Networks with point-to-point codes

Abbas El Gamal

Stanford University

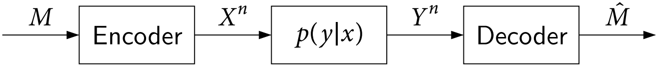
Shannon Memorial Lecture, UCSD 2013

Based on joint work with B. Bandemer, D. Tse, F. Bacelli, and Y.-H. Kim

"The fundamental problem of communication is that of reproducing at one point, either exactly or approximately, a message selected at another point."

C.E. Shannon (1948)

Shannon's point-to-point communication system



- Discrete memoryless channel (DMC) $p(y|x)$
- $(2^{nR}, n)$ block code:
  - Message: $M \sim \mathrm{Unif}[1:2^{nR}]$
  - Encoder: $x^n(m)$, $m \in [1:2^{nR}]$
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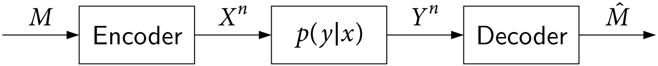

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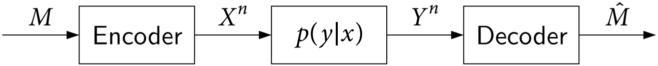

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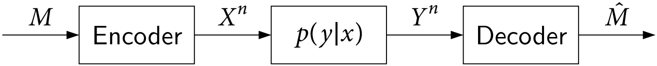
- Probability of error $P_e^{(n)} = \mathsf{P}\{\hat{M} \neq M\}$
- R achievable if $\exists$ $(2^{nR}, n)$ codes with $\lim_{n\to\infty} P_e^{(n)} = 0$
- Capacity C is supremum of all achievable rates

Shannon's channel coding theorem



Theorem (Shannon 1948)

$$C = \max_{p(x)} I(X; Y)$$

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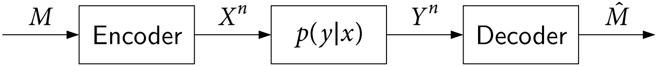

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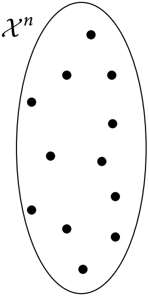
- Shannon gave an existential proof of achievability via random coding

Achievability using random point-to-point (ptp) codes

- Codebook generation: Fix $p(x)$
  - Randomly generate 2^{nR} sequences $x^n(m) \sim \prod_{i=1}^{n} p_X(x_i)$, $m \in [1:2^{nR}]$
  - Random codebook: $\mathcal{C}_n(p) = \{X^n(m) : m \in [1:2^{nR}]\}$



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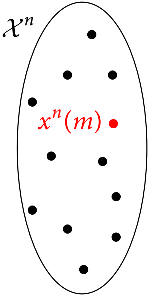

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- Encoding:
  - To send m, the encoder transmits $x^n(m)$
- Decoding:
  - Optimal decoding rule is maximum likelihood (MLD):

$$\hat{m}(y^n) = \arg \max_{m \in [1:2^{nR}]} \prod_{i=1}^n p_{Y|X}(y_i | x_i(m))$$

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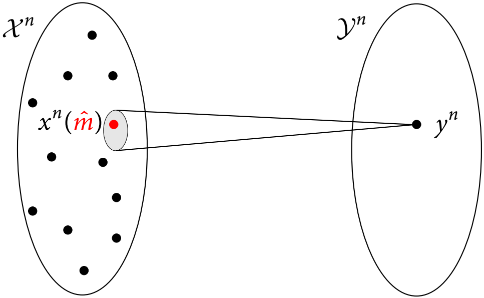
- Analysis of the probability of error, e.g., Gallager (1965):
  - Show that $\mathsf{E}[P_e^{(n)}(\mathcal{C}_n(p))] \to 0$ if $R < I(X;Y)$; hence good codes exist
  - Tight error exponents, but difficult to extend analysis to networks

Shannon's joint typicality proof

- Decoding:
  - Find unique message $\hat{m}$ such that $(x^n(\hat{m}), y^n) \in \mathcal{T}_\epsilon^{(n)}$
  - Otherwise declare an error



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- Several ways to define typicality, e.g., Orlitsky–Roche (2001):

$$\mathcal{T}_\epsilon^{(n)} = \{(x^n, y^n) \colon |\pi(x, y|x^n, y^n) - p(x, y)| \le \epsilon \cdot p(x, y) \text{ for all } (x, y)\},$$

where $\pi(x, y|x^n, y^n)$ is joint type of (x^n, y^n)

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- Converse for random ptp codes: Given $p(x)$ and a decoding rule
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  - If $\lim_{n\to\infty} \mathsf{E}[P_e^{(n)}(\mathcal{C}_n(p))] = 0$, then $R \le I(X;Y)$

  $\Rightarrow$ Joint typicality decoding achieves same rate as MLD
  - Extensions are useful for networks where MLD is difficult to analyze

Practical point-to-point codes

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- These codes are widely used in communication networks and storage
- Results in network information theory suggest we need network codes

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- How well do random ptp codes perform over networks?
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- Outline:
  - Brief introduction to network information theory
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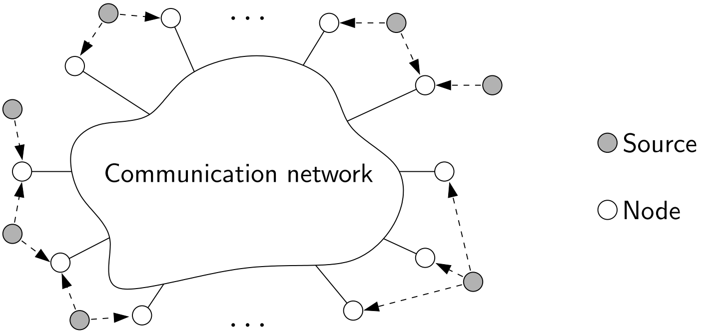
- Outline:
  - Brief introduction to network information theory
  - Discuss various network models with random ptp codes
- Preview of the results:
  - Random ptp codes with more sophisticated decoding can perform very well
  - Joint-typicality-based decoding continues to achieve same rates as MLD
  - There are settings where we may need to develop network codes

Network information theory

- Extends Shannon's point-to-point information theory to networks with:
  - Multiple sources and destinations
  - Multiple access, broadcast, and interference
  - Feedback and interactive communication
  - Cooperation (multihop)



