \langle -p(t), \phi(t, \cdot, \cdot) \rangle + \lambda_0 F(t, \cdot, \cdot) \} (x^*(t), u^*(t))$$

is measurable in  $t$ . Moreover, by [33, Example 14.15], the multifunction  $\Omega_i^*(t)$  is measurable. Thus, it follows from the implicit measurable function theorem [33, Theorem 14.16] that there exist measurable functions  $\alpha(\cdot) \in \Delta$  and  $\{(\beta^j(\cdot), \gamma^j(\cdot)) \in \mathcal{M}^*(\cdot) : j = 1, \dots, n+m+1\}$  such that for almost every  $t \in [t_0, t_1]$ ,

$$\begin{aligned} (\dot{p}(t), 0) &\in \partial^C \{ \langle -p(t), \phi(t, \cdot, \cdot) \rangle + \lambda_0 F(t, \cdot, \cdot) \} (x^*(t), u^*(t)) \\ &+ \sum_{i=1}^l \sum_{j=1}^{n+m+1} \alpha_j(t) \beta_i^j(t) \nabla_{x,u} G_i(t, x^*(t), u^*(t)) + \sum_{i=1}^l \sum_{j=1}^{n+m+1} \alpha_j(t) \gamma_i^j(t) \nabla_{x,u} H_i(t, x^*(t), u^*(t)). \end{aligned}$$

Let

$$\lambda_i^G(\cdot) := \sum_{j=1}^{n+m+1} \alpha_j(\cdot) \beta_i^j(\cdot), \quad \lambda_i^H(\cdot) := \sum_{j=1}^{n+m+1} \alpha_j(\cdot) \gamma_i^j(\cdot) \quad i = 1, \dots, l.$$

Then  $\lambda^G(\cdot), \lambda^H(\cdot)$  are measurable in  $t$ . Since  $(\beta_i^j(t), \gamma_i^j(t)) \in \mathcal{M}_i^*(t)$  for any  $j = 1, \dots, n+m+1$  and any  $i = 1, \dots, l$ , it is not hard to see that

$$\lambda_i^G(t) = 0 \quad i \in \mathcal{I}_t^{+0}(x^*, u^*), \quad \lambda_i^H(t) = 0 \quad i \in \mathcal{I}_t^{0+}(x^*, u^*).$$

Hence the desired result follows immediately. The proof is complete.  $\blacksquare$

By virtue of the Weierstrass condition for radius  $R(\cdot)$  (i.e., Theorem 3.1(iv)), we have that for almost every  $t \in [t_0, t_1]$ ,  $u^*(t)$  is a local minimizer of the following MPEC:

$$\begin{aligned} \min_{u \in U(t)} \quad & -\langle p(t), \phi(t, x^*(t), u) \rangle + \lambda_0 F(t, x^*(t), u) \\ \text{s.t.} \quad & g(t, x^*(t), u) \leq 0, \quad h(t, x^*(t), u) = 0, \\ & 0 \leq G(t, x^*(t), u) \perp H(t, x^*(t), u) \geq 0. \end{aligned} \quad (3.6)$$

Hence under MPEC constraint qualifications (see, e.g., [23, 35, 48, 49]), the C-, M- and S-stationarities may hold at  $u^*(t)$ . This and Theorem 3.1 motivate us to define the following stationarity conditions.

**Definition 3.1** *Let  $(x^*(\cdot), u^*(\cdot))$  be an admissible pair of the OCPEC. We say that the FJ type weak stationarity (W-stationarity) holds at  $(x^*(\cdot), u^*(\cdot))$  if there exist  $\lambda_0 \in \{0, 1\}$ , an arc  $p(\cdot)$ , and measurable functions  $\lambda^g(\cdot), \lambda^h(\cdot), \lambda^G(\cdot), \lambda^H(\cdot)$  such that Theorem 3.1(i)-(iv) hold.*

*We say that the FJ type C-stationarity holds at  $(x^*(\cdot), u^*(\cdot))$  if  $(x^*(\cdot), u^*(\cdot))$  is W-stationary with arc  $p(\cdot)$  and there exist measurable functions  $\eta^g(\cdot), \eta^h(\cdot), \eta^G(\cdot), \eta^H(\cdot)$  such that for almost every  $t \in [t_0, t_1]$ ,*

$$\begin{aligned} 0 \in \partial_u^L \{ & -\langle p(t), \phi(t, x^*(t), \cdot) \rangle + \lambda_0 F(t, x^*(t), \cdot) \}(u^*(t)) \\ & + \nabla_u \Psi(t, x^*(t), u^*(t); \eta^g(t), \eta^h(t), \eta^G(t), \eta^H(t)) + \mathcal{N}_{U(t)}^L(u^*(t)), \end{aligned} \quad (3.7)$$

$$\eta^g(t) \geq 0, \quad \eta_i^g(t) = 0 \quad i \in I_t^-(x^*, u^*), \quad (3.8)$$

$$\eta_i^G(t) = 0 \quad i \in \mathcal{I}_t^{+0}(x^*, u^*), \quad \eta_i^H(t) = 0 \quad i \in \mathcal{I}_t^{0+}(x^*, u^*), \quad (3.9)$$

and

$$\eta_i^G(t) \eta_i^H(t) \geq 0 \quad i \in \mathcal{I}_t^{00}(x^*, u^*).$$

*We say that the FJ type M-stationarity holds at  $(x^*(\cdot), u^*(\cdot))$  if  $(x^*(\cdot), u^*(\cdot))$  is W-stationary with arc  $p(\cdot)$  and there exist measurable functions  $\eta^g(\cdot), \eta^h(\cdot), \eta^G(\cdot), \eta^H(\cdot)$  such that for almost every  $t \in [t_0, t_1]$ , (3.7)–(3.9) hold and*

$$\eta_i^G(t) > 0, \eta_i^H(t) > 0 \text{ or } \eta_i^G(t) \eta_i^H(t) = 0 \quad i \in \mathcal{I}_t^{00}(x^*, u^*). \quad (3.10)$$

*We say that the FJ type S-stationarity holds at  $(x^*(\cdot), u^*(\cdot))$  if  $(x^*(\cdot), u^*(\cdot))$  is W-stationary with arc  $p(\cdot)$  and there exist measurable functions  $\eta^g(\cdot), \eta^h(\cdot), \eta^G(\cdot), \eta^H(\cdot)$  such that for almost every  $t \in [t_0, t_1]$ , (3.7)–(3.9) hold and*

$$\eta_i^G(t) \geq 0, \eta_i^H(t) \geq 0 \quad i \in \mathcal{I}_t^{00}(x^*, u^*).$$

We will refer them to the *W*-, *C*-, *M*-, and *S*-stationarities respectively if  $\lambda_0 = 1$ .

