OMITTING TYPES IN LOGIC OF METRIC STRUCTURES

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Abstract. The present paper is about omitting types in logic of metric structures introduced by Ben Yaacov, Berenstein, Henson and Usvyatsov. While a complete type is omissible in a model of a complete theory if and only if it is not principal, this is not true for incomplete types by a result of Ben Yaacov. We prove that there is no simple test for determining whether a type is omissible in a model of a theory $\mathbf{T}$ in a separable language. More precisely, we find a theory in a separable language such that the set of types omissible in some of its models is a complete $\mathbf{\Sigma}^1_2$ set and a complete theory in a separable language such that the set of types omissible in some of its models is a complete $\mathbf{\Pi}^1_1$ set. We also construct (i) a complete theory $\mathbf{T}$ and a countable set of types such that each of its finite sets is jointly omissible in a model of $\mathbf{T}$, but the whole set is not and (ii) a complete theory and two types that are separately omissible, but not jointly omissible, in its models.

The Omitting Types Theorem is one of the most useful methods for constructing models of first-order theories with prescribed properties (see [26], [23], or any general text on model theory). It implies, among other facts, the following.

(1) If $\mathbf{T}$ is a theory in a countable language, then the set of all n-types realized in every model of $\mathbf{T}$ is Borel in the logic topology on $S_n(\mathbf{T})$.
(2) If $\mathbf{T}$ is moreover complete, then any sequence $\mathbf{t}_n$, for $n \in \omega$, of types each of which can be omitted in a model of $\mathbf{T}$ can be simultaneously omitted in a model of $\mathbf{T}$.

We note that the types $\mathbf{t}_n$ in (2) are not required to be complete, but the theory $\mathbf{T}$ is.

While in classical logic the criterion for a given type to be omissible in a model of a given theory applies regardless of whether the type is complete or not, the situation in logic of metric structures is a bit more subtle. The omitting types theorem ([3, §12] or [22, Lecture 4]) has the following straightforward consequences (see Proposition 1.7 for a proof of (3) and Corollary 3.7 for a proof of (4)).

(3) If $\mathbf{T}$ is a theory in a separable language of logic of metric structures, then the set of all *complete* n-types realized in every model of $\mathbf{T}$ is Borel in the logic topology on $S_n(\mathbf{T})$.

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(4) If $\mathbf{T}$ is moreover complete, then any sequence $\mathbf{t}_n$, for $n \in \omega$, of *complete* types each of which can be omitted in a model of $\mathbf{T}$ can be simultaneously omitted in a model of $\mathbf{T}$.

Examples constructed by I. Ben-Yaacov ([2]) and T. Bice ([7]) demonstrate that omitting partial types in logic of metric structures is inherently more complicated than omitting complete types. We find a high lower bound for the descriptive complexity of the set of omissible types over certain theory in a separable language and thus show that (1) fails and confirm the intuition that the problem is essentially intractable. Our results are expressed in the language of descriptive set theory, an excellent reference to which is [24]; see also the beginning of §2.

Theorem 1. (5) *There is a complete theory* $\mathbf{T}$ *in a separable language such that the set of all types omissible over a model of* $\mathbf{T}$ *is* $\mathbf{\Pi}^1_1$-*complete.*

(6) *There is a theory* $\mathbf{T}$ *in a separable language such that the set of all types omissible in a model of* $\mathbf{T}$ *is* $\mathbf{\Sigma}^1_2$-*complete.*

(7) *There is a separable structure* M *in a separable language such that the set of all unary quantifier-free types realized in* M *is a complete* $\mathbf{\Sigma}^1_1$ *set.*

Proof. (5) is proved in Theorem 2.5, (6) is proved in Theorem 2.6, and (7) is Proposition 2.4. □

We also show that (2) fails in logic of metric structures (as customary in logic, ω denotes the least infinite ordinal identified with the set of natural numbers).

Theorem 2. *There are a complete theory* $\mathbf{T}$ *in a separable language and types* $\mathbf{s}_n$*, for* $n \in \omega$*, such that for every* k *there exists a model of* $\mathbf{T}$ *that omits all* $\mathbf{s}_n$ *for* $n \leq k$ *but no model of* $\mathbf{T}$ *simultaneously omits all* $\mathbf{s}_n$*.*

Proof. This is proved in §5. □

Theorem 2 should be compared to a consequence of [4, Corollary 4.7], that under additional conditions any countable set of types that are not omissible in a model of a complete separable theory $\mathbf{T}$ has a finite subset consisting of types that are not omissible in a model of $\mathbf{T}$.

Theorem 3. *There are a complete theory* $\mathbf{T}$ *in a separable language* L *and types* $\mathbf{s}$ *and* $\mathbf{t}$ *omissible in models of* $\mathbf{T}$ *such that no model of* $\mathbf{T}$ *simultaneously omits both of them.*

Proof. This is proved in §5. □

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1. Preliminaries

We assume that the reader is acquainted with logic of metric structures ([3], [22]). We strictly follow the outline of this logic given in [3]. In particular, all metric structures are required to have diameter 1 and all formulas are $[0,1]$-valued. All function and predicate symbols are equipped with a fixed modulus of uniform continuity. Every structure is a complete metric space in which interpretations of functional and relational symbols respect this modulus. It is a straightforward exercise to see that our results apply to the modification of this logic adapted to operator algebras presented in [17].

For a formula $\phi(\bar{x})$, structure of the same language M and a tuple $\bar{a}$ in M by $\phi(\bar{a})^M$ we denote the interpretation of $\phi(\bar{x})$ at $\bar{a}$ in M. In logic of metric structures there are two ways in which one can define theory of a structure M. We shall think of theory as a set of sentences, and accordingly set $\mathrm{Th}(M) = \{\phi : \phi^M = 0\}$. Alternatively, one may consider the functional $\phi \mapsto \phi^M$ as the theory of M.

We shall tacitly use completeness theorem for logic of metric structures whenever convenient ([6]).

1.1. Conditions and types.

A *closed condition* is an expression of the form $\phi(\bar{x}) = 0$ for a formula $\phi(\bar{x})$ and *type* is a set of conditions. (Open conditions will be defined in §1.1.1 below.) It is *realized* in structure M if there is a tuple $\bar{a}$ in M of appropriate sort such that $\phi(\bar{a})^M = 0$ for all conditions $\phi(\bar{x}) = 0$ in $\mathbf{t}$. A type $\mathbf{t}$ is *consistent* if it is consistent with $\mathbf{T} = \emptyset$. If free variables of every formula appearing in $\mathbf{t}$ are included in $\{x_0, \dots, x_{n-1}\}$ and n is minimal with this property, we say that $\mathbf{t}$ is an *n-type*.

1.1.1. *Continuous functional calculus.*

Recall that every L-formula $\phi(\bar{x})$ has a modulus of uniform continuity and that the set R_ϕ of all possible values of ϕ in all L-structures is a compact subset of $\mathbb{R}$ ([3]). If $\phi(\bar{x})$ is a formula and f is a continuous function on R_ϕ then $f(\phi(\bar{x}))$ is a formula.

Consider conditions of the form $\phi \in K$, where $K \subseteq \mathbb{R}$. Condition $\phi(\bar{x}) \in K$ is *closed* (*open*) if K is closed (open, respectively). Two conditions $\phi(\bar{x}) \in K$ and $\psi(\bar{x}) \in M$ are *equivalent* if in every L-structure A for every $\bar{a}$ of the appropriate sort one has $\phi(\bar{a})^A \in K$ if and only if $\psi(\bar{a})^A \in M$.

The following lemma will be used tacitly.

Lemma 1.1. *Every open condition is equivalent to a condition of the form $\phi(\bar{x}) < 1$. Every closed condition is equivalent to one of the form $\phi(\bar{x}) = 0$.*

Proof. Fix a condition $\psi(\bar{x}) \in K$. Since R_ψ is compact, one can use Tietze extension theorem to find $f\colon R_\psi \to [0,1]$ such that $f^{-1}(\{0\}) = K \cap R_\psi$ if K is closed or $f^{-1}(\{1\}) = R_\psi \setminus K$ is K is open. In either case $\phi = f(\psi)$ is as required. □

1.1.2. *Pairing types.* The reader will excuse us for making some easy observations for future reference.

Lemma 1.2. *If* $\mathbf{t}$ *and* $\mathbf{s}$ *are types over a consistent and complete theory then there are types* $\mathbf{t} \wedge \mathbf{s}$ *and* $\mathbf{t} \vee \mathbf{s}$ *such that for every* $M \models \mathbf{T}$ *we have that*

1. *M omits $\mathbf{t} \vee \mathbf{s}$ if and only if it omits both $\mathbf{t}$ and $\mathbf{s}$,*
2. *M omits $\mathbf{t} \wedge \mathbf{s}$ if and only if it omits at least one of $\mathbf{t}$ or $\mathbf{s}$.*

Proof. We may assume that $\mathbf{t}$ and $\mathbf{s}$ are an m-type and a k-type, respectively, in disjoint sets of variables.

By adding dummy conditions we may assume that the types are of the same cardinality κ and we enumerate conditions in $\mathbf{t}$ as $\phi_\xi(\bar{x}) = 0$ for $\xi < \kappa$ and conditions in $\mathbf{s}$ as $\psi_\xi(\bar{y}) = 0$ for $\xi < \kappa$. For a finite $F \subseteq \kappa$ let

$$\theta_F(\bar{x}, \bar{y}) := \max_{\xi \in F}(\phi_\xi(\bar{x}), \psi_\xi(\bar{y})),$$
$$\zeta_F(\bar{x}, \bar{y}) := \min\{\max_{\xi \in F} \phi_\xi(\bar{x}), \max_{\xi \in F} \psi_\xi(\bar{y})\}.$$

Let

$$\mathbf{t} \vee \mathbf{s} = \{\theta_F : F \subseteq \kappa, F \subseteq \kappa \text{ finite}\},$$
$$\mathbf{t} \wedge \mathbf{s} = \{\zeta_F : F \subseteq \kappa, F \subseteq \kappa \text{ finite}\}.$$

Then an $m+k$-tuple $\bar{a}, \bar{b}$ realizes $\mathbf{t} \vee \mathbf{s}$ if and only if for all ξ we have $\phi_\xi(\bar{a}) = 0 = \psi_\xi(\bar{b})$ if and only if $\bar{a}$ realizes $\mathbf{t}$ and $\bar{b}$ realizes $\mathbf{s}$.

An $m+k$-tuple $\bar{a}, \bar{b}$ realizes $\mathbf{t} \wedge \mathbf{s}$ if and only if for every finite F at least one of (i) $\max_{\xi \in F} \phi_\xi(\bar{a}) = 0$ or (ii) $\max_{\xi \in F} \psi_\xi(\bar{b}) = 0$ holds. For a cofinal set of F we have (i) or (ii) and therefore $\bar{a}$ realizes $\mathbf{t}$ or $\bar{b}$ realizes $\mathbf{s}$. □

1.1.3. *Type* $\mathbf{t}_\omega$. Assume $\mathbf{T}$ is a theory and $\mathbf{t} = \{\phi_j(\bar{x}) = 0 : j \in \omega\}$ is an n-type omissible in a model of $\mathbf{T}$. We shall assume $n = 1$ for simplicity.

To $\mathbf{t}$ we associate the type $\mathbf{t}_\omega$ in infinitely many variables x_j, for $j \in \omega$, consisting of all formulas of the form

($\mathbf{t}_\omega$1) $\phi_j(x_n) \leq \frac{1}{n}$ for all $j \leq n$, and
($\mathbf{t}_\omega$2) $d(x_j, x_{j+1}) \leq 2^{-j}$ for all $j \in \omega$.

We can think of $\mathbf{t}_\omega$ as an ω-type—an increasing union of n-types $\mathbf{t}_n$, where $\mathbf{t}_n$ is the restriction of $\mathbf{t}_\omega$ to x_j, for $j < n$. The following is clear.

Lemma 1.3. *Model M realizes $\mathbf{t}$ if and only if every (equivalently, some) dense subset D of its universe includes a sequence that realizes $\mathbf{t}_\omega$.* □

We note that a slightly finer fact is true. If D is an arbitrary dense subset of the universe of M then we can consider $D^{<\omega}$ as a tree with respect to the end-extension. Let $T_{D,\mathbf{t}}$ be the family of all $\bar{d} \in D^{<\omega}$ such that with n being the length of $\bar{d}$ we have that $M \models \mathbf{t}_n(\bar{d})$. Then M omits $\mathbf{t}$ if and only if the tree $T_{D,\mathbf{t}}$ is *well-founded* (i.e., it has no infinite branches)

1.2. **Spaces.** For a fixed separable language L we shall now introduce standard Borel spaces of formulas, structures, complete types, and incomplete types.

1.2.1. *Linear space of formulas.* For $n \in \omega$ let $\mathbb{F}_n(L)$ denote the set of formulas all of whose free variables are among $\{x_0, \dots, x_{n-1}\}$. If L is implicit from the context we shall write $\mathbb{F}_n$ instead of $\mathbb{F}_n(L)$. On $\mathbb{F}_n(L)$ one considers the norm

$$\|\phi\|_\infty = \sup_{M,\vec{a}} |\phi(\vec{a})^M|.$$

Here M ranges over all L-structures, $\vec{a}$ ranges over all n-tuples of elements in M, and $\phi(\vec{a})^M$ is the interpretation of ϕ at $\vec{a}$ in M.

As pointed out in [3], if L is countable then $\mathbb{F}_n(L)$ is separable with respect to this metric. In this situation we say that L *is separable* and make the standing assumption of separability of L throughout the present section (and most of the paper).

1.2.2. *Borel space of models.* One can consider structures of a fixed countable (discrete) language L as structures with universe ω. Thus the space of countable L-structures is equipped with the Cantor-set topology and a natural continuous action of the permutation group S_∞. This observation is a rich source of results on the interface between (classical first-order) model theory and descriptive set theory (see e.g., [20]). The space of separable metric structures of a fixed separable language L can be construed as a standard Borel space in more than one way. In [12] it was shown that every metric L-structure can be canonically extended to one whose universe is the Urysohn metric space. Borel space $\mathcal{M}(L)$ of all L-structures obtained in this way is not convenient for our purposes. We consider the space that was essentially introduced in [5, p. 2]. Although this space was denoted $\mathcal{M}(L)$ in [5], we use the notation $\hat{\mathcal{M}}(L)$ to avoid conflict with [12]. Space $\hat{\mathcal{M}}(L)$ is defined as follows.

For simplicity we consider the case when L has no predicate symbols. Let d_j, for $j \in \omega$, be a sequence of new constant symbols and let $L^+ = L \cup \{d_j : j \in \omega\}$. Let $\mathfrak{p}_j$, for $j \in \omega$, be an enumeration of a countable dense set of L^+-terms closed under application of function symbols from L. Space $\hat{\mathcal{M}}(L)$ is the space of all functions

$$\gamma\colon \omega^2 \to [0,1]$$

such that

(i) γ is a metric on ω,

(ii) γ respects the moduli of uniform continuity of all functions in L. In particular [3, (UC), p. 8] holds: if f is a function symbol with modulus of uniform continuity Δ and i, j, i' and j' are such that $\mathfrak{p}_{j'} = f(\mathfrak{p}_j)$ and $\mathfrak{p}_{i'} = f(\mathfrak{p}_i)$ then

$$\gamma(i,j) < \Delta(\varepsilon) \text{ implies } \gamma(i',j') \leq \varepsilon.$$

(An analogous condition holds for n-ary function symbols for $n \geq 2$.)

The set of $\gamma \in [0,1]^{\omega^2}$ satisfying (i) and (ii) is a closed subspace of the Hilbert cube, and $\hat{\mathcal{M}}(L)$ is equipped with the induced compact metric topology. For $\gamma \in \hat{\mathcal{M}}(L)$ we can consider the structure with universe ω and metric given by γ. This structure falls short of being an L-structure only because it is incomplete, and the completion $M(\gamma)$ of such structure is a separable L-structure. Every complete separable metric L-structure M is a completion of such countable structure. Also, every such M has many different representations in $\hat{\mathcal{M}}(L)$.

One can modify $\hat{\mathcal{M}}(L)$ to accommodate the case when L has predicate symbols. As a matter of fact, the version of space $\hat{\mathcal{M}}(L)$ in the case when L has only predicate symbols was considered in [5, p. 2]; let us recall the details. For each n-ary predicate symbol R add a function $\gamma_R \colon \omega^n \to [0,1]$ corresponding to the interpretation of R in $M(\gamma)$. The straightforward details are omitted.

Space $\hat{\mathcal{M}}(L)$ is similar to the space of separable C*-algebras $\hat{\Gamma}$ introduced in [19]. Although $\hat{\mathcal{M}}(L)$ is different from the Borel space of L-structures $\mathcal{M}(L)$ defined in [12], these two spaces are equivalent in the sense of [19]. The proof of this fact is analogous to the proof given in [12, §3] for the case of C*-algebras.