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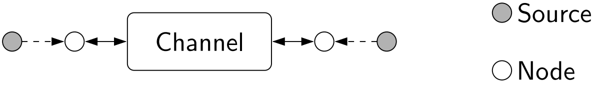

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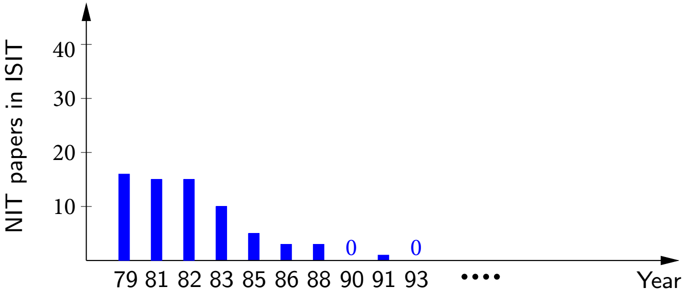

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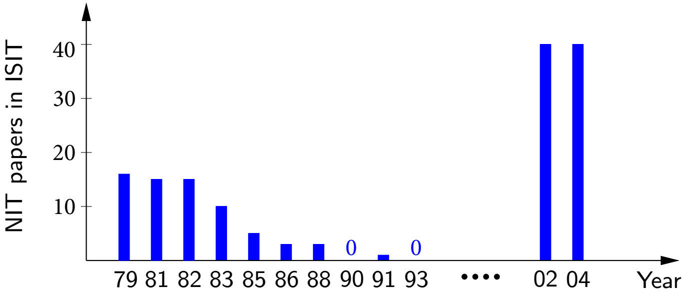

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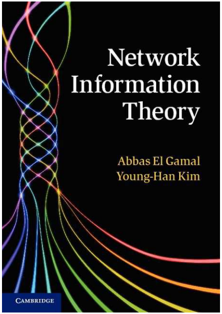
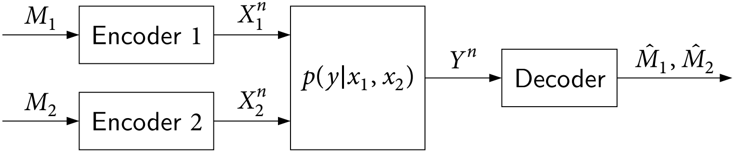
- State of the theory:



Network models with ptp codes

- Multiple access channel
- Interference channel
- Broadcast channel
- Relay channel
- Multicast networks
- Mutimessage networks

Multiple access channel (MAC)



- $(2^{nR_1}, 2^{nR_2}, n)$ block code:
  - Message pair: $(M_1, M_2) \sim \mathrm{Unif}([1:2^{nR_1}] \times [1:2^{nR_2}])$
  - Encoder $j = 1, 2$: $x_j^n(m_j)$, $m_j \in [1:2^{nR_j}]$
  - Decoder: $(\hat{m}_1, \hat{m}_2) \in [1:2^{nR_1}] \times [1:2^{nR_2}]$
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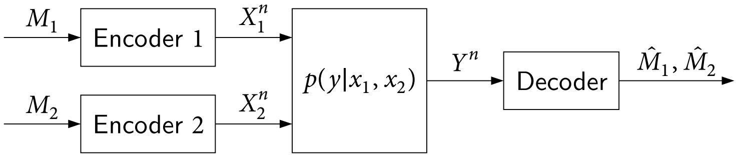

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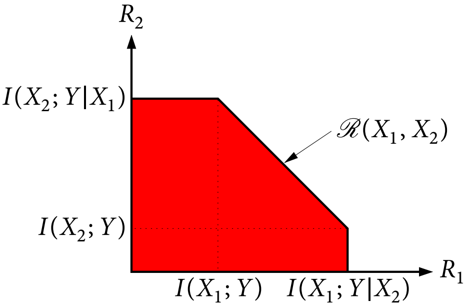
- Probability of error: $P_e^{(n)} = \mathsf{P}\{(\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)\}$
- (R_1, R_2) achievable if $\exists$ $(2^{nR_1}, 2^{nR_2}, n)$ codes with $\lim_{n\to\infty} P_e^{(n)} = 0$
- Capacity region $\mathscr{C}$: Closure of the set of achievable (R_1, R_2)

MAC capacity region

Theorem (Ahlswede 1971, Liao 1972)

The capacity region is the convex hull of $\bigcup_{p(x_1)p(x_2)} \mathscr{R}(X_1, X_2)$



MAC with random ptp codes

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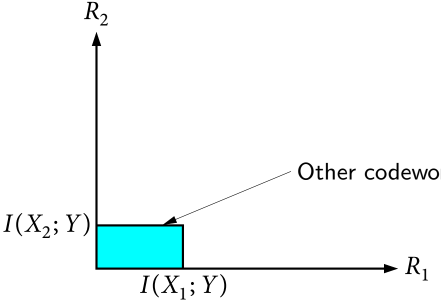
- Decoding: Use joint typicality decoding, treating the other codeword as noise:



MAC with random ptp codes

- Successive cancellation decoding (SCD):
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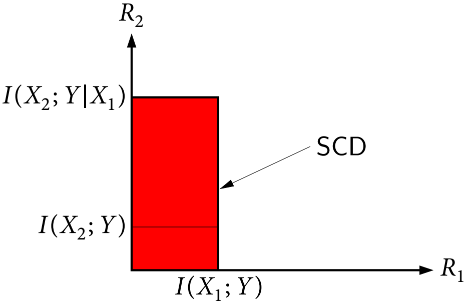
  - If such $\hat{m}_1$ is found, find unique $\hat{m}_2$: $(x_1^n(\hat{m}_1), x_2^n(\hat{m}_2), y^n) \in \mathcal{T}_\epsilon^{(n)}$



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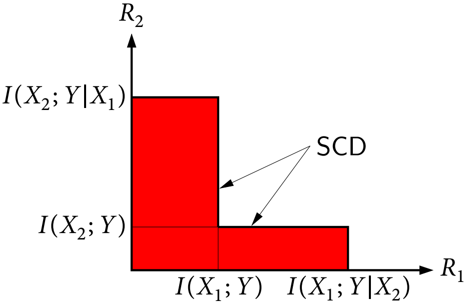
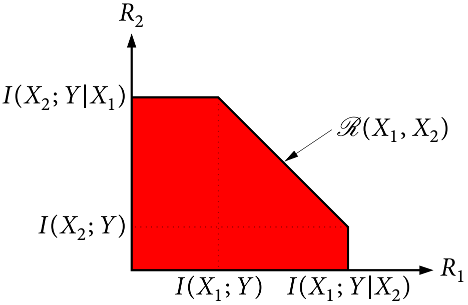
- Can achieve other corner point by reversing decoding order



