Research on Students' Reasoning about the Formal Definition of Limit: An Evolving Conceptual Analysis

Craig A. Swinyard & Elise Lockwood

Introduction

The concept of limit is fundamental to the study of calculus and to introductory analysis; this has been noted by many researchers (Artigue, 2000; Bezuidenhout, 2001; Cornu, 1991; Dorier, 1995). Artigue (ibid) views the role of limit in calculus as a unifying concept as "more important than its role as a productive tool for solving problems" (p.5). Cornu (ibid) echoes that sentiment, stating that the limit "holds a central position which permeates the whole of mathematical analysis – as a foundation of the theory of approximation, of continuity, and of differential and integral calculus" (p.153). Indeed, limits arise in these and many other mathematical contexts, including the convergence and divergence of infinite sequences and series, applications related to determining measurable quantities of geometric figures (e.g., arc length, area, and volume), and describing the behavior of real-valued functions.

The formal definition of limit is foundational as students proceed to more formal, rigorous mathematics. Indeed, the vast majority of topics encountered in an undergraduate analysis course, where students study the theoretical foundations of calculus, are built upon the formal definition. Continuity (both point-wise and uniform), derivatives, integrals, and Taylor series approximations are just a few of the topics studied in an analysis course for which limit serves as an indispensable component. Further, the formal definition of limit often serves as a starting point for developing facility with formal proof techniques, making sense of rigorous, formally-quantified mathematical statements, and transitioning to abstract thinking. Tall (1992) notes that the ability to think abstractly is a prerequisite for the transition to advanced mathematical thinking, and Ervynck (1981) cites the limit as an opportunity for students to develop the ability to think abstractly. For all of these reasons, the limit concept holds an important place in pedagogical considerations and as an object of research in mathematics education.

Literature Review

Though there are numerous ways to categorize the existing research on limit, we choose to separate the literature into two broad categories – informal and formal limit research. We define informal limit research here as research that does not have, as its focus, the ways in which students reason about the formal definition of limit. By formal limit research, we mean research that is focused on how students reason about or understand the formal definition. The vast majority of existing limit research consists of the former. These studies have focused largely on the fact that informal treatments of limit often result in students developing misconceptions (i.e., thinking of the limit as a bound, as something that cannot be reached, and/or as an approximation) based on their interpretation of colloquial language used in the classroom to describe limits (Ferrini-Mundy & Graham, 1994; Frid, 1994; Monaghan, 1991; Tall, 1992; Williams, 1991). Other studies have shown that informal methods can also result in an over-reliance on simplistic examples used initially to introduce the concept (Cornu, 1991; Davis & Vinner, 1986; Tall & Vinner, 1981; Tall, ibid). Many of the studies mentioned above emphasize what students *do not* know about the concept of limit, which we refer to as a *deficit* perspective. A much smaller segment of the informal limit research literature attempts to describe what students *do* understand about limits (Ferrini-Mundy & Graham, 1994;

Oehrtman, 2003; Oehrtman, 2004; Williams, 2001). Rather than viewing student thinking as deficient, these researchers describe initial student thinking as entailing natural informal conceptions that might facilitate the development of strong conceptual understanding.

In contrast to informal limit research, far fewer studies have explored how students reason about the formal definition of limit. Those that do, however, suggest that formal treatments of the concept also often prove unsuccessful (Cornu, 1991; Dorier, 1995; Gass, 1992; Tall, ibid; Tall & Vinner, 1981; Williams, 1991). The formal definition is rich with quantification and notation, and, according to Cornu (ibid), is cognitively sophisticated for first semester calculus students. Dorier (ibid) points out that the formal definition was "conceived for solving more sophisticated problems and for unifying all of them" (p.177), yet at the outset of calculus and introductory analysis, students likely have difficulty understanding the importance of a definition designed to unify problems they have yet to encounter. The message seems clear – the formal definition of limit is difficult for students to understand. What is less apparent from the bulk of the literature, however, is how students might come to reason coherently about this difficult concept. Studies by Cottrill et al. (1996), Larsen (2001), and Fernandez (2004) have attempted to answer this question. We discuss these studies below and in the subsequent section, as they have had a profound impact on the development of our own research.