In Definition 3.1, there are two sets of multipliers. The ideal situation is when these two sets of multipliers are identical almost everywhere. In the case where the multipliers  $\lambda^g(\cdot)$ ,  $\lambda^h(\cdot)$ ,  $\lambda^G(\cdot)$ ,  $\lambda^H(\cdot)$  and  $\eta^g(\cdot)$ ,  $\eta^h(\cdot)$ ,  $\eta^G(\cdot)$ ,  $\eta^H(\cdot)$  can be chosen as the same almost everywhere,  $(x^*(\cdot), u^*(\cdot))$  being *C*-, *M*-, *S*-stationarities becomes that  $(x^*(\cdot), u^*(\cdot))$  is *W*-stationary with multipliers satisfying the following extra sign conditions:

$$\begin{aligned} \lambda_i^G(t)\lambda_i^H(t) &\geq 0 \quad i \in \mathcal{I}_t^{00}(x^*, u^*) \quad \text{a.e.}, t \in [t_0, t_1]; \\ \lambda_i^G(t) > 0, \lambda_i^H(t) > 0 \text{ or } \lambda_i^G(t)\lambda_i^H(t) &= 0 \quad i \in \mathcal{I}_t^{00}(x^*, u^*) \quad \text{a.e.}, t \in [t_0, t_1]; \\ \lambda_i^G(t) &\geq 0, \lambda_i^H(t) \geq 0 \quad i \in \mathcal{I}_t^{00}(x^*, u^*) \quad \text{a.e.}, t \in [t_0, t_1], \end{aligned}$$

respectively. Although we hope that these two sets of multipliers can be chosen to be the same almost everywhere, the following example shows that it is not always possible.

**Example 3.1** Consider the problem

$$\begin{aligned} \min \quad & x(t_1) \\ \text{s.t.} \quad & \dot{x}(t) = u(t) \quad \text{a.e. } t \in [t_0, t_1], \\ & 0 \leq -u(t) \perp -x(t) - u^2(t) \geq 0 \quad \text{a.e. } t \in [t_0, t_1], \\ & x^3(t) \geq 0 \quad \text{a.e. } t \in [t_0, t_1], \\ & x(t_0) \leq 0, \end{aligned}$$

where  $x, u : \mathbb{R} \rightarrow \mathbb{R}$ . Since  $x(\cdot)$  is absolutely continuous and  $x(t) \geq 0$  for almost every  $t \in [t_0, t_1]$ , we must have  $x(t) \geq 0$  for every  $t \in [t_0, t_1]$ . Then it is not hard to verify that  $(x^*(\cdot), u^*(\cdot)) = (0, 0)$  is a minimizer of the above problem. By solving the *M*-stationarity condition at  $(x^*(\cdot), u^*(\cdot))$ , we have

$$p(t_0) \geq 0, p(t_1) = -1, \tag{3.11}$$

$$\dot{p}(t) = \lambda^H(t), -p(t) + \lambda^G(t) = 0 \quad \text{a.e. } t \in [t_0, t_1], \tag{3.12}$$

$$-p(t) + \eta^G(t) = 0, \eta^G(t) > 0, \eta^H(t) > 0 \text{ or } \eta^G(t)\eta^H(t) = 0 \quad \text{a.e. } t \in [t_0, t_1]. \tag{3.13}$$

Since  $p(\cdot)$  is absolutely continuous, by virtue of (3.11), there exists an interval  $[t', t''] \subseteq [t_0, t_1]$  with  $t' < t''$  such that

$$p(t) < 0, \dot{p}(t) < 0 \quad \forall t \in [t', t''].$$

This together with (3.12)–(3.13) implies

$$\lambda^G(t) < 0, \lambda^H(t) < 0 \quad \text{a.e. } t \in [t', t''],$$

$$\eta^G(t) < 0, \eta^H(t) = 0 \quad \text{a.e. } t \in [t', t''],$$

which implies that  $\lambda^H(t) \neq \eta^H(t)$  for almost every  $t \in [t', t'']$ .

We now show that the FJ type M-stationarity is necessary for optimality under certain constraint qualifications. We call a condition under which the M-stationary condition holds at any local minimizer of the MPEC an *MPEC constraint qualification for M-stationarity*. The reader is referred to [13,23,48,49] and the references within for MPEC constraint qualifications for M-stationarity.

**Theorem 3.2** *Let  $(x^*(\cdot), u^*(\cdot))$  be a local minimizer of radius  $R(\cdot)$  for the OCPEC and Assumptions 2.1 and 3.1 hold. Then the FJ type M-stationarity holds at  $(x^*(\cdot), u^*(\cdot))$  if for almost every  $t \in [t_0, t_1]$ , one of the following conditions holds:*

- (a) *The local error bound condition for the system representing  $S(t)$  as in (1.11) holds at  $(x^*(t), u^*(t))$  and an MPEC constraint qualification for M-stationarity holds at  $u^*(t)$  for problem (3.6);*
- (b) *The MPEC linear condition holds for problem (3.6), i.e., functions  $g(t, \cdot, \cdot)$ ,  $h(t, \cdot, \cdot)$ ,  $G(t, \cdot, \cdot)$ ,  $H(t, \cdot, \cdot)$  are linear in  $(x, u)$  and  $U(t)$  is a union of finitely many polyhedral sets;*
- (c) *The MPEC quasi-normality holds at  $u^*(t)$  for problem (3.6), i.e., there is no nonzero multiplier  $(\lambda, \nu, \mu, \nu)$  such that*