MAC with random ptp codes

- Simultaneous decoding (SD):
  - Find unique pair $(\hat{m}_1, \hat{m}_2)$: $(x_1^n(\hat{m}_1), x_2^n(\hat{m}_2), y^n) \in \mathcal{T}_\epsilon^{(n)}$



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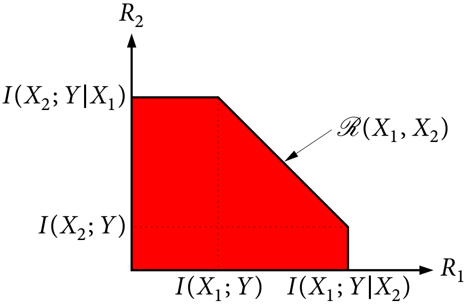
$\Rightarrow$ SD achieves same rates as MLD



Random ptp codes perform extremely well over MAC

Theorem (Ahlswede 1971, Liao 1972)

The capacity region is the convex hull of $\bigcup_{p(x_1)p(x_2)} \mathscr{R}(X_1, X_2)$

- $\mathscr{R}(X_1, X_2)$ achieved using random ptp codes + simultaneous decoding
- Rest of the capacity region is achieved using time-sharing

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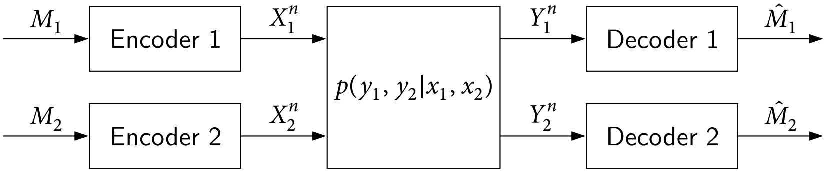
- Rest of the capacity region is achieved using time-sharing
- Results generalize to more than two senders

Interference channel



- First studied by Ahlswede (1974)

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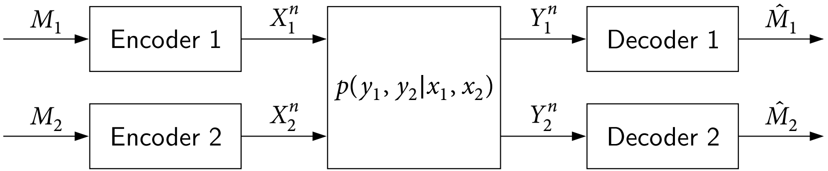

- First studied by Ahlswede (1974)
- $(2^{nR_1}, 2^{nR_2}, n)$ code, $P_e^{(n)}$, achievability, capacity region $\mathcal{C}$: Same as MAC
- Capacity region is not known in general
- Coding schemes that are optimal in some cases

Interference channel with random ptp codes

- Codebook generation: Fix $p = p(x_1)p(x_2)$
  - Randomly generate 2^{nR_1} sequences $x_1^n(m_1) \sim \prod_{i=1}^n p_{X_1}(x_{1i})$, $m_1 \in [1:2^{nR_1}]$
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- Encoding:
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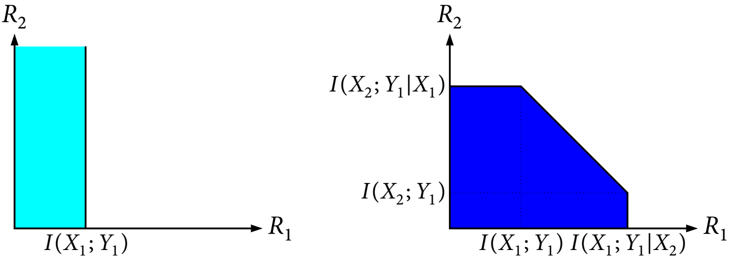
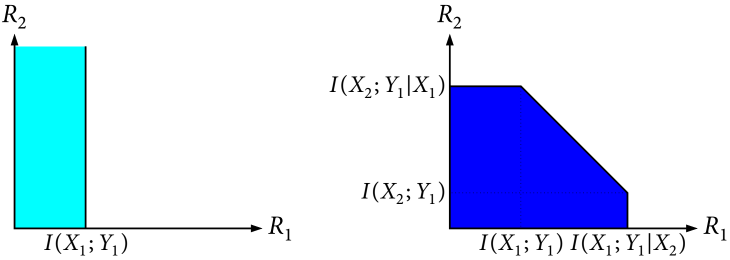
  - To send (m_1, m_2), transmit $x_1^n(m_1)$ and $x_2^n(m_2)$
- Decoding schemes we used for the MAC (receiver 1):



Treating interference as noise (IAN) | Simultaneous decoding (SD)

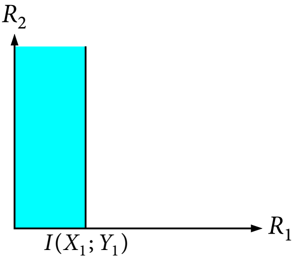
- Neither scheme uniformly outperforms the other



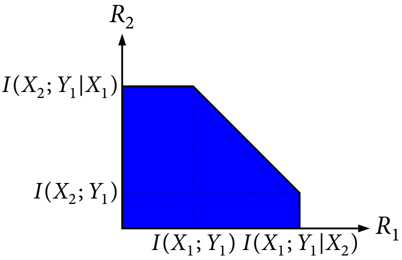
Treating interference as noise (IAN) Simultaneous decoding (SD)

Interference channel with random ptp codes

- Neither scheme uniformly outperforms the other
- IAN: Receiver 1 knows the codebook for M_2
  - If R_2 is not too high, it may be better to recover M_2 and achieve higher R_1



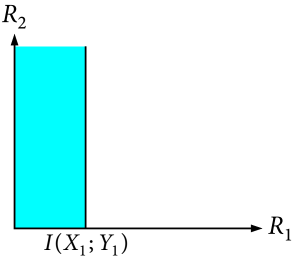
Treating interference as noise (IAN)



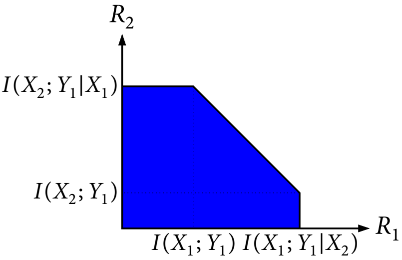
Simultaneous decoding (SD)