1.2.3. *Compact spaces of theories and types.* Consider space $\mathbb{F}_n(L)$ as in §1.2.1. On the space of linear functionals on $\mathbb{F}_n(L)$ we consider the topology of pointwise convergence (i.e., the logic topology, also known as the weak*-topology). Every L-structure M defines a linear functional on $\mathbb{F}_0$ by the evaluation of sentences,

$$\phi \mapsto \phi^M.$$

Such functionals are *consistent L-theories.*

Since $\mathbb{F}_0$ is normed by $\|\phi\|_\infty = \sup_{M,\vec{a}} |\phi(\vec{a})^M|$ (see §1.2.1), this functional has norm 1. By the compactness theorem for logic of metric structures the space of complete, consistent L-theories is compact in logic topology (see [3]).

Via the interpretation map $(M,\phi) \mapsto \phi^M$ the spaces $\hat{\mathcal{M}}(L)$ and $\mathbb{F}_0(L)$ are in duality (although note that $\hat{\mathcal{M}}(L)$ is not a linear space).

Special case of the following lemma in case of C*-algebras was proved in [19, Proposition 5.1]. Proof of the general case is virtually identical.

Lemma 1.4. *The function from $\hat{\mathcal{M}}(L)$ to the space of L-theories that associates the theory of M to M is Borel.* □

Proof of the following lemma is a straightforward computation (similar to one in the appendix of [19], where it was proved that separable C*-algebras that tensorially absorb the Jiang–Su algebra $\mathcal{Z}$ form a Borel set; at the hindsight this is an instance of our lemma; see Corollary 6.3).

Lemma 1.5. *If* $\mathbf{T}$ *is an* L*-theory then the set of* $\gamma \in \hat{\mathcal{M}}(L)$ *such that* $M(\gamma) \models \mathbf{T}$ *is Borel.* □

An L-model M and an n-tuple a_i, for $i < n$, in M define by interpretation a linear functional $\mathrm{tp}_M(\bar{a})$ by

$$\mathrm{tp}_M(\bar{a})(\phi(\bar{x})) = \phi(\bar{a})^M.$$

Again, by compactness the space $S_n(\mathbf{T})$ of all consistent complete n-types is compact in the logic topology.

1.2.4. *Metric on the space of complete types over a complete theory.* Let $\mathbf{T}$ be a complete L-theory. Following ([3, p. 44]) on the space $S_n(\mathbf{T})$ of complete n-types over $\mathbf{T}$ we define metric d by

$$d(\mathbf{t}, \mathbf{s}) = \inf\{\max_{i<n} d(a_i, b_i) : \text{ there exist } M \models T \text{ and } \bar{a} \text{ and } \bar{b} \text{ in } M \text{ such that } M \models \mathbf{t}(\bar{a}) \text{ and } M \models \mathbf{s}(\bar{b})\}.$$

(Since both types and $\mathbf{T}$ are complete, the triangle inequality is satisfied.) We denote the set of realizations of type $\mathbf{t}$ in model M by $\mathbf{t}(M)$.

Given an n-type $\mathbf{t}$ over a theory $\mathbf{T}$ and a new n-tuple of constants $\bar{c}$, we let $\mathbf{T}_{\mathbf{t}/\bar{c}}$ denote the theory in language $L \cup \{\bar{c}\}$ obtained by extending $\mathbf{T}$ with axioms asserting that $\bar{c}$ realizes $\mathbf{t}$. More precisely, one adds all conditions of the form $\phi(\bar{c}) = 0$ to $\mathbf{T}$, where $\phi(\bar{x}) = 0$ is a condition in $\mathbf{t}$. It is straightforward to check that if both $\mathbf{T}$ and $\mathbf{t}$ are complete then so is $\mathbf{T}_{\mathbf{t}/\bar{c}}$. One can iterate this definition and name realizations of more than one type, as in the proof of Lemma 1.6 below.

For a fixed n the set of types $(\mathbf{T}, \mathbf{t})$ where $\mathbf{T}$ is a complete theory and $\mathbf{t}$ is a complete type over $\mathbf{T}$ is endowed with a Polish topology as follows. We identifyi each pair $(\mathbf{T}, \mathbf{t})$ with the complete theory $\mathbf{T}_{\mathbf{t}/\bar{c}}$. Each theory obtained in this way is complete, and every complete theory in the language obtained by extending the language of $\mathbf{T}$ by adding constants $\bar{c}$ is equal to $\mathbf{T}_{\mathbf{t}/\bar{c}}$ for some pair $(\mathbf{T}, \mathbf{t})$.

Lemma 1.6. *For every* n *and* $\varepsilon \geq 0$ *the set* $\{(\mathbf{r}, \mathbf{s}) : d(\mathbf{r}, \mathbf{s}) > \varepsilon\} \subseteq S_n(\mathbf{T})^2$ *is open in the logic topology.*

Proof. Fix types $\mathbf{t}$ and $\mathbf{s}$ such that $d(\mathbf{t}, \mathbf{s}) > \varepsilon$. This is equivalent to stating that

$$\mathbf{T}_{\mathbf{t}/\bar{c}, \mathbf{s}/\bar{d}} \vdash d(\bar{c}, \bar{d}) > \varepsilon.$$

Then by compactness (or by [6]) there exists a finite set of open conditions $\mathbf{T}_0$ of $\mathbf{T}_{\mathbf{t}/\bar{c}, \mathbf{s}/\bar{d}}$ such that $\mathbf{T}_0 \vdash d(\bar{c}, \bar{d}) > \varepsilon$. This defines a logic open neighbourhood U of $(\mathbf{t}, \mathbf{s})$ in $S_n(\mathbf{T})$ such that $d(\mathbf{r}, \mathbf{s}) > \varepsilon$ for all $(\mathbf{r}, \mathbf{s}) \in U$. □

1.2.5. *The compact metric space of incomplete types.* Let $\mathbf{T}$ be a (not necessarily complete) theory in a separable language. An n-type $\mathbf{t}$ in $\mathbf{T}$ is a countable set of conditions (§1.1), but we can also identify it with the set of all complete types extending it. This set is closed (and therefore compact) in the logic topology. Fix separable language L, $n \geq 1$ and an L-theory $\mathbf{T}$. For $K \subseteq S_n(\mathbf{T})$ closed in the logic topology let (considering $\mathbf{s} \in S_n(\mathbf{T})$ as the set of conditions)

$$\mathbf{t}_K = \bigcap_{\mathbf{s} \in K} \mathbf{s}.$$

Then $\mathbf{t}_K$ is a type that includes $\mathbf{T}$. If $\mathbf{s} \notin K$, then there exists a condition $\phi(\bar{x}) < \varepsilon$ such that $\phi(\bar{x}) \leq \varepsilon/2$ belongs to $\mathbf{s}$ but not to any type in K. Therefore no type $\mathbf{s} \notin K$ extends $\mathbf{t}_K$ and we have (considering types as sets of closed conditions)

$$K = \{\mathbf{s} \in S_n(\mathbf{T}) : \mathbf{s} \supseteq \mathbf{t}_K\}.$$

We can therefore identify the space $S_n^-(\mathbf{T})$ of not necessarily complete types over $\mathbf{T}$ with the exponential space of $S_n(\mathbf{T})$, equipped with its compact metric topology given by the Hausdorff metric.

1.3. **Omitting complete types.**

1.3.1. *Principal types.* Assume $\mathbf{T}$ is a complete theory and $\mathbf{t}$ is an n-type in the signature of $\mathbf{T}$. As in §1.2.5, we identify $\mathbf{t}$ with the set of all complete types extending $\mathbf{t}$. This is a closed set in logic, and therefore also in metric, topology. An n-type $\mathbf{t}$ is *principal* (or *isolated*) if for every $\varepsilon > 0$ the set

$$B_\varepsilon(\mathbf{t}) = \{\mathbf{s} \in S_n(\mathbf{T}) : \inf\{d(\mathbf{t}', \mathbf{s}) : \mathbf{t}' \in S_n(\mathbf{T}), \mathbf{t} \subseteq \mathbf{t}'\} < \varepsilon\}$$

is not nowhere dense in the logic topology (with respect to $\mathbf{T}$). If a type is not principal, then the proof of the omitting types theorem given in [3, §12] shows that $\mathbf{T}$ has a separable model omitting $\mathbf{T}$. If $\mathbf{t}$ is principal and complete, then ε-balls as in the definition can be chosen so that they form a decreasing chain. The intersection of this chain gives a realization of $\mathbf{t}$ in every model of $\mathbf{T}$. This paper is about the case not covered by these observations: principal, but not complete, types over a complete theory in a separable language.

1.3.2. *The set of omissible complete types is Borel.* A type is omissible in a model of $\mathbf{T}$ if and only if there exists a countable metric space with interpretations for all symbols in L whose completion is a model of $\mathbf{T}$ (the original set is not literally a model since its universe is not necessarily a complete metric space) such that no Cauchy sequence of its elements has a limit that realizes $\mathbf{t}$ (equivalently, no subset of this set realizes type $\mathbf{t}_\omega$ as defined in §1.1.3). This condition is $\mathbf{\Sigma}^1_2$, but by the following well-known result under additional assumptions it is Borel.

In the following proposition we consider the logic topology on the space of all complete theories in a fixed separable language L. For $n \in \omega$ consider the space of pairs $(\mathbf{T}, \mathbf{t})$ where $\mathbf{T}$ is an L-theory and $\mathbf{t}$ is a complete n-type over $\mathbf{T}$ with respect to the logic topology defined before Lemma 1.6.

Proposition 1.7. *For every $n \in \omega$ the following sets are Borel.*

1. *The set of all pairs $(\mathbf{T}, \mathbf{t})$ such that $\mathbf{T}$ is a complete theory and $\mathbf{t}$ is a complete n-type realized in every model of $\mathbf{T}$.*
2. *The set of all pairs such that $\mathbf{T}$ is a theory and $\mathbf{t}$ is an n-type realized in some model of $\mathbf{T}$.*

Proof. (1) By the Omitting Types Theorem ([3, §12], [22], or Corollary 3.7) type $\mathbf{t}$ has to be realized in every model of $\mathbf{T}$ if and only if it is principal (principal types were defined in §1.3.1).

Since the logic topology is second countable, expressing the fact that $B_\varepsilon(\mathbf{t})$ is nowhere dense requires only quantification over a countable set. It therefore suffices to show that the set $\{\mathbf{s} : d(\mathbf{t}, \mathbf{s}) \geq \varepsilon\}$ is Borel, and this follows from Lemma 1.6.

(2) By the compactness theorem ([3, Theorem 5.8]) the condition that a type $\mathbf{t}$ is consistent with theory $\mathbf{T}$ is finitary, and therefore Borel. □

Completeness assumption on types in Proposition 1.7 is necessary by Theorem 1, but see also Proposition 4.7.

1.4. **A test for elementary equivalence.** We include a general test for elementary equivalence used in §5 below. A subset Y of a metric space is *ε-dense* if for every point $x \in X$ there exists $y \in Y$ such that $d(x, y) < \varepsilon$.

Lemma 1.8. *Assume A and B are structures of language L such that for every finite $L_0 \subseteq L$ and every $\varepsilon > 0$ there are ε-dense substructures A_0 and B_0 of L_0-reducts of A and B, respectively, which are isomorphic. Then A and B are elementarily equivalent.*

Proof. By [3, Proposition 6.9] every formula can be uniformly approximated by formulas in prenex normal form. It will therefore suffice to show that every formula in prenex normal form has the same value in A and B. Let ϕ be an L-sentence in prenex normal form

$$\sup_{x_0} \inf_{x_1} \dots \sup_{x_{2n-2}} \inf_{x_{2n-1}} \psi(\bar{x})$$

where $\psi(\bar{x})$ is quantifier-free and let $L_0 \subseteq L$ be a finite subset consisting only of symbols that appear in ϕ. For $\delta > 0$ fix $\varepsilon > 0$ small enough so that perturbing variables in $\bar{x}$ by $\leq \varepsilon$ does not change the value of $\psi(\bar{x})$ by more than $\delta/2$. It is then straightforward to check that

$$|\phi^A - \phi^{A_0}| \leq \delta/2 \qquad \text{and} \qquad |\phi^B - \phi^{B_0}| \leq \delta/2.$$

Since $\delta > 0$ was arbitrary and A_0 and B_0 are isomorphic, we conclude that $\phi^A = \phi^B$. Since ϕ was arbitrary, we conclude that A and B are elementarily equivalent. □

Lemma 1.8 can also be proved by using EF-games ([22]) and it admits a number of yet unexplored possibilities for generalizations. For example, one could define a variant of Gromov–Hausdorff distance on structures in a given language L and show that for any fixed sentence ϕ the computation of ϕ is continuous with respect to this metric. See also [15, Corollary 2.1].

2. Complexity of spaces of types

From now on, *all types are assumed to be partial and consistent*. Following [3] we write $r \dot{-} s$ for $\max(0, r-s)$. Recall that a subset A of a Polish space X is $\mathbf{\Pi}^1_1$ if it is a complement of a continuous image of a Borel subset of a Polish space. A $\mathbf{\Pi}^1_1$-set is $\mathbf{\Pi}^1_1$-*complete* if for every $\mathbf{\Pi}^1_1$ subset B of a Polish space Y there exists a continuous $f\colon Y \to X$ such that $B = f^{-1}(A)$. For more information see [24].

Whenever we say that type $\mathbf{t}$ is omissible in a model of $\mathbf{T}$ it is assumed to be consistent with $\mathbf{T}$.

2.1. Basic complexity results.

Recall that the space $S_n(\mathbf{T})$ of complete n-types over a complete theory $\mathbf{T}$ is a compact metric space (§1.2.3) and that the space $S_n^-(\mathbf{T})$ of not necessarily complete n-types over $\mathbf{T}$ is identified with the compact metric space of its closed subsets (§1.2.5).

Lemma 2.1. *Assume M is a separable model in a separable language and $n \in \omega$.*

1. *The set of all complete n-types realized in M is $\mathbf{\Sigma}^1_1$.*
2. *The set of all not necessarily complete n-types realized in M is $\mathbf{\Sigma}^1_1$*

Proof. (1) Let $\hat{\mathcal{M}}(L)$ be the space of L-models (§1.2.2). as in §1.2.2. For $M \in \hat{\mathcal{M}}(L)$ let $a(M, j)$, for $j \in \omega$, be an enumeration of a countable subset of its universe used for the coding of M as a member of $\hat{\mathcal{M}}(L)$. Define $f\colon \hat{\mathcal{M}}(L) \times (\omega^\omega)^n \to S_n(\mathbf{T}) \cup \{*\}$ by $f(M, x(0), \dots, x(n-1)) = *$ if sequence $\{a(M, x(i)(j)) : j \in \omega\}$ is not Cauchy for some $i < n$. Otherwise, we let $f(M, x(0), \dots, x(n-1))$ be the limit of types of $x(0)(j), \cdots x(n-1)(j)$, for $j \in \omega$, in M. This function is Borel and the image of $\{M\} \times (\omega^\omega)^n$ is the set of n-types realized in M.

(2) The set $A \subseteq S_n(\mathbf{T})$ of all types omitted in M is $\mathbf{\Pi}^1_1$. We need to show that the set $\{K \in S_n^-(\mathbf{T}) : K \subseteq A\}$ is also $\mathbf{\Pi}^1_1$. This is standard but we include an argument for the convenience of the reader. The set

$$Z = \{(x, K) \in S_n(\mathbf{T}) \times S_n^-(\mathbf{T}) : x \in K\}$$

is closed and $K \subseteq A$ if and only if $(\forall x)((x, K) \in Z \to x \in A)$, giving the required $\mathbf{\Pi}^1_1$ definition. □

If $\mathbf{T}$ has a prime model then a type is omissible in a model of $\mathbf{T}$ if and only if it is omitted in its prime model, and we have an immediate corollary to (2) above.

Corollary 2.2. *If* $\mathbf{T}$ *is a complete theory in a separable language with a prime model then the set of types omissible in a model of* $\mathbf{T}$ *is* $\mathbf{\Pi}^1_1$. □

Note that the set of complete types realized in M is always dense in the logic topology and therefore if M does not realize all types then the set of complete types realized in M is not closed in logic topology. Compare the following with Proposition 1.7.

Lemma 2.3. *If* $\mathbf{T}$ *is a (not necessarily complete) theory in a separable language, then the set of all types omissible in a model of* $\mathbf{T}$ *is* $\mathbf{\Sigma}^1_2$.

Proof. This is a consequence of (2) of Lemma 2.1. □

The following result (cf. Question 6.1) was inspired by [7].

