Research Statement

Bin Zhao

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I do research in number theory and arithmetic geometry, and particularly in the area related to Galois representations and deformation of abelian varieties.

It is a celebrated theorem of Serre and Tate that the deformation theory of an abelian variety A over a field of characteristic $p > 0$ is equivalent that of its Barsotti-Tate group. When the abelian variety A is ordinary, there is a structure of formal group on the infinitesimal deformation space of $A_{/k}$, and any deformation of A is determined by the Serre-Tate coordinates on the deformation space ([5]). Moreover, the Serre-Tate deformation theory can be applied to determine the local structure of Shimura varieties of Hodge type at ordinary locus ([11],[14]). My research has been forcused on applying the Serre-Tate theory to two types of questions in arithmetic geometry: the first is the study of the local behavior of Galois representations attached to nearly ordinary Hilbert modular forms; the second is the study of special cases of the Mumford-Tate conjecture. I will give a summary of my work ([19],[20]) toward these two directions below.

1 Local behavior of Hilbert modular Galois representations

Let F be a totally real field inside an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and let $F_{\mathbb{A}}$ be the adele ring of F. Fix a rational prime p and a prime $\mathfrak{p}$ of F over p. Let $\mathfrak{m}$ be an integral ideal of F and $S_k(\mathfrak{m}, \mathbb{C})$ be the space of Hilbert modular cusp forms of parallel weight k and level $\mathfrak{m}$. For each prime $\mathfrak{q}$ of F and fractional ideal $\mathfrak{n}$ of F prime to $\mathfrak{m}$, let $T(\mathfrak{q})$ and $\langle\mathfrak{n}\rangle$ be the Hecke operator acting on $S_k(\mathfrak{m}, \mathbb{C})$. Suppose that we have a Hilbert modular form $f \in S_k(\mathfrak{m}, \mathbb{C})$ such that f is an eigenvector for all the above Hecke operators, i.e. there exist $c(\mathfrak{q}, \mathfrak{f}), d(\mathfrak{n}, f)$ such that $T(\mathfrak{q})(f) = c(\mathfrak{q}, \mathfrak{f})f$ and $\langle\mathfrak{n}\rangle(f) = d(\mathfrak{n}, f)f$. Let K_f be the field generated over $\mathbb{Q}$ by these eigenvalues $c(\mathfrak{q}, \mathfrak{f}), d(\mathfrak{n}, f)$, which is known to be a number field. Let λ be a prime of K_f over the rational prime p and let $K_{f,\lambda}$ be the completion of K_f at λ. Then it is well known that there is a continuous representation $\rho_{f,\lambda} : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to \mathrm{GL}_2(K_{f,\lambda})$ attached to f. Moreover, if f is nearly $\mathfrak{p}$-ordinary in the sense that $c(\mathfrak{p}, f) \in K_f \subset K_{f,\lambda}$ is a p-adic unit, then the restriction of $\rho_{f,\lambda}$ to the decomposition group $D_{\mathfrak{p}}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ at $\mathfrak{p}$ is reducible and of the following shape:

$$\rho_{f,\lambda}|_{D_{\mathfrak{p}}} \sim \begin{pmatrix} \epsilon_1 & * \\ 0 & \epsilon_2 \end{pmatrix},$$

where ϵ_1, ϵ_2 are two characters of $D_{\mathfrak{p}}$. When f is a binary theta series of the norm form of a CM quadratic extension M of F (we say that f has complex multiplication in this case, see [7] section 6.2 for the construction), then the local representation $\rho_{f,\lambda}|_{D_{\mathfrak{p}}}$ splits. When $F = \mathbb{Q}$, R.Greenberg asked whether the local representation $\rho_{f,\lambda}|_{D_{\mathfrak{p}}}$ splits only when f has complex multiplication. This question has been studied by B.Balasubramanyam, E.Ghate and V.Vatsal in [1] and [4].

To state my result, I need to impose the following assumption:

(H) if the degree $[F : \mathbb{Q}]$ is even, there exists some finite place v of F such that π_v is square integrable, where $\pi_f = \otimes_w \pi_w$ is the automorphic representation of $\mathrm{GL}_2(F_{\mathbb{A}})$ attached to f.