Interference channel with random ptp codes

- Neither scheme uniformly outperforms the other
- IAN: Receiver 1 knows the codebook for M_2
  - If R_2 is not too high, it may be better to recover M_2 and achieve higher R_1
- SD: Receiver 1 doesn't really want to recover M_2
  - Insisting on recovering M_2 when R_2 is high artificially limits R_1



Treating interference as noise (IAN)



Simultaneous decoding (SD)

Interference channel with random ptp codes

- Simultaneous nonunique decoding:
  - Receiver 1 finds unique $\hat{m}_1$: $(X_1^n(\hat{m}_1), X_2^n(m_2), Y_1^n) \in \mathcal{T}_\epsilon^{(n)}$ for some m_2

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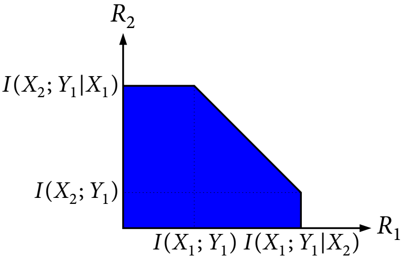
- (R_1, R_2) achievable using SD $\Rightarrow$ achievable using SND



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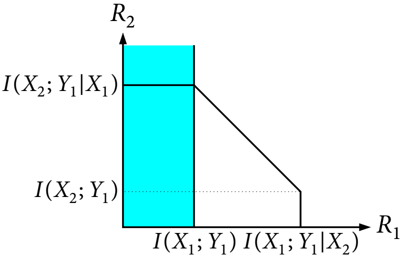
- (R_1, R_2) achievable using IAN $\Rightarrow$ achievable using SND



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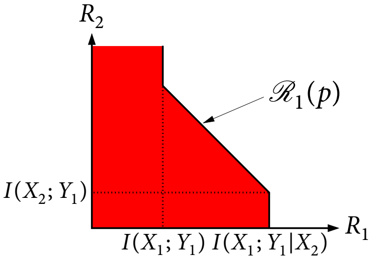
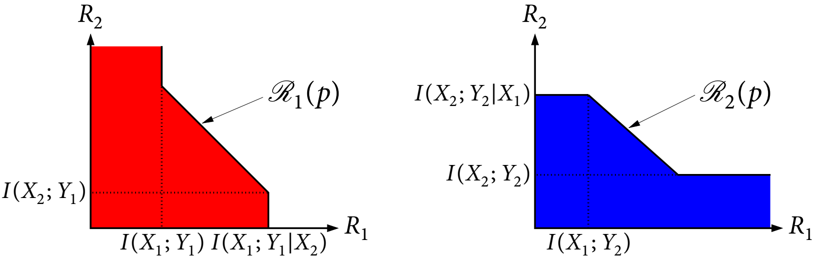
- (R_1, R_2) achievable using IAN $\Rightarrow$ achievable using SND

$\Rightarrow$ SND achieves union of regions for SD and IAN



Interference channel with random ptp codes

- Similar analysis can be performed for receiver 2, yielding $\mathscr{R}_2(p)$



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Theorem (Bandemer–EG–Kim 2012)

The optimal rate region with random ptp codes $p(x_1)p(x_2)$ is:

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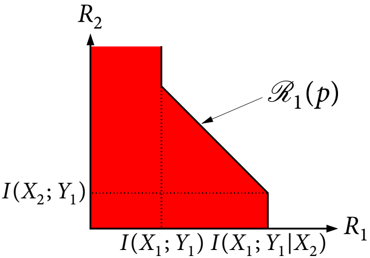

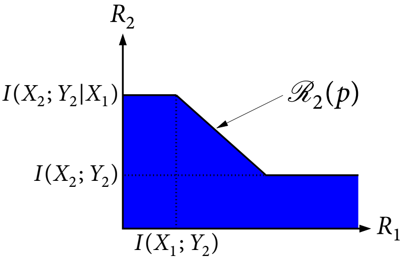

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- SND cannot achieve more than union of SD and IAN regions
- SND achieves same rates as MLD

Interference channel with random ptp codes

- Similar analysis can be performed for receiver 2, yielding $\mathscr{R}_2(p)$

Theorem (Bandemer–EG–Kim 2012)

The optimal rate region with random ptp codes $p(x_1)p(x_2)$ is:

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- Optimal when interference is strong (Costa–EG 1987)
- Generalizes result for deterministic IC (Bandemer–EG 2011)
- Generalizes result for Gaussian IC (Baccelli–EG–Tse 2011)

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- Optimal when interference is strong (Costa–EG 1987)
- Generalizes result for deterministic IC (Bandemer–EG 2011)
- Generalizes result for Gaussian IC (Baccelli–EG–Tse 2011)
- But, random ptp codes can do better

Han–Kobayashi (1981) coding scheme

- Idea: Decode part of interference and treat rest as noise:
  - Split message M_j, $j = 1, 2$, into public message M_{j0} and private message M_{jj}

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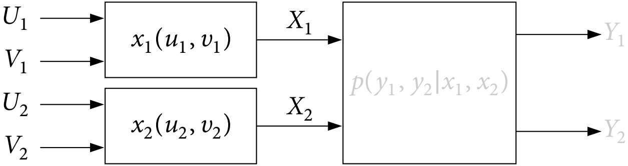
- Idea: Decode part of interference and treat rest as noise:
  - Split message M_j, $j = 1, 2$, into public message M_{j0} and private message M_{jj}
  - Use superposition encoding (Cover 1972)
  - Fix $p(u_1)p(u_2)p(v_1)p(v_2)$, functions $x_1(u_1, v_1)$, $x_2(u_2, v_2)$



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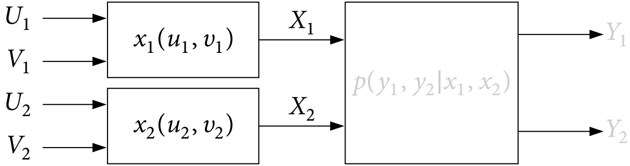

- Codebook generation: For $j = 1, 2$
  - Randomly generate $2^{nR_{j0}}$ sequences $u_j^n(m_{j0}) \sim \prod_{i=1}^n p_{U_j}(u_{ji})$, $m_{j0} \in [1 : 2^{nR_{j0}}]$
  - Randomly generate $2^{nR_{jj}}$ sequences $v_j^n(m_{jj}) \sim \prod_{i=1}^n p_{V_j}(u_{ji})$, $m_{jj} \in [1 : 2^{nR_{jj}}]$