Proposition 2.4. *There are a separable language* L *and a separable* L*-model* M *such that the set of quantifier-free unary types realized in* M *is a complete* $\mathbf{\Sigma}^1_1$ *set.*

Proof. Language L has only one unary predicate symbol f, interpreted as a 1-Lipshitz function. Identify ω^ω with $[0,1]\setminus\mathbb{Q}$ and consider it with a complete metric such that the identity map from ω^ω into $[0,1]$ is 1-Lipshitz. Fix a closed $X\subseteq\omega^\omega\times\omega^\omega$. Consider X with the max-metric induced from ω^ω and interpret f as the projection to the x-axis. Then f is 1-Lipshitz by the choice of metric on ω^ω.

The only atomic formulas in L are $f(x)$ and $d(x,y)$. If $\phi(x)$ is a quantifier-free formula with only one free variable, then the only atomic subformulas of ϕ are of the form $f(x)$ and $d(x,x)$. The latter is identically equal to 0, and therefore the quantifier-free type of an element of M is completely determined by its projection to the x-axis. Choosing X so that its projection is a complete analytic set completes the proof. □

2.2. **Theory of the Baire space.** Let $L_{\mathcal{N}}$ be a language with a single sort D_1. The intended interpretation of D_1 is $\omega^{<\omega}\sqcup\omega^\omega$. Language $L_{\mathcal{N}}$ is equipped with the following.

(1) Constant symbols for all elements of $\omega^{<\omega}$ (we shall identify $t\in\omega^{<\omega}$ with the corresponding constant).
(2) Unary function symbols f_k for $k\in\omega$.

The interpretation of each f_k is required to be 1-Lipshitz. Theory $\mathbf{T}_{\mathcal{N}}$ is the theory of $L_{\mathcal{N}}$-model $\mathcal{N}$ described as follows. The universe of $\mathcal{N}$ is the set $\omega^{<\omega}\sqcup\omega^\omega$.

The metric on $\omega^{<\omega}\sqcup\omega^\omega$ is the standard Baire space metric,

$$d(s,t)=1/(\Delta(s,t)+1).$$

If $s\sqsubseteq t$ then $d(s,t)=1/(|s|+1)$. For $s\in\omega^{<\omega}$ we write $|s|=n$ if $s\in\omega^n$ and for $k\leq|s|$ we denote the initial segment of s with length k by $s\restriction k$.

For $k \in \omega$ function f_k is interpreted as

$$f_k(s) = \begin{cases} s \restriction k & \text{if } k \leq |s| \\ s & \text{if } k > |s|. \end{cases}$$

Clearly $\omega^{<\omega}$ is a dense subset of D_1^M which is closed under all f_k.

2.2.1. *Well-foundedness.* Every model N of $\mathbf{T}_\mathcal{N}$ has a dense F_σ set

$$T_N = \{a \in D_1^N : f_k(a) = a \text{ for some } k\}.$$

With the ordering defined by $a \sqsubseteq b$ if and only if $a = f_k(b)$ for some k this set is a tree of height ω. Moreover, elements of $N \setminus T_N$ are in a natural bijective correspondence to branches of this tree, because $f_k(x) = f_k(y)$ for all k implies $d(x, y) = 0$ and therefore $x = y$.

Note that $\mathcal{N}$ has a dense subset consisting of elements that are interpretations of constant symbols. Therefore every model N of $\mathbf{T}_\mathcal{N}$ has $\mathcal{N}$ as an elementary submodel, hence $\mathcal{N}$ is the prime model of $\mathbf{T}_\mathcal{N}$.

2.2.2. *Type* $\mathbf{s}_0$ *and the standard model.* Let us describe a unary type $\mathbf{s}_0$ in the expanded language $L_\mathcal{N} \cup \{h\}$ such that the only model of the theory of an appropriate expansion of M to $L_\mathcal{N}$ omitting $\mathbf{s}_0$ is the standard model itself.

To $L_\mathcal{N}$ we add the following.

(3) Unary function symbol h.

Fix an enumeration s_n, for $n \in \omega$, of $\omega^{<\omega}$. Then h is interpreted as follows (here $\langle n \rangle$ denotes an element of $\omega^{<\omega}$ of length 1 whose only digit is n).

$$h(x) = \begin{cases} s_n, & \text{if } x = \langle n \rangle \text{ for some } n \\ f_1(x), & \text{otherwise.} \end{cases}$$

Then h is 3-Lipshitz because $d(x, y) \leq 1/3$ implies $h(x) = f_1(x) = h(y)$. Let

$$\psi := \sup_x \inf_y d(x, h(y)) + |f_1(y) - y|.$$

Then $\psi^\mathcal{N} = 0$ since the range of $h \restriction \{y : f_1(y) = y\}$ is $\omega^{<\omega}$ and $\omega^{<\omega}$ is a dense subset of $\mathcal{N}$. We get that $\psi \in \mathbf{T}_\mathcal{N}$. Hence in every model of $\mathbf{T}_\mathcal{N}$ the range of $\{h(x) : f_1(x) = x\}$ is dense in the model.

Let $\mathbf{s}_0(x)$ be the type consisting of conditions

$$d(f_1(x), \langle n \rangle) = 1$$

for all $n \in \omega$. Then every finite subset of $\mathbf{s}_0$ is realized in $\mathcal{N}$ by a large enough $\langle m \rangle$. On the other hand, $\mathcal{N}$ clearly omits $\mathbf{s}_0$. Let $\mathcal{M}$ be any model of $\mathbf{T}_\mathcal{N}$ which omits $\mathbf{s}_0$. hence the set $\{y \in \mathcal{M} : f_1(y) = y\}$ is equal to $\{\langle n \rangle : n \in \omega\}$. It follows that in $\mathcal{M}$ the set $\{h(x) : f_1(x) = x\}$ is exactly $\omega^{<\omega}$. Hence the model $\mathcal{N}$, which is the prime model of $\mathbf{T}_\mathcal{N}$, is dense in $\mathcal{M}$ and $\mathcal{M}$ and $\mathcal{N}$ are isometrically isomorphic.

2.3. **$\mathbf{\Pi}^1_1$-complete.** Fix a complete theory $\mathbf{T}$ in a separable language. The set of (not necessarily complete) n-types omissible in a model of $\mathbf{T}$ is $\mathbf{\Sigma}^1_2$, by counting quantifiers.

Theorem 2.5. *There is a complete theory $\mathbf{T}_2$ in a separable language L_2 such that the space of all 2-types $\mathbf{t}$ omissible in a model of $\mathbf{T}_2$ is $\mathbf{\Pi}^1_1$-complete.*

Proof. Let L_2 be a two-sorted language with sorts D_1 and D_2, including $L_{\mathcal{N}}$. The intended interpretation of D_2 is the space $\mathcal{T}$ of all subtrees of $\omega^{<\omega}$. Language L_2 is equipped with the following

(4) constant symbols S_n, for $n \in \omega$, for all finite-width subtrees of $\omega^{<\omega}$ all of whose branches have eventually zero value. [1]
(5) Binary predicate symbol Elm of sort $D_1 \times D_2$.

Let $\mathcal{T}$ denote the space of all subtrees of $\omega^{<\omega}$. The interpretation of Elm is required to be 1-Lipshitz. Theory $\mathbf{T}_2$ is theory of the L_2-model $\mathcal{N}_2$ described as follows. The universe of $\mathcal{N}_2$ is the set $\omega^{<\omega} \sqcup \omega^\omega \sqcup \mathcal{T}$ and $\mathcal{N}_2$ is an extension of the $L_{\mathcal{N}}$-model $\mathcal{N}$ as described in §2.2.

Metric on $\mathcal{T}$ is defined as (let $\delta(a,b) = \min\{k : a \cap k^{\leq k} \neq b \cap k^{\leq k}\}$)

$$d(a,b) = 1/(\delta(a,b)+1).$$

Note that $D_2^{\mathcal{N}_2}$ is a compact metric space in which the interpretations of constant symbols from (4) form a countable dense set.

In order to interpret predicate Elm we introduce an auxiliary function $\ell \colon \omega^{<\omega} \to \omega$ via

$$\ell(t) = \max(\{|t|\} \cup \operatorname{range}(t)).$$

Now define Elm on $\omega^{<\omega} \times \mathcal{T}$ via

$$\operatorname{Elm}(t,S) = \begin{cases} 0, & \text{if } t \in S \\ 1/\ell(t), & \text{if } t \notin S. \end{cases}$$

The predicate Elm is Lipshitz because $d(S,T) \leq 1/k$ and $d(s,t) \leq 1/k$ implies that $\operatorname{Elm}(s,S) \leq 1/m$ iff $\operatorname{Elm}(t,T) \leq 1/m$ for all $m \leq k$. By continuity we have $\operatorname{Elm}(x,S) = 0$ for all $x \in \omega^\omega$ and all $S \in \mathcal{T}$.

Let $\mathbf{T}_2$ be the theory of $\mathcal{N}_2$. Just like in the case of $\mathcal{N}$ (§2.2) definable elements of $\mathcal{N}_2$ form a dense subset, and therefore $\mathcal{N}_2$ is a prime model of $\mathbf{T}_2$. By Corollary 2.2 the set of types omissible in a model of $\mathbf{T}$ is $\mathbf{\Pi}^1_1$.

Let $S \in \mathcal{T}$. We let $\mathbf{t}^S$ be the partial type in x, y of sort $D_1 \times D_2$ consisting of the following conditions

(6) $\operatorname{Elm}(f_k(x), y) = 0$ for all k.
(7) $\operatorname{Elm}(t,y) = 0$ if $t \in S$ and $\operatorname{Elm}(t,y) = 1/\ell(t)$ if $t \notin S$, for all $t \in \omega^{<\omega}$.
(8) $d(S_n, y) = \varepsilon_n$, where $\varepsilon_n = d(S_n, S)$, for all n.
(9) $|d(f_k(x), x) - 1/(k+1)| = 0$.

[1]These constant symbols are included only for convenience. Their addition is of no consequence since $\mathcal{T}$ is compact.

A realization of $\mathbf{t}^S$ is a pair (b, c) such that c is a tree whose standard part $c^0 = c \cap \omega^\omega$ satisfies $d(c^0, S) = 0$. Hence $c^0 = S$ and $\{f_k(b) : k \in \omega\} \cap \omega^{<\omega}$ is included in S. Clearly map $\mathcal{T} \ni S \mapsto \mathbf{t}^S \in S_2^-(\mathbf{T}_2)$ (see §1.2.5 is continuous.

Since the set of well-founded trees in $\mathcal{T}$ is $\mathbf{\Pi}^1_1$-complete, it only remains to check that $\mathbf{t}^S$ is omissible in a model of $\mathbf{T}_2$ if and only if S is well-founded.

If S is well-founded then the standard model $\mathcal{N}_2$ of $\mathbf{T}_2$ omits $\mathbf{t}^S$. This is because if (b, a) realizes $\mathbf{t}^S$ then $a = S$, and therefore $b \in \omega^\omega$ has to be a 'true' branch of S.

Now assume S is ill-founded and let b be its branch. Let N be a model of $\mathbf{T}_2$. Then $\mathcal{N}_2$ is an elementary submodel of N and since $\mathcal{N}_2$ realizes $\mathbf{t}^S$ by (b, S), so does N. □

2.4. **$\mathbf{\Sigma}^1_2$-complete.** The following theorem is logically incomparable with Theorem 2.5 since its conclusion stronger but the theory $\mathbf{T}_3$ in it is not required to be complete.

Theorem 2.6. *There is a theory $\mathbf{T}_3$ in a separable language L_3 such that the space of all 1-types $\mathbf{t}$ omissible in a model of $\mathbf{T}_3$ is $\mathbf{\Sigma}^1_2$-complete.*

Proof. By Lemma 1.2 it will suffice to show that the space of all triples $(\mathbf{T}, \mathbf{t}, \mathbf{s})$ where $\mathbf{T}$ is a consistent L-theory and $\mathbf{t}$ and $\mathbf{s}$ are 1-types consistent with $\mathbf{T}$ and simultaneously omissible in a model of $\mathbf{T}$ is $\mathbf{\Sigma}^1_2$-complete. Type $\mathbf{s}$ will be $\mathbf{s}_0$ as defined in §2.2.2.

Let $\mathcal{T}^2$ denote the space of all subtrees of $\omega^{<\omega} \times \omega^{<\omega}$. For $R \in \mathcal{T}^2$ and $x \in \omega^\omega$ let

$$R_x = \{s \in \omega^{<\omega} : (s, x \restriction |s|) \in R\}$$

be the projection of R. Then the subspace of all $R \in \mathcal{T}^2$ such that for some x the tree R_x is well-founded is a complete $\mathbf{\Sigma}^1_2$ set (see [24]).

The language L_3 is a three-sorted language with sorts D_1, D_2 and D_3 which includes L_2 from Theorem 2.5 and function symbol h as interpreted in §2.2.2. The intended interpretation of D_3 is $\mathcal{T}^2$. In addition to (1)–(4) in L_3 have the following.

(10) constant symbol c of sort D_1.
(11) constant symbols R_n, for $n \in \omega$, for all finite-width subtrees of $\omega^{<\omega} \times \omega^{<\omega}$ all of whose branches have eventually zero value.
(12) Predicate Elm is extended to sort $D_1 \times D_1 \times D_3$.

Theory $\mathbf{T}_3$ is the theory of the L_3-model $\mathcal{N}_3$ described as follows. Its universe is equal to $\omega^{<\omega} \sqcup \omega^\omega \sqcup \mathcal{T} \sqcup \mathcal{T}^2$ and it includes model $\mathcal{N}_1$ as defined in the proof of Theorem 2.5.

Metric d on $\mathcal{T}^2$ defined as

$$d(R, S) = \inf\{1/k : R \cap (k^{\leq k})^2 = S \cap (k^{\leq k})^2\}$$

turns $\mathcal{T}^2$ into a compact metric space.

Now define Elm on $\omega^{<\omega} \times \omega^{<\omega} \times \mathcal{T}^2$ via

$$\mathrm{Elm}(s, t, R) = \begin{cases} 0, & \text{if } (s, t) \in R \\ 1/(\max(\ell(s), \ell(t)), & \text{if } (s, t) \notin R \end{cases}$$

(the case when $|s| \neq |t|$ seems ok). The extended predicate Elm is Lipshitz because $d(S,T) \leq 1/k$, $d(s_1,t_1) \leq 1/k$ and $d(s_2,t_2) \leq 1/k$ implies that $\mathrm{Elm}(s_1,s_2,S) = 1/m$ iff $\mathrm{Elm}(t_1,t_2,T) = 1/m$ for all $m \leq k$. By continuity we have $\mathrm{Elm}(x,y,S) = 0$ for all $x, y \in \omega^\omega$ and all $S \in \mathcal{T}$.

Let $\mathbf{T}_3$ be the theory of $\mathcal{N}_3$. Note that $\mathbf{T}_3$ is not complete only because it provides no information on interpretation of constant c. For a tree $R \in \mathcal{T}^2$ type $\mathbf{t}^R(x)$ consists of the following conditions.

(10) $\mathrm{Elm}(f_k(x), f_k(c), R) = 0$ for all $k \in \omega$.
(11) $|d(f_k(x), x) - 1/(k+1)| = 0$.

Condition (11) assures that $x \in \omega^\omega$ and condition (10) assures that x is a branch of R_c.

Again, the map $\mathcal{T}^2 \ni R \mapsto \mathbf{t}^R \in S_2^-(\mathbf{T}_3)$ (see §1.2.5) is clearly continuous. We claim that $\mathbf{t}^R$ and $\mathbf{s}$ are simultaneously omissible if and only if there exists real $a \in \omega^\omega$ such that R_a is well-founded. If there is such a real, then the model of $\mathbf{T}_3$ obtained by interpreting c as a omits both $\mathbf{t}^R$ and $\mathbf{s}$. Now assume there is no such real and let N be a model of $\mathbf{T}_3$ in which $\mathbf{s}$ is omitted. Then by the choice of $\mathbf{s} = \mathbf{s}_0$ (§2.2.2) the reduct of N to $L_{\mathcal{N}}$ is isometrically isomorphic to $\mathcal{N}$. If a is the interpretation of c in N, then $c \in \omega^\omega$. Therefore the tree R_c is ill-founded, and $\mathbf{t}^R$ is realized. □

3. Forcing and omitting types

Our study of generic models is motivated by potential applications to operator algebras (see [14], [15], §4 and §6). Results related to our results were obtained in [4] and [11], similarly inspired by Keisler's classic [26]. Both of these papers study a version of Keisler's forcing adapted to the infinitary version of the logic of metric structures. In the classical situation a type (complete or partial) is omitted in the generic model if and only if it is omissible. In the context of logic of metric structures this statement remains true for complete types (by [3] or Proposition 3.1) but not for partial types (see Corollary 5.4). There are several good sources for model-theoretic forcing in the context of logic of metric structures ([9], [11], [4], [21, Appendix A]). Since the present paper is a companion to [16] meant to be self-contained and accessible to non-logicians, we include some of the basics for the reader's convenience. The forcing construction described below is also known as the *Henkin construction*.