Cottrill et al. (1996) provide what they call a *genetic decomposition* of how students might reason about the limit concept. This genetic decomposition describes the process a student might experience as he or she constructs a formal understanding of limit. Cottrill and his colleagues suggest that the concept of limit might eventually be thought of as a schema that is the collection of actions, processes and objects[1]. The hypothesized framework for how students may come to understand the limit concept is as follows:

1. The action of evaluating *f* at a single point *x* that is considered to be close to, or even equal to, *a*.
2. The action of evaluating the function *f* at a few points, each successive point closer to *a* than was the previous point.
3. Construction of a coordinated schema as follows.
   (a) Interiorization of the action of Step 2 to construct a domain process in which *x* approaches *a*.
   (b) Construction of a range process in which *y* approaches *L*.
   (c) Coordination of (a), (b) via *f*. That is, the function *f* is applied to the process of *x* approaching *a* to obtain the process of *f*(*x*) approaching *L*.
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.
5. Reconstruct the processes of Step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.
7. A completed ε-δ conception applied to a specific situation.

Evidence led Cottrill et al. to believe that the construction of a coordinated schema happens in a three-part process, the substeps listed under step three above. The majority of their analysis focused on this third step, as well as the two preceding steps.

None of the students' thinking in the study conducted by Cottrill et al. (1996) evolved to the point of having a formal conceptual understanding of limit. There was also no evidence that students were able to move to the level of thinking of the limit as a schema – that is, a "coherent collection of actions, processes, objects and other schemas that are linked in some way and brought to bear upon a problem situation" (p.172). Further aspects of this framework will be discussed in the next section.

In summary, the majority of the literature focuses on informal, rather than formal, understanding of the limit concept. While these studies indicate that students have great difficulty reasoning coherently about the formal definition, it remains to be seen how

[1] This theoretical perspective is commonly referred to as APOS theory (action-process-object-schema). For a detailed description, see Dubinsky (1992).

students might successfully come to understand it. Therefore, it is to this area of limit research, in which there is a paucity of studies, that we direct our investigations. The research by Cottrill et al. (1996) serves as the primary framework upon which our research is based.

Origin of Research Questions

We believe the genetic decomposition suggested by Cottrill et al. (1996) serves as a particularly useful starting point for studying students' formal conceptions of limit. Their research suggests that to develop a formal understanding of limit, one must merely formalize one's informal notions of the concept. In the decomposition outlined above, doing so amounts to formalizing the first three steps, specifically by reconstructing the coordinated schema described in step 3c in terms of intervals and inequalities. We argue however, that the formalization process is not so straightforward – formal understanding does require one to think in terms of intervals and inequalities, but the transition to formal thinking is not merely a reconstruction of what is described in the first three stages of the genetic decomposition. Research by Larsen (2001) substantiates this perspective. Most students in Larsen's study did not make connections between their formal understandings and the rest of their concept image (Vinner, 1991), which was comprised mostly of informal conceptions described in the first three steps of the genetic decomposition. Larsen suggests that "the formal definition is structurally different from the dynamic conception as described by the first four steps of the genetic decomposition," thus making it "unlikely that a student could successfully interpret the syntax in terms of their dynamic conception" (p.29). In light of Larsen's findings, we recommend that a clearer distinction be made between informal and formal understanding of limit. In informal understanding, the goal is generally to ***find*** a candidate for the limit. Formal understanding, on the other hand, typically addresses how one might ***validate*** the choice of a candidate. Finding and validating are two different processes[2]. In calculus courses, students are taught a variety of strategies for finding candidates for limits – direct substitution, algebraic manipulation, and tabular and graphical inspection. However, none of these satisfy the formal definition's requirement of validation. Cottrill et al. (1996) provide evidence that when students select a candidate for the limit of a function, they do so in a "forward" manner. By "forward," it is meant that students focus their attention first on inputs (*x*'s), and then on corresponding outputs (*y*'s). The selection of a candidate is made based on what numeric value the *y*'s are getting close to as *x*'s get closer to *a*. If students use the algebraic function or tabular approach to find the limit, they plug in *x*'s and look at corresponding *y*'s. If students use a graphical approach, an "up and over" technique is likely applied, in which students allow inputs along the *x*-axis to get closer to *a*, and then look at the corresponding outputs (the *y*'s).

