

## STRICT CONVEXITY OF SOME SUBSETS OF HANKEL OPERATORS

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ABSTRACT. We find some extreme points in the unit ball of the set of Hankel operators and show that the unit ball of the set of compact Hankel operators is strictly convex. We use this result to show that the collection of  $N \times N$  lower triangular Toeplitz contractions is strictly convex. We also find some extreme points in certain reduced Cowen sets and discuss cases in which they are or are not strictly convex.

### 1. INTRODUCTION

M. Chō, R. Curto, and W. Lee [2] recently showed that the set  $\mathcal{T}_N$  of all lower triangular contractive Toeplitz matrices, matrices of the form

$$(1.1) \quad \begin{pmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_1 & c_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{N-2} & & \ddots & \ddots & 0 \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{pmatrix},$$

is strictly convex. Their proof relied on the solution to the Carathéodory-Schur interpolation problem. In this paper, we will find some extreme points in the unit ball of the set of Hankel operators and use these to show that the unit ball of the set of compact Hankel operators is strictly convex. The result for lower triangular Toeplitz contractions can then be viewed as a special case. We will also examine, as in [2], the strict convexity of certain Cowen sets related to hyponormal Toeplitz operators. We thus give a complete answer to a question of Cowen [3].

The exploration of extreme points in the unit ball of the set of Hankel operators may have interest beyond those aspects discussed here. We will use  $H_f$  to denote the Hankel operator with  $L^\infty$  symbol  $f$ , given by

$$H_f : H^2 \longrightarrow (H^2)^\perp, \quad H_f h = (I - P) f h$$

where  $P$  is the orthogonal projection from  $L^2$  to  $H^2$ . Nehari's Theorem (see [9]) tells us that  $\|H_f\| = \text{dist}_{L^\infty}(f, H^\infty)$ . This lets us see the set of Hankel operators

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as isometrically isomorphic to  $L^\infty/H^\infty$ , and extreme points in the unit ball of the set of Hankel operators correspond to extreme points in the unit ball of  $L^\infty/H^\infty$ . There is also a natural isometric isomorphism between  $L^\infty/H^\infty$  and  $(H_0^1)^*$ .

## 2. PRELIMINARIES

We will need to make use of the following lemma of V. Adamjan, D. Arov, and M. Krein [1].

**Lemma 1.** *Assume  $x$  is a vector in  $H^2$  satisfying  $\|H_f x\| = \|H_f\| \|x\|$  and set  $y = H_f x$ . Then  $\|H_f\|^{-1} (e^{it} y/x)$  is unimodular on the unit circle.  $q = e^{it} y/x$  is the only function satisfying*

$$H_f = H_q \text{ and } \|q\|_\infty = \|H_f\|.$$

For an inner function  $\theta$ , let  $\mathcal{H}(\theta) = H^2 \ominus \theta H^2$ . The following lemma is from C. Foias, A. Tannenbaum and G. Zames [6]. For completeness, we outline a proof.

**Lemma 2.** *Let  $\theta$  be an inner function and  $w$  be an analytic rational function, such that  $\theta$  and  $w$  are coprime. The essential norm of  $H_{\bar{\theta}w}$  is equal to*

$$(2.1) \quad \max\{|w(z)| : z \text{ is any essential singularity of } \theta\}.$$

*Proof.* Since  $H_{\bar{\theta}w} = H_{\bar{\theta}} T_w$ , where  $T_w$  is the Toeplitz operator with symbol  $w$ , and  $H_{\bar{\theta}}^* H_{\bar{\theta}}$  is the projection of  $H^2$  onto  $\mathcal{H}(\theta)$ ,

$$H_{\bar{\theta}w}^* H_{\bar{\theta}w} = T_w^* H_{\bar{\theta}}^* H_{\bar{\theta}} T_w = w^*(T)w(T)$$

where  $T$  is the model operator on  $H(\theta)$ ,  $T = P_{\mathcal{H}(\theta)} z|_{\mathcal{H}(\theta)}$ , and  $w(T)$  is the functional calculus. The essential norm of  $w(T)$  is given by (2.1), see [8].  $\square$

## 3. SUBSETS OF HANKEL OPERATORS

Now we prove the main theorem of this section and discuss some of its consequences.