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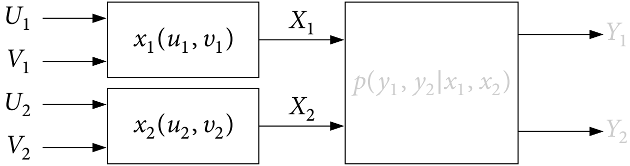

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- Encoding: To send m_j, $j = 1, 2$, transmit $x_j(u_{ji}(m_{j0}), v_{ji}(m_{jj}))$, $i \in [1 : n]$

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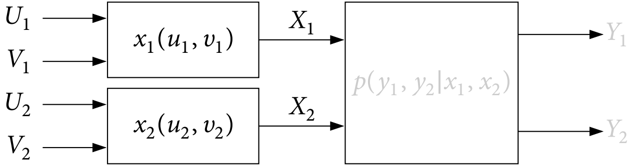

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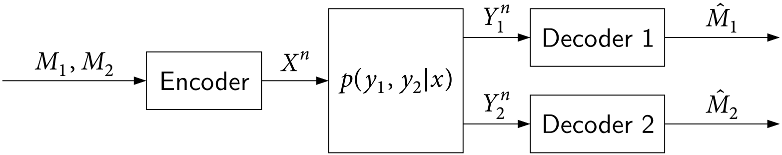
- Encoding: To send m_j, $j = 1, 2$, transmit $x_j(u_{ji}(m_{j0}), v_{ji}(m_{jj}))$, $i \in [1 : n]$
- Decoding: SND is rate optimal (Bandemer–EG–Kim 2012)

Summary

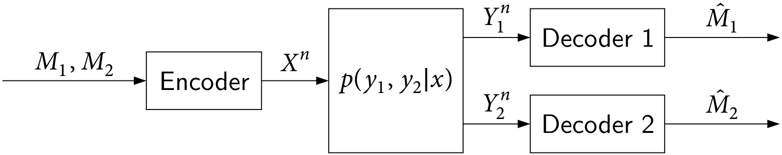
- Multiple access channel:
- Interference channel:
  - Random ptp codes + superposition + SND achieve H–K bound
  - SND achieves the same rates as MLD
  - We don't know how to do better than H–K
- Broadcast channel
- Multihop networks
- Multimessage networks

Broadcast channel



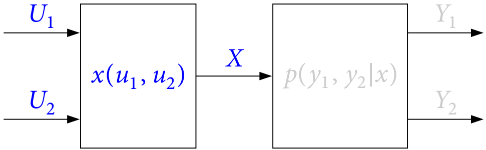
- First studied by Cover (1972)

Broadcast channel



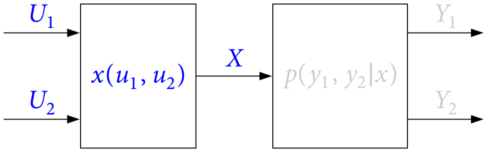
- First studied by Cover (1972)
- $(2^{nR_1}, 2^{nR_2}, n)$ code, $P_e^{(n)}$, achievability, capacity region $\mathcal{C}$: Same as MAC
- Capacity region is not known in general
- Coding schemes that are optimal in some cases

- Use superposition coding: Fix $p(u_1)p(u_2)$, function $x(u_1, u_2)$



Broadcast channel with random ptp codes

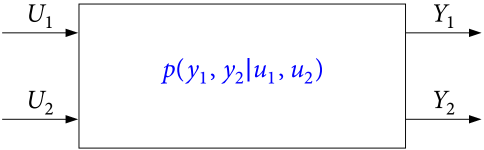
- Use superposition coding: Fix $p(u_1)p(u_2)$, function $x(u_1, u_2)$



- Codebook generation:
  - Randomly generate 2^{nR_1} sequences $u_1^n(m_1) \sim \prod_{i=1}^n p_{U_1}(u_{1i})$, $m_1 \in [1:2^{nR_1}]$
  - Randomly generate 2^{nR_2} sequences $u_2^n(m_2) \sim \prod_{i=1}^n p_{U_2}(u_{2i})$, $m_2 \in [1:2^{nR_2}]$
- Encoding:
  - To send (m_1, m_2), transmit $x(u_{1i}(m_1), u_{2i}(m_2))$, $i \in [1:n]$

Broadcast channel with random ptp codes

- Use superposition coding: Fix $p(u_1)p(u_2)$, function $x(u_1, u_2)$



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- Encoding:
  - To send (m_1, m_2), transmit $x(u_{1i}(m_1), u_{2i}(m_2)),\ i \in [1:n]$
- Exactly the same as scheme for IC without superposition $(X_j \leftarrow U_j)$!

Broadcast channel with random ptp codes

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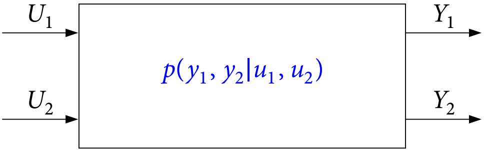

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- Encoding:
  - To send (m_1, m_2), transmit $x(u_{1i}(m_1), u_{2i}(m_2)),\ i \in [1:n]$
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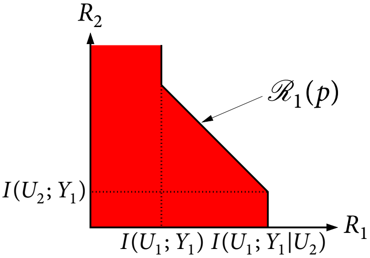
$\Rightarrow$ SND is rate optimal

Broadcast channel with random ptp codes

Theorem (Bandemer–EG–Kim 2012)

The optimal rate region with random ptp codes $p(u_1)p(u_2)$, $x(u_1, u_2)$ is:

$$\mathscr{R}(p) = \mathscr{R}_1(p) \cap \mathscr{R}_2(p)$$



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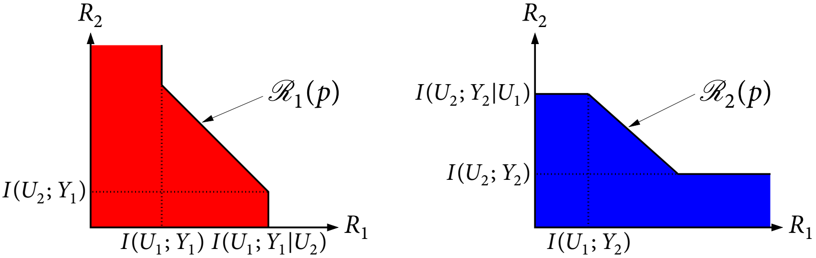