- $0 \in \nabla_u \Psi(t, x^*(t), u^*(t); \lambda, \nu, \mu, \nu) + \mathcal{N}_{U(t)}^L(u^*(t))$ ,
- $\lambda \geq 0$ ,  $\lambda_i = 0$   $i \in I_t^-(x^*, u^*)$ ,  $\mu_i = 0$   $i \in \mathcal{I}_t^{+0}(x^*, u^*)$ ,  $\nu_i = 0$   $i \in \mathcal{I}_t^{0+}(x^*, u^*)$ ,  
 $\mu_i > 0, \nu_i > 0$  or  $\mu_i \nu_i = 0$   $i \in \mathcal{I}_t^{00}(x^*, u^*)$ ,
- *there exists a sequence  $\{u^k\} \subseteq U(t)$  converging to  $u^*(t)$  such that for each  $k$ ,*

$$\begin{aligned} \lambda_i > 0 &\implies g_i(t, x^*(t), u^k) > 0, & \nu_i \neq 0 &\implies \nu_i h_i(t, x^*(t), u^k) > 0, \\ \mu_i \neq 0 &\implies \mu_i G_i(t, x^*(t), u^k) < 0, & \nu_i \neq 0 &\implies \nu_i H_i(t, x^*(t), u^k) < 0. \end{aligned}$$

**Proof.** The conclusion under condition (a) follows from Theorem 3.1 and the discussion on the stationarity of problem (3.6). For almost every  $t \in [t_0, t_1]$ , the local error bound condition for the system representing  $S(t)$  as in (1.11) holds at  $(x^*(t), u^*(t))$  under either the MPEC linear condition in condition (b) or the MPEC quasi-normality in condition (c). Then it follows from Theorem 3.1 that  $(x^*(\cdot), u^*(\cdot))$  is W-stationary under conditions (a) or (b). Moreover, both the MPEC linear condition and the MPEC quasi-normality imply that for almost every  $t \in [t_0, t_1]$ , there exist  $\eta^g(t), \eta^h(t), \eta^G(t), \eta^H(t)$  such that (3.7)–(3.10) hold (see [48, Theorem 2.2] and [23, Theorem 3.3]). Moreover by the implicit measurable function theorem [33, Theorem 14.16],  $\eta^g(\cdot), \eta^h(\cdot), \eta^G(\cdot), \eta^H(\cdot)$  can be chosen measurably. Therefore the FJ type M-stationarity holds at  $(x^*(\cdot), u^*(\cdot))$  under conditions (b) and (c).  $\blacksquare$

We next derive the FJ type S-stationarity under the MPEC LICQ. It should be noted that the MPEC LICQ is generic and hence not a stringent assumption by [36].

**Theorem 3.3** *Let  $(x^*(\cdot), u^*(\cdot))$  be a local minimizer of radius  $R(\cdot)$  for the OCPEC and Assumptions 2.1 and 3.2 hold. Assume that for almost every  $t \in [t_0, t_1]$ ,  $U(t) = \mathbb{R}^m$  and functions  $F(t, \cdot, \cdot)$ ,  $\phi(t, \cdot, \cdot)$  are strictly differentiable at  $(x^*(t), u^*(t))$ . If for almost every  $t \in [t_0, t_1]$ , the MPEC LICQ holds at  $u^*(t)$  for problem (3.6), i.e., the family of gradients*

$$\begin{aligned} & \{\nabla_u g_i(t, x^*(t), u^*(t)) : i \in \mathcal{I}_t^0(x^*, u^*)\} \cup \{\nabla_u h_i(t, x^*(t), u^*(t)) : i = 1, \dots, l_2\} \\ & \cup \{\nabla_u G_i(t, x^*(t), u^*(t)) : i \in \mathcal{I}_t^{00}(x^*, u^*)\} \cup \{\nabla_u H_i(t, x^*(t), u^*(t)) : i \in \mathcal{I}_t^0(x^*, u^*)\} \end{aligned}$$

where  $\mathcal{I}_t^0(x^*, u^*) := \mathcal{I}_t^{0+}(x^*, u^*) \cup \mathcal{I}_t^{00}(x^*, u^*)$  and  $\mathcal{I}_t^0(x^*, u^*) := \mathcal{I}_t^{+0}(x^*, u^*) \cup \mathcal{I}_t^{00}(x^*, u^*)$ , is linearly independent, then the FJ type  $S$ -stationarity holds at  $(x^*(\cdot), u^*(\cdot))$ . Moreover, the multipliers  $\eta^g(\cdot), \eta^h(\cdot), \eta^G(\cdot), \eta^H(\cdot)$  can be taken as equal to  $\lambda^g(\cdot), \lambda^h(\cdot), \lambda^G(\cdot), \lambda^H(\cdot)$  almost everywhere. That is, there exist a number  $\lambda_0 \in \{0, 1\}$ , an arc  $p(\cdot)$ , and measurable functions  $\lambda^g(\cdot), \lambda^h(\cdot), \lambda^G(\cdot), \lambda^H(\cdot)$  such that  $(x^*(\cdot), u^*(\cdot))$  is  $W$ -stationary and for almost every  $t \in [t_0, t_1]$ , the following extra sign condition holds:

$$\lambda_i^G(t) \geq 0, \lambda_i^H(t) \geq 0 \quad i \in \mathcal{I}_t^{00}(x^*, u^*).$$

**Proof.** Under the MPEC LICQ assumption, we have that for almost every  $t \in [t_0, t_1]$ , the local error bound condition holds at  $(x^*(t), u^*(t))$  by Proposition 2.2. Thus it follows from Theorem 3.1 that  $(x^*(\cdot), u^*(\cdot))$  is  $W$ -stationary. Moreover, since the MPEC LICQ holds at  $u^*(t)$  for almost every  $t \in [t_0, t_1]$ , it then follows from [35, Theorem 2] that there exist measurable functions  $\eta^g(\cdot), \eta^h(\cdot), \eta^G(\cdot), \eta^H(\cdot)$  such that for almost every  $t \in [t_0, t_1]$ , (3.7)–(3.9) hold and

$$\eta_i^G(t) \geq 0, \eta_i^H(t) \geq 0 \quad i \in \mathcal{I}_t^{00}(x^*, u^*).$$

Thus, the first part of the theorem is derived immediately. Moreover, by the MPEC-LICQ assumption, it is not hard to see that  $\lambda^g(t) = \eta^g(t)$ ,  $\lambda^h(t) = \eta^h(t)$ ,  $\lambda^G(t) = \eta^G(t)$ ,  $\lambda^H(t) = \eta^H(t)$  for almost every  $t \in [t_0, t_1]$ . Therefore, the second part of the theorem follows immediately. ■

