Isoperimetry for product of heavy tails distributions

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Extending an approach by Bobkov we obtain some isoperimetric inequalities for product of heavy tails distributions.

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Consider a separable metric space (X,d) equipped with a probability measure μ which is not a Dirac mass at a point. In this note we study the following isoperimetric inequality

$$\mu_s(\partial A) \geq J\left(\mu(A)\right) \qquad A \subset X \text{ Borel} \tag{1}$$

where $J : [0,1] \to \mathbb{R}^+$ is symmetric around $1/2$ and where the surface measure is defined by the Minkowski content $\mu_s(\partial A) = \liminf_{\varepsilon \to 0} \frac{\mu(A_\varepsilon \setminus A)}{\varepsilon}$ with $A_\varepsilon = \{x \in X : d(x,A) < \varepsilon\}$. For any function $f : X \to \mathbb{R}$ we define the modulus of the gradient of f by $|\nabla f|(x) = \limsup_{d(x,y)\to 0} \frac{|f(y)-f(x)|}{d(x,y)}$ with the convention that $|\nabla f|(x) = 0$ as soon as x is an isolated point of X. We define similarly on the product space X^n equipped with the distance $d_n(x,y) = \sqrt{\sum_{i=1}^n d(x_i,y_i)^2}$ and the n-fold product measure $\mu^n = \mu \otimes \cdots \otimes \mu$, the modulus of the gradient of $f : X^n \to \mathbb{R}$. A function is said to be "locally Lipschitz" if its Lipschitz constant is finite on every ball of X (or X^n). We assume that the product structure is of Euclidean type for the gradient, *i.e.*, for any locally Lipschitz function $f : X^n \to \mathbb{R}$, μ^n almost surely, $|\nabla f|^2(x) = \sum_{i=1}^n |\nabla_{x_i} f|^2(x)$ where $|\nabla_{x_i} f|$ is the modulus of the gradient of the function $X \ni x_i \mapsto f(x)$ with $(x_j)_{j \neq i}$ fixed. This is for example the case when $X = \mathbb{R}^k$ and μ is any absolutely continuous probability measure with respect to the Lebesgue measure.

Isoperimetric Inequalities are related to some Sobolev-type inequalities and to the concentration of measure phenomenon. Thus it has a lot of applications in high dimension (*e.g.* semi-group contraction properties, convergence to equilibrium of Markov processes etc.). It is therefore interesting to understand how Inequality (1) evolves on the product X^n. We refer to Refs. 1, 2, 3 and 4 for survey papers on the isoperimetric inequalities for probability measures and a more complete bibliography of the field and to Refs. 5 and 6 for an introduction to Sobolev-type inequalities and their applications.

When $J(s) = h_\mu \min(s, 1-s)$ then (1) is the celebrated Cheeger's isoperimetric inequality: $\mu_s(\partial A) \geq h_\mu \min(\mu(A), 1-\mu(A))$ and $h_\mu > 0$ is the Cheeger's constant. See also Refs. 7 and 8.

Cheeger[9] proved that the constant h_μ is related to the spectral gap of the Laplacian on compact Riemannian manifolds. See Refs. 10, 11 and 12 for more references and related results. In this particular case, Bobkov and Houdré[13] proved that if the above Cheeger's isoperimetric inequality holds on X, then the same isoperimetric inequality holds on X^n with $h_{\mu^n} \geq h_\mu/(2\sqrt{6})$. In this work we extend their result in the following way:

Theorem 0.1. *Assume that for any Borel set $A \subset X$:*

$$\mu_s(\partial A) \geq J\left(\mu(A)\right)$$

for some $J : [0,1] \to \mathbb{R}^+$ symmetric around $1/2$. Assume that $s \mapsto J(s)/s$ is non-decreasing on $(0, \frac{1}{2})$. Then, for any integer $n \geq 1$,

$$\mu_s^n(\partial A) \geq \frac{n}{2} J\left(\frac{h}{4\sqrt{6}n}\min(\mu^n(A), 1-\mu^n(A))\right) \qquad \forall A \subset X^n \text{ Borel.}$$