- Includes superposition coding region (Cover 1972, Bergmans 1973):
  - Optimal for degraded, less noisy, more capable BCs
- Also includes Cover–van der Meulen (1975) region

Broadcast channel with random ptp codes

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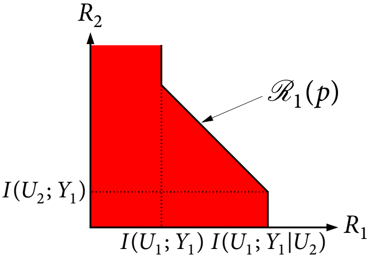
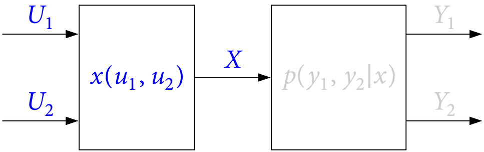


- Includes superposition coding region (Cover 1972, Bergmans 1973):
  - Optimal for degraded, less noisy, more capable BCs
- Also includes Cover–van der Meulen (1975) region
- We can do better using schemes beyond ptp codes

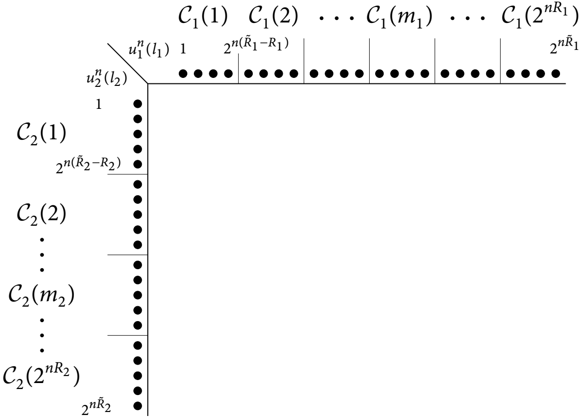
Marton (1979) coding scheme

- Motivation: Since (M_1, M_2) is available at sender, jointly code them
- Fix $p(u_1, u_2)$ (instead of $p(u_1)p(u_2)$) and function $x(u_1, u_2)$



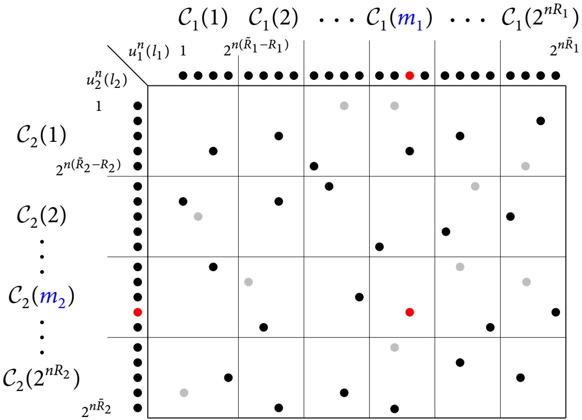
Marton (1979) coding scheme

- Codebook generation: Fix $p(u_1, u_2)$ and function $x(u_1, u_2)$
  - Randomly generate ptp codebooks $u_j^n(l_j)$, $l_j \in [1:2^{n\tilde{R}_j}]$, $\tilde{R}_j > R_j$, $j = 1, 2$
  - Partition each into subcodebooks $\mathcal{C}_j(m_j)$, $m_j \in [1:2^{nR_j}]$, $j = 1, 2$



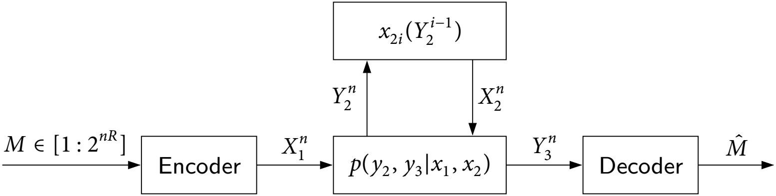
Marton (1979) coding scheme

- Encoding: To send (m_1, m_2):
  - Find pair $(u_1^n(l_1), u_2^n(l_2)) \in \mathcal{T}_\epsilon^{(n)}$, $l_j \in \mathcal{C}_j(m_j)$, $j = 1, 2$
  - Transmit $x(u_{1i}(l_1), u_{2i}(l_2))$, $i \in [1:n]$



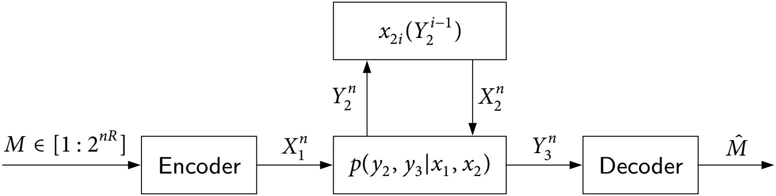
Summary

- Multiple access channel:
  - Random ptp codes achieve capacity region
- Interference channel
  - Random ptp codes achieve best known inner bound
- Broadcast channel:
  - Can do better than ptp codes using Marton coding
- Relay channel
- Multicast networks
- Multimessage networks



- First studied by van der Meulen (1971)

Relay channel



- First studied by van der Meulen (1971)
- Capacity C is not known in general
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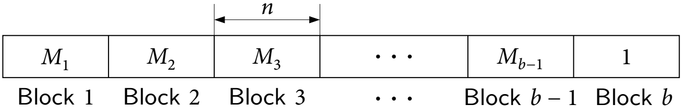
RC with random ptp codes: Multihop

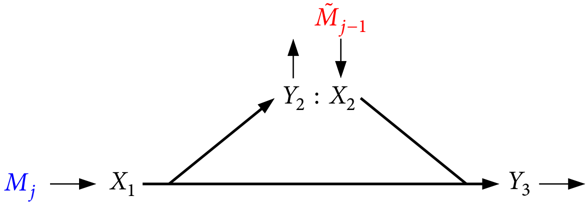
- Send $b-1$ messages over b n-transmission blocks