It is worth noting, then, that the *validation* of a limit requires that one begin with a given candidate. Hence, the formal definition is dependent upon a candidate having already been selected. The key to validating a candidate, however, is the ability to reverse one's thinking (i.e., think in a "backwards" manner). Instead of going from *x*'s to *y*'s, a student must first consider what is taking place along the *y*-axis.

> In order to understand the definition of a limit, a student must coordinate an entire interval of output values, imagine reversing the function process and determine the corresponding region of input values. The action of a function on these values

[2] Fernandez (2004) and Juter (2006) have also suggested that *validating* limits involves a process distinct from the process of *finding* limits. Their perspectives, in addition to Larsen's perspective discussed here, have assisted us in articulating our own thinking on the distinction between these two processes. Thompson's insights (P.W. Thompson, personal communication, March 10, 2006) have also benefited our thinking on this matter.

> must be considered simultaneously since another process (one of reducing the size of the neighborhood in the range) must be applied while coordinating the results (Carlson, Oehrtman, & Thompson, 2007, p.160).

Thus, the process of validating a candidate requires a student to recognize that his/her customary ritual of first considering input values is no longer appropriate. Instead the student must think in a reverse manner, considering first a range of output values around the candidate, projecting back to the *x*-axis, and subsequently determining an interval around the limit value that will produce outputs within the pre-selected *y*-interval. Larsen's research (2001) suggests that the intricacies involved in this "backwards" process are arguably far more complex for students than merely formalizing a "forwards" process. The very complex nature of the formal definition makes it highly unlikely that a student with a strong "forwards" view of functions would be able to conceive of a new concept in such a "backwards" way, particularly when the focus during a first term calculus course is on *finding* limits, not *verifying* them.

In summary, we view the genetic decomposition offered by Cottrill et al. (1996) as a helpful framework from which to develop our own research. Instead of working from a deficit perspective, their research provides a positive description of student thought. Specifically, their work provides evidence of how students reason about the informal/forwards process of finding limits (steps 1-3 of their genetic decomposition); however, due to the complexity of the concept, there is a dearth of data describing how students reason about the formal/backwards process of validating limits. Thus, it seems that more research is needed to elucidate the latter stages of their genetic decomposition. The overarching purpose of the research reported here is to generate such insights and to move toward the elaboration of a cognitive model of what might be entailed in coming to understand this formal definition. Specifically, the intent of this research is:

1. To develop insight into students' reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit, and;
2. To inform the design of principled instruction that might support students' attempts to reinvent the formal definition of limit

To be clear, the first objective listed above is at the foreground of our study. Further, this first objective is set against the broader background goal of contributing to an epistemological analysis (Thompson & Saldanha, 2000) of the concept of limit of a real-valued function and its formal definition. Also, while other studies (e.g., Larsen, 2001; Fernandez, 2004) have sought to describe how students *interpret* the formal definition, our research seeks to address this need by focusing on how students reason about the formal definition of limit in the context of *reinvention*. We contend that interpreting the definition could result in a very different type of reasoning than would attempting to reinvent the definition. Indeed, the formal definition of limit constructed by Cauchy, and subsequently Weierstrass, was motivated by a need to specify the local behavior of functions in a precise manner. Neither mathematician's respective definition was a reformulation or interpretation of the traditional formal definition – on the contrary, these mathematicians constructed their definitions in response to an inherent need to classify functional behavior. We feel, then, that we might learn a great deal about how students reason about the formal definition of limit if we engage them in activities designed to foster their reinvention of the formal definition of limit.

Theoretical Perspective

Ernst von Glasersfeld (1995), drawing on Piaget's genetic epistemology (1971, 1977), developed a psychological theory of knowing which is known as ***radical constructivism*** (RC). Two central tenets of RC are:

1. Knowledge is not passively received either through the senses or by way of communication, but is actively built up by the cognizing subject.
2. The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability and serves the subject's organization of the experiential world, not the discovery of an objective ontological reality (von Glasersfeld, 1995, p.51).