**Theorem 1.** *If  $\|H_f\| = 1$  and there is an  $x$  in  $H^2$  with  $\|H_f x\| = \|H_f\| \|x\|$ , then  $H_f$  is an extreme point in the unit ball of the set of Hankel operators.*

*Proof.* We must show that if  $\|H_f\| = 1$  and  $H_f$  attains its norm, and we write  $H_f = \frac{1}{2}(H_g + H_k)$  for  $H_g$  and  $H_k$  in the unit ball of the set of Hankel operators, then we must have  $H_f = H_g = H_k$ . Assume we can write  $H_f = \frac{1}{2}(H_g + H_k)$  as above, for some  $g, k \in L^\infty$ . It is clear that since  $\|H_f\| = 1$ , we must also have  $\|H_g\| = 1$  and  $\|H_k\| = 1$ . Then by Nehari's Theorem, there must be functions  $q_1$  and  $q_2$  with

$$\begin{aligned} \|H_g\| &= \inf_{q \in H^\infty} \|g - q\|_\infty = \|g - q_1\|_\infty = 1 \text{ and} \\ \|H_k\| &= \inf_{q \in H^\infty} \|k - q\|_\infty = \|k - q_2\|_\infty = 1. \end{aligned}$$

Using Lemma 1, let  $p$  the unique unimodular function such that  $H_f = H_p$  and  $\|H_f\| = \|p\|_\infty = 1$ .

We can then write

$$H_f = H_p = \frac{1}{2}(H_{g-q_1} + H_{k-q_2}) = H_{\frac{1}{2}(g+k-q_1-q_2)}.$$

Then there is some  $h \in H^\infty$  with

$$p = \frac{1}{2}(g + k - q_1 - q_2) + h.$$

We then have

$$\begin{aligned} 1 &\geq \left\| \frac{1}{2}(g - q_1) + \frac{1}{2}(k - q_2) \right\|_\infty \\ &= \|p - h\|_\infty \\ &\geq \inf_{q \in H^\infty} \|p - q\|_\infty = \|p\|_\infty = 1. \end{aligned}$$

The inequalities above must all be equalities, i.e., we have  $1 = \inf_{q \in H^\infty} \|p - q\|_\infty = \|H_p\| = \|p\|_\infty$ . Lemma 1 tells us there is a unique function  $q$  (which will be unimodular) with  $H_p = H_q$  and  $\|H_p\| = \|q\|_\infty = 1$ , and  $p$  is such a function, so  $p$  is unimodular. Since  $H_{p-h} = H_p$  and  $\|p - h\|_\infty = 1$ , by uniqueness we must have  $h = 0$ , and

$$\frac{1}{2}(g - q_1) + \frac{1}{2}(k - q_2) = p$$

is thus unimodular. We can thus conclude that  $\frac{1}{2}(g - q_1) = \frac{1}{2}(k - q_2)$ , and thus  $H_g = H_k$ , from which it follows that both are equal to  $H_f$ .  $\square$

**Corollary 1.** *The set of all compact Hankel operators is strictly convex.*

*Proof.* It is easy to see that this set is convex. All compact Hankel operators attain their norms, and, by the theorem, any Hankel operator of norm 1 which attains its norm is an extreme point in the unit ball of all Hankel operators, and thus an extreme point in any subset. It can then be seen that the boundary of the set of compact Hankel operators consists entirely of extreme points, so it is strictly convex.  $\square$

**Corollary 2.** *For any  $f \in C(T)$ ,  $G = \{H_{fq} : q \in H^\infty \text{ and } \|H_{fq}\| \leq 1\}$  is a strictly convex set. In particular, if  $\theta$  is rational,  $\{H_{\bar{\theta}q} : q \in H^\infty \text{ and } \|H_{\bar{\theta}q}\| \leq 1\}$  is strictly convex.*

*Proof.*  $G$  is a set of compact Hankel operators. We must show that every boundary point of  $G$  is an extreme point of  $G$ . Boundary points of  $G$  must be operators of norm 1 and are thus boundary points of the unit ball of the set of all compact Hankel operators. They are thus extreme points in that set, and consequently in the subset  $G$ .  $\square$

**Corollary 3.** *If  $f \in H^\infty$  is such that  $\|H_f\|_e < \|H_f\| = 1$ , then  $H_f$  is an extreme point in the unit ball of the set of Hankel operators.*

*Proof.* If  $\|H_f\|_e < \|H_f\|$ , then  $H_f$  attains its norm.  $\square$

**Example:** Consider the Hankel operators  $H_{\bar{\theta}\bar{z}(1+\alpha\theta)}$  for an inner function  $\theta$  and complex constant  $\alpha$ . It is shown by the authors in [7] that  $\gamma = \left\| H_{\bar{\theta}\bar{z}(1+\alpha\theta)} \right\|$  is given by

$$\gamma^2 = \frac{2 + 2 \operatorname{Re} [\alpha\theta(0)] + |\alpha|^2 + \sqrt{4|\alpha|^2(1 + 2 \operatorname{Re} [\alpha\theta(0)]) + |\alpha|^4 - 4|\operatorname{Im} [\alpha\theta(0)]|^2}}{2}.$$

If  $\theta$  is rational, then Corollary 2 tells us that  $\frac{1}{\gamma}H_{\overline{\theta z}(1+\alpha\theta)}$  is an extreme point in the unit ball of the set of Hankel operators. Note that  $H_{\overline{\theta z}(1+\alpha\theta)} = H_{\overline{\theta z}} + H_{\alpha\overline{z}}$  is a rank one perturbation of  $H_{\overline{\theta z}}$ , so they have the same essential norm.