My result can be stated as follows:

Theorem 1. *([19]) Under the assumption (H), if the Hilbert modular form f is parallel of weight 2 and does not have complex multiplication, then the local Galois representation $\rho_{f,\lambda}|_{D_\mathfrak{p}}$ does not split.*

The proof is based on Hida's result on the local indecomposability of Tate modules of non CM abelian varieties with real multiplication ([8]). It gives a positive answer to Greenberg's question for all weight 2 elliptic modular eigenforms. Also it implies results in [1] under the assumption (H).

The above result also has some interesting consequences. Let N be an integer prime to p. For each integer k we use $M_k^\dagger(\Gamma_1(N))$ to denote the space of overconvergent p-adic modular forms of level N over $\mathbb{C}_p$. In [2], Coleman constructed a linear map $\theta^{k-1} : M_{2-k}^\dagger(\Gamma_1(N)) \to M_k^\dagger(\Gamma_1(N))$ such that the effect of θ^{k-1} on the q-expansions is given by the differential operator $(q\frac{d}{dq})^{k-1}$. He proved that for $k \geq 2$, every classical CM cusp eigenform of weight k and slope $k-1$ is in the image of map θ^{k-1} and he asked whether there is a non-CM classical eigenform in the image of θ^{k-1}. As a corollary of Theorem 1, I can give a partial answer to Coleman's question when $k=2$:

Proposition 1. *Let g be a weight two normalized classical cusp eigenform on $\Gamma_1(N)$ with the Hecke field K_g. Suppose that there exists a prime $\mathfrak{p}$ of K_g over the rational prime p such that g is $\mathfrak{p}$-ordinary, and the associated slope one oldform f is in the image of the operator θ. Then g is a CM eigenform.*

2 Mumford-Tate conjecture for abelian fourfolds

Let F be a number field with an algebraic closure $\bar{F}$ and a complex embedding $i : F \hookrightarrow \mathbb{C}$. Let A be an abelian variety of dimension d over F.

The singular homology group $V = H_1(A(\mathbb{C}), \mathbb{Q})$ is a $2d$-dimensional vector space over $\mathbb{Q}$. We have the Hodge decomposition $V_\mathbb{C} = V \otimes_\mathbb{Q} \mathbb{C} = V^{-1,0} \oplus V^{0,-1}$, such that $\overline{V^{-1,0}} = V^{0,-1}$. We define a cocharacter $\mu_\infty : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{Aut}_\mathbb{C}(V_\mathbb{C})$ such that any $z \in \mathbb{C}^\times$ acts on $V^{-1,0}$ by multiplication by z^{-1} and acts trivially on $V^{0,-1}$.

Definition 2. *The Mumford-Tate group of the abelian variety $A_{/F}$ is the smallest algebraic subgroup $\mathrm{MT}(A) \subset \mathrm{Aut}_\mathbb{Q}(V)$ defined over $\mathbb{Q}$ such that the cocharacter μ_∞ factors through $\mathrm{MT}(A) \times_\mathbb{Q} \mathbb{C}$.*

For any rational prime l, let $T_lA(\bar{F})$ be the l-adic Tate module of A and set $V_l = T_lA(\bar{F}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, which is a $2d$-dimensional vector space over $\mathbb{Q}_l$. Then we have a Galois representation:

$$\rho_l : \mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}_{\mathbb{Q}_l}(V_l).$$

We define $G_{l/\mathbb{Q}_l}$ as the Zariski closure of the image of ρ_l inside $\mathrm{Aut}_{\mathbb{Q}_l}(V_l)$ and let $G^\circ_{l/\mathbb{Q}_l}$ be its identity connected component. Under the comparison isomorphism $V \otimes_\mathbb{Q} \mathbb{Q}_l \cong V_l$, the Mumford-Tate conjecture predicts that:

Conjecture 3. *For any prime l, we have the equality $G^\circ_{l/\mathbb{Q}_l} = \mathrm{MT}(A) \times_\mathbb{Q} \mathbb{Q}_l$.*

In [3], Deligne proved the following:

Theorem 2. *For any prime l, we have the inclusion $G^\circ_{l/\mathbb{Q}_l} \subseteq \mathrm{MT}(A) \times_\mathbb{Q} \mathbb{Q}_l$.*