In our study, we drew upon RC in two important ways. First, RC served as a guiding framework methodologically, both in regards to the dynamic we intended to create between participants and in regards to how we selected participants. For example, the instructional sequence was designed to create a dynamic in which students might experience frequent perturbations, thus providing them with opportunities to make cognitive accommodations. In this way, the students were motivated to organize their experiential world and thus actively build up knowledge. Second, RC served as a lens through which we analyzed the data generated in the study. For example, in our analyses of the data, we paid particular attention to students' personal interpretations of the tasks, looking for evidence of how they compared with those targeted in instruction. In so doing, we could make subsequent revisions for future iterations of the research cycle, and our research findings could be cast as inferences about student reasoning given particular interpretations of instructional tasks. Given RC's perspective on ontological reality, we believe the intention of data analysis is not to generate statements of fact about how students reason about or understand limits, but rather to generate *viable interpretations* of students' reasoning and understanding.

In addition to the overarching perspective of radical constructivism, we briefly describe aspects drawn from the perspectives of *developmental research* and the *teaching experiment methodology* that guided the instructional design for our study. The goal of *developmental research* is to "design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy" (Gravemeijer, 1998, p.279). Developmental research in education typically unfolds in cycles that are driven by two reflexively related phases – a *developmental phase* and a *research phase*. The former is characterized by the development of instructional activities designed to assist students in progressing toward previously identified understandings related to a particular mathematical concept. The instructional activities are developed based on a *local instructional theory*. The latter research phase is characterized by analysis of student activity and reasoning as they engage in the instructional activities. This analysis, in turn, then serves as a guide in further developing the local instructional theory and in refining the instructional activities to be implemented in subsequent research cycles.

A heuristic commonly associated with developmental research is *guided reinvention*. This well-established heuristic has been employed in numerous content areas of post-secondary mathematics education (see Larsen, 2004; Marrongelle & Rasmussen, 2006; Weber & Larsen, 2005). Guided reinvention is described by Gravemeijer et al. (2000) as "a process by which students formalize their informal understandings and intuitions" (p.237). An important aspect of this process is the identification of plausible instructional starting points from which students might naturally formalize their informal understandings and intuitions.

The focus of our research is on modeling student thinking, along the lines articulated by proponents of the *teaching experiment methodology* (Steffe & Thompson, 2000). An important aspect of the teaching experiment methodology is the distinction that is made between *students' mathematics* and *mathematics of students*, which Steffe and Thompson explicate below.

> [W]e have to accept the student's mathematical reality as being distinct from ours. We call those mathematical realities "students' mathematics," whatever they might be. Students' mathematics is indicated by what they say and do as they engage in mathematical activity, and a basic goal of the researchers in a teaching experiment is to construct models of students' mathematics. "Mathematics of students" refers to these models, and it includes the modifications students make in their ways of operating (p.269).

Hence, the researcher's central purpose in a teaching experiment is to construct a model of student thinking or reasoning in relation to a particular concept or idea. In this way, the teaching experiment methodology is an appropriate framework to help us address the second purpose of our research described above.

In summary, we intend to model students' reasoning in relation to their engagement in instruction designed to guide them in reinventing the formal definition of limit. The *teaching experiment methodology* serves as an orienting and guiding framework for this central research objective. Our research objective is also in line with the goal of *developmental research* – to design instructional activities that allow students to autonomously build upon their informal knowledge as they develop more sophisticated, abstract, formal knowledge. Finally, the *guided reinvention principle* orients our selection of starting points for instruction – students' informal interpretations of the concept will guide the development of instruction.

Method

The study was conducted during the summer of 2006, with four introductory analysis students (two females and two males) from a large, urban university. We chose introductory analysis students because we presumed they would a) have been exposed to limit and functions in a variety of contexts; b) have a strong informal sense of limit; and, c) not yet have seen the formal definition of limit. The four students were selected based on the extent to which they demonstrated a willingness to communicate their conceptual understandings of calculus concepts. The study consisted of interviewing each student twice. These interviews were videotaped and took place over the course of two weeks. Each interview lasted approximately 55 to 75 minutes. The interviews were conducted by the first author and videotaped by the second author. Students A and D were interviewed individually, while Students B and C were interviewed as a pair. All four students were filmed a second time in the same groupings during the following week. Each interview proceeded in a similar format – the researchers presented tasks (described below) and asked questions related to those tasks and the students talked and worked through responses on a whiteboard.