3.0.1. *Syntax.* A natural definition of the provability relation $\vdash$ was given in [6], where a completeness theorem was proven. For a theory $\mathbf{T}$, new constants $\bar{d}$ and conditions $\phi(\bar{d}) < \varepsilon$ and $\psi(\bar{d}) < \delta$ of the same sort we have

$$\mathbf{T} \cup \{\phi(\bar{x}) < \varepsilon\} \vdash \psi(\bar{x}) < \delta$$

if and only if for every model M of $\mathbf{T}$ and every n-tuple $\bar{a}$ of elements of M of the same type as $\bar{d}$ one has that $\phi(\bar{a})^M < \varepsilon$ implies $\psi(\bar{a})^M < \delta$.

3.0.2. *Forcing notion.* Fix a (not necessarily complete) theory $\mathbf{T}$ in language L and a set of L-formulas Σ with the following closure properties.

(Σ1) Σ includes all quantifier free formulas,
(Σ2) Σ is closed under taking subformulas and under the change of variables,
(Σ3) if $k \in \omega$ and $\phi_i(\bar{x})$, for $0 \leq i < k$, are in Σ and $f \colon \mathbb{R}^k \to [0, \infty)$ is a continuous function then $f(\phi_0(\bar{x}), \dots, \psi_{k-1}(\bar{x}))$ is in Σ.

Two most interesting cases are when Σ is the set of all quantifier-free formulas and when Σ is the set of all formulas. Forcing notion $\mathbb{P}_{\mathbf{T},\Sigma}$ defined below is similar to the ones defined and discussed in detail in [4], [11] and [15], but we sketch definitions for the reader's convenience.

We postulate a simplifying assumption that L has a single sort with a single domain of quantification. If this is not the case, forcing can be modified by adding an infinite supply of constants (like d_j, for $j \in \omega$ below) for every domain of quantification. For example, in case of C*-algebras, tracial von Neumann algebras, or other Banach algebras one adds constants d_j^n, for $j \in \omega$, for elements of the n-ball for every $n \geq 1$ (see the axiomatizations of C*-algebras and tracial von Neumann algebras in [17]). We omit the straightforward details.

Let d_j, for $j \in \omega$, be a sequence of new constant symbols and let $L^+ = L \cup \{d_j : j \in \omega\}$. If F is an n-tuple of natural numbers, then $\bar{d}_F$ denotes the n-tuple $(d_i : i \in F)$. Conditions in $\mathbb{P}_{\mathbf{T},\Sigma}$ correspond to open conditions as defined in §1.1.1. They are triples

$$p = (\psi^p, F^p, \varepsilon^p)$$

(we shall write (ψ, F, ε) whenever p is clear from the context) where ψ is an n-ary formula in Σ, F is an n-tuple of natural numbers, $\varepsilon > 0$, and $\psi(\bar{d}_F) < \varepsilon$ is a condition consistent with $\mathbf{T}$. Note that $\psi(\bar{d}_F)$ is an L^+-sentence. We shall write $\bar{d}^p$ instead of $\bar{d}_{F^p}$. The ordering on $\mathbb{P}_{\mathbf{T},\Sigma}$ is defined by

$$p \geq q \qquad \text{if} \qquad F^p \subseteq F^q \text{ and } \mathbf{T} \cup \{\psi^q(\bar{d}^q) < \varepsilon^q\} \vdash \psi^p(\bar{d}^p) < \varepsilon^p.$$

If $p \geq q$ then we say that q *extends* p or that q is *stronger than* p. By Lemma 1.1 every condition is equivalent to some p such that $\varepsilon^p = 1$. Conditions p and q are *incompatible*, $p \not\perp q$, if no condition extends both p and q.

In the terminology of [23] and [15], if Σ consists of all quantifier-free formulas then $\mathbb{P}_{\mathbf{T},\Sigma}$ is the *Robinson forcing*, also known as *finite forcing.* If Σ consists of all formulas, then $\mathbb{P}_{\mathbf{T},\Sigma}$ is the *infinite forcing.* In the latter case, we shall write $\mathbb{P}_{\mathbf{T}}$ for $\mathbb{P}_{\mathbf{T},\Sigma}$.

We shall identify condition $p = (\psi, F, \varepsilon)$ in $\mathbb{P}_{\mathbf{T},\Sigma}$ with the expression $\psi(\bar{d}^p) < \varepsilon$ and use notations $T + p$ and $T \cup \{\psi(\bar{d}^p) < \varepsilon\}$ interchangeably.

A recap of the standard forcing terminology ([27], [29]) is in order. Subset G of $\mathbb{P}_{\mathbf{T},\Sigma}$ is a *filter* if every two elements of G have a common extension in G. A subset $\mathbf{D}$ of $\mathbb{P}_{\mathbf{T},\Sigma}$ is *dense* if every $q \in \mathbb{P}_{\mathbf{T},\Sigma}$ has an extension in $\mathbf{D}$. It is *dense below* some $p \in \mathbb{P}_{\mathbf{T},\Sigma}$ if every $q \leq p$ has an extension in $\mathbf{D}$. If $\mathbf{F}$ is a family of dense subsets of $\mathbb{P}_{\mathbf{T},\Sigma}$ then a filter G is $\mathbf{F}$-*generic* if $G \cap \mathbf{D} \neq \emptyset$

for all $\mathbf{D} \in \mathbf{F}$. A straightforward diagonalization argument shows that one can always find an $\mathbf{F}$-generic filter if $\mathbf{F}$ is countable.

For a formula $\phi(\bar{d}_F)$ in Σ and $\varepsilon > 0$ the set

$$\mathbf{D}_{\phi(\bar{d}_F),\varepsilon} = \{p \in \mathbb{P}_{\mathbf{T},\Sigma} : (\exists r \in \mathbb{R})\mathbf{T} + p \vdash |\phi(\bar{d}_F) - r| < \varepsilon\}.$$

is dense in $\mathbb{P}_{\mathbf{T},\Sigma}$ since every p can be extended to a condition of the form

$$(\max(\psi^p, |\phi(\bar{d}_F) - r|), F^p, \min(\varepsilon^p, \varepsilon))$$

for some r. The fact that $\mathbf{D}_{\phi(\bar{d}_F),\varepsilon}$ is dense in $\mathbb{P}_{\mathbf{T},\Sigma}$ for all ϕ and $\varepsilon > 0$ even if $\phi \notin \Sigma$ is also true (it follows from Cohen's forcing lemma, see [27] or [29]), but we shall not need it. If L is separable then G meets all dense sets of the form $\mathbf{D}_{\phi(\bar{d}_F),\varepsilon}$ if and only if it meets all dense sets of the form $\mathbf{D}_{\phi_j(\bar{d}_F),1/k}$ where $\phi_j(\bar{x})$, for $j \in \omega$, is a set of formulas dense in d_∞ metric and $k \in \omega$.

A formula is an $\forall\exists$-*formula* if it is of the form

$$\sup_{\bar{x}} \inf_{\bar{y}} \psi(\bar{x}, \bar{y}, \bar{z})$$

where ψ is quantifier-free. Theory $\mathbf{T}$ is $\forall\exists$-*axiomatizable* if it is axiomatizable by a set of $\forall\exists$-sentences. Typical algebraic theories are $\forall\exists$-axiomatizable when expressed in the 'right' language. In the following proposition we do not need to assume that $\mathbf{T}$ is complete.

Proposition 3.1. *Assume $\mathbf{T}$ is an L-theory and Σ is a set of L-formulas satisfying ($\Sigma 1$)–($\Sigma 3$). Then there is a family $\mathbf{F}$ of dense subsets of $\mathbb{P}_{\mathbf{T},\Sigma}$ such that to an $\mathbf{F}$-generic filter G one can associate L^+-structure M_G satisfying the following.*

1. *M_G has the interpretations of $\{d_j : j \in \omega\}$ as a dense subset.*
2. *Every condition $p \in G$ is satisfied in M_G.*
3. *If Σ is the set of all L-formulas then $M_G \models \mathbf{T}$.*
4. *If Σ is the set of all quantifier-free L-formulas and $\mathbf{T}$ is $\forall\exists$-axiomatizable then $M_G \models \mathbf{T}$.*

If L is separable, then $\mathbf{F}$ can be chosen to be countable.

Proof. (1) In addition to meeting all $\mathbf{D}_{\phi(\bar{d}_F),\varepsilon}$ as defined above we need G to meet other dense sets such as

$$\mathbf{E}_{\phi(\bar{d}_F,x)} = \{p \in \mathbb{P}_{\mathbf{T},\Sigma} : \mathbf{T} + p \vdash \inf_x \phi(\bar{d}, x) \geq r \\ \text{or } (\exists j)\mathbf{T} + p \vdash \phi(\bar{d}_F, d_j) < r\}$$

as well as dense sets $\mathbf{D}_{d,i,j,k}$, $\mathbf{D}_{p,j}$, $\mathbf{E}_{\phi(\bar{d}_F,x),k}$ defined below. It is straightforward to check that each of these sets is dense and that separability of L implies there are only countably many relevant dense sets of this form (details are very similar to the ones given in [9], [11], [4], [21, Appendix A]).

We shall construct a countable metric model with the universe $M_G^0 = \{\mathbf{d}_j : j \in \omega\}$ such that for every $p \in \mathbb{P}_{\mathbf{T}}$ we have

$$p \Vdash \psi^p(\mathbf{d}^p) < \varepsilon^p.$$

Model M_G will be metric completion of this countable model.

On M^0_G define metric d as follows.

$$d(\mathbf{d}_i, \mathbf{d}_j) = r \Leftrightarrow \{p \in \mathbb{P}_{\mathbf{T}} : \mathbf{T} + p \vdash |d(d_i, d_j) - r| < \varepsilon\} \in G \text{ for all } \varepsilon > 0.$$

Since for all i, j and $k > 0$ the set

$$\mathbf{D}_{d,i,j,k} = \{p \in \mathbb{P}_{\mathbf{T},\Sigma} : (\exists r)\mathbf{T} + p \vdash |d(d_i, d_j) - r| < 1/k\}$$

is dense, if G is a sufficiently generic filter, then this defines a metric on M^0_G. For every k-ary predicate symbol P in L one defines interpretation of P as a function from $(M^0_G)^k$ to $\mathbb{R}$ in an analogous manner. The universe of M is the metric completion of M^0_G. For every function symbol f in the language of $\mathbf{T}$ one can now define an interpretation of f as a function from $(M^0_G)^k$ (where k is the arity of f) into M. All predicates and functions obtained in this way are uniformly continuous since the uniform continuity modulus is built into the language, and we can therefore continuously extend them to predicates and functions on M.

(2) is true by the construction of M_G.

(3) Assume Σ consists of all L-formulas, hence $\mathbb{P}_{\mathbf{T},\Sigma}$ is $\mathbb{P}_{\mathbf{T}}$. We claim that $p \Vdash \psi^p(\mathbf{d}^p) < \varepsilon^p$ for every $p \in \mathbb{P}_{\mathbf{T}}$. This is proved by induction on the complexity of formula ψ. The atomic case is immediate from the definition. Assume that the claim is true for all proper subformulas of ψ. If ψ is formed by applying a continuous function to other formulas, the claim is immediate (details are similar to those provided in [4] or [15]).

If p is a condition such that ψ^p is of the form $\inf_x \phi(x)$ for some ϕ, then the set

$$\mathbf{D}_{p,1} = \{q : \mathbf{T} + q \vdash \phi(d_j) < \varepsilon^p \text{ for some } j \in F^q \text{ or } q \not\perp p\}$$

is dense, and therefore it is forced that M^0_G contains a witness for the condition $\psi^p < \varepsilon^p$.

Now assume ψ is of the form $\sup_x \phi(x)$ for some ϕ. Then for every condition q extending p and every $j \in F^q$ we have that $\mathbf{T} + q \vdash \phi(d_j) < \varepsilon^p$. Moreover, the set

$$\mathbf{D}_{p,2} = \{q : \psi^q < \varepsilon^q \vdash \psi^p < \varepsilon^p - \delta \text{ for some } \delta > 0 \text{ or } q \not\perp p\}$$

is dense in $\mathbb{P}_{\mathbf{T}}$. Therefore $\mathbb{P}_{\mathbf{T}}$ forces that there exists $\delta > 0$ such that for all j the generic theory proves that $\phi(d_j) < \varepsilon^p - \delta$. By the continuity of ϕ^p in M_G, the conclusion follows.

This concludes proof that the generic model M_G is a model of $\mathbb{P}_{\mathbf{T}}$.

(4) Assume Σ includes all quantifier-free formulas. The argument from the proof of (3) shows that for every quantifier-free formula $\phi(\bar{x}, \bar{y})$ such that $\sup_{\bar{x}} \inf_{\bar{y}} \phi(\bar{x}, \bar{y}) = 0$ is in $\mathbf{T}$ for every F and $k > 0$ the set

$$\mathbf{E}_{\phi(\bar{d}_F, x),k} = \{q \in \mathbb{P}_{\mathbf{T},\Sigma} : (\exists j)\mathbf{T} + q \vdash \phi(\bar{d}_F, d_j) < 1/k\}$$

is dense. If G intersects $\mathbf{E}_{\phi(\bar{d}_F,x),k}$ for a d_∞-dense set of ϕ then M_G satisfies all $\forall\exists$-consequences of $\mathbf{T}$. Therefore (4) follows. □

A notable (and well-known to logicians) consequence of Proposition 3.1 is that if language L is separable then the generic set is required to meet only fixed countable family of dense sets. Therefore models of $\mathbf{T}$ as required can be constructed by a straightforward recursive construction with no need to pass to the generic extension. This remark applies to all other theorems about generic models in present section and in §4.

Type $\mathbf{t}(\bar{x})$ is *non-principal* (cf. §1.3.1) if there exists $\varepsilon > 0$ such that its metric ε-neighbourhood (§1.2.4) is nowhere dense in the logic topology. This is equivalent to stating that for every condition $p(\bar{x})$ there exists a stronger condition $q(\bar{x})$ such that in every model M of $\mathbf{T}$, every tuple $\bar{a}$ in M of the appropriate sort satisfies (with $\mathbf{t}(M) = \{\bar{b} \in M : M \models \mathbf{t}(\bar{b})\}$) $\operatorname{dist}(\bar{a}, \mathbf{t}(M)) \geq \varepsilon$. This also applies to incomplete types, when identified with closed subsets of the space of complete types. If Σ is a set of L-formulas satisfying (Σ1)–(Σ3) we say that a type $\mathbf{t}(\bar{x})$ is Σ-*non-principal* if for every condition $p(\bar{x})$ in $\mathbb{P}_{\mathbf{T},\Sigma}$ there exists a stronger condition $q(\bar{x})$ in $\mathbb{P}_{\mathbf{T},\Sigma}$ such that in every model M of $\mathbf{T}$, every tuple $\bar{a}$ in M of the appropriate sort satisfies $\operatorname{dist}(\bar{a}, \mathbf{t}(M)) \geq \varepsilon$.

Proof of the following is analogous to the proof of Proposition 3.1.

Proposition 3.2. *Assume* $\mathbf{T}$ *is a complete L-theory and* Σ *is a set of L-formulas satisfying* (Σ1)–(Σ3). *If* $\mathbf{t}$ *is a Σ-non-principal type then there is a family* $\mathbf{F_t}$ *of dense subsets of* $\mathbb{P}_{\mathbf{T}}$ *such that if G is* $\mathbf{F_t}$*-generic then* M_G *omits* $\mathbf{t}$. *If L is countable then we can choose* $\mathbf{F_t}$ *to be countable.* □

Proposition 3.2 implies that a complete type over a complete theory $\mathbf{T}$ is omissible if and only if $\mathbb{P}_{\mathbf{T}}$ forces that M_G omits it ([3, §12] or [22, Lecture 4]).

3.1. **Omitting types in generic model M_G.** Recall that a model M of a complete theory $\mathbf{T}$ is *atomic* if the set of realizations of principal types is dense in M (see [3, p. 79]). This is equivalent to every element of M having a principal type. Recall that $\mathbb{P}_{\mathbf{T}}$ denotes the 'infinite' forcing of the form $\mathbb{P}_{\mathbf{T},\Sigma}$, the case when Σ is the set of all formulas of the language of $\mathbf{T}$.

Lemma 3.3. *If* $\mathbf{T}$ *is a complete theory in a separable language that has an atomic model N then* $\mathbb{P}_{\mathbf{T}}$ *forces that* $M_G \cong N$.