When $X = \mathbb{R}$ and $d\mu(x) = e^{-\Phi(|x|)}dx$ with Φ convex, the optimal function J in (1) is known[14] to be $I = F'_\mu \circ F_\mu$, with $F_\mu(x) := \mu(-\infty, x)$, and is concave. Since $J(0) = 0$, $s \mapsto J(s)/s$ is non-increasing. Hence our result does not apply to log-concave distributions. For results in this direction, see Refs. 15, 16, 17, 18, 19 and 20. When Φ is concave, it is known[21] that the optimal J in (1) is $J(t) = \min(I(t), 2I(\min(t, 1-t)/2))$ (I as above). It follows easily[22] that $s \mapsto J(s)/s$ is non-decreasing on $(0, \frac{1}{2})$. Hence typical examples of application of our result are sub-exponential laws $d\mu_p(x) = e^{-|x|^p}/(2\Gamma(1+(1/p)))$, $p \in (0,1)$, Cauchy-type distributions $dm_\alpha(x) = \frac{\alpha}{2(1+|x|)^{1+\alpha}}$, $\alpha > 0$ and more generally κ-concave probability measures ($\kappa \leq 0$). In all these cases our result is optimal,see Refs. 22,23.

Note that the standard Cheeger's inequality, corresponding to $J(s) = h_\mu \min(s, 1-s)$, enters the framework of Theorem 0.1 and leads to a weak version of Bobkov and Houdré's result. Namely, with the notation above, we get that $h_{\mu^n} \geq h_\mu/(8\sqrt{6})$. We are off by a factor 4, for technical reasons. This can anyway be improved to $h_{\mu^n} \geq h_\mu/(4\sqrt{6})$, see Remark 0.1 below.

Isoperimetric inequalities for product of heavy tails distributions are also obtained in Ref. 22. However our result is by nature very different from Ref. 22 and more intrinsic in the sense that we start with an isoperimetric inequality on X and derive from it an isoperimetric inequality on X^n. Also our approach, based on Bobkov's ideas,[24] is very elementary.

The proof of Theorem 0.1 relies on Sobolev-type inequalities which are known to be equivalent to isoperimetric inequalities. Indeed, it is easy to prove (see *e.g.* Ref. 25) that (1) is equivalent to the following weak Cheeger inequality: for any $f : X \to \mathbb{R}$ locally Lipschitz,

$$\int |f - m(f)| d\mu \leq \beta(s) \int |\nabla f| d\mu + s \operatorname{Osc}(f) \qquad \forall s \in (0, 1/2) \tag{2}$$

where $m(f)$ is a the median of f under μ and $\operatorname{Osc}(f) = \sup f - \inf f$. More precisely (1) implies (2) with $\beta(s) = \sup_{s \leq t \leq \frac{1}{2}} \frac{t-s}{J(t)}$, and (2) implies (1) with $J(t) = \sup_{0<s\leq t} \frac{t-s}{\beta(s)}$ for t $\in (0, \frac{1}{2})$ and $J(t) = J(1-t)$. Unfortunately, the weak Cheeger inequality (2) does not behave in a proper way on

product spaces, due to the $\mathbb{L}^1$ norm of the gradient. Bobkov proposed an alternative functional form of (and equivalent to) the isoperimetric inequalities:

$$I\left(\int f d\mu\right) \leq \int \sqrt{I(f)^2 + C^2|\nabla f|^2} d\mu \tag{3}$$

where $I : [0,1] \to \mathbb{R}^+$ and $f : X \to [0,1]$. Such inequalities enjoy the tensorisation property and was used by Bobkov[26] as an alternative proof of the Gaussian dimension free isoperimetric inequality of Sudakov and Tsirel'son[27] and Borell[28] and in Ref. 24 as shorter proof of Bobkov-Houdré's result mentioned above. See also Refs. 15, 29, 30, 31 and 32 for related results. Since our result are dimension dependent we shall prove a weak form of (3):