If  $\theta$  is irrational,  $H_{\overline{\theta z}}^*H_{\overline{\theta z}}$  is the projection onto the infinite dimensional space  $\mathcal{H}(z\theta)$ , so  $\|H_{\overline{\theta z}}\|_e = \|H_{\overline{\theta z}(1+\alpha\theta)}\|_e = 1$ . Let us now choose  $\alpha$  be any number which makes  $\gamma > 1$ . The operator  $\frac{1}{\gamma}H_{\overline{\theta z}(1+\alpha\theta)}$  will then be a Hankel operator of unit norm, but we will have  $\|\frac{1}{\gamma}H_{\overline{\theta z}(1+\alpha\theta)}\|_e = \frac{1}{\gamma} < 1$ . This tells us then by Corollary 3 that  $\frac{1}{\gamma}H_{\overline{\theta z}(1+\alpha\theta)}$  is an extreme point in the unit ball of the set of Hankel operators.

**Theorem 2.** *The unit ball of the set of all Hankel operators is not strictly convex, i.e., not every Hankel operator of norm 1 is an extreme point of the unit ball.*

*Proof.* Let  $m(z)$  be any inner function with one essential singularities at some nonempty set  $\mathcal{N}$ . Let  $w_1$  and  $w_2$  be distinct rational functions with  $\|w_1\|_\infty = \|w_2\|_\infty = |w_1(z_0)| = |w_2(z_0)| = 1$  for some  $z_0 \in \mathcal{N}$ . For example,  $w_1 = 1$  and  $w_2 = \overline{z_0}z$ . Then

$$\left\| \overline{H_{m(z)(\lambda w_1(z) + (1-\lambda)w_2(z))}} \right\| = 1 \text{ for all } 0 \leq \lambda \leq 1$$

since the essential norm of  $\overline{H_{m(z)(\lambda w_1(z) + (1-\lambda)w_2(z))}}$  is equal to 1 by Lemma 2, and

$$\left\| \overline{H_{m(z)(\lambda w_1(z) + (1-\lambda)w_2(z))}} \right\| \leq \left\| \overline{m(z)}(\lambda w_1(z) + (1-\lambda)w_2(z)) \right\|_\infty \leq 1.$$

This shows that the Hankel operators  $\overline{H_{m(z)(\lambda w_1(z) + (1-\lambda)w_2(z))}}$  for  $0 < \lambda < 1$  are boundary points in the unit ball of Hankel operators which are not extreme points. It is also easy to see that  $\overline{H_{m(z)w_1(z)}} \neq \overline{H_{m(z)w_2(z)}}$  since  $\overline{H_{m(z)(w_1(z) - w_2(z))}} = 0$  if and only if  $\overline{m(z)}(w_1(z) - w_2(z))$  is analytic, and that is impossible here. Thus the unit ball of the set of all Hankel operators is not strictly convex.  $\square$

**Corollary 4.** *If  $\theta$  is an irrational inner function,  $\left\{ H_{\overline{\theta q}} : q \in H^\infty \text{ and } \|H_{\overline{\theta q}}\| \leq 1 \right\}$  is not a strictly convex set.*

#### 4. FINITE TOEPLITZ MATRICES

Let  $C \in \mathcal{T}_N$  be a contractive lower-triangular Toeplitz matrix. Let

$$(4.1) \quad H = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be the  $N \times N$  Hankel matrix corresponding to the Hankel operator  $H_{\overline{z^N}}$ . It is clear that  $H$  is unitary, and that  $HC$  is thus a contractive Hankel matrix.

We can now give a different proof of the theorem of Chō, Curto, and Lee [2]:

**Theorem 3.**  *$\mathcal{T}_N$  is strictly convex.*

*Proof.* That  $\mathcal{T}_N$  is convex is easy to see. To show it is strictly convex, it is sufficient to show that for any two elements  $C_1$  and  $C_2$  in  $\partial\mathcal{T}_N$ , if  $\frac{1}{2}(C_1 + C_2) \in \partial\mathcal{T}_N$  (i.e., has norm 1), then  $C_1 = C_2$ . If this were not the case, then there would be  $C_1$  and  $C_2$  in  $\partial\mathcal{T}_N$  with  $C_1 \neq C_2$  and  $\frac{1}{2}(C_1 + C_2) \in \partial\mathcal{T}_N$ . We would then have