Proof. It suffices to prove that type $\mathbf{t}$ it is omissible in a model of $\mathbf{T}$ if and only if $\mathbb{P}_{\mathbf{T}}$ forces that M_G omits $\mathbf{t}$. We claim that $\mathbb{P}_{\mathbf{T}}$ forces that for every $\delta > 0$ and every n the tuple $\mathbf{d}_1, \dots, \mathbf{d}_n$ is forced to be within δ of an n-tuple realizing a principal type (N^n is equipped with metric $d(\bar{a}, \bar{b}) = \max_{i<n} d(a_i, b_i)$).

Fix δ and F and p such that $F^p \supseteq \{i : i \leq n\}$. Then $N \models \inf_{\bar{x}} \psi^p(\bar{x}) < \varepsilon^p$, and therefore $N \models \psi^p(\bar{a}) < \varepsilon^p$ for some n-tuple $\bar{a}$ in N. Since $\bar{a}$ realizes a principal type $\mathbf{t}$ in N, we can find a formula ϕ such that in every model M of $\mathbf{T}$ for all $\bar{x}$ in M we have that $\phi(\bar{x}) < \delta$ implies there is an n-tuple $\bar{c}$ in M satisfying $\mathbf{t}$ such that $d(\bar{c}, \bar{x}) < \delta$. Then the condition $\max(\phi, \psi^p) < \min(\varepsilon^p, \delta)$ extends p and forces that $\mathbf{d}^F$ is within δ of a realization of $\mathbf{t}$ in M_G.

Therefore it is forced that a dense subset of n-tuples in M_G realize a principal type. Since limit of principal types is principal, M_G is forced to be atomic. □

Lemma 3.3 can be recast as the assertion that if $\mathbf{T}$ has an atomic model then every omissible type is forced to be omitted in the generic model. The assumption that $\mathbf{T}$ has an atomic model cannot be dropped from this assertion (Corollary 5.4).

3.2. **Forcing with 'certifying structures'.** Let $\mathbf{T}$ be a not necessarily complete theory, let Σ be a set of formulas in the language of $\mathbf{T}$ satisfying closure properties (1)–(3) as in §3.0.2 and let $\mathfrak{M}$ be some nonempty set of models of $\mathbf{T}$. Forcing $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ is defined as follows. Conditions are triples

$$p = (\psi^p, F^p, \varepsilon^p)$$

(we shall write (ψ, F, ε) whenever p is clear from the context) where ψ is an n-ary formula, F is an n-tuple of natural numbers, $\varepsilon > 0$, and $\psi(\bar{d}_F) < \varepsilon$ is a condition satisfied in some model from $\mathfrak{M}$. We shall write $\bar{d}^p$ instead of $\bar{d}_{F^p}$. Preorder $\leq$ on $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ is defined via

> $p \geq q$ if $F^p \subseteq F^q$ and for every $M \in \mathfrak{M}$ and $\bar{a}$ in M of the appropriate sort, if $\psi^q(\bar{a})^M < \varepsilon^q$ then $\psi^p(\bar{a}) < \varepsilon^p$.

If $\mathbf{T}$ is a complete theory, then every condition consistent with $\mathbf{T}$ is certified in every model of $\mathbf{T}$ and $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ is isomorphic to $\mathbb{P}_{\mathbf{T},\Sigma}$ if $\mathfrak{M}$ is any nonempty set of models of $\mathbf{T}$.

Proofs of the following proposition is analogous to the proof of Proposition 3.1 and is therefore omitted.

Proposition 3.4. *Assume $\mathbf{T}$ is a (not necessarily complete) L-theory and that either Σ consists of all L-formulas or that $\mathbf{T}$ is $\forall\exists$-axiomatizable and Σ includes all quantifier-free formulas. Then forcing $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ generically adds a model M_G of $\mathbf{T}$ which has the interpretations of $\{d_j : j \in \omega\}$ as a dense subset.* □

3.3. **Set-theoretic analysis.** Subsections §3.3 and §3.4 require acquaintance with forcing (e.g., [27] or [29]). Readers not interested in forcing may skip ahead to §4. Recall that the poset for adding a Cohen real has nonempty rational intervals in $[0,1]$ as its conditions, ordered by the inclusion (see e.g., [27]).

Lemma 3.5. *If $\mathbf{T}$ is theory in a separable language then $\mathbb{P}_{\mathbf{T},\Sigma}$ has a countable dense subset, and is therefore equivalent to the standard forcing for adding a Cohen real.*

Proof. This is a standard continuous functional calculus trick (§1.1.1). By fixing a countable uniformly dense family of 'propositional connectives' of the form $f\colon [0,1]^n \to [0,1]$ and considering only formulas built from the language of $\mathbf{T}$ and constants d_j, for $j \in \omega$, using these connectives, one obtains a countable set of formulas $\mathbf{D}$ dense in the uniform metric

$$d_\infty(\phi, \psi) = \sup_{M,\bar{a}}(\phi(\bar{a}) - \psi(\bar{a}))^M$$

where M ranges over all models of $\mathbf{T}$ and $\bar{a}$ ranges over all tuples in M of the appropriate sort. For each $m \in \omega$ let $f_m(t) = \max(t - 1/m, 0)$. Consider the set $\mathbf{C}$ of all conditions of the form $f_m(\phi(\bar{d})) < 1/n$, for m, n in ω and $\phi \in \mathbf{D}$. We claim that this countable set is dense in $\mathbb{P}_{\mathbf{T},\Sigma}$. Take a condition $\psi(\bar{d}) < \varepsilon$. Fix $n > 1/\varepsilon$ and let $\phi(\bar{x}) \in \mathbf{D}$ be such that $d_\infty(\phi, \psi) < 1/(2n)$. Then $f_{2n}(\phi(\bar{d})) < 1/(2n)$ is a condition in $\mathbf{C}$ stronger than $\psi(\bar{d}) < \varepsilon$. □

Proof of the following lemma is analogous to the proof of Lemma 3.5.

Lemma 3.6. *If $\mathbf{T}$ is a theory in a separable language then $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ has a countable dense subset, and is therefore equivalent to the Cohen forcing.* □

Recall that $\mathrm{cov}(\mathcal{M})$ denotes the minimal number of sets of first category required to cover the real line ([1]).

Corollary 3.7. *If $\mathbf{T}$ is a complete theory in a separable language, $\kappa < \mathrm{cov}(\mathcal{M})$, and $\mathbf{t}_\gamma$ for $\gamma < \kappa$ is a set of complete non-principal types over $\mathbf{T}$, then $\mathbf{T}$ has a separable model that omits all $\mathbf{t}_\gamma$.*

Proof. By Proposition 3.1 and Proposition 3.2 $\mathbb{P}_{\mathbf{T}}$ forces that M_G is a model of $\mathbf{T}$ which omits each $\mathbf{t}_\gamma$. Pick a transitive model $\mathcal{N}$ of a large enough fragment of ZFC containing $\mathbf{T}$ and all $\mathbf{t}_\gamma$ for $\gamma < \kappa$ such that the cardinality of $\mathcal{N}$ is κ. Countable dense subset of $\mathbb{P}_{\mathbf{T}}$ defined in Lemma 3.5 is included in $\mathcal{N}$. Since $\kappa < \mathrm{cov}(\mathcal{M})$ and $\mathbb{P}_{\mathbf{T}}$ is equivalent to the Cohen forcing ([1]) we can choose a filter $G \subseteq \mathbb{P}_{\mathbf{T}}$ that meets all κ dense subsets of $\mathbb{P}_{\mathbf{T}}$ that belong to $\mathcal{N}$. We claim that M_G omits each $\mathbf{t}_\gamma$. This is because the assertion that M_G omits a fixed type is $\mathbf{\Pi}^1_1$ and therefore absolute between $\mathcal{N}[G]$ and the universe. □

3.4. **Strong homogeneity of $\mathbb{P}_{\mathbf{T},\Sigma}$.** Cohen forcing is *homogeneous* in the sense that for any two conditions p and q there exists an automorphism Φ such that $\Phi(p)$ is compatible with q. Therefore $\mathbb{P}_{\mathbf{T},\Sigma}$ is homogeneous by Lemma 3.5. We shall prove a refinement of this fact in case when $\mathbf{T}$ is complete, showing that automorphism Φ can be chosen to preserve the relevant logical structure. Recall that S_∞ denotes the group of permutations of ω. To a permutation $h \in S_\infty$ we associate an automorphism α_h of $\mathbb{P}_{\mathbf{T},\Sigma}$ which sends d_j to $d_{h(j)}$ for all $j \in \omega$. More explicitly,

$$\alpha_h((\psi(\bar{x}), F, \varepsilon)) := (\psi(\bar{x}), h(F), \varepsilon).$$

Lemma 3.8. *Assume $\mathbf{T}$ is a complete L-theory and Σ is a set of L-formulas satisfying (Σ1)–(Σ3). For any two conditions p_1 and p_2 in $\mathbb{P}_{\mathbf{T},\Sigma}$ there is $h \in S_\infty$ such that p_1 and $\alpha_h(p_2)$ are compatible.*

Proof. Let p_j be $(\psi_j, F_j, \varepsilon_j)$ for $j = 1$ and $j = 2$. We shall write $d(j)$ and $x(j)$ for d_{F_j} and x_{F_j}, respectively. By Lemma 1.1 we may assume $\varepsilon_1 = \varepsilon_2 = \varepsilon$. Since $\mathbf{T}$ is complete we have $\mathbf{T} \vdash \inf_{\bar{x}(j)} \psi_j(\bar{x}(j)) < \varepsilon$ for $j = 1, 2$ and therefore

$$\mathbf{T} \vdash \max(\inf_{\bar{x}(1)} \psi_1(\bar{x}(1)), \inf_{\bar{x}(2)} \psi_2(\bar{x}(2))) < \varepsilon.$$

Let h be such that $h[F_2]$ is disjoint from F_1. Then

$$q = (\max(\psi_1(\bar{d}(1)), \psi_2(\bar{d}_{h(F_2)}), F_1 \cup h(F_2), \varepsilon)$$

is a condition in $\mathbb{P}_{\mathbf{T},\Sigma}$ which extends both p_1 and $h(p_2)$. □

In the following M_G^0 denotes the countable dense submodel of M_G as defined in the proof of Proposition 3.1.

Corollary 3.9. *Assume* $\mathbf{T}$ *is a complete* L*-theory and* Σ *is a set of* L*-formulas satisfying (*$\Sigma 1$*)–(*$\Sigma 3$*). If* $\Theta(\bar{x}, \bar{y})$ *is a statement of ZFC with parameters in the ground model, then* $\mathbb{P}_{\mathbf{T},\Sigma}$ *either forces* $\Theta(M_G, M_G^0)$ *or it forces* $\neg\Theta(M_G, M_G^0)$*.*

In particular, for every type $\mathbf{t}$ $\mathbb{P}_{\mathbf{T},\Sigma}$ *either forces that* M_G *realizes* $\mathbf{t}$ *or it forces that* M_G *omits* $\mathbf{t}$*.*

Proof. Fix a condition p which decides $\Theta(M_G, M_G^0)$. If q is any other condition then by Lemma 3.8 there exists an $h \in S_\infty$ such that $\alpha_h(p)$ is compatible with q. But α_h is an automorphism of $\mathbb{P}$ that sends M_G^0 to itself and M_G to itself, and therefore $\alpha_h(p)$ forces $\Theta(M_G, M_G^0)$ if and only if p does and $\alpha_h(p)$ forces $\neg\Theta(M_G, M_G^0)$ if and only if p does. This implies that every condition in $\mathbb{P}$ decides $\Theta(M_G, M_G^0)$ the same way that p does. □

Lemma 3.10. *If* $\mathbf{T}$ *is a theory in a separable language then the set of all types forced by* $\mathbb{P}_{\mathbf{T}}$ *to be omitted in the generic model* M_G *is a* $\mathbf{\Pi}^1_1$*-set.*

Moreover, for every type $\mathbf{t}$ *we have that* $\mathbb{P}_{\mathbf{T}}$ *either forces* $\mathbf{t}$ *is realized in* M_G *or it forces that* $\mathbf{t}$ *is omitted in* M_G*.*

Proof. The set of names $\dot{h}$ for a function from $\omega \to \omega$ can be identified with a Borel subset of $\mathbb{P}_0 \times \omega^2$, where $\mathbb{P}_0$ is a fixed countable dense subset of $\mathbb{P}_{\mathbf{T}}$ as in Lemma 3.5.

By Lemma 1.3 type $\mathbf{t}$ is forced to be omitted if and only if $\mathbf{t}_\omega$ (see §1.1.3) is forced to be omitted by every subsequence of the generic sequence $\{d_j(G) : j \in \omega\}$. This is equivalent to saying that for every name $\dot{h}$ for a function from ω to ω, for every $p \in \mathbb{P}_{\mathbf{T}}$ there exists $q \leq p$ and n such that the following holds.

> q decides $\dot{h}(i)$ for $i \leq n$ and this n-tuple is not an initial segment of a sequence satisfying $\mathbf{t}_\omega$.

The latter condition is Borel. Since all quantifiers, except one on $\dot{h}$, range over a countable set this set is $\mathbf{\Pi}^1_1$.

The last sentence is an immediate consequence of Corolary 3.9. □

4. Uniform sequences of types

Separable language L is fixed throughout this section. For $m \in \omega$ a sequence $\mathbf{t}_n$ of m-ary types is *uniform* if there are m-ary formulas $\phi_i(\bar{x})$ for $i \in \omega$ such that

(1) $\mathbf{t}_n(\bar{x}) = \{\phi_i(\bar{x}) \geq 2^{-n} : i \in \omega\}$ for every n, and
(2) all ϕ_i have the same modulus of uniform continuity.

If $\mathbf{t}_n$ and ϕ_i are as above, then

$$\psi(\bar{x}) := \inf_{i \in \omega} \phi_i(\bar{x})$$

is an $L_{\omega_1,\omega}$ formula (see [4]) whose interpretation is uniformly continuous in every L-structure A and $\sup_{\bar{x}} \psi(\bar{x})^A = 0$ if and only if A omits all $\mathbf{t}_n$ for $n \in \omega$. Theorem 4.2 below may be related to omitting types theorems of [4] stated in terms of $L_{\omega_1\omega}$.

For simplicity of notation in the following we consider a single-sorted language.

Lemma 4.1. *Assume $\mathbf{t}_n$, for $n \in \omega$, is a uniform sequence of m-types in L. If M is an L-structure then the set*

$$Z = \{\bar{a} \in M^m : \text{for all } n,\ \mathbf{t}_n \text{ is not realized by } \bar{a} \text{ in } M\}$$

is a closed subset of M^m. In particular, if D is a dense subset of M, then all $\mathbf{t}_n$ are omitted in M if and only if none of them is realized by any m-tuple of elements of D.

Proof. Set Z is closed as the zero set of the interpretation of the continuous infinitary formula $\inf_i \phi_i(\bar{a})$. The last sentence of the lemma follows immediately. □

Syntactic characterization of omissible uniform sequences of types given below is analogous to the syntactic characterization of complete omissible types given in [3]. As Itaï Ben Yaacov and Todor Tsankov pointed out, the set X of complete types extending a type in a uniform sequence of types is metrically open (§1.2.3) and therefore by a standard argument (see [3, §12] or [22, Lecture 4]) types in X are simultaneously omissible iff X is meager in the logic topology (§1.2.3). We spell out details of the proof below since we will need a similar argument in case when theory $\mathbf{T}$ is not necessarily complete in Theorem 4.3. Principal types were defined in §1.3.1.

Theorem 4.2. *Assume $\mathbf{T}$ is a complete theory in a separable language L. If for every $m \in \omega$ we have a uniform sequence of types*

$$\mathbf{t}_n^m = \{\phi_j^m(\bar{x}) \geq 2^{-n} : j \in \omega\}, \text{ for } n \in \omega,$$

then the following are equivalent.

(1) *None of the types $\mathbf{t}_n^m$, for $m, n \in \omega$, is principal.*
(2) *$\mathbf{T}$ has a model omitting all $\mathbf{t}_n^m$, for all $n \in \omega$.*

(3) *There are no* $\delta > 0$*,* $m \in \omega$*, and condition* $\psi(\bar{x}) < \varepsilon$ *such that* $\mathbf{T} \vdash \inf_{\bar{x}} \psi(\bar{x}) < \varepsilon$ *and* $\mathbf{T} + \psi(\bar{x}) < \varepsilon \vdash \phi_j^m(\bar{x}) \geq \delta$ *for every* $j \in \omega$*.*

Proof. To see that (1) implies (2) assume that none of the $\mathbf{t}_n^m$ for $n \in \omega$ is principal, By Proposition 3.2 and Lemma 4.1 one produces a model of $\mathbf{T}$ that omits all $\mathbf{t}_n^m$.