For problem  $(P_s)$ , if  $S(t) = \mathbb{R}^n \times U(t)$  for almost every  $t \in [t_0, t_1]$  (which corresponds to the case of standard optimal control problem without mixed constraints), then the bounded slope condition (2.4) holds automatically for almost every  $t \in [t_0, t_1]$  since in this case, (2.4) becomes

$$(x, u) \in S_*^{\epsilon, R}(t), \beta \in \mathcal{N}_{U(t)}^P(u) \implies k_S(t) \|\beta\| \geq 0,$$

which holds trivially if  $k_S(t) \geq 0$ . If there exists  $t_0$  such that  $S(t_0) = X(t_0) \times \mathbb{R}^m$  where  $X(t_0)$  is a closed subset in  $\mathbb{R}^n$  and

$$\text{dist}_{\text{bd } X(t_0)}(x^*(t_0)) = \|x^*(t_0) - \bar{x}^*(t_0)\| \leq \epsilon, \quad \bar{x}^*(t_0) \in \text{bd } X(t_0),$$

then (2.4) at time  $t_0$  never hold since there exists  $0 \neq \alpha \in \mathcal{N}_{X(t_0)}^P(\bar{x}^*(t_0))$  by [33, Exercise 6.19]. If the set of such a point  $t_0$  is not a set of measure zero, then the required bounded

slope condition in Theorem 2.1 does not hold. As a consequence, the bounded slope condition can hardly hold for the case of the pure state constraint  $S(t) = X(t) \times \mathbb{R}^m$  where  $X(t)$  is closed. In general the bound slope condition is a strong condition and it is also hard to verify. In the rest of this section, we investigate sufficient conditions for the bounded slope condition to hold in our problem setting.

**Proposition 3.1** *Assume that the local error bound condition for the system representing  $S(t)$  as in (1.11) holds at every  $(x, u) \in S_*^{\epsilon, R}(t)$  and*

$$\begin{cases} (x, u) \in S_*^{\epsilon, R}(t), \zeta \in \mathcal{N}_{U(t)}^L(u), \\ \lambda \geq 0, \lambda_i = 0 \ i \in I_t^-(x, u), \\ \mu_i = 0 \ i \in \mathcal{I}_t^{+0}(x, u), \nu_i = 0 \ i \in \mathcal{I}_t^{0+}(x, u), \mu_i > 0, \nu_i > 0 \text{ or } \mu_i \nu_i = 0 \ i \in \mathcal{I}_t^{00}(x, u), \end{cases} \\ \implies \|\nabla_x \Psi(t, x, u; \lambda, \nu, \mu, \nu)\| \leq k_S(t) \|\nabla_u \Psi(t, x, u; \lambda, \nu, \mu, \nu) + \zeta\|. \quad (3.14)$$

Then the bounded slope condition (2.4) holds.

**Proof.** Let  $(x, u) \in S_*^{\epsilon, R}(t)$  and  $(\alpha, \beta) \in \mathcal{N}_{S(t)}^L(x, u)$ . Since the local error bound condition holds at  $(x, u)$ , it follows from [22, Proposition 3.4] that

$$\begin{aligned} (\alpha, \beta) \in \{ \nabla_{x,u} \Psi(t, x, u; \lambda, \nu, \mu, \nu) : \lambda \in \mathcal{N}_{\mathbb{R}_+^1}(g(t, x, u)), (\mu, \nu) \in \mathcal{N}_{Ci}(G(t, x, u), H(t, x, u)) \} \\ + \{0\} \times \mathcal{N}_{U(t)}^L(u). \end{aligned}$$

By Proposition 2.1, there exist  $\lambda, \nu, \mu, \nu$  such that

$$\begin{aligned} (\alpha, \beta) \in \nabla_{x,u} \Psi(t, x, u; \lambda, \nu, \mu, \nu) + \{0\} \times \mathcal{N}_{U(t)}^L(u), \\ \lambda \geq 0, \lambda_i = 0 \ i \in I_t^-(x, u), \\ \mu_i = 0 \ i \in \mathcal{I}_t^{+0}(x, u), \nu_i = 0 \ i \in \mathcal{I}_t^{0+}(x, u), \mu_i > 0, \nu_i > 0 \text{ or } \mu_i \nu_i = 0 \ i \in \mathcal{I}_t^{00}(x, u). \end{aligned}$$

It follows that there exists  $\zeta \in \mathcal{N}_{U(t)}^L(u)$  such that

$$\alpha = \nabla_x \Psi(t, x, u; \lambda, \nu, \mu, \nu), \quad \beta = \nabla_u \Psi(t, x, u; \lambda, \nu, \mu, \nu) + \zeta.$$

Thus, by condition (3.14), we have  $\|\alpha\| \leq k_S(t) \|\beta\|$ . The proof is complete.  $\blacksquare$

A sufficient condition for condition (3.14) to hold is the following stronger condition which is similar to the  $M_*^{\epsilon, R}$  condition given in [7]: There exists a measurable function  $\kappa(\cdot)$  such that for almost every  $t \in [t_0, t_1]$ ,

$$\begin{cases} (x, u) \in S_*^{\epsilon, R}(t), \zeta \in \mathcal{N}_{U(t)}^L(u), \\ \lambda \geq 0, \lambda_i = 0 \ i \in I_t^-(x, u), \\ \mu_i = 0 \ i \in \mathcal{I}_t^{+0}(x, u), \nu_i = 0 \ i \in \mathcal{I}_t^{0+}(x, u), \mu_i > 0, \nu_i > 0 \text{ or } \mu_i \nu_i = 0 \ i \in \mathcal{I}_t^{00}(x, u), \end{cases} \\ \implies \|(\lambda, \nu, \mu, \nu)\| \leq \kappa(t) \|\nabla_u \Psi(t, x, u; \lambda, \nu, \mu, \nu) + \zeta\|. \quad (3.15)$$

**Assumption 3.2** *There exist measurable functions  $k_x^g(\cdot), k_x^h(\cdot), k_x^G(\cdot), k_x^H(\cdot)$  such that for almost every  $t \in [t_0, t_1]$ ,*

$$\begin{aligned} \|g(t, x_1, u) - g(t, x_2, u)\| &\leq k_x^g(t)\|x_1 - x_2\| \quad \forall (x_1, u), (x_2, u) \in S_*^{\epsilon, R}(t), \\ \|h(t, x_1, u) - h(t, x_2, u)\| &\leq k_x^h(t)\|x_1 - x_2\| \quad \forall (x_1, u), (x_2, u) \in S_*^{\epsilon, R}(t), \\ \|G(t, x_1, u) - G(t, x_2, u)\| &\leq k_x^G(t)\|x_1 - x_2\| \quad \forall (x_1, u), (x_2, u) \in S_*^{\epsilon, R}(t), \\ \|H(t, x_1, u) - H(t, x_2, u)\| &\leq k_x^H(t)\|x_1 - x_2\| \quad \forall (x_1, u), (x_2, u) \in S_*^{\epsilon, R}(t). \end{aligned}$$