Theorem 0.2. *Let $I(t) = 4t(1-t)$, $t \in [0,1]$. Assume that for any Borel set $A \subset X$, $\mu_s(\partial A) \geq J(\mu(A))$ for some $J : [0,1] \to \mathbb{R}^+$ symmetric around $1/2$. Let $C = 4\sqrt{6}$ and $\beta(s) = \sup_{s \leq t \leq \frac{1}{2}} \frac{t-s}{J(t)}$ for $s \in (0, \frac{1}{2})$. Then, for any $n \geq 1$, any locally Lipschitz function $f : X^n \to [0,1]$ and any $s \in (0, \frac{1}{2})$,*

$$I\left(\int_{X^n} f d\mu^n\right) \leq \int_{X^n} \sqrt{I(f)^2 + 4C^2\beta^2(s)|\nabla f|^2} d\mu^n + Cns \operatorname{Osc}(f). \tag{4}$$

Theorem 0.1 will easily follow from Theorem 0.2 by approximating indicator functions of sets by locally Lipschitz functions taking values in $[0,1]$.

Our starting point is the following one dimensional functional inequality, derived from (2).

Lemma 0.1. *Let $\Phi(x) = \sqrt{1+x^2} - 1$, $x \in \mathbb{R}$. Assume that for any Borel set $A \subset X$, $\mu_s(\partial A) \geq J(\mu(A))$ for some $J : [0,1] \to \mathbb{R}^+$ symmetric around $1/2$. Let $\beta(s) = \sup_{s \leq t \leq \frac{1}{2}} \frac{t-s}{J(t)}$ for $s \in (0, \frac{1}{2})$. Then for all locally Lipschitz functions $f : X \to \mathbb{R}$ with $m(f) = 0$,*

$$\int_X \Phi(f) d\mu \leq \int \Phi\left(4\beta(s)|\nabla f|\right) d\mu + 2s \operatorname{Osc}(f) \qquad \forall s \in (0, 1/2). \tag{5}$$

Proof. By the above discussion (see (2)), the assumption $\mu_s(\partial A) \geq J(\mu(A))$ implies that all locally Lipschitz functions $f : X \to \mathbb{R}$ with $m(f) = 0$ satisfy

$$\int_X |f| d\mu \leq \beta(s) \int_X |\nabla f| d\mu + s \operatorname{Osc}(f) \qquad \forall s \in (0, 1/2). \tag{6}$$

Now consider a bounded function f, locally Lipschitz, with $m(f) = 0$. Set $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Then $m(f_+) = m(f_-) = 0$ and thus $m(\Phi(f_+)) = m(\Phi(f_-)) = 0$. Hence, applying twice (6) to $\Phi(f_+)$ and $\Phi(f_-)$, we have for all $s \in (0, \frac{1}{2})$

$$\int_X \Phi(f_+) d\mu \leq \beta(s) \int_X \Phi'(|f|)|\nabla f| \chi_{\{f>0\}} d\mu + s \operatorname{Osc}(\Phi(f_+)),$$
$$\int_X \Phi(f_-) d\mu \leq \beta(s) \int_X \Phi'(|f|)|\nabla f| \chi_{\{f<0\}} d\mu + s \operatorname{Osc}(\Phi(f_-)).$$

Summing up we arrive at

$$\begin{aligned}\int_X \Phi(f)d\mu &\le \beta(s)\int_X \Phi'(|f|)|\nabla f|d\mu + s\left(\mathrm{Osc}(\Phi(f_+)) + \mathrm{Osc}(\Phi(f_-))\right)\\ &\le \beta(s)\int_X \Phi'(|f|)|\nabla f|d\mu + s\,\mathrm{Osc}(f) \qquad (7)\end{aligned}$$

where in the last line we used the fact that $\Phi(x) \le |x|$ and thus that

$$\mathrm{Osc}(\Phi(f_+)) + \mathrm{Osc}(\Phi(f_-)) = \Phi(\sup f) + \Phi(|\inf f|) \le \sup f + |\inf f| = \mathrm{Osc}(f).$$