Now assume (1) fails and let m and n be such that $\mathbf{t}_n^m$ is principal. Since all ϕ_j^m have the same modulus of uniform continuity we can find $\delta > 0$ such that for all $\bar{x}$ and $\bar{y}$ of the appropriate sort $\max_i d(x_i, y_i) < \delta$ implies $|\phi_j^m(\bar{x}) - \phi_j^m(\bar{y})| \leq 2^{-n-1}$ for all j. Since $\mathbf{t}_n^m$ is principal, there is a condition $\psi(\bar{d}) < \varepsilon$ with the property that in a sufficiently saturated model M of $\mathbf{T}$

$$\{\bar{a} : \psi(\bar{a}^M) < \varepsilon\} \subseteq \{\bar{a} : \mathrm{dist}(\bar{a}, \mathbf{t}_n^m(M)) < \delta\}.$$

We therefore have $\mathbf{T} + \psi(\bar{x}) < \varepsilon \vdash \phi_j^m(\bar{x}) \geq 2^{-n} - 2^{-n-1} \geq 2^{-n-1}$ for all j and (3) fails.

Now assume (3) fails. Then we have condition $\psi(\bar{x}) < \varepsilon$ and $\delta > 0$ such that $\mathbf{T} \vdash \inf_{\bar{x}} \psi(\bar{x}) < \varepsilon$ and $\mathbf{T} + \psi(\bar{x}) < \varepsilon \vdash \phi_j^m(\bar{x}) \geq \delta$ for all j. If $2^{-n} < \delta$ then clearly every model of $\mathbf{T}$ realizes $\mathbf{t}_n^m$, hence (2) fails. □

We state an extension of Theorem 4.2 to the case when theory $\mathbf{T}$ is not necessarily complete.

Theorem 4.3. *Assume* $\mathbf{T}$ *is a not necessarily complete theory in a separable language. If for every* $m \in \omega$ *we have a uniform sequence of types*

$$\mathbf{t}_n^m = \{\phi_j^m(\bar{x}) \geq 2^{-n} : j \in \omega\}, \text{ for } n \in \omega,$$

then the following are equivalent.

(1) $\mathbf{T}$ *has a model omitting all* $\mathbf{t}_n^m$*, for all* m *and* n *in* ω*.*
(2) *There are no* $\delta > 0$*, finite* $F \subseteq \omega$ *and conditions* $\psi_m(\bar{x}) < \varepsilon$ *for* $m \in F$ *such that* $\mathbf{T} \vdash \inf_{\bar{x}} \min_{m \in F} \psi_m(\bar{x}) < \varepsilon$ *and* $\mathbf{T} + \psi_m(\bar{x}) < \varepsilon \vdash \phi_j^m(\bar{x}) \geq \delta$ *for every* $j \in \omega$*.*

Proof. By Theorem 4.2 it suffices to show that $\mathbf{T}$ satisfies (2) if and only if it can be extended to a complete theory that still satisfies (2). Only the direct implication requires a proof and the proof is analogous to the proof in the first order case. Let θ_k, for $k \in \omega$, enumerate a countable dense set of L-sentences. By Lemma 1.1 for a closed interval $V \subseteq \mathbb{R}$ and sentence θ condition $\theta \in V$ is equivalent to one of the form $\theta' = 0$ for some θ'.

Assume $\mathbf{T}$ satisfies (2). We shall find an increasing chain of theories $\mathbf{T} = \mathbf{T}_0 \subseteq \mathbf{T}_1 \subseteq \mathbf{T}_2 \subseteq \dots$ and closed intervals $U_{nk} \subseteq \mathbb{R}$ of diameter at most 2^{-n} for $k \leq n$ such that for all $k \leq n$ we have

$$\mathbf{T}_n \vdash \theta_k \in U_{nk}$$

and $\mathbf{T}_n$ still satisfies (2).

Assume that for some $k \leq n+1$ both $\mathbf{T}_n$ and U_{nl} for $l \leq k \leq n$ as required were chosen. Let $\mathcal{V}$ be a finite cover of U_{nk} by closed intervals of diameter $\leq 2^{-n+1}$. We claim that there exists $V \in \mathcal{V}$ such that the theory $\mathbf{T}_n \cup \{\theta_k \in V\}$ (identified with a theory by using the closed case of Lemma 1.1) still satisfies (2). Assume otherwise. Then by the 'open' case of Lemma 1.1, for every $V \in \mathcal{V}$ there are a finite set F_V, conditions of the form $\psi_{V,j}(\bar{x}) < 1$, as well as $k(V,j) \in \omega$, and $m(V,j) \in \omega$ (for $j \in F_V$) such that $\mathbf{T}_n \vdash \inf_{\bar{x}} \psi_{V,j}(\bar{x}) < 1$ and for every $V \in \mathcal{V}$ and every $j \in F_V$ we have

$$\mathbf{T}_n + \psi_{V,j}(\bar{x}) < 1 \vdash \phi_i^{m(V,j)}(\bar{x}) \geq 2^{-k(V,j)}$$

for all $i \in \omega$. We shall show that (2) is violated with $\delta = \min_{V \in \mathcal{V}, j \in F_V} 2^{-k(V,j)}$. Since $\mathcal{V}$ is finite, $\theta' = \min_{V \in \mathcal{V}} \operatorname{dist}(\theta_k, V)$ is an L-sentence and $\mathbf{T}_n \vdash \theta' = 0$. Therefore

$$\mathbf{T}_n \vdash \min_{v \in \mathcal{V}, j \in F_V} \psi_{V,j}(\bar{x}) < 1$$

and we have $\mathbf{T}_n + \psi_{V,j}(\bar{x}) < 1 \vdash \phi_i^{m(V,j)}(\bar{x}) \geq \delta$, contradicting our assumption that $\mathbf{T}_n$ satisfies (2).

We can therefore find $V \in \mathcal{V}$ such that adding condition $\theta_k \in V$ to $\mathbf{T}$ preserves (2). By repeating this successively for all $k \leq n$ we obtain $\mathbf{T}_{n+1}$ as required.

Once all $\mathbf{T}_n$ are constructed, theory $\mathbf{T}_\infty = \bigcup_n \mathbf{T}_n$ is a complete theory that satisfies (2), as required. □

Unlike Theorem 4.2, the second clause of Theorem 4.3 allows for the possibility that $\mathbf{T}$ has models that omit sequence $\mathbf{t}_n^m$, for $n \in \omega$, for every m separately, but no single model of $\mathbf{T}$ omits these types. Since $\mathbf{T}$ is not assumed to be complete this possibility cannot be ruled out.

4.1. **Uniform sequences of types and forcing.** A class $\mathfrak{M}$ of models is *uniformly definable by a sequence of types* if there is a set of sequences of uniform types $\langle \mathbf{t}_n^m : n \in \omega \rangle$, for $m \in I$ such that A is in $\mathfrak{M}$ if and only if it omits all of these types.

If Σ is the set of all formulas of the language of $\mathbf{T}$ then we denote $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ by $\mathbb{P}_{\mathbf{T},\mathfrak{M}}$.

Proposition 4.4. *Assume $\mathfrak{M}$ is a nonempty class of models of a theory $\mathbf{T}$. If $\mathbf{t}_n$, for $n \in \omega$, is a uniform sequence of types that are omitted in every model in $\mathfrak{M}$ then $\mathbb{P}_{\mathbf{T},\mathfrak{M}}$ forces that M_G omits all $\mathbf{t}_n$.*

Proof. Let D denote the set of interpretations of constants $\{d_j : j \in \omega\}$ in M_G. Assume that some condition p forces that a tuple $F \subseteq F^p$ realizes $\mathbf{t}_n$ for some $n \in \omega$. But there is $M \in \mathfrak{M}$ and a tuple $\bar{a}$ in M of the appropriate sort such that $M \models \psi^p(\bar{a}) < \varepsilon^p$. Since M omits $\mathbf{t}_n$, we can extend p to a condition that decides that F does not satisfy some condition in $\mathbf{t}_n$, contradicting our assumption on p. Proposition now folllows by Lemma 4.1. □

Theorem 4.2 and Proposition 4.4 are potentially useful because some of the most important properties of C*-algebras are uniformly definable by a sequence of types. This includes nuclearity, nuclear dimension, decomposition rank, and being TAF, AF or UHF ([10], [16]). These types are particularly simple and we include a straightforward technical sharpening of Proposition 4.4 with an eye to potential applications.

A uniform sequence of types $\mathbf{t}_n = \{\phi_j(\bar{x}) \geq 2^{-n} : j \in \omega\}$, for $n \in \omega$, is *universal* if every $\phi_j(\bar{x})$ is of the form $\inf_{\bar{y}} \psi(\bar{y}, \bar{x})$ for some quantifier-free formula $\psi(\bar{y}, \bar{x})$. (By Lemma 1.1 condition of the form $\inf_{\bar{y}} \psi(\bar{y}, \bar{x}) \geq 2^{-n}$ is equivalent to a condition of the form $\sup_{\bar{y}} \psi'(\bar{y}, \bar{x}) = 0$, and formulas on the left-hand side of this expression are commonly recognized as universal.) The following theorem probably follows from the results from [4] but we include it for the reader's convenience.

Proposition 4.5. *Assume $\mathfrak{M}$ is a nonempty class of models of a theory $\mathbf{T}$ and let Σ be the set of all quantifier-free formulas of the language of $\mathbf{T}$. If $\mathbf{t}_n$, for $n \in \omega$, is a uniform sequence of universal types that are omitted in every model in $\mathfrak{M}$ then $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ forces that M_G omits all $\mathbf{t}_n$.*

If $\mathbf{T}$ is $\forall\exists$-axiomatizable then $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ forces that the generic model satisfies $\mathbf{T}$.

Proof. Let $G \subseteq \mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$ be generic and let D be as in the proof of Proposition 4.4. Assume that some condition p forces that a tuple $F \subseteq F^p$ realizes $\mathbf{t}_n$ for some $n \in \omega$. But there is $M \in \mathfrak{M}$ and a tuple $\bar{a}$ in M of the appropriate sort such that $M \models \psi^p(\bar{a}) < \varepsilon^p$. Since M omits $\mathbf{t}_n$, there exists a condition $\inf_{\bar{y}} \psi(\bar{y}, \bar{x}) \geq 2^{-n}$ in $\mathbf{t}_n$ such that ψ is quantifier-free and $\inf_{\bar{y}} \psi(\bar{y}, \bar{a})^M < 2^{-n} - \varepsilon$ for some $\varepsilon > 0$. If F' is disjoint from F and of cardinality $|\bar{y}|$, by the open case of Lemma 1.1 we have that and $\psi(\bar{d}_{F'}, \bar{d}_F) < 2^{-n} - \varepsilon$ is equivalent to a condition in $\mathbb{P}_{\mathbf{T},\Sigma,\mathfrak{M}}$. Then M certifies that this condition is compatible with p, and it decides that d_F does not realize $\bar{t}_n$. Thus types are omitted by Lemma 4.1.

Since $\mathbf{T}$ is $\forall\exists$-axiomatizable, the generic model satisfies $\mathbf{T}$ by (4) of Proposition 3.1. □

If $\mathfrak{M}$ is a class of L-models then $M \in \mathfrak{M}$ is *existentially closed for* $\mathfrak{M}$ if whenever $N \in \mathfrak{M}$ is such that M is isomorphic to a submodel of N, $\bar{a} \in M$, and $\phi(\bar{x}, \bar{y})$ is a quantifier-free L-formula then

$$\inf_{\bar{y}} \phi(\bar{a}, \bar{y})^M = \inf_{\bar{y}} \phi(\bar{a}, \bar{y})^N.$$

Existentially closed C*-algebras and II_1 factors were studied in [21] and [15], respectively.

Corollary 4.6. *Assume $\mathbf{T}$ is an $\forall\exists$-axiomatizable theory in a separable language and $\mathbf{t}_n$, for $n \in \omega$, is a uniform sequence of universal types. If the class $\mathfrak{M}$ of all models of $\mathbf{T}$ that omit all $\mathbf{t}_n$, for $n \in \omega$, is nonempty then it contains a model that is existentially closed for $\mathfrak{M}$.*

Proof. By Proposition 4.5 the generic model is forced to omit all $\mathbf{t}_n$ and satisfy $\mathbf{T}$ and is therefore forced to belong to $\mathfrak{M}$. By [21, Lemma A.4] and [21, Lemma A.7] it is existentially closed. □

4.2. **Some Borel sets.** We record some complexity results related to uniform sequences of types (cf. Proposition 1.7 and Theorem 1).

By Lemma 4.1, M omits a uniform sequence of types if and only if each of the elements of its countable dense set omits this sequence. We therefore have the following consequence (see §6.1 for its applications).

Proposition 4.7. *Assume* $\mathbf{T}$ *is a theory in a separable language* L*. Then every class of separable models of* $\mathbf{T}$ *uniformly definable by a sequence of types forms a Borel subset of the space of models of* $\mathbf{T}$*.* □

For a separable language L compact metric topology on the space of L-theories and compact metric topology on the space of L-types were defined in §1.2.3. The space of sequences of L-types is considered with the product Borel structure. We have an analogue of Proposition 1.7.

Corollary 4.8. *For a separable language* L *and every* n *the following sets are Borel.*

(1) *The set of uniform sequences of* n*-types.*
(2) *The set of all pairs* $(\mathbf{T}, (\mathbf{t}_j : j \in \omega))$ *such that* $\mathbf{T}$ *is a complete theory and* $(\mathbf{t}_n : n \in \omega)$ *is a uniform sequence of types omissible in a model of* $\mathbf{T}$*.*
(3) *The set of all pairs such that* $\mathbf{T}$ *is a theory and* $(\mathbf{t}_j : j \in \omega)$ *is a uniform type realized in some model of* $\mathbf{T}$*.*

Proof. (1) is clear from the definition.

(2) and (3) follow from Theorem 4.2 and the proof of Proposition 1.7. □

5. Simultaneous omission of types

We prove Theorem 2 by constructing an example of a separable complete theory $\mathbf{T}$ and types $\mathbf{s}_n$, for $n \in \omega$, such that for every k there exists a model of $\mathbf{T}$ that omits all $\mathbf{s}_n$ for $n \leq k$ but no model of $\mathbf{T}$ simultaneously omits all $\mathbf{s}_n$ (where $\mathbf{s}_0$ is not to be confused with the type $\mathbf{s}_0$ defined in §2.2.2).

We shall prove that all other types $\mathbf{s}_n$, for $n > 0$, are simultaneously omissible in a single model M of $\mathbf{T}$. As a matter of fact, we shall first define M. Let $\operatorname{rank}(T)$ denote the rank of a well-founded tree T and let ρ_T denote the rank function on T. Hence $\operatorname{rank}(T) = \sup_{t \in T} \rho_T(t)$. We write $\operatorname{rank}(T) = \infty$ if T is ill-founded.

For a sequence T_i for $i \in \omega$ of trees we denote their disjoint sum by $\bigoplus_i T_i$. Thus $\operatorname{rank}(\bigoplus_i T_i) = \sup_i \operatorname{rank}(T_i)$. We write $\bigoplus_\omega T$ for $\bigoplus_i T_i$ if all T_i are equal to T.

If S and T are trees then $S^\frown T$ denotes the tree obtained by adding ω copies of T to every node of S. Formally, we identify S and T with trees of finite sequences from a large enough set and with $s^\frown t$ denoting the concatenation of s and t we let (assuming that S and $\bigoplus_\omega T$ are disjoint)

$$S^\frown T = \{s^\frown t : s \in S, t \in \bigoplus_\omega T\}$$

with the natural end-extension ordering. Then

$$\mathrm{rank}(S^\frown T) = \mathrm{rank}(S) + \mathrm{rank}(T)$$

because $\rho_{S^\frown T}(s) = \mathrm{rank}_{\mathbf{t}}(T)$ for every end-node s of S.

Let T_1 denote the tree of all strictly decreasing sequences of natural numbers:

$$T_1 = \{s \colon n \to \omega : n \in \omega, s(i) > s(i+1) \text{ for } 0 \leq i < n-1\}$$

ordered by the extension. Hence T_1 is a well-founded tree of rank ω.

Also let T_2 be the 'wider' version of T_1 defined as (a function $s \colon n \to \omega \times \omega$ is identified with a pair of functions $s_0 \colon n \to \omega$ and $s_1 \colon n \to \omega$):

$$T_2 = \{s \colon n \to \omega \times \omega : n \in \omega, s_0(i) > s_0(i+1) \text{ for } 0 \leq i < n-1\}.$$

This is also a well-founded tree of rank ω. For every node t in T_2 and every immediate successor s of t there are infinitely many immediate successors s' of t such that there is an automorphism of T_2 swapping s and s' (because there are infinitely many s' such that $s'_0 = s_0$).