**Proposition 3.2** *Let Assumption 3.2 and condition (3.15) hold. Then the local error bound condition for the system representing  $S(t)$  as in (1.11) holds at every  $(x, u) \in S_*^{\epsilon, R}(t)$  and the bounded slope condition (2.4) holds with  $k_S(t) = \kappa(t)(k_x^g(t) + k_x^h(t) + k_x^G(t) + k_x^H(t))$ .*

**Proof.** Let  $(x, u) \in S_*^{\epsilon, R}(t)$ . It is not hard to verify that condition (3.15) implies that

$$\begin{cases} 0 \in \nabla_u \Psi(t, x, u; \lambda, v, \mu, \nu) + \mathcal{N}_{U(t)}^L(u), \\ \lambda \geq 0, \lambda_i = 0 \quad i \in I_t^-(x, u), \\ \mu_i = 0 \quad i \in \mathcal{I}_t^{+0}(x, u), \nu_i = 0 \quad i \in \mathcal{I}_t^{0+}(x, u), \mu_i > 0, \nu_i > 0 \text{ or } \mu_i \nu_i = 0 \quad i \in \mathcal{I}_t^{00}(x, u), \end{cases} \\ \implies (\lambda, v, \mu, \nu) = 0.$$

This indicates that the MPEC quasi-normality holds at  $(x, u)$  and then by Proposition 2.2, the local error bound condition holds at  $(x, u)$ . Moreover, by Assumption 3.2, we can have

$$\|\nabla_x \Psi(t, x, u; \lambda, v, \mu, \nu)\| \leq k_x^\Psi(t)\|\lambda, v, \mu, \nu\|,$$

where  $k_x^\Psi(t) := k_x^g(t) + k_x^h(t) + k_x^G(t) + k_x^H(t)$ . It then follows from condition (3.15) that

$$\|\nabla_x \Psi(t, x, u; \lambda, v, \mu, \nu)\| \leq \kappa(t)k_x^\Psi(t)\|\nabla_u \Psi(t, x, u; \lambda, v, \mu, \nu) + \zeta\|$$

and hence condition (3.14) holds with  $k_S(t) := \kappa(t)k_x^\Psi(t)$ . Consequently, the bounded slope condition follows from Proposition 3.1 immediately. The proof is complete.  $\blacksquare$

In general, it is not easy to guarantee the integrability of the measurable function  $k_S(\cdot)$  in the bounded slope condition (2.4). We next consider a special case where mappings  $g(\cdot), h(\cdot), G(\cdot), H(\cdot), U(\cdot)$  are all autonomous (i.e., independent of  $t$ ). In this case, we will give some sufficient conditions to ensure that function  $k_S(\cdot)$  can be a positive constant function which is clearly integrable. We denote  $U(t) \equiv U, S(t) \equiv S, \mathbb{S}(x) := \{u : (x, u) \in S\}$ , and

$$C_*^{\epsilon, R} := \text{cl} \{(t, x, u) \in [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m : (x, u) \in S_*^{\epsilon, R}(t)\}.$$

Note that  $u^*(\cdot)$  may be unbounded over  $[t_0, t_1]$  and thus  $C_*^{\epsilon, R}$  may be unbounded.

**Proposition 3.3** *Let all the mappings  $g(\cdot), h(\cdot), G(\cdot), H(\cdot), U(\cdot)$  be autonomous. Assume that  $C_*^{\epsilon, R}$  is compact for some  $\epsilon > 0$  and  $D^*\mathbb{S}(x, u)(0) = \{0\}$  for every  $(x, u)$  such that*

$(t, x, u) \in C_*^{\epsilon, R}$ . Then for every  $t \in [t_0, t_1]$ , the bounded slope condition (2.4) holds with certain positive constant  $\pi > 0$ . A sufficient condition for  $D^*\mathbb{S}(x, u)(0) = \{0\}$  to hold is the local error bound condition for the system representing  $S(t)$  as in (1.11) at  $(x, u)$  and the implication

$$\begin{cases} 0 \in \nabla_u \Psi(t, x, u; \lambda, v, \mu, \nu) + \mathcal{N}_U^L(u), \\ \lambda \geq 0, \lambda_i = 0 \ i \in I_t^-(x, u), \\ \mu_i = 0 \ i \in \mathcal{I}_t^{+0}(x, u), \nu_i = 0 \ i \in \mathcal{I}_t^{0+}(x, u), \mu_i > 0, \nu_i > 0 \text{ or } \mu_i \nu_i = 0 \ i \in \mathcal{I}_t^{00}(x, u), \end{cases} \\ \implies \nabla_x \Psi(t, x, u; \lambda, v, \mu, \nu) = 0. \quad (3.16)$$

**Proof.** We prove the first part of this result by contradiction. Assume to the contrary that there exist sequences  $\{t_k\} \subseteq [t_0, t_1]$ ,  $\{(x^k, u^k)\} \subseteq S_*^{\epsilon, R}(t_k)$ , and  $(\alpha^k, \beta^k) \in \mathcal{N}_S^L(x^k, u^k)$  such that  $\|\alpha^k\| > k\|\beta^k\|$ . Without loss of generality, we assume that  $\|\alpha^k\| = 1$  and  $\alpha^k \rightarrow \alpha$  with  $\|\alpha\| = 1$ . Since  $\|\alpha^k\| > k\|\beta^k\|$ , it also holds that  $\beta^k \rightarrow 0$ . Since  $C_*^{\epsilon, R}$  is compact, we may assume that  $(t_k, x^k, u^k) \rightarrow (t, x, u) \in C_*^{\epsilon, R}$ . Since the limiting normal cone is closed as a multifunction, we have  $(\alpha, 0) \in \mathcal{N}_S^L(x, u)$  and it then follows from the definition of coderivative that  $\alpha \in D^*\mathbb{S}(x, u)(0)$  with  $\|\alpha\| = 1$ , which contradicts the relation  $D^*\mathbb{S}(x, u)(0) = \{0\}$ . The proof of the first part of the theorem is complete.