Hankel matrices of unit norm  $HC_1$  and  $HC_2$  with  $\frac{1}{2}(HC_1 + HC_2)$  again of unit norm. The same would be true of the corresponding finite-rank (and thus compact) Hankel operators, and this would violate the strict convexity of the unit ball of the space of compact Hankel operators.  $\square$

## 5. COWEN SETS

We can use the Hankel operator methods to get several results concerning the extreme points of the reduced Cowen set (see [3], [4]) for a function  $f \in H^\infty$ . Recall that an operator  $A$  is hyponormal if its self-commutator  $[A^*, A] = A^*A - AA^*$  is a positive operator. For a function  $f \in H^\infty$ , the Cowen set is

$$G_f = \{g \in H^\infty : T_{f+\bar{g}} \text{ is hyponormal}\}.$$

Since the hyponormality of  $T_{f+\bar{g}}$  is independent of the constant term in Fourier series for  $f + \bar{g}$ , we will assume, without loss of generality, that  $g(0) = 0$ , and define the reduced Cowen set

$$G'_f = \{g \in zH^\infty : T_{f+\bar{g}} \text{ is hyponormal}\}.$$

It is shown in [4] that the Cowen set  $G_f$  is weakly compact and convex, and hence the reduced Cowen set  $G'_f$  is also weakly compact and convex. In 1988, Cowen posed the following:

**Problem 1** ([3, Question 3]). *What are the extreme points of  $G'_f$ ? In particular, if  $g \in G'_f$  but  $\lambda g \notin G'_f$  for all  $|\lambda| > 1$ , is  $g$  an extreme point?*

We will provide some answers to these questions. To do so, we will define

$$B_f = \{g \in G'_f : \lambda g \notin G'_f \text{ for all } |\lambda| > 1\}.$$

The second question, then, is whether it is true that

$$B_f = \text{ext } G'_f.$$

Our analysis will depend on whether  $\bar{f}$  is or is not of bounded type, i.e., can be expressed as the quotient of a bounded analytic function and an inner function. In the first case, we can write  $\bar{f} = \frac{h}{\theta}$  for  $h \in H^\infty$  and  $\theta$  inner. If  $h$  and  $\theta$  are coprime, we then have  $\ker H_{\bar{f}} = \theta H^2$ . In the other case, we will have  $\ker H_{\bar{f}} = \{0\}$ . We will prove, over the course of this section, the following theorem.

### Theorem 4.

1. If  $\ker H_{\bar{f}} = \{0\}$ , then  $B_f \supsetneq \text{ext } G'_f$ .
2. If  $\ker H_{\bar{f}} = \theta H^2$  for an irrational inner function  $\theta$ , then  $B_f \supsetneq \text{ext } G'_f$ .
3. If  $\ker H_{\bar{f}} = \theta H^2$  for a rational inner function  $\theta$ , then  $B_f = \text{ext } G'_f$ .

This will completely answer the second question in the problem above. The proof of part 1 of the above theorem will be given after Proposition 1. The proof of part 2 will be given at the end of Section 5.2 and the proof of part 3 at the beginning of Section 5.3.

Since there is no bounded projection of  $L^\infty$  onto  $H^\infty$ , we will need the following lemma whose proof is a direct computation and is omitted.

**Lemma 3.** *If  $f \in H^\infty$  and  $k$  in  $L^\infty$  is a rational function, then both  $P[\bar{k}f]$  and  $P[k\bar{f}]$  are in  $H^\infty$ .*

For an inner function  $\theta$ , recall  $\mathcal{H}(\theta) = H^2 \ominus \theta H^2$ .

**Lemma 4.** *Let  $\theta$  be an inner function. Let  $a \in H^\infty$  be such  $\bar{\theta}a = \bar{f}$  for some  $f \in H^\infty$ . If  $k$  is in  $H^\infty$  or  $\theta$  is rational, then  $P_{\mathcal{H}(\theta)}(ka)$  is in  $H^\infty$ .*

*Proof.* An easy calculation shows that  $P_{\mathcal{H}(\theta)}h = h - \theta P(\bar{\theta}h)$  for  $h \in H^2$ , so

$$\begin{aligned} P_{\mathcal{H}(\theta)}(ka) &= ka - \theta P(\bar{\theta}ka) \\ &= ka - \theta P(k\bar{f}). \end{aligned}$$

The result follows from Lemma 3 by noting  $f, ka \in H^\infty$ .  $\square$

**5.1. The set  $G'_f$  when  $\ker H_{\bar{f}} = \{0\}$ .** We will restrict our attention here to  $f \in H^\infty$  such that  $\ker H_{\bar{f}} = \{0\}$ . We will be interested in those  $g$  in  $H_0^\infty$  ( $:= zH^\infty$ ) making  $T_{f+\bar{g}}$  hyponormal, so for such a  $g$ , let  $g = zg_1$  for  $g_1 \in H^\infty$ .