Now, using the Young inequality $xy \le \Phi(2x) + \Phi^*(y/2)$ with $x = \beta(s)|\nabla f|/h$ and $y = \Phi'(|f|)$, where $\Phi^*(y) = \sup_u\{uy - \Phi(u)\}$, we have

$$\beta(s)\int_X \Phi'(|f|)|\nabla f|d\mu \le \int_X \Phi\left(2\beta(s)|\nabla f|\right)d\mu + \int_X \Phi^*\left(\Phi'(|f|)/2\right)d\mu. \qquad (8)$$

A simple computation gives that $\Phi^*(y) = 1 - \sqrt{1-y^2}$ for $|y| \le 1$. Hence, since $|\Phi'| \le 1$, we have $\Phi^*(\Phi'(x)) = 1 - \sqrt{1 - \frac{x^2}{1+x^2}} = \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}} \le \Phi(x)$ for any $x \in \mathbb{R}$. In turn, using the convexity of Φ^*, we get

$$\int_X \Phi^*\left(\frac{\Phi'(|f|)}{2}\right)d\mu \le \frac{1}{2}\int_X \Phi^*(\Phi'(|f|))d\mu \le \frac{1}{2}\int_X \Phi(f)d\mu.$$

Plugging this bound into (8), it follows from (7), after simplifications, that

$$\int_X \Phi(f)d\mu \le 2\int_X \Phi\left(\frac{2\beta(s)}{h}|\nabla f|\right)d\mu + 2s\,\mathrm{Osc}(f).$$

This ends the proof since $2\Phi(x) \le \Phi(2x)$ for $x \ge 0$, by convexity of Φ. $\square$

Proof of Theorem 0.2. Let $\Phi(x) = \sqrt{1+x^2} - 1$, let $n = 1$ and consider a locally Lipschitz function $f : X \to [0,1]$. By Lemma 0.1 applied to $\sqrt{24}(f - m(f))$ we have for any $s \in (0, \frac{1}{2})$

$$\int_X \Phi(\sqrt{24}(f-m))d\mu \le \int_X \Phi\left(4\sqrt{24}\beta(s)|\nabla f|\right)d\mu + 2\sqrt{24}s\,\mathrm{Osc}(f).$$

Now observe that for $|t| \le 1$, $\Phi(\sqrt{24}t) \ge 4t^2$. Hence,

$$\int_X \Phi(\sqrt{24}(f-m))d\mu \ge 4\int_X (f-m)^2 \ge 4\,\mathrm{Var}_\mu(f) = I\left(\int_X f d\mu\right) - \int_X I(f)d\mu.$$

Let $C = 2\sqrt{24} = 4\sqrt{6}$. For any $s \in (0, \frac{1}{2})$ it follows that

$$\begin{aligned}I\left(\int_X f d\mu\right) &\le \int_X \left[\Phi\left(2C\beta(s)|\nabla f|\right) + I(f)\right]d\mu + Cs\,\mathrm{Osc}(f)\\ &= \int_X \left[\sqrt{1 + \left(2C\beta(s)|\nabla f|\right)^2} - 1 + I(f)\right]d\mu + Cs\,\mathrm{Osc}(f)\\ &\le \int_X \left[\sqrt{I(f)^2 + \left(2C\beta(s)|\nabla f|\right)^2}\right]d\mu + Cs\,\mathrm{Osc}(f)\end{aligned}$$

since $u = I(f) \le 1$ and the function $u \mapsto \sqrt{u^2+v^2} - u$ is non-increasing in $u \ge 0$ (for any v). Inequality (4) follows in dimension $n = 1$ and from Lemma 0.2 below, in any dimension. □