Next we show the second part of the theorem. For any  $\alpha \in D^*\mathbb{S}(x, u)(0)$ , by the definition of coderivative, we have  $(\alpha, 0) \in \mathcal{N}_S^L(x, u)$ . Then, as in the proof of Proposition 3.1, we have

$$\begin{aligned} (\alpha, 0) &\in \nabla_{x,u} \Psi(t, x, u; \lambda, v, \mu, \nu) + \{0\} \times \mathcal{N}_U^L(u), \\ \lambda &\geq 0, \lambda_i = 0 \ i \in I_t^-(x, u), \\ \mu_i &= 0 \ i \in \mathcal{I}_t^{+0}(x, u), \nu_i = 0 \ i \in \mathcal{I}_t^{0+}(x, u), \mu_i > 0, \nu_i > 0 \text{ or } \mu_i \nu_i = 0 \ i \in \mathcal{I}_t^{00}(x, u). \end{aligned}$$

Then we have

$$\alpha = \nabla_x \Psi(t, x, u; \lambda, v, \mu, \nu), \quad 0 \in \nabla_u \Psi(t, x, u; \lambda, v, \mu, \nu) + \mathcal{N}_U^L(u),$$

which together with condition (3.16) implies that  $\alpha = 0$ . The proof of the second part of the theorem is complete.  $\blacksquare$

The following condition that is stronger than condition (3.16) is similar to the so-called MFC proposed in [7]:

$$\begin{cases} 0 \in \nabla_u \Psi(t, x, u; \lambda, v, \mu, \nu) + \mathcal{N}_U^L(u), \\ \lambda \geq 0, \lambda_i = 0 \ i \in I_t^-(x, u), \\ \mu_i = 0 \ i \in \mathcal{I}_t^{+0}(x, u), \nu_i = 0 \ i \in \mathcal{I}_t^{0+}(x, u), \mu_i > 0, \nu_i > 0 \text{ or } \mu_i \nu_i = 0 \ i \in \mathcal{I}_t^{00}(x, u), \end{cases} \\ \implies (\lambda, v, \mu, \nu) = 0, \quad (3.17)$$

which implies the local error bound condition at  $(x, u)$  by Proposition 2.2. Then the following result follows from Proposition 3.3 immediately.

**Corollary 3.1** *Let all the mappings  $g(\cdot), h(\cdot), G(\cdot), H(\cdot), U(\cdot)$  be autonomous. Assume that  $C_*^{\epsilon, R}$  is compact for some  $\epsilon > 0$  and condition (3.17) holds for every  $(x, u)$  such that  $(t, x, u) \in C_*^{\epsilon, R}$ . Then for every  $t \in [t_0, t_1]$ , the bounded slope condition (2.4) holds with certain positive constant  $\pi > 0$ .*

In Proposition 3.3 and Corollary 3.1, the qualification conditions (3.16) and (3.17) are required to hold over some neighborhood of the optimal process  $(x^*(\cdot), u^*(\cdot))$ . In order to weaken this requirement, Clarke and De Pinho [7, Definition 4.7] introduced the following concept.

**Definition 3.2** *We say that  $(t, x^*(t), u)$  is an admissible cluster point of  $(x^*(\cdot), u^*(\cdot))$  if there exist a sequence  $\{t^k\} \subseteq [t_0, t_1]$  converging to  $t$  and corresponding points  $(x^k, u^k) \in S(t^k)$  such that  $\lim_{k \rightarrow \infty} x^k \rightarrow x^*(t)$  and  $\lim_{k \rightarrow \infty} u^k = \lim_{k \rightarrow \infty} u^*(t^k) = u$ .*

Based on Definition 3.2, we have the following sufficient condition for the bounded slope condition to hold with certain positive constant.

**Proposition 3.4** *Let  $R(\cdot) \equiv r > 0$  be a positive constant function and all the mappings  $g(\cdot), h(\cdot), G(\cdot), H(\cdot), U(\cdot)$  autonomous. Assume that for all admissible cluster points  $(t, x^*(t), u)$  of  $(x^*(\cdot), u^*(\cdot))$ , condition (3.16) and the local error bound condition hold at  $(x^*(t), u)$  or the stronger condition (3.17) holds at  $(x^*(t), u)$ . Then for every  $t \in [t_0, t_1]$ , the bounded slope condition (2.4) holds with some radius  $\eta \in (0, r)$  and  $k_S(t) \equiv \pi$  for some constant  $\pi > 0$ .*

**Proof.** Mimicking the proof of Proposition 3.3, we can show that there exist  $\epsilon_1 \in (0, \epsilon)$ ,  $\eta \in (0, r)$  and  $\pi > 0$  such that for every  $t \in [t_0, t_1]$ , the following bounded slope condition holds:

$$(x, u) \in S_*^{\epsilon_1, \eta}(t), (\alpha, \beta) \in \mathcal{N}_S^P(x, u) \implies \|\alpha\| \leq \pi \|\beta\|.$$

The proof is complete. ■

Although all the mappings  $g(\cdot), h(\cdot), G(\cdot), H(\cdot), U(\cdot)$  are assumed to be autonomous in Propositions 3.3–3.4 and Corollary 3.1, their results can be applied to the non-autonomous case if  $U(t) \equiv U$  is autonomous and we treat the time variable  $t$  as a state variable. We now illustrate how this can be done. Since

$$\sigma(t) = t \quad \forall t \in [t_0, t_1] \iff \dot{\sigma}(t) = 1 \quad \forall t \in [t_0, t_1], \quad \sigma(t_0) = t_0, \quad (3.18)$$

it is clear that the OCPEC is equivalent to the following optimal control problem:

$$\begin{aligned}
\min \quad & J(x(\cdot), u(\cdot)) := \int_{t_0}^{t_1} F(\sigma(t), x(t), u(t)) dt + f(x(t_0), x(t_1)), \\
\text{s.t.} \quad & \dot{x}(t) = \phi(\sigma(t), x(t), u(t)), \quad \dot{\sigma}(t) = 1 \quad \text{a.e. } t \in [t_0, t_1], \\
& g(\sigma(t), x(t), u(t)) \leq 0, \quad h(\sigma(t), x(t), u(t)) = 0 \quad \text{a.e. } t \in [t_0, t_1], \\
& 0 \leq G(\sigma(t), x(t), u(t)) \perp H(\sigma(t), x(t), u(t)) \geq 0 \quad \text{a.e. } t \in [t_0, t_1], \\
& u(t) \in U \quad \text{a.e. } t \in [t_0, t_1], \\
& \sigma(t_0) = t_0, \quad (x(t_0), x(t_1)) \in E.
\end{aligned}$$

It is easy to see that  $(\sigma(\cdot), x^*(\cdot), u^*(\cdot))$  is a local minimizer of the above problem if  $(x^*(\cdot), u^*(\cdot))$  is a local minimizer of the OCPEC and  $\sigma(\cdot)$  is defined in (3.18). Thus, the results in Propositions 3.3–3.4 and Corollary 3.1 can be applied to the above problem to obtain the desired result immediately.