My work is concerned with the case that $A_{/F}$ is an absolutely simple abelian fourfolds with trivial endomorphism algebra, i.e. $\mathrm{End}^\circ(A_{/\bar{F}}) = \mathbb{Q}$. In this case, from the calculation of Moonen and Zarhin [12], there are two possibilities for the Lie algebra of $\mathrm{MT}(A)_{/\mathbb{Q}}$ together with its action on V (resp. the Lie algebra of $G^\circ_{l/\mathbb{Q}_l}$ together with its action on V_l) when base changed to an algebraic closure of $\mathbb{Q}$ (resp. $\mathbb{Q}_l$):

1. $\mathfrak{c} \oplus \mathfrak{sp}_4$ with the standard representation, where $\mathfrak{c}$ is the 1-dimensional center $\mathfrak{c}$ of the Lie algebra;

2. $\mathfrak{c} \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, with the 1-dimensional center $\mathfrak{c}$, and the representation of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is the tensor product of the standard representation of $\mathfrak{sl}_2$.

It is proved by Noot [16] that the second case happens to the Lie algebra of $\mathrm{MT}(A)_{/\mathbb{Q}}$ together with its action on V if and only if $A_{/F}$ comes from an analytic family of abelian varieties constructed by Mumford in [10]. Noot also proved in [15] that if the Lie algebra of $G^{\circ}_{l/\mathbb{Q}_l}$ together with its action on V_l is of the second case for one prime l, it is true for all primes l's, and we call $A_{/F}$ an abelian variety with Galois representation of Mumford's type in this case.

My study on the Mumford-Tate conjecture for abelian fourfolds is based on two ingredients: formal linearity of Shimura varieties of Hodge type and the relationship between the Serre-Tate coordinates and the local Galois representation of an ordinary abelian variety.

If the abelian fourfold $A_{/F}$ comes from Mumford's analytic family of abelian varieties, we can find a Shimura curve C defined over some number field F' and a closed immersion $C \hookrightarrow \mathcal{A}_{4,1,n}$ into the Siegel moduli space $\mathcal{A}_{4,1,n}$ which parameterizes isomorphism classes of principally polarized abelian fourfolds with a suitable level n structure, such that C contains the abelian variety $A_{/F}$. To simplify the notations, by extending the base field if necessary, we can assume that $F = F'$. We assume that $A_{/F}$ has good ordinary reduction at a finite place v of F, whose residue characteristic does not divide n. Let $\mathcal{O}_F$ be the ring of integers of F and the Siegel moduli space $\mathcal{A}_{4,1,n}$ has an integral model defined over $R = \mathcal{O}_{F,(v)}$, which is the localization of $\mathcal{O}_F$ at the place v . The special fiber of $A_{/F}$ at v gives a closed ordinary point x_v on $\mathcal{A}_{4,1,n/R}$. Let $\tilde{C}_{/R}$ be the Zariski closure of $C_{/F}$ inside $\mathcal{A}_{4,1,n/R}$. From the formal linearity of Shimura subvarieties of Hodge type ([15]), the formal completion of $\tilde{C}$ at x_v should be a rank 1 formal torus.

In [20], I studied the relationship between the Serre-Tate coordinates of an abelian variety with its local Galois representation. To be more precise, let $A_{/F}$ be an abelian variety defined over a number field of dimension d with a principal polarization λ. Suppose that $A_{/F}$ has good ordinary reduction at a finite place v. Let k_v be the residue field of F at v with characteristic $p = p_v$ and T_pA be the p-adic Tate module of $A_{/F}$. The polarization λ induces a nondegenerate alternating form on T_pA, and after fixing a $\mathbb{Z}_p$-basis of $\mathrm{T}_p\mu_{p^\infty}$, we can choose a symplectic basis $\{v_1^{\circ}, \dots, v_d^{\circ}, v_1^{et}, \dots, v_d^{et}\}$ of the p-adic Tate module T_pA, such that $\{v_1^{et}, \dots, v_d^{et}\}$ corresponds to a basis of T_pA_v under the reduction map, and the local Galois representation is of the shape:

$$\begin{aligned} \rho_p : \quad I_v &\to \mathrm{GSp}_{2d}(\mathbb{Z}_p) \\ \sigma &\mapsto \begin{pmatrix} \chi_p(\sigma) \cdot \mathrm{I}_d & B(\sigma) \\ 0 & \mathrm{I}_d \end{pmatrix} \end{aligned}$$

here $I_v \subseteq \mathrm{Gal}(\bar{F}/F)$ is the inertia group at v, $\chi_p : I_v \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character, I_d is the $d \times d$ identity matrix, and $B(\sigma) = (b_{ij}(\sigma))_{1 \le i,j \le d}$ is a $d \times d$ matrix. By direct calculation, the maps $b_{ij} : I_v \to \mathbb{Z}_p$ are 1-cocycles valued in $\mathbb{Z}_p(\chi_p)$ for all $1 \le i, j \le 4$ and give elements in the cohomology group $H^1(I_v, \mathbb{Z}_p(\chi_p))$. The local Galois representation ρ_p and the Serre-Tate coordinates of $A_{/F}$ are related by the following:

Theorem 3. *([20] Remark 2.1) Under the isomorphism*

$$H^1(I_v, \mathbb{Z}_p(\chi_p)) \cong H^1(I_v, \mathrm{T}_p\mu_{p^\infty}) \cong \widehat{(F_v^{ur})^\times},$$

where F_v^{ur} is the maximal unramified extension of the v-adic completion of F and $\widehat{(F_v^{ur})^\times}$ is the pro-p-completion of the multiplicative group $(F_v^{ur})^\times$, the 1-cocycles b_{ij}'s correspond to the Serre-Tate coordinates $q(A_{/F}; u_i, u_j)$ of $A_{/F}$.

We now start with an abelian variety $A_{/F}$ with Galois representation of Mumford's type (so we make no assumption on the existence of the Shimura curve C as above). Pink ([17]) proved that there is a subset Ω of finite places of F with Dirichlet density 1 at which $A_{/F}$ has good ordinary reduction. In [20], combining previous results developed by Noot ([16]) with Theorem 3, I was able to determine explicitly the rank 1 formal subtorus in the local deformation space of the special fiber A_{v/k_v} of $A_{/F}$ at v which contains $A_{/F}$ for infinitely many places $v \in \Omega$, because under the ordinary assumption the image of the p-inertia group under the p-adic Galois representation is a rank 1-module over $\mathbb{Z}_p$. By considering the CM points on the rank 1 formal subtorus, I prove the following:

Theorem 4. *([20] Theorem 1) Let $A_{/F}$ be an abelian variety of dimension 4 over a number field F. Suppose that $\mathrm{End}(A_{/\bar{F}}) = \mathbb{Z}$. If for some prime l, the group $G^{\circ}_{l/\mathbb{Q}_l}$ together with its action on V_l belongs to the second case listed above, then the same is true for the the group $\mathrm{MT}(A)_{/\mathbb{Q}}$ together with its action on V, i.e. the Mumford-Tate conjecture holds for $A_{/F}$.*

3 Future project

My further project contains two parts:

1. the first project is to extend my result on the local behavior of Hilbert modular Galois representations to higher weights as well as to remove the assumption (H) in Theorem 1. In [1] and [4], Balasubramanyam, Ghate and Vatsal used Hida family of ordinary modular forms to get a statistical result for hight weight modular forms. I would like to give a more detailed analysis on the variation of local Galois representations on a Hida family as to prove the previous results. This question has intimate relationship with the properties of Λ-adic Barsotti-Tate groups constructed in [6] and [9], which is another subject I am interested in;

2. the second project is to establish new cases of Mumford-Tate conjecture. For an abelian variety $A_{/F}$ as in Theorem 4, the argument in [20] consists of four key steps:

   (a) to show the existence of places at which $A_{/F}$ has good ordinary reduction;

   (b) the analysis of the isogeny type of the reduction A_{v/k_v} of the abelian variety $A_{/F}$ at a good ordinary place v;

   (c) the determination of the rank 1 formal subtorus in the deformation space of A_{v/k_v} which contains $A_{/F}$;

   (d) using CM points on the formal subtorus determined in the previous step to construct the Mumford-Tate group of $A_{/F}$.

   As is clear from the proof in [20], there is much room for a generalization of our argument to abelian varieties of higher dimensions. In [18], Mumford's construction of analytic family of abelian fourfolds was generalized to that of abelian varieties of dimension 4^k for all $k \geq 1$. The properties of ordinary reduction of these abelian varieties have been discussed in [13], which establish step (a). The rest three steps are generalization of the argument in [15] and [20] with appropriate modification. Hence the method in [20] can be applied to these abelian varieties without essential changes.