It is useful to note that  $T_{f+\bar{g}}$  is hyponormal (see [4]) if and only if

$$H_{\bar{g}}^* H_{\bar{g}} \leq H_{\bar{f}}^* H_{\bar{f}}.$$

By Cowen's characterization [4],  $T_{f+\bar{g}}$  is hyponormal if and only if there is some function  $k \in H^\infty$  with  $\|k\|_\infty \leq 1$  with

$$\bar{g} - k\bar{f} = h \quad \text{for some } h \in H^2.$$

For any such  $k$ , write

$$\bar{z}g_1 = h + k\bar{f}$$

or

$$(5.1) \quad g_1 = \bar{z}h + \bar{z}k\bar{f}.$$

This is the same as to say that  $g_1 = P[\bar{z}k\bar{f}]$ . Thus we can conclude that  $g \in G'_f$  if and only if  $g \in H^\infty$  and  $g \in \widehat{G}_f$  where the set  $\widehat{G}_f$  is defined by

$$\widehat{G}_f = \{zP[\bar{z}k\bar{f}] : \|k\|_\infty \leq 1\} = \left\{g \in zH^2 : H_{\bar{g}}^* H_{\bar{g}} \leq H_{\bar{f}}^* H_{\bar{f}}\right\},$$

and we have proved:

$$(5.2) \quad G'_f = \widehat{G}_f \cap H^\infty \quad \text{and} \quad B_f = \{zP[\bar{z}k\bar{f}] : \|k\|_\infty = 1\} \cap H^\infty.$$

Let  $(H^\infty)_1$  denote the closed unit ball of  $H^\infty$ . We will now define a map  $\phi : (H^\infty)_1 \rightarrow \widehat{G}_f$  by

$$\phi(k) = zP[\bar{z}k\bar{f}].$$

$\phi$  is linear, and it is easy to see that  $\phi$  maps  $(H^\infty)_1$  onto  $\widehat{G}_f$ .

**Claim 1.**  *$\phi$  is a bijection.*

*Proof.* We must show that  $\phi$  is one-to-one. If  $\phi(k_1) = \phi(k_2)$ , then  $P[\bar{z}k_1\bar{f}] = P[\bar{z}k_2\bar{f}]$ , i.e.,  $P[\bar{z}(k_1 - k_2)\bar{f}] = 0$ , or

$$\bar{z}(k_2 - k_1)\bar{f} = \bar{z}h, \quad \text{for some } h \in H^2.$$

This is equivalent to

$$(k_2 - k_1)\bar{f} = h,$$

which means that  $k_2 - k_1$  would be in the kernel of  $H_{\bar{f}}$ , which we are assuming to be  $\{0\}$ .  $\square$

Any bijective linear operator will preserve extreme points, so we know that the extreme points of  $\widehat{G}_f$  are precisely the images of the extreme points of  $(H^\infty)_1$ , which are known to be those  $k$  with

$$\int_0^{2\pi} \log(1 - |k(e^{i\theta})|) d\theta = -\infty.$$

Since  $G'_f = \widehat{G}_f \cap H^\infty$ , extreme points of  $G'_f$  must include all  $\phi(k)$  for  $k$  an extreme point in the closed unit ball of  $H^\infty$ , when  $\phi(k) = zP[\overline{z}k f]$  is also in  $H^\infty$ . For example, if  $k$  is a finite Blaschke product, by Lemma 3,  $\phi(k) = zP[\overline{z}k f]$  is indeed in  $H^\infty$ . Whether there are any other extreme points of  $G'_f$ , we cannot say. Summarizing the above discussion we have

**Proposition 1.** *If  $\ker H_{\overline{f}} = \{0\}$ , then*

$$B_f = \{zP[\overline{z}k f] : k \in H^\infty \text{ satisfying } \|k\|_\infty = 1\} \cap H^\infty,$$

$$\text{ext } G'_f \supseteq \left\{ zP[\overline{z}k f] : k \in H^\infty \text{ satisfying } \int_0^{2\pi} \log(1 - |k(e^{i\theta})|) d\theta = -\infty \right\} \cap H^\infty.$$

*In particular if  $\theta$  is a finite Blaschke product, then  $zP[\overline{z}\theta f] \in \text{ext } G'_f$ . If  $\theta_1$  and  $\theta_2$  are two distinct finite Blaschke products, then  $zP[\overline{z}k f] \notin \text{ext } G'_f$  for  $k = \lambda\theta_1 + (1 - \lambda)\theta_2$  with  $0 < \lambda < 1$ .*