- Fix $p(x_1)p(x_2)$, generate (for each block) ptp codes for sender and relay

RC with random ptp codes: Multihop

- Send $b-1$ messages over b n-transmission blocks



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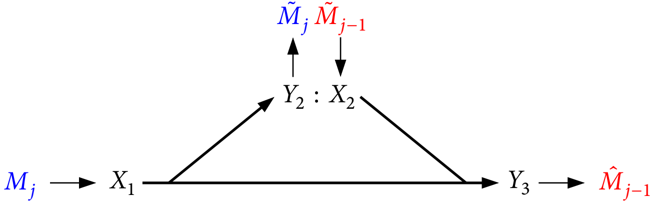

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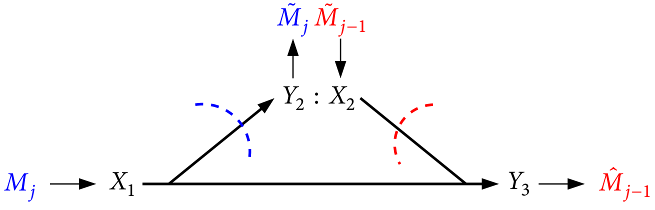

- This achieves:

$$R = \min\{I(X_1; Y_2 | X_2),\ I(X_2; Y_3)\}$$

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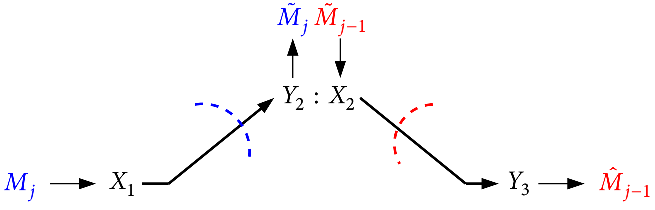


- This achieves:

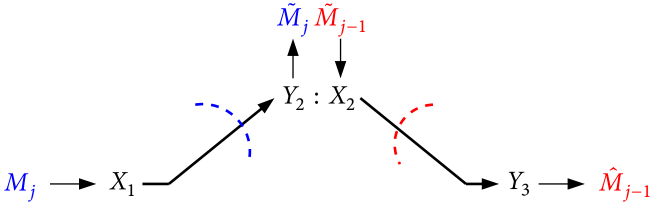
$$R = \min\{I(X_1; Y_2|X_2),\ I(X_2; Y_3)\}$$

- Optimal for a cascade of two DMCs: $p(y_2, y_3|x_1, x_2) = p(y_2|x_1)p(y_3|x_2)$

$$C = \min\left\{I(X_1; Y_2),\ I(X_2; Y_3)\right\}$$

RC with random ptp codes: Multihop

- Send $b-1$ messages over b n-transmission blocks



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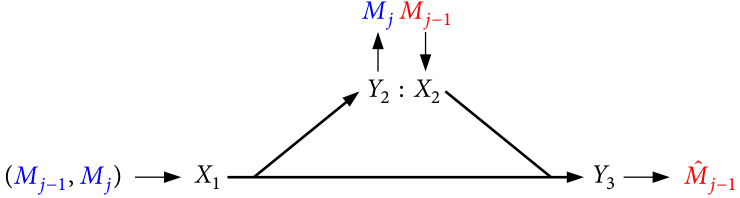
- But we can do better via superposition and more sophisticated decoding

RC with random ptp codes: Decode–forward (DF)

- The sender knows what the relay knows:
  - They can coherently cooperate via superposition coding at the sender
  - Fix $p(x_2)p(u)$ and $x_1(u, x_2)$, generate ptp codes for relay and sender
- Receiver uses backward decoding (Zeng–Kuhlmann–Buzo 1989)



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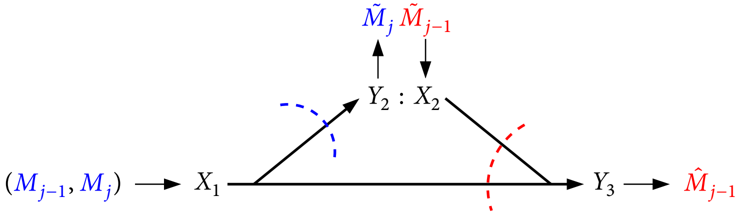

- Achieves (Cover–EG 1979):

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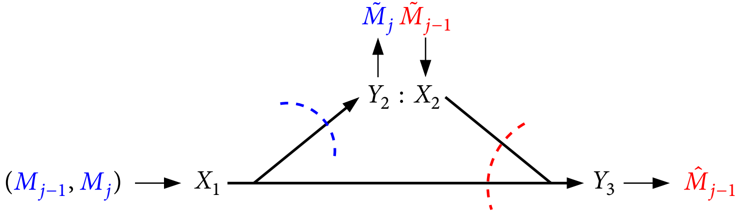


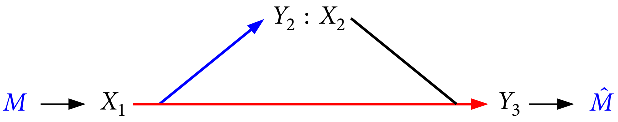
- Achieves (Cover–EG 1979):

$$R = \min\{I(X_1; Y_2|X_2),\ I(X_1, X_2; Y_3)\}$$

- Optimal for physically degraded RC: $(X_1, X_2) \to (Y_2, X_2) \to Y_3$

Compress–forward (Cover–EG 1979)

- DF does not do well when $X_1 \to Y_2$ is not better than $X_1 \to Y_3$



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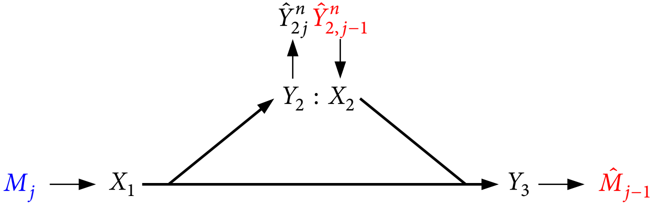
- Can do better by relaying compressed version of Y_2^n instead of DF



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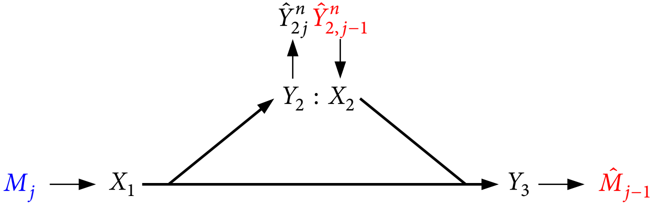
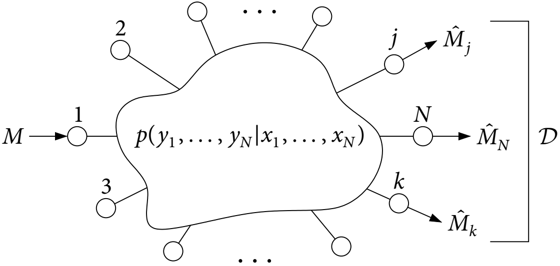


