COMPLEXITY OF COUNTING PLANAR TILINGS BY TWO BARS

KYLE MEYER

Abstract. We show that the problem of determining the number of ways of tiling a planar figure with a horizontal and a vertical bar is #P-complete. We build off of the results of Beauquier, Nivat, Remila, and Robson in [1] in which they showed that the problem determining existence of such tilings was shown to be NP-complete.

1. Introduction

For this problem we consider the plane as a grid of unit squares, and we define figures to be subsets of these squares. We consider the complexity of counting how many ways a figure can be tiled by a $1 \times l$ rectangle and an $m \times 1$ rectangle where l and m are both at least 2. In the case that $l = m = 2$ this problem can be formulated in terms of counting perfect matchings of planar graphs which Kasteleyn showed could be done in polynomial time [3]. Thus we look at the case either l or m is at least 3; in particular we will show:

Theorem 1.1. *Counting the number of ways to tile a planar figure with $1 \times l$ and $m \times 1$ rectangles is #P-complete if at least one of l and m is at least 3 and the other is at least 2.*

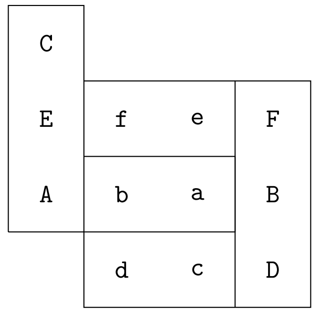
Our proof is a modification of that given in [1] that was used to show that the associated decision problem is NP-complete. We can associate a bipartite graph to a conjunction of boolean clauses by letting the clauses and variables be nodes and having edges connecting clauses to the variables they contain. The proof in [1] reduces from planar 3-CNF Sat by converting the associated planar graph into a figure which is tileable if and only if the planar 3-CNF expression is satisfiable. We will do a similar reduction from planar1-Ex3MonoSat.

Definition 1.2. *Planar 1-Ex3MonoSat A boolean expression is **1-Ex3Mono** if it the conjunction of a series of clauses such that each clause contains exactly 3 non-negated variables and the clause is true if exactly one of those variables is true. Such an expression is **planar** if the associated graph is planar. Finally, planar 1-Ex3MonoSat is the question: Given a planar 1-Ex3Mono expression, is there an assignment of boolean values to each variable such that all clauses in the expression are true?*

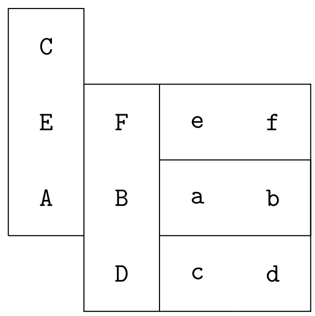
Counting solutions of planar 1-Ex3MonoSat expressions was shown to be #P-complete by Hunt, Marathe, Radhakrishnan, and Stearns in [2].

2. Proof of the case $l = 2$, $m = 3$

We will first prove our claim for the case $l = 2$,$m = 3$, and then extend these results to $l \geq 2$, $m \geq 3$. In order to do this we introduce variable figures, clause



Example 1 of Notation for a Tiling of a Figure
Figure 1



Example 2 of Notation for a Tiling of a Figure
Figure 2

figures, and edge figures that correspond to the variable nodes, clause nodes, and edges respectively of the planar graph associated to a planar 1-Ex3Mono expression. Following the convention of [1], we will depict our figures by the letters **a-f**, where the letter we use is determined by the position of the square mod 2 horizontally and mod 3 vertically, and where the letter is capitalized if it is covered by the vertical bar and lower-case if it is covered by the horizontal bar(see Figures 1 and 2).

The first figure we constuct is a wire. A wire will connect a variable figure to a clause figure and will transmit the variable values from the variable figure to the clause figure. Wires can be tiled in two ways corresponding to true and false. The true tiling leaves a square on each end untiled, while the false tiling covers all the squares. We have two types of wires; those that starts on **a** and those that start on **b**, and each of these will end on the other letter, and more specifically we can extend the wire so that it ends on any of that letter, we show a wire of the first type with both true and false tilings in Figures 3 and 4. Two wires, one of each type, run along side each other form an **edge figure**.

We next construct the **variable figure**. This figure will connect to one edge figure for each clause in which the corresponding variable appears, thus the size varies depending on how many clauses a variable is in. There are only two ways to tile this figure: one way will force the connecting edge figures to have a true tiling and the other will force them to have a false tiling. These two ways correspond to

```
    Abababa·
    C
    E
·baba
```

Wire with True Tiling

Figure 3

```
    abababab
    C
    E
ababA
```

Wire with False Tiling

Figure 4

```
A    B  A    B  A    B
C    D  C    D  C    D
E    F  E    F  E    F
ab  ab  ab  ab  ab  ab
 dcdc    dcdc    dcdc
ef  efefef  efefef  ef
A                    B
C                    D
EfefefefefefefefefefeF
```

Variable Figure with True Tiling

Figure 5

```
.    .  .    .  .    .
C    D  C    D  C    D
E    F  E    F  E    F
AB  AB  AB  AB  AB  AB
 DcdC    DcdC    DcdC
EF  EfefeF  EfefEF  EF
A                    B
C                    D
efefefefefefefefefefef
```