We now prove part 1 of Theorem 4. Assume  $\ker H_{\overline{f}} = \{0\}$ .  $B_f \supseteq \text{ext } G'_f$  follows from (5.2). Let  $\theta_1$  and  $\theta_2$  be two distinct Blaschke such that  $k(z) = \frac{1}{2}\theta_1 + \frac{1}{2}\theta_2$  satisfying  $\|k\|_\infty = 1$ . For example  $\theta_1 = 1$ ,  $\theta_2 = z$ . By the above proposition, we have  $zP[\overline{z}\theta f] \in B_f$  and  $P[\overline{z}k f] \notin \text{ext } G'_f$ . Therefore

$$B_f \not\supseteq \text{ext } G'_f.$$

We now give examples of  $f$  such that  $G'_f = \widehat{G}_f \cap H^\infty = \widehat{G}_f$  so the extreme points in  $G'_f$  are identified as above.

**Lemma 5.** *If  $f = \sum_{n=0}^\infty f_n z^n \in H^\infty$  satisfying  $M = \sum_{n=0}^\infty |f_n| \sqrt{n+1} < \infty$ , then*

$$\|P[\overline{k} f]\|_\infty \leq M \|k\|_\infty \quad \text{for } k \in H^\infty.$$

*Proof.* We write

$$P[\overline{k} f] = \sum_{n=0}^\infty f_n P[\overline{k} z^n].$$

Let  $k = \sum_{j=0}^\infty k_j z^j$ . It is clear that  $\|k\|_2^2 = \sum_{j=0}^\infty |k_j|^2 \leq \|k\|_\infty^2$ . Therefore

$$|P[\overline{k} z^n]|^2 = \left| \sum_{j=0}^n \overline{k_{n-j}} z^j \right|^2 \leq \left( \sum_{j=0}^n |k_{n-j}|^2 \right) \left( \sum_{j=0}^n |z^j|^2 \right) \leq (n+1) \|k\|_\infty^2$$

and

$$|P[\overline{k} f]| = \left| \sum_{n=0}^\infty f_n P[\overline{k} z^n] \right| \leq \sum_{n=0}^\infty |f_n| \sqrt{n+1} \|k\|_\infty = M \|k\|_\infty.$$

The proof is complete.  $\square$

**Example:**

If  $f$  satisfies the condition in the above proposition, then  $G'_f = \widehat{G}_f \cap H^\infty = \widehat{G}_f$ . For example, let  $f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ , then  $f(z)$  satisfies the condition in the above lemma. Also since  $\overline{f(z)} = e^{\bar{z}}$  is not meromorphic in  $\mathbb{D}$ ,  $e^{\bar{z}} \neq \frac{h}{\theta}$  for any  $H^2$  function  $h$  and inner function  $\theta$ . Therefore  $\ker H_{\bar{f}} = \{0\}$  and

$$\text{ext } G'_f = \{zP[\bar{z}ke^z] : k \text{ is an extreme point of } (H^\infty)_1\}.$$

**5.2. The set  $G'_f$  when  $\ker H_{\bar{f}} = \theta H^2$ .** In this section we will assume  $\ker H_{\bar{f}} = \theta H^2$  for some inner function  $\theta$ .

When  $T_{f+\bar{g}}$  is hyponormal, we will, without loss of generality, assume that  $f(0) = g(0) = 0$ . It is shown in [7] that, we can write  $\bar{f} = \bar{\theta}a$  for some  $a \in \mathcal{H}(\theta) \cap H^\infty$ . Let  $g \in G'_f$ . Since  $T_{f+\bar{g}}$  is hyponormal,  $\bar{g} = \bar{\theta}b$  for some  $b \in \mathcal{H}(\theta) \cap H^\infty$ . We then know that  $T_{f+\bar{g}}$  is hyponormal if and only if there is an  $H^\infty$  function  $k$  with  $\|k\|_\infty \leq 1$  and

$$\begin{aligned} \bar{g} - k\bar{f} &= h \quad \text{for some } h \in H^2 \\ \iff \bar{\theta}b - k\bar{\theta}a &= h \\ \iff b &= \theta h + ka. \end{aligned}$$