The purpose of the first interview was for students to reinvent the formal definition of limit at infinity ($\lim_{x\to\infty} f(x) = L$); the purpose of the second interview was for students to reinvent the formal definition of limit at a point ($\lim_{x\to a} f(x) = L$). Ultimately, the purpose of the task in the first interview was to provide scaffolding for the task in the second interview. In both of these tasks the students were asked to come up with the formal definition of limit as it pertained to the respective situation. The rationale was that in the first scenario, students would only have to describe closeness along the *y*-axis; and in the second scenario, students would have to describe closeness along both axes. We hoped that this sequence would provide a natural progression allowing students to use their first definition in reinventing the second.

Data Analysis

Analysis of the videos proceeded in two stages. In the first viewing, descriptive notes on each interview were taken so as to: a) characterize what the students were being asked to do in each interview; b) make inferences about the students' interpretations of what they were being asked to do in each interview; and c) conjecture potential conceptual entailments of students' reasoning about the formal definition. In the second viewing, the videos were reviewed for evidence that might substantiate the conjectures. Throughout the analysis of data, both researchers were frequently engaged in discussion in an effort to reach consensus about the data.

Discussion/Results

Analyses of data from the first iteration of our research cycle (in conjunction with an a priori mathematical-conceptual analysis of the limit concept and its formal definition) led us to identify components of the formal definition that we conjecture are primary in reasoning coherently about it and should thus be targeted in instruction. These 'conjectured conceptual entailments' may serve to elucidate the latter steps of the genetic decomposition proposed by Cottrill et al. (1996). We list them below and subsequently discuss each of them in connection to the data.

Conjectured conceptual entailments of the formal definition of limit:

1) The purposes of a mathematical definition
2) The purpose of this particular definition
3) The individual components of the definition (i.e., an understanding of how to describe getting close to both L and a)
4) The role of an implication in the definition
5) The quantifier on x is universal (i.e., it has to work for ALL $x \neq a$ within delta of a, not for just a single x)
6) The types of quantifiers in the definition (i.e., existential and universal)
7) The role of quantifier structure in the definition

1) *Purposes of a mathematical definition*

We conjecture that the successful reinvention of *any* mathematical definition necessarily requires some understanding of the purposes of a mathematical definition. In particular, a mathematical definition is designed to describe an idea in a precise manner, so that examples of the idea are retained while counterexamples of the idea are excluded. Those students in our study who demonstrated an awareness of the purposes of a mathematical definition were most successful in their reinvention. For instance, in the first interview, Student A appeared to have a clear understanding that his goal was to characterize precisely what it means for a function to have a finite limit at infinity. In the excerpt below, the interviewer had written Student A's definition for limit at infinity on the board, "For all ε, there exists an a such that $x>a$ implies $50-\varepsilon < L < 50+\varepsilon$," and began asking Student A questions about his definition. Note, in particular, the highlighted portions where Student A appears to exhibit understanding of the purposes of a mathematical definition.

I[3]: Can epsilon equal zero?
SA: Uh, no.
I: And why can epsilon not equal zero?
SA: Because then…if it's for all epsilon greater or ***equal*** to zero, then it's true for zero.

[3] For all interview transcripts included in this report, I stands for the interviewer, which in all cases was the first author, while SA stands for Student A, SB stands for Student B, and so forth.

I: Okay.

SA: If it's true for zero that means that umm, there exists a number *a* such that *x* is greater than *a* means that it's trapped between 50 and 50, which means that it's constant for everything after that. So that would not be true for this one [***referring to the dampened sine wave***] because this is never constant. It's always fluctuating a little bit.

I: Okay. So you don't want to include epsilon equaling zero in your definition because that does work for functions that become constant…

SA: Yeah, constant, but others don't.

I: Do all functions whose limit equal 50 become constant?