We close this section by noting the equivalence of the S-stationarity for the OCPEC and the classical necessary optimality condition for the OCPEC treated as an optimal control problem with mixed inequality constraints (1.8). The proof for the following result is similar to [11, Proposition 4.1] and we omit the proof here.

**Proposition 3.5**  *$(x^*(\cdot), u^*(\cdot))$  is a FJ type stationary solution of the OCPEC treated as an optimal control problem with mixed inequality constraints (1.8) if and only if  $(x^*(\cdot), u^*(\cdot))$  is a FJ type S-stationary solution of the OCPEC for which those two sets of multipliers can be chosen to be the same.*

## 4 A simple example

In this section, we consider a simple class of the OCPEC in which the “best” control needs to be chosen from the DCP (1.3) so that the final state  $x(t_1)$  will reach some prescribed target  $\mathcal{T}$  from an given initial state  $x(t_0)$ , that is, we need to minimize the distance  $\|x(t_1) - \mathcal{T}\|$ . Mathematically, the problem considered in this section is

$$\begin{aligned}
\min \quad & \frac{1}{2} \|x(t_1) - \mathcal{T}\|^2 \\
& \dot{x}(t) = \phi(t, x(t), u(t)) \quad \text{a.e. } t \in [t_0, t_1], \\
& 0 \leq u(t) \perp \Upsilon(t, x(t), u(t)) \geq 0 \quad \text{a.e. } t \in [t_0, t_1], \\
& x(t_0) \in E_0,
\end{aligned} \tag{4.1}$$

where  $E_0 \subseteq \mathbb{R}^n$  is a closed subset. In this case,

$$S(t) := \{(x, u) : (u, \Upsilon(t, x, u)) \in \mathcal{C}^m\}, \tag{4.2}$$

$$S_*^{\epsilon, R}(t) := \{(x, u) \in S(t) : \|x - x^*(t)\| \leq \epsilon, \|u - u^*(t)\| \leq R(t)\}. \tag{4.3}$$

For simplicity, we assume that for almost every  $t \in [t_0, t_1]$ , functions  $\Upsilon(t, \cdot, \cdot), \phi(t, \cdot, \cdot)$  are strictly differentiable in  $(x, u)$  and there exists a measurable function  $k_x^\Upsilon(\cdot)$  such that for almost every  $t \in [t_0, t_1]$ ,

$$\|\Upsilon(t, x_1, u) - \Upsilon(t, x_2, u)\| \leq k_x^\Upsilon(t) \|x_1 - x_2\| \quad \forall (x_1, u), (x_2, u) \in S_*^{\epsilon, R}(t).$$

In the following, we apply the derived results in Section 3 to problem (4.1). Note that since the final point  $x(t_1)$  in problem (4.1) is free,  $\lambda_0$  can be chosen as 1 by Remark 2.1. The following result follows immediately from Proposition 3.2 and Theorems 3.1-3.3. Note that  $k_S(t) = \kappa(t)k_x^\Upsilon(t)$  for almost every  $t \in [t_0, t_1]$  in this case.

**Theorem 4.1** *Let  $(x^*(\cdot), u^*(\cdot))$  be a local minimizer of radius  $R(\cdot)$  for problem (4.1). Assume that there exists a measurable function  $\kappa(\cdot)$  such that for almost every  $t \in [t_0, t_1]$ ,*

$$\left\{ \begin{array}{l} (x, u) \in S_*^{\epsilon, R}(t), \\ \mu_i = 0 \ i \in \mathcal{I}_t^{+0}(x, u), \ \nu_i = 0 \ i \in \mathcal{I}_t^{0+}(x, u), \implies \|(\mu, \nu)\| \leq \kappa(t) \|\mu + \nabla_u \Upsilon(t, x, u)\nu\|. \\ \mu_i > 0, \ \nu_i > 0 \text{ or } \mu_i \nu_i = 0 \ i \in \mathcal{I}_t^{00}(x, u), \end{array} \right.$$

Assume also that functions  $k_x^\phi(\cdot), \kappa(\cdot)k_x^\Upsilon(\cdot)k_u^\phi(\cdot)$  are integrable and there exists a positive number  $\eta > 0$  such that  $R(t) \geq \eta k_S(t)$  for almost every  $t \in [t_0, t_1]$ . Then  $(x^*(\cdot), u^*(\cdot))$  is a  $W$ -stationary point, i.e., there exist an arc  $p(\cdot)$  and measurable functions  $\lambda^G(\cdot), \lambda^H(\cdot)$  such that for almost every  $t \in [t_0, t_1]$ ,

- 1)  $p(t_0) \in \mathcal{N}_{E_0}^L(x^*(t_0)), \ -p(t_1) = x^*(t_1) - \mathcal{T},$
- 2)  $(\dot{p}(t), 0) = -\nabla_{x,u} \phi(t, x^*(t), u^*(t))p(t) - (0, \lambda^G(t)) - \nabla_{x,u} \Upsilon(t, x^*(t), u^*(t))\lambda^H(t),$   
 $\lambda_i^G(t) = 0 \ i \in \mathcal{I}_t^{+0}(x^*, u^*), \ \lambda_i^H(t) = 0 \ i \in \mathcal{I}_t^{0+}(x^*, u^*),$
- 3)  $(x^*(t), u) \in S(t), \|u - u^*(t)\| < R(t) \implies \langle p(t), \phi(t, x^*(t), u) \rangle \leq \langle p(t), \phi(t, x^*(t), u^*(t)) \rangle.$