   On the other hand, let $A_{/F}$ be an abelian variety of dimension g over a number field F such that $\mathrm{End}(A_{/\bar{F}}) = \mathbb{Z}$. In [17], Pink proved that the Mumford-Tate conjecture holds for $A_{/F}$ under the assumption that $2g$ is neither a k-th power for any odd $k > 1$ nor of the form $\binom{2k}{k}$ for any odd $k > 1$. In fact, Pink proved that the only candidate of the Mumford-Tate group and the p-adic monodromy groups of $A_{/F}$ is GSp_{2g} while more candidates show up when the dimension g does not satisfy his assumption. I would like to start with the case when g is of the form $\binom{2k}{k}$ for some odd $k > 1$. In this case, there are only two possibilities for the the Mumford-Tate group as well as the p-adic monodromy groups of $A_{/F}$. From Deligne's result (Theorem 2), to establish the Mumford-Tate conjecture in this case, it is enough to prove: if the p-adic monodromy group of $A_{/F}$ is not GSp_{2g}, neither is the Mumford-Tate group of $A_{/F}$. As Pink also discussed about the places of ordinary reduction of $A_{/F}$ in [17], his result can be used to establish step (a) in our method. As we are dealing with Shimura varieties of higher dimensions in this setting, in step (c) we need to determine a formal subtorus of rank bigger than 1. To generalize the rest three steps, we need to make more effort on characterizing this formal subtorus.

References

[1] B. Balasubramanyam, E. Ghate and V. Vatsal. On local Galois representations associated to ordinary Hilbert modular forms, to appear in Manuscripta Math.

[2] R. Coleman. Classical and overconvergent modular forms. Invent. Math. **124** (1996), no. 1-3, 215-241.

[3] P. Deligne, J.S. Milne. Tannakian categories, in: Hodge cycles, motives, and Shimura varieties (P.Deligne et al., Eds.), ch. II, Lect. Notes Math. **900** (1982), 101-228.

[4] E. Ghate and V . Vatsal. On the local behaviour of ordinary Λ-adic representations, Ann.Inst. Fourier (Grenoble) **54** (2004), 2143-2162

[5] N.M. Katz. Serre-Tate local moduli. Algebraic surfaces (Orsay, 1976-78), 138-202, Lecture Notes in Math., 868, Springer, Berlin-New York, 1981.

[6] H. Hida. Λ-adic p-divisible groups I,II. lecture notes at CRM (Montreal) in September 2005.

[7] H. Hida. Anticyclotomic main conjectures. Doc. Math. (2006), Extra Vol., 465-532.

[8] H. Hida, Local indecomposability of Tate modules of non CM abelian varieties with real multiplication, J. Amer. Math. Soc. 26 (2013), 853-877.

[9] H. Hida. Λ-adic p-divisible groups. to appear in Parcific J. Math in a memorial volume of J.Rogawski.

[10] D. Mumford. A note of Shimura's paper "Discontinuous groups and abelian varieties". Math. Ann. **181** (1969) 345-351.

[11] B. Moonen. Linearity properties of Shimura varieties, II Compositio Mathematica **114** (1998) 3-35.

[12] B. Moonen, Yu.G. Zarhin. Hodge classes and Tate classes on simple abelian fourfolds. Duke Math. J. **77** (1995), no. 3, 553-581.

[13] R. Noot. Abelian varieties-Galois representations and properties of ordinary reduction. Compositio Mathematica **97** (1995), 161-171.

[14] R. Noot. Models of Shimura varieties in mixed characteristic. J. Algebraic Geom. **5** (1996), no. 1, 187-207.

[15] R. Noot. Abelian varieties with l-adic Galois representation of Mumford's type, J.reine angew. Math. **519** (2000), 155-169.

[16] R. Noot. On Mumford's families of abelian varieties. Journal of Pure and Applied Algebra **157** (2001), 87-106.

[17] R. Pink. l-adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture, J. Reine Angew. Math. **495** (1998), 187-237.

[18] S. Tankeev. On algebraic cycles on surfaces and abelian varieties, Math. USSR Izvestia, **18** (1982), 349-380.

[19] B. Zhao. Local Indecomposability of Hilbert Modular Galois Representations, to appear in Ann.Inst. Fourier (Grenoble).

[20] B. Zhao. On the Mumford-Tate conjecture of abelian fourfolds, preprint available at http://www.math.ucla.edu/ zhaobin/MTAV4.pdf, to be submitted.