5.0.1. *Language L_t and model M.* Consider language L_t in the logic of metric structures that includes the language $L_{\mathcal{N}}$ of Baire space as defined in §2.2 and has unary predicate symbols $P_{i,j}$ for i, j in ω. For

$$\vec{k} = (k_s \in \omega, \text{ for } s \in T_2),$$

define an L_t-structure $M = M(\vec{k})$ as follows. The underlying set is the tree T_2. Function symbols f_k for $k \in \omega$ are interpreted as in §2.2, by $f_k(a) = b$ if b is the unique element of the k-th level below a if there is such b or $f_k(a) = a$ otherwise.

Predicates $P_{i,j}$ with range $\{0, 1\}$ are interpreted so that the following holds for all t in T_2.

- (P1) $P_{i,j}(t) = 0$ implies $|t| = i$.
- (P2) $P_{i,j}(t) = 0$ implies $P_{i,j'}(t) = 1$ for all $j' \neq j$.
- (P3) for $i = |t| + 1$ and $j \geq k_t$ there are infinitely many immediate successors s of t such that $P_{i,j}(s) = 0$.
- (P4) for $i = |t| + 1$ and $j < k_t$ there no immediate successors s of t such that $P_{i,j}(s) = 0$.

We think of $P_{i,j}(t) = 0$ as t being coloured in colour (i, j), and note that every node in T_2 is coloured.

The formal definition of P_{ij} is in order (here t^- stands for the immediate predecessor of $t \in T_2$ and $t = (t_0, t_1)$)

$$P_{i,j}(t) = \begin{cases} 0, & \text{if } |t| = i \text{ and } j - k_{t^-} = t_1(i-1) \\ 1, & \text{otherwise.} \end{cases}$$

For the sake of definiteness and convenience we also set

$$P_{0,0}(\langle\rangle) = 0.$$

Fix a countable indexed family $\vec{k}^i$ for $i \in \omega$ of functions from T_2 into ω. Later on, in §5.0.4, we shall make additional requirements on the choice of $\vec{k}$.

We define model M to be

$$M := \bigoplus_n M(\vec{k}^n)^\frown T_2.$$

The *bottom part* of M is $\bigoplus_n M(\vec{k}^n)$ and the *top part* of M consists of copies of T_2 added to all nodes of the bottom part. Bottom part is taken with all of its structure, in particular predicates $P_{i,j}$ are imported literally from it. For all i, j and all nodes t of the top part we let

$$P_{i,j}(t) = 1.$$

Then each $P_{i,j}$ has interpretation that is i-Lipshitz, and nodes of the top part are not coloured at all.

5.0.2. *Nodes, successors and branches.* Let

$$\mathbf{T} = \mathrm{Th}(M).$$

Any model N of $\mathbf{T}$ has two kinds of elements (not distinguished by sorts in L_t). Ranges of functions f_m for $m \in \omega$ comprise *nodes* of N. If $f_m(a) = a$ then $d(a, b) < 1/m$ implies $a = b$, and therefore nodes form a discrete subset of N. The *height* of a node a, denoted $|a|$, is the least m such that $f_m(a) = a$. By elementarity, N has the unique node of height 0, denoted $\langle\rangle$. Height of a node $a \neq \langle\rangle$ is the unique $m \geq 1$ such that

$$d(f_m(a), a) + \left|\frac{1}{m} - f_{m-1}(a)\right| = 0.$$

All non-nodes of N (if any) satisfy $d(f_m(a), a) = 1/m$ for all m. These are the *branches* of N. Model M has no branches.

For a node a in N and $k > m = |a|$ the set

$$\mathrm{Succ}_k(a) = \{b \in N : f_m(b) = a \text{ and } |b| = k\}$$

is easily checked to be a definable subset of N (see [3]). In particular, adding quantifiers of the form $\sup_{z \in \mathrm{Succ}_k(x)}$ does not change the expressive power of the language (see [3]).

5.0.3. *Types* $\mathbf{s}_m$. Now that model M and $\mathbf{T}$ are defined, we proceed to define 1-types $\mathbf{s}_m(x)$ for $m \in \omega$. Let $\mathbf{s}_0(x)$ be the 1-type of an infinite branch of the tree underlying this model. That is, it consists of all conditions of the form $d(x, f_n(x)) = 1/n$ for $n \in \omega$.

Type $\mathbf{s}_m(x)$ is realized in the canonical model M by any a which is a terminal node of the bottom part of M and belongs to its mth level.

In symbols, for $m > 0$ type $\mathbf{s}_m(x)$ asserts the following.

(S1) x is on the mth level of the tree: $d(f_m(x), x) = 0$ and $d(f_{m-1}(x), x) = 1/m$.

(S2) On every level of the tree there is a node above x. This can be expressed by formulas $\phi_k(x)$ for $m < k$ stating that there exists z above the kth level such that $f_m(z) = x$:

$$\phi_k(x) := \inf_z d(f_m(z), x) + \left| \frac{1}{k} - d(f_k(z), z) \right|.$$

(S3) All successors of x are uncoloured. This can be expressed by conditions

$$\sup_{y \in \mathrm{Succ}_{m+1}(x)} 1 \dot{-} P_{m+1,n}(y) = 0$$

for $n \in \omega$.

We pause to record some straightforward facts about these types for $m \geq 1$. Type $\mathbf{s}_m(x)$ is not realized by any a is in the top part of M, because such a has rank $< \omega$ and therefore fails (S2). Type $\mathbf{s}_m(x)$ is also not realized by any a which is not a terminal node of the bottom part of M, since then it has a coloured successor and therefore fails (S3).

We shall prove that $\mathbf{s}_m(x)$ is generically omissible. The following is the key lemma towards this goal ($\mathbb{P}_{\mathbf{T}}$ denotes the forcing defined in §3 and $\bar{x}, y, z$ stand for assorted d_j's).

Lemma 5.1. *Assume $\phi(\bar{x}, y, z) < \varepsilon$ is a condition in $\mathbb{P}_{\mathbf{T}}$ which forces the following statements.*

(a) *$P_{m,n}(y) = 0$ for some m and n,*

(b) *z is an immediate successor of y.*

Then this condition can be extended to a condition that in addition implies

(c) *$P_{m+1,k}(z) = 0$ for some k.*

Proof. We can assume that ϕ is in the prenex normal form since such formulas are uniformly dense in the space of all formulas by [3, Proposition 6.9], We may also assume that $\varepsilon < 1$. Since $\phi(\bar{x}, y, z) < \varepsilon$ is a consistent condition, we can find a tuple $\bar{a}, b, c$ in M that realizes it. Let

$$\varepsilon' := (\varepsilon - \phi(\bar{a}, b, c)^M)/3.$$

Let $\psi_j(\bar{x}, y, z, \bar{t})$, for $j < m$, be the list of all atomic subformulas of $\phi(\bar{x}, y, z)$. Thus $\phi(\bar{x}, y, z)$ is of the form (variables in ψ_j suppressed for readability)

$$Qt_1 Qt_2 \dots Qt_l f(\psi_0, \dots, \psi_{m-1})$$

for some continuous function f. Since the interpretation of $f(\psi_0, \dots, \psi_{m-1})$ is uniformly continuous and its modulus of continuity does not depend on the interpretation, we can find $\delta > 0$ such that changing values of all variables occurring in any ψ_j by $< \delta$ affects the change of the value of $\phi(\bar{x}, y, z)$ by $< \varepsilon'$. Let $l > 1/\delta$ be such that all pairs i, j for which predicate $P_{i,j}$ occurs in some $\psi_j(\bar{x}, y, z)$ satisfy $\max(i, j) < l$. By increasing l we may also assume that all projection functions f_i occurring in some $\psi_j(\bar{x}, y, z)$ satisfy $i < l$ and that $\bar{a}, b, c$ belong to one of the first l levels of M. Let L_0 be the reduct of L_t to the language containing only $P_{i,j}$ for $\max(i, j) < l$ and f_i for $i < l$.

Let $M \restriction (l, L_0)$ be the submodel of the L_0-reduct of M consisting only of its first l levels.

If d is any node in the intersection of $M \restriction (l, L_0)$ and the bottom part of M, then its rank in M is at least ω. Moreover, for any two such nodes d and d' on the same level of M the trees $\{e \in M \restriction (l, L_0) : d \sqsubseteq e\}$ and $\{e \in M \restriction (l, L_0) : d' \sqsubseteq e\}$ are isomorphic. Whether an isomorphism between these trees extends to an automorphism of $M \restriction (l, L_0)$ depends only on whether the $P_{i,j}$ labels are matched for $\max(i, j) < l$.

We claim that for any tuple $\bar{p}, d, e$ in $M \restriction (l, L_0)$ of the same sort as $\bar{x}, y, z$ we have

$$(*) \qquad |\phi(\bar{p}, d, e)^{M \restriction (l, L_0)} - \phi(\bar{p}, d, e)^M| < \varepsilon'.$$

Let us prove this. For every tuple $\bar{q}$ in M such that $\bar{p}, d, e, \bar{q}$ is of the same sort as $\bar{x}, y, z, \bar{t}$ (where $\bar{t}$ are variables occurring freely in formulas ψ_j but not in ϕ), we have $d((t_i, f_l(t_i)) < \delta$ for all i. Since $f_l(t_i) \in M \restriction (l, L_0)$, and of course $\bar{p}, d$, and e are in $M \restriction (l, L_0)$, this means that for every choice of values in M for variables in the body of ϕ there is a choice of values for these variables in $M \restriction (l, L_0)$ at the distance $< \delta$. Now (*) follows easily by the choice of δ, ε' and the argument from the first few lines of [18, Lemma 1.8].

For $p \in \omega$ we write

$$N_p := M(\vec{k}^p)^\frown T_2,$$

and consider it as a submodel of M; hence $M = \bigoplus_{p \in \omega} N_p$. Note that for each $s \in N_p$ all successors of s in M belong to N_p.

Recall that we have previously fixed $\bar{a}, b, c$ such that $\phi(\bar{a}, b, c)^M < \varepsilon$. Since $P_{m,n}(b) = 0$, b belongs to the bottom part of M. Let r be such that $b \in N_r$.

Now take a look at $M \restriction (l, L_0)$ and note that c is an immediate successor of b by (b). We shall consider three cases.

First, if $P_{m+1,j}(c) = 0$ for some $j < l$, then all of the above work was unnecessary and we can extend condition ϕ as required so that it implies $f_{m+1}(z) = z$, $f_m(z) = y$ and $P_{m+1,j}(z) = 0$.

The second case is a generalization of the first case. Assume $P_{m+1,j}(e) = 0$ for some j and some immediate successor e of b. Then b is not a terminal node of the bottom part of M. We can choose $j > m$ and consider an automorphism of $M \restriction (l, L_0)$ that swaps c with an immediate successor e of b satisfying $P_{m+1,j}(e) = 0$ in M. Then we can extend condition ϕ as required so that it implies $f_{m+1}(z) = z$, $f_m(z) = y$ and $P_{m+1,j}(z) = 0$.

It remains to consider the third case, when neither of the above two cases applies. This means that none of the immediate successors of b in $M \restriction (l, L_0)$ satisfies any of the $P_{m+1,j}$ for $j < l$. Choose p such that $\vec{k}^p$ and $\vec{k}^r$ agree everywhere except that $k^p_b > l$. The map θ that swaps $N_p \restriction (l, L_0)$ and $N_r \restriction (l, L_0)$ and is equal to the identity on $N_q \restriction (l, L_0)$ for all $q \notin \{p, r\}$ does not move any of the labelled nodes and is therefore an automorphism of $M \restriction (l, L_0)$. By applying (*) twice we obtain

$$|\phi(\bar{a}, b, c)^M - \phi(\theta(\bar{a}), \theta(b), \theta(c))^M| < 2\varepsilon'$$

and in particular $\phi(\theta(\bar{a}), \theta(b), \theta(c)) < \varepsilon$. Now we can choose $j > k_b^p > l$ and an immediate successor c' of $\theta(b)$ such that c' and $\theta(c)$ have the same L_0-type over $\bar{a}$ and b but $P_{m+1,j}(c') = 0$. Therefore the condition

$$\max(\phi(\bar{x}, y), P_{m+1,j}(z)) < \varepsilon$$

is satisfied in M by $\theta(\bar{a}), \theta(b), c'$ and since $\varepsilon < 1$ it forces $P_{m+1,j}(z) = 0$. Now we can extend our condition as required and complete the proof. □

Generic model M_G was defined in Proposition 3.1.

Lemma 5.2. *Forcing $\mathbb{P}_{\mathbf{T}}$ forces that all nodes of M_G are coloured.*

Proof. Since $P_{0,0}(\langle\rangle) = 0$ in M, by elementarity in every model of $\mathbf{T}$ the unique node of height 0 is coloured. Therefore $\mathbb{P}_{\mathbf{T}}$ forces that the coloured nodes include a nonempty initial segment of M_G. Since nodes in M_G (and any other model of $\mathbf{T}$) are isolated points, every node of M_G is an interpretation of some d_j. Lemma 5.1 implies that no coloured node of M_G has a non-coloured isolated successor, and this concludes the proof. □

We put the ongoing proof of Theorem 2 on hold for a moment and record an immediate consequence of the above analysis.

Corollary 5.3. *Forcing $\mathbb{P}_{\mathbf{T}}$ forces that all $\mathbf{s}_m$ for $m \geq 1$ are omitted in M_G and that the nodes of M_G form an ill-founded tree. In particular, M_G realizes $\mathbf{s}_0$.* □

Corollary 5.4. *There exists a complete theory $\mathbf{T}$ in a separable language and a type $\mathbf{t}(x)$ which is omissible in a model of $\mathbf{T}$ but $\mathbb{P}_{\mathbf{T}}$ forces that M_G realizes $\mathbf{t}$.*

Proof. Type $\mathbf{s}_0$ is omissible in model M of $\mathbf{T}$, but by Corollary 5.3 it is realized in the generic model. □

Corollary 5.5. *There are a complete theory $\mathbf{T}$ in a separable language and types $\mathbf{s}_n$, for $n \in \omega$, such that for every n there exists a model of $\mathbf{T}$ that omits all $\mathbf{s}_n$ but no model of $\mathbf{T}$ simultaneously omits all $\mathbf{s}_n$.*

Proof. With $M, \mathbf{T}$ and $\mathbf{s}_m$ as above, M omits type $\mathbf{s}_0$ by the construction. On the other hand, Corollary 5.3 implies that the generic model omits all $\mathbf{s}_m$ for $m \geq 1$.

Assume N is an L_t-model elementarily equivalent to M. Then by elementarity every 'named' node in N has the property that arbitrarily high levels of N above this node are nonempty. Now assume N omits $\mathbf{s}_m$ for all sufficiently large m. Then the 'bottom part' of N (consisting of all nodes labelled by some $P_{m,j}$) is ill-founded and therefore N realizes $\mathbf{s}_0$ as defined in the beginning of §5. Therefore no model of $\mathbf{T}$ omits all $\mathbf{s}_m$ for $m \in \omega$. □

5.0.4. *Model* $M[l]$. With Corollary 5.5 in our hands, in order to complete the proof of Theorem 2 we only need to describe a model $M[l]$ of $\mathbf{T}$ that omits $\mathbf{s}_j$ for all $j < l$, for every $l \in \omega$. This will require a more careful choice of the sequence $\vec{k}$, as described after an upcoming batch of definitions.

Fix $l \in \omega$. Let $M(\vec{k}, l)$ be the submodel of $M(\vec{k})$ obtained by pruning all nodes s that have no extensions to the lth level. Therefore (recall that the elements of $M(\vec{k})$ are pairs $s = (s_0, s_1)$)

$$M(\vec{k}, l) = \{s \in M(\vec{k}) : s_0(|s| - 1) \geq l - |s|\}.$$

For a model N of L_t, $m \in \omega$ and finite sublanguage L_0 of L we let denote the model consisting of the first m levels of N by $N \restriction m$ and its L_0-reduct by $N \restriction (m, L_0)$.

We postpone the description of our choice of $\vec{k}$ until required properties are exhibited in the following two lemmas.