Lemma 0.2. *Let $I : [0,1] \to \mathbb{R}^+$. Assume that for some constants $a, b > 0$ and all locally Lipschitz function $f : X \to [0,1]$ we have*

$$I\left(\int_X f d\mu\right) \le \int_X \sqrt{I(f)^2 + a^2|\nabla f|^2} d\mu + b\,\mathrm{Osc}(f).$$

Then, for any $n \ge 1$ and any locally Lipschitz function $f : X^n \to [0,1]$

$$I\left(\int_{X^n} f d\mu^n\right) \le \int_{X^n} \sqrt{I(f)^2 + a^2|\nabla f|^2} d\mu^n + bn\,\mathrm{Osc}(f).$$

Proof. The proof is by induction. Let $f : X^{n+1} \to [0,1]$. For simplicity we decompose any element of X^{n+1} as $(y,x) \in X^n \times X$. Let $g(x) = \int_{X^n} f(y,x) d\mu^n(y)$. Then we have,

$$I\left(\int_{X^{n+1}} f d\mu^{n+1}\right) = I\left(\int_X g(x) d\mu(x)\right) \le \int_X \sqrt{I(g(x))^2 + a^2|\nabla g|^2(x)} d\mu(x) + b\,\mathrm{Osc}(g).$$

Note that $\mathrm{Osc}(g) \le \mathrm{Osc}(f)$. Also $|\nabla g|(x) \le \int_{X^n} |\nabla_x f|(x,y) d\mu^n(y)$ for any $x \in X$, where $|\nabla_x f|$ is the modulus of the gradient of $x \mapsto f(y,x)$ with fixed $y \in X^n$. Furthermore,

$$I(g(x)) = I\left(\int_{X^n} f(y,x) d\mu^n(y)\right) \le \int_{X^n} \sqrt{I(f)^2(y,x) + a^2|\nabla_y f|^2(y,x)} d\mu^n(y) + bn\,\mathrm{Osc}(f)$$

where $|\nabla_y f|$ is the modulus of the gradient of $y \mapsto f(y,x)$ with fixed $x \in X$. Hence, using the following Hölder-Minkowski inequality $\sqrt{(\int u)^2 + (\int v)^2} \le \int \sqrt{u^2+v^2}$, where the integral is over X^n with respect to $d\mu^n(y)$, with $u = \sqrt{I(f)^2 + a^2|\nabla_y f|^2} + bn\,\mathrm{Osc}(f)$ and $v = C|\nabla_x f|$, we end up with

$$\begin{aligned} I\left(\int_{X^n+1} f d\mu^{n+1}\right) &\le \int_X \int_{X^n} \sqrt{\left(\sqrt{I(f)^2 + a^2|\nabla_y f|^2} + bn\,\mathrm{Osc}(f)\right)^2 + a^2|\nabla_x f|^2 d\mu^n d\mu} + b\,\mathrm{Osc}(f) \\ &\le \int_{X^{n+1}} \sqrt{I(f)^2 + a^2\left(|\nabla_y f|^2 + |\nabla_x f|^2\right)} d\mu^{n+1} + b(n+1)\,\mathrm{Osc}(f) \end{aligned}$$

where in the last line we used the following inequality, $\sqrt{(\alpha+\beta)^2 + \gamma^2} \le \sqrt{\alpha^2+\gamma^2} + \beta$, valid for any $\alpha, \beta, \gamma \ge 0$, that we applied to $\alpha = \sqrt{I(f)^2 + a^2|\nabla_y f|^2}$, $\beta = bn\,\mathrm{Osc}(f)$ and $\gamma = a|\nabla_x f|$. This ends the proof. □

The next result is a more general version of Theorem 0.1 (without the assumption on $J(s)/s$).