Since  $b \in \mathcal{H}(\theta)$ , we can conclude that  $b = P_{\mathcal{H}(\theta)}(ka)$ . Now let  $k$  be any solution to  $b = \theta h_1 + ka$  (for some  $h_1 \in H^2$ ). Then if  $l$  is any function in  $H^\infty$  (not necessarily with  $\|l\|_\infty \leq 1$ ) satisfying  $b = \theta h + la$  for some  $h \in H^2$ , then

$$\theta(h - h_1) + (l - k)a = 0$$

or

$$(l - k)a = \theta(h_1 - h).$$

Since  $a \in \mathcal{H}(\theta)$ ,  $\theta$  must divide  $l - k$ , so we can write  $l = k + \theta x$  for some  $x \in H^\infty$ . This is in contrast to (5.1), where if  $\ker H_{\bar{f}} = \{0\}$ , then the  $k \in H^\infty$  is unique. We can thus conclude that there is some solution  $k$  of norm at most 1 in  $b = \theta h + ka$  (and thus  $T_{f+\bar{g}}$  is hyponormal) if and only if there is some  $l = k + \theta x$  of norm at most 1, i.e.,  $\inf_{x \in H^\infty} \|k + \theta x\|_\infty \leq 1$ . By Nehari's Theorem,

$$\inf_{x \in H^\infty} \|k + \theta x\|_\infty = \inf_{x \in H^\infty} \|\bar{\theta}k + x\|_\infty = \|H_{\bar{\theta}k}\|.$$

Putting the above facts together, we see that for  $g \in H_0^\infty$  ( $:= zH^\infty$ ),  $T_{f+\bar{g}}$  is hyponormal if and only if there is some  $k \in H^\infty$  with  $\|H_{\bar{\theta}k}\| \leq 1$ , and in this case,  $g = \theta \bar{b}$  where  $b = P_{\mathcal{H}(\theta)}(ka)$ . Equivalently,

$$g = \theta \overline{P_{\mathcal{H}(\theta)}(ka)} = \theta \overline{P_{\mathcal{H}(\theta)}(\theta k \bar{\theta} a)} = \theta \overline{P_{\mathcal{H}(\theta)}(\theta k \bar{f})}.$$

Now let

$$\widehat{G}_f = \left\{ \theta \overline{P_{\mathcal{H}(\theta)}(ka)} : \|H_{\bar{\theta}k}\| \leq 1 \right\}.$$

We conclude that

$$G'_f = \widehat{G}_f \cap H^\infty \text{ and } B_f = \left\{ \theta \overline{P_{\mathcal{H}(\theta)}(ka)} : \|H_{\bar{\theta}k}\| = 1 \right\} \cap H^\infty.$$

Define

$$\widetilde{G}_f = \{H_{\bar{\theta}k} : \|H_{\bar{\theta}k}\| \leq 1\}.$$



For  $k \in H^\infty$ , let  $\phi$  be defined by

$$\phi : \overline{\theta P_{\mathcal{H}(\theta)}(ka)} \mapsto H_{\overline{\theta k}}.$$

**Claim 2.**  $\phi$  is a linear bijection of  $\widehat{G}_f$  onto  $\widetilde{G}_f$ .

*Proof.* Let  $k_1$  and  $k_2$  be in  $H^\infty$ .

$$\begin{aligned} P_{\mathcal{H}(\theta)}(k_1 a) &= P_{\mathcal{H}(\theta)}(k_2 a) \\ \iff k_1 a &= k_2 a + \theta h \text{ for some } h \in H^2 \\ \iff (k_1 - k_2) a &= \theta h \\ \iff \theta \text{ divides } (k_1 - k_2), &\text{ since } a \in \mathcal{H}(\theta) \\ \iff k_1 &= k_2 + \theta h_1 \text{ for some } h_1 \in H^2 \\ \iff \overline{\theta k_1} &= \overline{\theta k_2} + h_1 \\ \iff H_{\overline{\theta k_1}} &= H_{\overline{\theta k_2}}. \end{aligned}$$

This tells us that  $\phi$  is well-defined, and, further, that it is injective. It is easily seen to be surjective by the definition of  $\widetilde{G}_f$  and  $\widehat{G}_f$ .  $\square$

Again using the fact that a bijective map preserves the extreme points, we conclude:

**Proposition 2.** *The extreme points of  $\widehat{G}_f$  are exactly*

$$\left\{ \overline{\theta P_{\mathcal{H}(\theta)}(ka)} : H_{\overline{\theta k}} \text{ is an extreme point of } \widetilde{G}_f \right\}.$$

*The extreme points of  $G'_f = \widehat{G}_f \cap H^\infty$  include*

$$\left\{ \overline{\theta P_{\mathcal{H}(\theta)}(ka)} : H_{\overline{\theta k}} \text{ is an extreme point of } \widetilde{G}_f \right\} \cap H^\infty.$$

We can now prove part 2 of Theorem 4. Assume  $\ker H_{\overline{f}} = \theta H^2$  for an irrational inner function  $\theta$ . Take for example a function  $f$  which has  $\theta$  an inner function with one essential singularity at  $z = 1$ . Let  $k = \lambda z + (1 - \lambda) z^2$  for  $0 < \lambda < 1$ . We then know, by the proof to Theorem 2, that  $H_{\overline{\theta k}}$  is a boundary point but not an extreme point in  $\widetilde{G}_f$ , so we know that  $g = \overline{\theta P_{\mathcal{H}(\theta)}(ka)}$  is not an extreme point of  $\widehat{G}_f$ . By Lemma 4,  $g \in H^\infty$ , so it is a nonextreme point in  $G'_f$  as well. This  $g$  is in  $B_f$  since it is the image under  $\phi$  of a boundary point. Thus  $B_f \not\supseteq \text{ext } G'_f$ .