Lemma 5.6. *For every $\vec{k}$, finite subset L_0 of L_t, and $m > l$ in ω there is $\vec{k}'$ such that*

$$(M(\vec{k}, l)^\frown T_2) \restriction (m, L_0) \cong (M(\vec{k}')^\frown T_2) \restriction (m, L_0).$$

Proof. Let r be large enough so that if $P_{i,j}$ appears in L_0 then $j < r$. Define k'_s to be equal to k_s if s has an extension to the l-th level of $M(\vec{k}, l)^\frown T_2$ and $k'_s = r$ otherwise. Since our original models have a large supply of uncoloured nodes and nodes s such that $k'_s = r$ are uncoloured in L_0-reduct, it is easily seen that this $\vec{k}'$ is as required. □

Lemma 5.7. *For every $\vec{k}, L_0, m$ and l as in Lemma 5.6 there is a function $\vec{k}''$ such that*

$$(M(\vec{k})^\frown T_2) \restriction (m, L_0) \cong (M(\vec{k}'', l)^\frown T_2) \restriction (m, L_0).$$

Proof. Let r be as in the proof of Lemma 5.6. Consider the tree $T^* = T_2{}^\frown T_2$, with its lower and upper parts defined naturally but no other structure. The restriction T^*_m of T to its first m levels is isomorphic to $\omega^{<m}$. By a back-and-forth argument we can find an automorphism π of T^*_m such that if s is in the lower part of T^*_m then $\pi(s)$ is in the lower part of T^*_m and it moreover has an extension to the l-th level of the lower part of T^*_m. Define $\vec{k}''$ on T_2 by

$$k''_s = k_{\pi^{-1}(s)}$$

if s (when considered as an element of the lower part of T^*) is equal to $\pi(t)$ for some t in the lower part of T^*_m and by

$$k''_s = r$$

otherwise. It is easily seen that π is an isomorphism of $(M(\vec{k})^\frown T_2) \restriction (m, L_0)$ onto $(M(\vec{k}'', l)^\frown T_2) \restriction (m, L_0)$. □

Using Lemma 5.6 and Lemma 5.7 we construct a sequence $\vec{k}^n$ of functions on T_2 such that for every $n \in \omega$ the following hold.

(k1) $\vec{k}^n = \vec{k}^i$ for infinitely many i.
(k2) If $\vec{k}^*$ agrees with $\vec{k}^n$ except on finitely many places then $\vec{k}^* = \vec{k}^i$ for some i.
(k3) For all $m > l$ there is i such that

$$(M(\vec{k}^n, l)^\frown T_2) \restriction (m, L_0) \cong (M(\vec{k}^i)^\frown T_2) \restriction (m, L_0).$$

(k4) For all $m > l$ there is i such that

$$(M(\vec{k}^n)^\frown T_2) \restriction (m, L_0) \cong (M(\vec{k}^i, l)^\frown T_2) \restriction (m, L_0).$$

Using $\vec{k}$ satisfying these conditions we define M as before,

$$M := \bigoplus_n M(\vec{k}^n)^\frown T_2$$

and for $l \in \omega$ define $M[l]$ to be

$$M[l] := \textstyle\bigoplus_n M(\vec{k}^n, l)^\frown T_2.$$

Then $M[l]$ omits $\mathbf{s}_m$ for all $m < l$ because the bottom part of $M[l]$ has no terminal nodes.

Lemma 5.8. *For all $m > l$ in ω and every finite subset L_0 of language L_t we have $M[l] \restriction (m, L_0) \cong M \restriction (m, L_0)$*

Proof. By the back-and-forth argument matching the direct summands of M with the isomorphic direct summands of $M[l]$, using properties of $\vec{k}$. □

Proof of Theorem 2. We shall prove that $\mathbf{T}$ and types $\mathbf{s}_n$, for $n \in \omega$ as defined above are such that $\mathbf{T}$ is complete and such that for every k there exists a model of $\mathbf{T}$ that omits all $\mathbf{s}_n$ for $n \leq k$ but no model of $\mathbf{T}$ simultaneously omits all $\mathbf{s}_n$.

We claim that $M[l]$ as defined before Lemma 5.8 is elementarily equivalent to M for all $l \in \omega$. For every $k \geq 1$ models $M[l] \restriction k$ and $M \restriction k$ as defined in Lemma 5.8 are $1/k$-dense in L_0 reducts of $M[l]$ and M, respectively. By Lemma 5.8 $M[l] \restriction k \cong M \restriction k$ for all k, and therefore Lemma 1.8 implies $M[l]$ is elementarily equivalent to M for all l.

Since $M[l]$ omits $\mathbf{s}_0$ and all $\mathbf{s}_m$ for $m < l$, and it is a model of $\mathbf{T}$ by the above, this concludes the proof. □

Proof of Theorem 3. We now construct a complete theory $\mathbf{T}_4$ in a separable language L_4 and types $\mathbf{s}$ and $\mathbf{t}$ omissible in models of $\mathbf{T}_4$ such that no model of $\mathbf{T}_4$ simultaneously omits both of them.

Start from T_1, T_2, L_t, M and $\mathbf{T}$ as in the proof of Theorem 2. Add a new sort X to L_t, a unary function g from X into X, and a unary function h from X to M. Both g and h are to be interpreted as 1-Lipshitz functions. This describes language L_4 and we proceed to describe L_4-model M_4. The L_t-part of M_4 is isomorphic to M. The underlying set for the interpretation of X is tree $T_2^\frown T_2$, taken with the discrete $\{0, 1\}$-valued metric. As it is isomorphic to the tree comprising the universe of M, we interpret h as the natural isomorphism function from X onto M. Finally, the interpretation of g sends $\langle\rangle$ to itself and every other node to its immediate predecessor.

This describes model M_4 and theory $\mathbf{T}_4 = \mathrm{Th}(M_4)$. The universe of every other model N of $\mathbf{T}_4$ consists of a model M^N of $\mathbf{T}$ and a discrete set X^N. It is equipped with functions $h^N \colon X^N \to M^N$ and $g^N \colon X^N \to X^N$.

We shall need some facts about $\mathbf{T}$ and nodes in M. A node in M has height $\leq m$ if and only if the open $1/m$-ball centered at it has no other elements. If $N \models \mathbf{T}$ then *nodes* of N are its isolated points and a node a has height $\leq m$ if and only if $\mathrm{P}_{\leq \mathrm{m}}(a)^N = 0$. By a straightforward computation in M for every node a and $m \geq 1$ the following are equivalent

(1) $|a| \leq m$.
(2) $(\forall b)(d(a,b) < \frac{1}{m} \to a = b)$.
(3) $\sup_y \min(\frac{1}{m} \dot{-} d(a,y), d(a,y)) = 0$.

Therefore the predicate

$$\mathrm{P}_{\leq \mathrm{m}}(x) := \sup_y \min(\frac{1}{m} \dot{-} d(x,y), d(x,y))$$

has nodes of height $\leq m$ in M as its zero set and satisfies $\mathrm{P}_{\leq \mathrm{m}}(b) \geq \frac{1}{m(m+1)}$ for every other element b of M. Also, predicate

$$\mathrm{P}_{>\mathrm{m}}(x) := \frac{1}{m(m+1)} \dot{-} \mathrm{P}_{\leq \mathrm{m}}(x)$$

has nodes of height $\geq m+1$ and branches as its zero set and satisfies

$$\mathrm{P}_{>\mathrm{m}}(b) \geq \frac{1}{m(m+1)}$$

for every node b of height $\leq m$ in every model N of $\mathbf{T}$.

Hence both the set of nodes of height $\leq m$ and its complement are definable (see [3, Definition 9.1]) in every model of $\mathbf{T}$.

We shall prove that the following hold for every $N \models \mathbf{T}_4$ and a and b in X^N.

(4) If $a \neq b$ and $h(a)$ and $h(b)$ are nodes of M^N then $h(a) \neq h(b)$.
(5) For every b in X^N such that $h(b)$ is a node of height $m+1$ we have $h(g(b)) = f_m(h(b))$.
(6) Every node of M^N is in the range of h.
(7) The set of immediate successors of $h(a)$ is equal to

$$\{h(c) : c \in X^N \text{ and } g(c) = a\}.$$

Proof. We fix $m \geq 1$ throughout the proof and prove (4)–(7) for nodes of height $\leq m$.

(4) The following sentence is in $\mathrm{Th}(M_4)$:

$$\sup_{x \in X, y \in X} \min(\mathrm{P}_{>\mathrm{m}}(h(x)), \mathrm{P}_{>\mathrm{m}}(h(y)), d(x,y), \frac{1}{m} \dot{-} d(h(x), h(y))).$$

Thus in every model N of $\mathbf{T}_4$, for all $m \geq 1$ and a and b in X^N such that $\max(|h(a)|, |h(b)|) \leq m$ we have $h(a) = h(b)$ or $d(h(a), h(b)) \geq 1/m$. This implies (4).

(5) The following sentence is in $\mathrm{Th}(M_4)$:

$$\sup_{x \in X} \min(\mathrm{P}_{>\mathrm{m}+1}(h(x)), \mathrm{P}_{\leq \mathrm{m}}(h(x)), d(h(g(x)), f_m(h(g(x))))).$$

Thus in every model N of $\mathbf{T}_4$, for every $b \in X^N$ such that $|h(b)| = m + 1$ we have $h(g(b)) = f_m(h(b)$.

(6) The following sentence is in $\mathrm{Th}(M_4)$ (x ranges over the L_t-part):

$$\sup_x \min(\mathrm{P}_{>\mathrm{m}}(x), \inf_{y \in X} d(h(y), x)).$$

Since the set of nodes of height $\leq m$ is discrete, in every model N of $\mathbf{T}_4$ (6) holds.

(7) The following sentence is in $\mathrm{Th}(M_4)$ (y ranges over the L_t-part of M_4):

$$\sup_{x \in X, y} \min(\mathrm{P}_{>\mathrm{m}}(h(x)), \mathrm{P}_{\leq \text{m-1}}(h(x)), \mathrm{P}_{>\mathrm{m}+1}(y), \mathrm{P}_{\leq \mathrm{m}}(y),$$
$$\frac{1}{m} \dot{-} d(f_m(y), h(x)), \inf_{z \in X} \max(d(h(z), y), d(g(z), x))).$$

Again, since the set of nodes of height $\leq m + 1$ is discrete, this implies (7) holds in every model of $\mathbf{T}_4$. □

We proceed to define types $\mathbf{s}$ and $\mathbf{t}$. Type $\mathbf{s}(x)$ is the same as $\mathbf{s}_0(x)$ in the proof of Theorem 2: the type of an infinite branch in M^N. Type $\mathbf{t}(x)$, for x of sort X, asserts the following (g^k denotes the kth iterate of g).

(T1) For every $k \geq 1$ there exists y of sort X such that $g^k(y) = x$:

$$\inf_{y \in X} d(x, g^k(y)) = 0.$$

(T2) For all immediate successors of x (i.e., y such that $g(y) = x$), $h(y)$ is not coloured:

$$\inf_{y \in X} \max(d(x, g(y)), P_{m,n}(h(y))) = 1$$

for all m and n.

In a model N of $\mathbf{T}_4$ type $\mathbf{t}$ is realized by every a such that $h(a)$ is a branch of M^N. However, this is of no concern to us as if N omits $\mathbf{s}$ then M^N has no branches.

Claim 5.9. *Assume N is a model of $\mathbf{T}_4$, $a \in X^N$, and $h(a)$ is a node of M^N of height m. Then a realizes $\mathbf{t}$ in N if and only if $h(a)$ realizes $\mathbf{s}_m$ in M.*

Proof. Fix a such that $h(a)$ is a node of height m. As $h(a)$ satisfies (S1) by our assumption, we only need to check that it satisfies (S2) and (S3) if and only if it satisfies (T1) and (T2).

Suppose b is in X^N and $g^k(b) = a$ for some $k \geq 1$. Then $h(b)$ is a node of height $m + k$ and $f_m(h(b)) = a$. Thus if a satisfies (T1) then $h(a)$

satisfies (S2). Now assume $h(a)$ satisfies (S2) and fix x of height $m+k$ such that $f_m(b) = a$. By (6) $x = h(b)$ for some b, and then by iterating (5) we have $g^k(b) = a$. Therefore (T1) holds for a.

By (7), a satisfies (T2) if and only if $h(a)$ satisfies (S3). □

Type $\mathbf{s}$ is clearly omitted in M_4 and type $\mathbf{t}$ is omitted in $\mathbb{P}_{\mathbf{T}_4}$-generic model. The proof of the latter fact is virtually identical to the proof of Lemma 5.1, Lemma 5.2 and Corollary 5.3. Alternatively, one can prove that the L_t-part of the $\mathbb{P}_{\mathbf{T}_4}$-generic model is $\mathbb{P}_{\mathbf{T}}$-generic and use these facts directly.

Assume N omits $\mathbf{s}$. Then M^N omits $\mathbf{s}_0$ and therefore all elements of M^N are nodes and by Theorem 2 it realizes $\mathbf{s}_m$ for some $m \geq 1$. By (6) there is $a \in X^N$ such that $h(a)$ realizes $\mathbf{s}_m$. By Claim 5.9, a realizes $\mathbf{t}$ in N. □

6. Concluding remarks

According to [9] and [11, Definition 4.12], a type $\mathbf{t}(\bar{x})$ is metrically principal over a theory $\mathbf{T}$ (we consider the case when L is the fragment consisting of all finitary sentences) if and only if for every $\delta > 0$ the type $\mathbf{t}^\delta(\bar{x})$, asserting that every finite subset of $\mathbf{t}$ is realizable by an n-tuple within δ of $\bar{x}$, is principal.

For example, type $\mathbf{t}$ defined in the proof of Proposition 2.5 is metrically principal over $\mathbf{T}^S$ if the tree S has height ω. This is because $\mathbf{t}_{1/n}$ is realized by any node of S that is not and end-node. This gives an example of an omissible metrically principal type. A simple argument shows that a complete metrically principal type cannot be omissible.

The following questions are left open after Theorem 1 and Proposition 2.4.

Question 6.1. *Is there a separable model M in a separable language such that the set of types realized in M is a complete $\mathbf{\Sigma}^1_1$-set?*

Question 6.2. *What are the possible complexities of the set of types omissible in a model of a set of types omissible in a model of a complete theory $\mathbf{T}$ in a separable language?*

In particular, is there such $\mathbf{T}$ for which the set of types omissible in some model of $\mathbf{T}$ is $\mathbf{\Sigma}^1_2$-complete?

Significant notions of the first-order model theory such as stability can be expressed in terms of cardinalities of sets of types (see [30]). This carries over to logic of metric structures (see [3], [17, §5]). Complexity of the set of not necessarily complete types over a given theory may give some information about theories in logic of metric structures. In particular, it is plausible that in case when $\mathbf{T}$ is natural theory (e.g., theory of a C*-algebra) the set of types omissible in a model of $\mathbf{T}$ is Borel. Also, conclusions of Theorem 2 and Theorem 3 give dividing lines for complexities of metric theories. Each theory used in our counterexamples interprets the Baire space as presented in §2.2. Is this a necessary condition for pathological behaviour of metric theories?

6.1. **C*-algebras.** The original motivation for this study came from the model-theoretic study of C*-algebras. Answers to some of the most prominent open problems in the theory of C*-algebras depend on whether nuclear C*-algebras can be constructed in a novel way. While C*-algebras are axiomatizable in logic of metric structures, essentially none of the important classes of C*-algebras is axiomatizable. This is because every UHF algebra is by [17, §6.1] elementarily equivalent to a non-exact C*-algebras (it suffices to say that UHF algebras form the most restrictive important class of non-type I C*-algebras while the exact algebras form the largest reasonably well-behaved and well-studied class of C*-algebras).

C*-algebras which are UHF, AF ([10]), nuclear, of finite nuclear dimension, of finite decomposition rank, or TAF ([16]) are uniformly definable by a sequence of universal types. Therefore Theorem 4.2, Theorem 4.3, Proposition 4.4, Proposition 4.5 and Corolary 4.6 open possibilities for constructing C*-algebras with prescribed first-order properties in these classes. Also, some of the deepest recent results on classification of C*-algebras have equivalent formulation in the language of (metric) first-order logic (see [13], the introduction to [28], and [14] or [16]).

We have a machine for construction of C*-algebras with properties prescribed by a given theory. These are generic algebras obtained by Henkin construction as described in §3 and §4. Although they are assembled from finite pieces corresponding to conditions of $\mathbb{P}_{\mathbf{T}}$ or one of its variations, they are not obviously obtained from matrix algebras and abelian algebras by applying basic operations of taking inductive limits, crossed products by $\mathbb{Z}$, stable isomorphisms, quotients, extensions, hereditary subalgebras, or KK-equivalence (cf. the bootstrap class problem, [8, IV.3.1.16 and V.1.5.4]). At present no method for assuring that the C*-algebras obtained by using the Henkin construction do not belong to the (large or small) bootstrap class is known. Results of [16] (combined with §3 and §4) reduce several prominent open problems on classification of C*-algebras to problems about the existence of theories with certain properties.

In [25] Kechris defined a Borel space of C*-algebras and proved that the nuclear C*-algebras form its Borel subset. We give a generalization of this result. Recall that a Borel structure on the space of models was defined in §1.2.2. Although this space is different from one used by Kechris, these representations of space of separable C*-algebras are equivalent ([19]).

Corollary 6.3. *The following sets of C*-algebras are Borel subsets of the standard Borel space of C*-algebras: UHF, AF, nuclear, nuclear dimension $\leq n$ for $n \leq \aleph_0$, decomposition rank $\leq n$ for $n \leq \aleph_0$, tracially AF, simple.*

Proof. Since each of the sets of C*-algebras listed above is uniformly definable by a sequence of types by [10] and [16], the conclusion follows by Proposition 4.7. □

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