SA: No….That would be sort of a useless definition.

I: Okay, why would it be useless? What do you mean by that?

SA: Because if there's a useful function…if you want to find that something converges to something, then if there's some function that after a while it just equals π, then why do you mess with the other stuff? Why don't you just take umm, $x>a$ and then you'd have it exactly…If that's your definition then, well almost nothing would converge.

Student A's comments imply that he is aware that a definition of "limit at infinity" that allows ε to equal zero would over-restrict his characterization of the concept he is trying to capture. That is, such a definition would only describe functions which are constant for $x>a$. However, he is aware of other functions (e.g., dampened sine waves) which are not constant for $x>a$ yet have a finite limit at infinity. In contrast, other students, when asked to define what it means for a function to have a finite limit at infinity (and, later, at a finite point), appeared confused about the purpose of their efforts, and consequently struggled to propose a viable definition. We conjecture that instruction designed to support the development of this first conceptual entailment would greatly benefit students in developing subsequent conceptual entailments listed above.

2) *Purpose of this particular definition*

We remind the reader that there is an important distinction between the process of ***finding*** candidates for limits and subsequently ***verifying*** that such a candidate is indeed the limit. There are a variety of approaches used for *finding* candidates for limits (i.e., algebraic techniques that allow for direct substitution, as well as graphical and tabular techniques that help make potential candidates evident), but the formal definition is not one of them. Indeed, the stated definition only gives a mechanism for *verifying* that a given candidate L is a limit; it gives no mechanism for *finding* candidates for L. We hypothesize that students are often unaware of this distinction and that understanding the purpose of the formal definition of limit (i.e., its role as a mechanism for verifying) is a conceptual entailment of coherent understanding. The excerpt below illustrates one student's difficulty to understand the purpose of the formal definition of limit. In the paired interview (Students B and C), Student C had written down a formal definition of limit, and the interviewer asked the other student (Student B) what he thought of this definition.

SB: This [***referring to Student C's written formal definition***] is like a memorized definition, I suppose; it doesn't mean - it's not as meaningful.…I mean to us pictorially if you look at it, [the limit] is the point that everything kind of approaches as it gets to the place you're interested in.

Later, during the same interview, the students continued to reason about their reinvented formal definition of limit, and ultimately appeared to both reason coherently about the definition. And yet, at the end of the interview when the first author asked Student B to reflect upon the written, formal definition in relation to the traditional ε-δ picture to which Students B and C had been referring, Student B did not acknowledge any increased appreciation for the usefulness of the definition.

I: If you look at this situation, and you look at this [***the definition***], how do you think of – as you read this – how do you think of this in light of the picture?

SB: I wouldn't.

I: Okay. How would you think of – if you were describing limits, how would you describe them?

SB: I would, I think that to me it seems easier to say that there's a sequence of numbers on that function that is approaching – or a sequence of outputs I guess you would call it for the function that approaches the limit. But I mean pictorially this [referring to the definition] wouldn't necessarily mean anything to me.

In the excerpts above, it appears that Student B is thinking in a "forwards" manner, considering *x*-values first and only then corresponding *y*-values. As we have discussed in earlier sections of this report, we believe the formal definition requires one to think in a "backwards" manner. We conjecture that students may view the formal definition as useless if they fail to recognize its role as a mechanism for verifying candidates for limits, as well as the corresponding mental shift (from thinking in a "forwards" manner to thinking in a "backwards" manner) that is needed when reasoning about the formal definition.

3) *Individual components of the definition*
4) *Role of an implication in the definition*
5) *The variable x is universally quantified*

For the sake of brevity, we describe the next three conceptual entailments together. Evidence from our study suggests that an important element of reinventing the formal definition of limit is carefully describing the individual components of the definition. Specifically, it seems important for students to characterize what it means for function values $f(x)$ to get arbitrarily close to the limit L as x gets sufficiently close to a. In the study, students either used absolute value statements like those traditionally employed in the formal definition, or synonymous inequality statements, to describe closeness to a and L along the *x*- and *y*-axes respectively. Evidence from the study suggests that students made use of their definitions of limit at infinity (which requires one to describe closeness *only* along the *y*-axis) from the first task when constructing their definitions of limit at a point in the second task. As students constructed their definitions of limit at a point, they simply made adjustments to their first definitions to incorporate the idea of closeness along the *x*-axis. This lends credence to including both tasks in our study in subsequent instructional sequences.