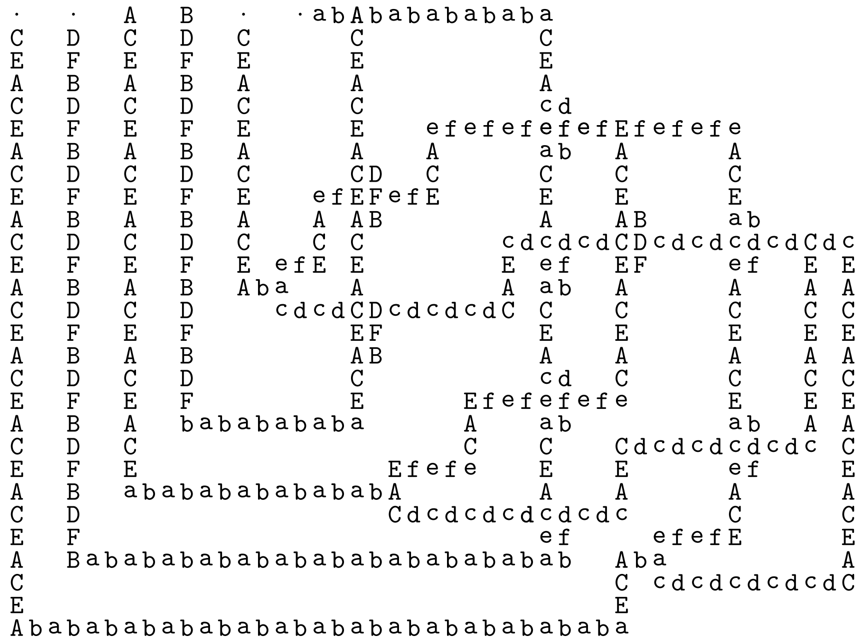
Variable Figure with False Tiling

Figure 6

a variable being true or false and are shown in Figures 5 and 6 for the case of 3 connecting edge figures.

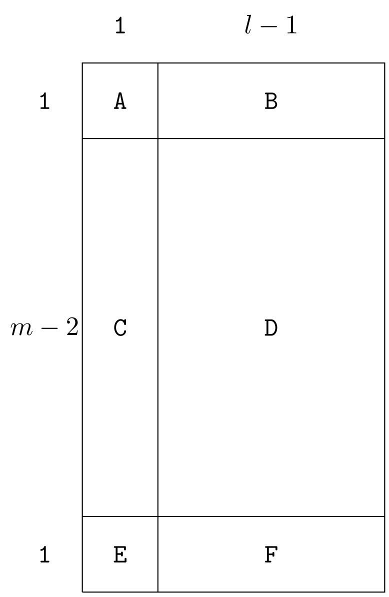
Our last figure, shown in Figure 7, is the **clause figure**. It takes in three input edge figures and is tileable if and only if exactly one of them has a true value. This is the step in which our reduction is not parsimonious, that is, it does not preserve the number of solutions. Specifically, there are crossing of wires in this figure such that if both wires have false values, then the crossing is tileable in two ways, either by three horizontal bars or two vertical ones. But we have constructed this figure in such a way that wires corresponding to each set of two of the three variables has three crossings. Thus each clause figure can be tiled $2^3 = 8$ ways if it can be tiled.

We have thus constructed the three figures that we need, and we can connect them all because of the planarity of the expression. Thus we have reduced the counting problem for planar 1-Ex3MonoSat to a planar tiling problem with bars of length 2 and 3 in which the number of tilings is 8^c times the number of solutions for the 1-Ex3MonoSat expression, where c is the number of clauses. Thus the problem is #P-complete.



Clause Figure

Figure 7



Extension from $l=2, m=3$ to any $l \geq 2$, $m \geq 3$

Figure 8

3. Extension for $l \geq 2$, $m \geq 3$

We now extend these results to longer bars; a horizontal bar of length $l \geq 2$ and a vertical bar of length $m \geq 3$. We do this replacing each letter **a-f** by a rectangle as shown in Figure 7. It is clear that every tiling of the original figure will give rise to a tiling of the expanded figure, but for general figures this expansion is not parsimonious. We claim that this expansion is parsimonious for the figures that we are constructing. To show this we just need to check that the tilings for each component figure is parsimonious. We start with the variable component. To see that this is parsimonious, first consider the bottom left **e**. This **e** is a single square, and thus it can be covered either by a horizontal or a vertical bar. Once that is chosen, then we can work around the figure and see that there is only one way to tile the rest of the figure. Similarly with wires, once we have either a true or false value for the wire given by the variable figure we work along the wire and there is only one way to tile the wire. The last figure is the clause figure, but the clause figure is only composed of wires with 3 different types of small figures, and it is easy to see that each of them have a parsimonious expansion, and thus the whole figure does. This complete the proof.

4. Future Directions

One possible direction for future research is to restrict this problem to simply connected regions. As mentioned above, in the general case determining the existence of a tiling is NP-complete, but in the simply-connected case Kenyon and Kenyon showed in [4], that determining the existence of a tiling can be done in linear time. Thus there is a fundamental difference between these two cases. Additionally, Pak and Yang showed in [5] that there is a large set of rectangles for which tiling simply connected regions is NP-complete and #P-complete, so it would be interesting to see if it is possible to reduce the size of that set to 2. Additionally it would be interesting to extend these results to more general pairs of rectangles in both the simply and non-simply connected cases.

5. Acknowledgments

I would like to thank Jed Yang and Kevin Dilks for their advice and guidance in writing this paper. I would also like to thank University of Minnesota, Minneapolis for hosting this REU. Additionally I thank the RTG grant NSF/DMS-1148634 for providing funding for this research.

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Department of Mathematics, Northeastern University, Boston, Massachusetts 02115
E-mail address: `meyer.ky@husky.neu.edu`