Corollary 0.1. *Assume that for any Borel set $A \subset X$, $\mu_s(\partial A) \ge J(\mu(A))$ for some function $J : [0,1] \to \mathbb{R}^+$ symmetric around $1/2$. Let $\beta(s) = \sup_{s \le t \le \frac{1}{2}} \frac{t-s}{J(t)}$ for $s \in (0, \frac{1}{2})$ and $H(t) = \sup_{0 < s \le t} \frac{t-s}{\beta(s)}$ for $t \in (0, \frac{1}{2})$. Then, for any $n \ge 1$ and any Borel set $A \subset X^n$,*

$$\mu_s^n(\partial A) \ge \frac{n}{2} H\left(\frac{1}{2\sqrt{6}n} \min(\mu^n(A), 1 - \mu^n(A))\right).$$

Remark 0.1. When $J(s) = h_\mu s$, then $H(t) = h_\mu t$. In this case, using the notation of the introduction, we get that $h_{\mu^n} \geq h_\mu/(4\sqrt{6})$. This is Bobkov-Houdré's result[13] (see also Ref. 24) with a worst constant (off by a factor 2).

Proof. Fix $n \geq 1$ and a Borel set $A \subset X^n$. Thanks to Theorem 0.2 and the inequality $\sqrt{a^2+b^2} \leq |a| + |b|$ we have for any locally Lipschitz function $f : X^n \to [0,1]$ and all $s \in (0, 1/2)$,

$$\begin{aligned} I\left(\int_{X^n} f d\mu^n\right) &\leq \int_{X^n} \sqrt{I(f)^2 + 4C^2\beta^2(s)|\nabla f|^2} d\mu^n + Cns\,\mathrm{Osc}(f) \\ &\leq \int_{X^n} I(f) d\mu^n + 2C\beta(s) \int_{X^n} |\nabla f| d\mu^n + Cns\,\mathrm{Osc}(f), \end{aligned}$$

where $I(t) = 4t(1-t)$, $t \in [0,1]$ and $C = 2\sqrt{24}$. Approximating the indicator function χ_A of A by locally Lipschitz functions on X^n with values in $[0,1]$ (see [13, Lemma 3.5]), we get that

$$4\mu^n(A)(1-\mu^n(A)) \leq 2C\beta(s)\mu_s^n(\partial A) + Cns \qquad \forall s \in (0,1/2).$$

Since $2t(1-t) \geq \min(t, 1-t)$, we end up with

$$\sup_{s\in(0,1/2)} \frac{2\min(\mu^n(A), 1-\mu^n(A)) - Cns}{2C\beta(s)} \leq \mu_s^n(\partial A).$$

This leads to the expected result after some rearrangements. □

Theorem 0.1 is a direct consequence of the previous corollary together with Lemma 0.3 below that compares H to J under the extra assumption that the function $s \mapsto J(s)/s$ is non-decreasing on $(0, 1/2)$ (we omit the proof).

Lemma 0.3. *Let J, β and H as in Corollary 0.1. Assume furthermore that the function $s \mapsto J(s)/s$ is non-decreasing on $(0, 1/2)$. Then, for all $t \in (0, \frac{1}{2})$, $H(t) \geq J\left(\frac{t}{2}\right)$.*

Proof. Since $s \mapsto J(s)/s$ is non-decreasing, we have

$$\beta(s) = \sup_{s\leq t\leq \frac{1}{2}} \frac{t-s}{J(t)} = \sup_{s\leq t\leq \frac{1}{2}} \frac{t}{J(t)}\frac{t-s}{t} \leq \frac{s}{J(s)} \sup_{s\leq t\leq \frac{1}{2}} \frac{t-s}{t} = \frac{s(1-2s)}{J(s)} \leq \frac{s}{J(s)}.$$

Therefore, choosing $s = t/2$ we get $H(t) = \sup_{0<s\leq t} \dfrac{t-s}{\beta(s)} \geq \sup_{0<s\leq t} \dfrac{(t-s)J(s)}{s} \geq J(t/2)$. □

Note that by similar reasoning it is also possible to prove (under the assumptions of Lemma 0.3) that $H(t) \leq 2J(2t)$ for $t \in (0, 1/4)$.

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