If in addition, the MPEC quasi-normality holds at  $u^*(t) \in \{u : (u, \Upsilon(t, x^*(t), u)) \in \mathcal{C}^m\}$  for almost every  $t \in [t_0, t_1]$ , then  $(x^*(\cdot), u^*(\cdot))$  is an  $M$ -stationary point, i.e., all the above results 1), 2) and 3) hold and there exist measurable functions  $\eta^G(\cdot), \eta^H(\cdot)$  such that

$$\begin{aligned} \nabla_u \phi(t, x^*(t), u^*(t))p(t) + \eta^G(t) + \nabla_u \Upsilon(t, x^*(t), u^*(t))\eta^H(t) &= 0, \\ \eta_i^G(t) = 0 \quad i \in \mathcal{I}_t^{+0}(x^*, u^*), \quad \eta_i^H(t) = 0 \quad i \in \mathcal{I}_t^{0+}(x^*, u^*), \\ \eta_i^G(t) > 0, \eta_i^H(t) > 0 \text{ or } \eta_i^G(t)\eta_i^H(t) = 0 \quad i \in \mathcal{I}_t^{00}(x^*, u^*). \end{aligned}$$

If in addition, the MPEC-LICQ holds at  $u^*(t) \in \{u : (u, \Upsilon(t, x^*(t), u)) \in \mathcal{C}^m\}$  for almost every  $t \in [t_0, t_1]$ , then  $(x^*(\cdot), u^*(\cdot))$  is an  $S$ -stationary point, that is, for almost every  $t \in [t_0, t_1]$ , all the above results 1), 2) and 3) hold and

$$\lambda_i^G(t) \geq 0, \ \lambda_i^H(t) \geq 0 \quad i \in \mathcal{I}_t^{00}(x^*, u^*).$$

In the rest of this section, we focus on a proper specialization of the DCP (1.3) where

$$\phi(t, x, u) := Ax + Bu + c, \quad \Upsilon(t, x, u) := Cx + Du + q$$

for some matrices  $A, B, C, D$  and vectors  $c, q$  of appropriate sizes. Note that in this case, functions  $\phi(\cdot), \Upsilon(\cdot)$  are autonomous. Then problem (4.1) reduces to

$$\begin{aligned} \min \quad & \frac{1}{2} \|x(t_1) - \mathcal{T}\|^2 \\ & \dot{x}(t) = Ax(t) + Bu(t) + c \quad \text{a.e. } t \in [t_0, t_1], \\ & 0 \leq u(t) \perp Cx(t) + Du(t) + q \geq 0 \quad \text{a.e. } t \in [t_0, t_1], \\ & x(t_0) \in E_0. \end{aligned} \tag{4.4}$$

In this case, let  $R(\cdot) \equiv r > 0$  be a positive constant function and set

$$S(t) \equiv S := \{(x, u) : (u, Cx + Du + q) \in \mathcal{C}^m\}, \tag{4.5}$$

$$S_*^{\epsilon, r}(t) := \{(x, u) \in S : \|x - x^*(t)\| \leq \epsilon, \|u - u^*(t)\| \leq r\},$$

$$C_*^{\epsilon, r} := \text{cl} \{(t, x, u) \in [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m : (x, u) \in S_*^{\epsilon, r}(t)\}. \tag{4.6}$$

The following result follows from Proposition 3.3 and Theorems 3.2–3.3 immediately. Note that if  $u^*$  is bounded over  $[t_0, t_1]$ , then the compactness of  $C_*^{\epsilon, r}$  is superfluous and moreover, the local error bound condition for the system representing  $S(t)$  as in (4.5) holds since functions  $\phi(t, \cdot, \cdot), \Upsilon(t, \cdot, \cdot)$  are linear in  $(x, u)$ .

**Theorem 4.2** *Let  $(x^*(\cdot), u^*(\cdot))$  be a local minimizer of positive constant radius  $R(\cdot) \equiv r > 0$  for problem (4.4) and  $C_*^{\epsilon, r}$  defined in (4.6) be compact. Assume that for almost every  $t \in [t_0, t_1]$  and for all  $(t, x, u) \in C_*^{\epsilon, r}$ ,*

$$\begin{cases} \mu + D^T \nu = 0, \\ \mu_i = 0 \ i \in \mathcal{I}_t^{+0}(x, u), \ \nu_i = 0 \ i \in \mathcal{I}_t^{0+}(x, u), \implies C^T \mu = 0. \\ \mu_i > 0, \ \nu_i > 0 \text{ or } \mu_i \nu_i = 0 \ i \in \mathcal{I}_t^{00}(x, u), \end{cases}$$

*Then  $(x^*(\cdot), u^*(\cdot))$  is an  $M$ -stationary point, i.e., there exist an arc  $p(\cdot)$  and measurable functions  $\lambda^G(\cdot), \lambda^H(\cdot)$  and  $\eta^G(\cdot), \eta^H(\cdot)$  such that for almost every  $t \in [t_0, t_1]$ ,*

- 1)  $p(t_0) \in \mathcal{N}_{E_0}^L(x^*(t_0)), \ -p(t_1) = x^*(t_1) - \mathcal{T},$
- 2)  $-A^T p(t) - C^T \lambda^H(t) = \dot{p}(t), \ B^T p(t) + \lambda^G(t) + D^T \lambda^H(t) = 0,$   
 $\lambda_i^G(t) = 0 \ i \in \mathcal{I}_t^{+0}(x^*, u^*), \ \lambda_i^H(t) = 0 \ i \in \mathcal{I}_t^{0+}(x^*, u^*),$
- 3)  $\langle B^T p(t), u - u^*(t) \rangle \leq 0 \quad \forall u : (x^*(t), u) \in S, \|u - u^*(t)\| < r,$
- 4)  $B^T p(t) + \eta^G(t) + D^T \eta^H(t) = 0,$   
 $\eta_i^G(t) = 0 \ i \in \mathcal{I}_t^{+0}(x^*, u^*), \ \eta_i^H(t) = 0 \ i \in \mathcal{I}_t^{0+}(x^*, u^*),$   
 $\eta_i^G(t) > 0, \eta_i^H(t) > 0 \text{ or } \eta_i^G(t) \eta_i^H(t) = 0 \ i \in \mathcal{I}_t^{00}(x^*, u^*).$

If in addition, the MPEC-LICQ holds at  $u^*(t) \in \{u : (u, Cx^*(t) + Du + q) \in \mathcal{C}^m\}$  for almost every  $t \in [t_0, t_1]$ , then  $(x^*(\cdot), u^*(\cdot))$  is an  $S$ -stationary point, that is, for almost every  $t \in [t_0, t_1]$ , all the above results 1), 2) and 3) hold and

$$\lambda_i^G(t) \geq 0, \lambda_i^H(t) \geq 0 \quad i \in \mathcal{I}_t^{00}(x^*, u^*).$$

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