**5.3. The set  $G'_f$  when  $\ker H_{\overline{f}} = \theta H^2$  for rational  $\theta$ .** We will now prove part 3 of Theorem 4. We consider the case when  $\theta$  is a rational inner function (a finite Blaschke product). In this case, Lemma 4 tells us that  $\overline{\theta P_{\mathcal{H}(\theta)}(ka)}$  is always in  $H^\infty$ , therefore  $G'_f = \widehat{G}_f \cap H^\infty = \widetilde{G}_f$ . In fact,  $G'_f$  consists of rational functions and is a compact subset of  $H^\infty$ . Furthermore, by Theorem 2, the extreme points of  $G'_f$  are the pre-images under  $\phi$  of the extreme points of  $\widetilde{G}_f$ . Since here  $\widetilde{G}_f$  is of the form in Corollary 2, the extreme points of  $\widetilde{G}_f$  coincide with its boundary points. That is

$$\begin{aligned} \text{ext } G'_f &= \left\{ \overline{\theta P_{\mathcal{H}(\theta)}(ka)} : H_{\overline{\theta k}} \text{ is an extreme point of } \widetilde{G}_f \right\} \\ &= \left\{ \overline{\theta P_{\mathcal{H}(\theta)}(ka)} : \|H_{\overline{\theta k}}\| = 1 \right\} = \left\{ \overline{\theta P_{\mathcal{H}(\theta)}(\theta k \overline{f})} : \|H_{\overline{\theta k}}\| = 1 \right\} = B_f. \end{aligned}$$

Chō, Curto, and Lee [2] show that for  $f(z) = \sum_{j=0}^N a_j z^j$  an analytic polynomial,  $g$  must be a polynomial of degree at most  $N$ , and the answer to the second question posed in Problem 1 above is yes, and that all boundary points are extreme points, i.e., that the reduced Cowen set is strictly convex. These can be seen as special cases of the work above.

When  $f$  is an analytic polynomial of degree  $N$ , we have  $\theta = z^N$ , and

$$\text{ext } G'_f = \left\{ g = z^N \overline{P_{\mathcal{H}(z^N)}(kz^N \bar{f})} : \|H_{z^N k}\| = 1 \right\}.$$

We can rewrite this by using, as in Lemma 4,  $P_{\mathcal{H}(z^N)}(kz^N \bar{f}) = kz^N \bar{f} - z^N P(k\bar{f})$ , so

$$(5.3) \quad \bar{g} = z^{-N} P_{\mathcal{H}(z^N)}(kz^N \bar{f}) = k\bar{f} - P(k\bar{f}) = H_{\bar{f}} k.$$

When  $f(z) = a_0 + \cdots + a_N z^N$  and  $g(z) = b_1 z + \cdots + b_N z^N$ , we let  $k(z) = \bar{c}_0 + \cdots + \bar{c}_{N-1} z^{N-1}$  and by using the matrix  $H$  in (4.1), we see that  $\|H_{z^N k}\| = 1$  if and only if the Toeplitz matrix

$$(5.4) \quad \begin{pmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_1 & c_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{N-2} & \ddots & \ddots & \ddots & 0 \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{pmatrix}$$

has norm 1.

What we have shown here is then seen to be equivalent to what Chō, Curto, and Lee [2] show, specifically, they prove:

**Theorem 5.** *If  $f$  is an analytic polynomial of degree  $N$ , then  $G'_f$  is strictly convex and*

$$\text{ext } G'_f = \{g(z) = b_1 z + \cdots + b_N z^N : \|C\| = 1\},$$

where  $C$  is the Toeplitz matrix in (5.4) corresponding to  $\varphi = f + \bar{g} = \bar{b}_N z^{-N} + \cdots + \bar{b}_1 z^{-1} + a_0 + \cdots + a_N z^N$  in the sense that the entries  $c_0, c_1, \dots, c_{N-1}$  must satisfy (5.5) below.

$$(5.5) \quad \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{N-1} & \bar{a}_N \\ \bar{a}_2 & \bar{a}_3 & \cdots & \bar{a}_N & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{a}_N & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{c}_0 \\ \bar{c}_1 \\ \vdots \\ \bar{c}_{N-1} \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_N \end{pmatrix}.$$

Note that (5.5) can be written as  $H_{\bar{f}} k = \bar{g}$ , as in (5.3).

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