- Does not use ptp codes as defined
- Optimal for some RCs (Aleksic–Razaghi–Yu 2009)

Summary

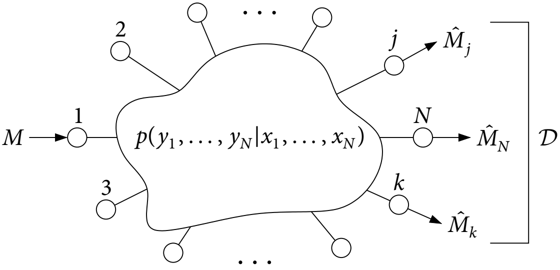
- Multiple access channel:
- Interference channel:
- Broadcast channel
- Relay channel
  - Random ptp codes + superposition + backward decoding achieve DF bound
  - Can sometimes do better using schemes beyond ptp codes, e.g., CF
- Multicast networks
- Multimessage networks:

Multicast network



- Capacity is not known in general (relay channel is special case)

Multicast network



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- DF can be extended (Xie–Kumar 2005, Kramer–Gastpar–Gupta 2005)
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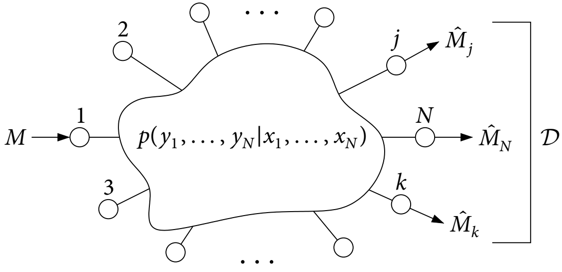
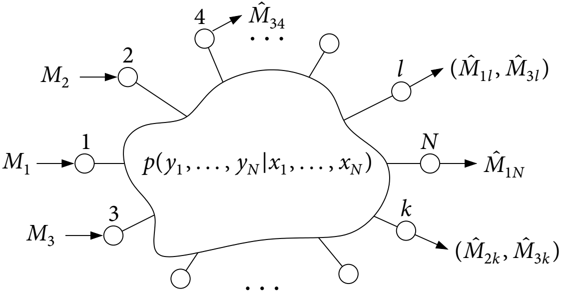


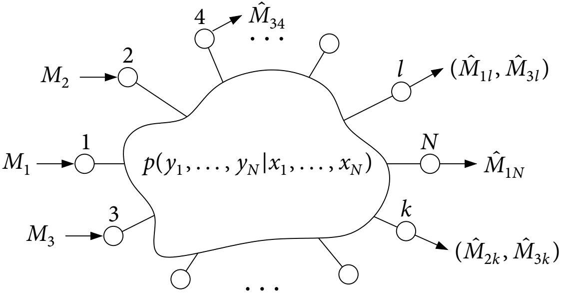
- Capacity is not known in general (relay channel is special case)
- DF can be extended (Xie–Kumar 2005, Kramer–Gastpar–Gupta 2005)
- CF can be extended (noisy network coding (EG–Kim 2011))
- Gaussian multicast network:
  - CF achieves within $0.63N$ bit of capacity (Lim–Kim–EG–Chung 2011)
  - DF can have unbounded gap to capacity

Multimessage network



- Node j wishes to send message M_j to set of destination nodes $\mathcal{D}_j$

Multimessage network



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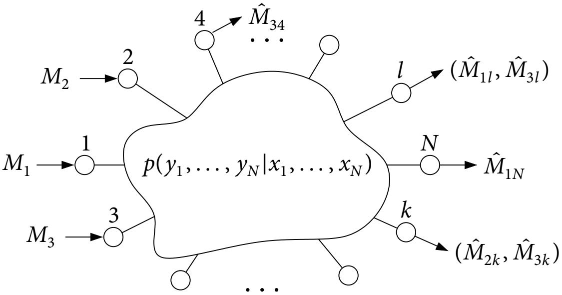

- Node j wishes to send message M_j to set of destination nodes $\mathcal{D}_j$
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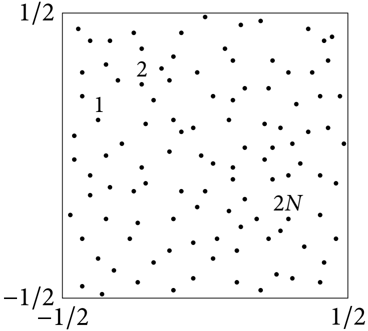
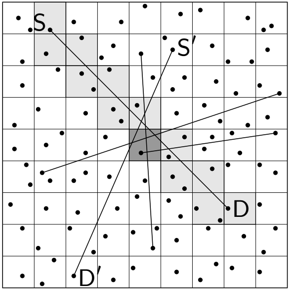
- How well do ptp codes perform?

Capacity scaling laws



- Random wireless network (Gupta–Kumar 2000):
  - Multi-unicast: Send message M_j at rate R from node $j \in [1:N]$ to $k = j + N$
  - Gaussian network model: Path loss exponent $\nu \geq 2$, average power constraint
  - What is the scaling law for symmetric capacity C (highest achievable R)?



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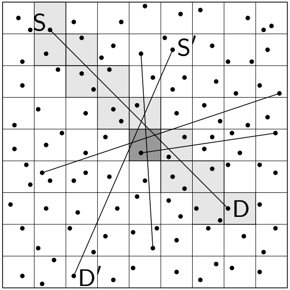

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- Ptp codes + cellular time division + multihop: $C = \Omega(N^{-1/2})$
- Upper bound (Lévêque–Telatar 2005): $C = O(N^{-1/2+1/\nu} \log N)$

Conclusion and final remarks

- Random ptp codes with sophisticated decoding can perform well:
  - Achieve capacity region of MAC
  - Achieve the Han–Kobayashi bound for IC (best known)
  - Achieve the superposition coding and Cover–van der Meulen bounds for BC
  - Achieve decode–forward bound for multicast networks
  - Achieve close to optimal scaling law for random multi-unicast networks

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  - Work on constructing better decoders
  - Find better practical codes for broadcasting and relaying
- The best answer, to quote Tom Cover, is:

"Theory is the first term in the Taylor series expansion of practice."

Thank You!

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