Evidence from the study also suggests that once students describe the individual components of the definition, it is important that they understand the role of the implication in the definition. That is, it appeared to be important that students become aware that restricting the distance between *x* and *a* ***results*** in the ability to restrict the distance between *f*(*x*) and L. Likewise, it was evident that coherent reasoning relied on students' understanding that the implication introduces a universal quantifier on *x*, meaning that the implication holds not just for a single *x*-value in a δ-neighborhood of *a*, but for every *x*-value in that δ-neighborhood (except possibly *a*). Initially, some student's definitions only required that a single *x*-value in a δ-neighborhood of *a* result in a corresponding function value in a pre-determined ε-neighborhood of L.

6) *Types of quantifiers*
7) *Role of quantifier structure in the definition*

Results from the study corroborate findings regarding the struggle students experience in understanding mathematically quantified statements (Dubinsky, Elterman, & Gong, 1989; Dubinsky & Yiparaki, 2000; S. Larsen, personal communication, December 20, 2006; Zaslavsky & Shir, 2005). Specifically, students had difficulty determining both the types of quantifiers appropriate for ε and δ (i.e., existential vs. universal) and the appropriate quantification structure (i.e., "For all/there exists" vs. "There exists/for all"). Some students believed no difference exists between the two quantification structures and did not understand the role quantification structure plays in the definition. These students struggled to reinvent a coherent formal definition. One student seemed to be quite cognizant of the difference between quantification structures and had little difficulty reasoning coherently about his reinvented definition of limit. Of the conceptual entailments discussed in this section, we conjecture that issues with quantification may be most difficult for students to resolve.

Looking Ahead

The seven conceptual entailments discussed in the previous section guided our formulation of a 'hypothetical learning trajectory' (Simon, 1995) and led to the development of an instructional sequence intended to support students' reinvention of the formal definition of limit under instructional guidance. Specifically, the initial instructional sequence was refined to support coherent student reasoning with respect to the conceptual entailments listed above. The first author will carry out a second iteration of the research cycle during the spring and summer of 2007. In this second iteration of the research cycle, students will be engaged in a newly-refined instructional sequence with two central goals: 1) to inform and refine the initial conceptual entailments listed above (e.g., confirm their viability and explain interrelations among them); and 2) to inform the design of principled instruction in relation to these conceptual entailments. We view the research reported here, as well as the forthcoming second iteration of the research cycle, as potentially elucidating the latter steps of the genetic decomposition proposed by Cottrill et al. (1996).

References

Artigue, M. (2000). Teaching and learning calculus: What can be learned from education research and curricular changes in France? In E. Dubinsky & A. Schoenfeld & J.J. Kaput (Eds.) *Research in collegiate mathematics education IV* (Vol. 8, pp.1-15). Providence: American mathematical society.

Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first-year students. *International journal of mathematical education in science and technology, 32*, 4, 487-500.

Carlson, M. P., Oehrtman, M. C., & Thompson, P. W. (2007). Key aspects of knowing and learning the concept of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 150-171). Washington, DC: Mathematical Association of America.

Cornu, B. (1991). Limits. In D. Tall (Ed.) *Advanced Mathematical Thinking*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 153-166.

Cottrill, J., Dubinsky, E., Nichols, D., Schwinngendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of mathematical behavior, 15*, 167-192.

Davis, R., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of mathematical behavior, 5,* 281-303.

Dorier, J. (1995). Meta level in the teaching of unifying and generalizing concepts in mathematics. *Educational studies in mathematics, 29,* 175-197.

Dubinsky, E. (1992). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-126). Dordrecht, The Netherlands: Kluwer Academic.

Dubinsky, E., Elterman, F., & Gong, C. (1988). The student's construction of quantification. *For the learning of mathematics – An international journal of mathematics education, 8*, 44-51.

Dubinsky, E. and Yiparaki, O. (2000). On student understanding of AE and EA quantification. *CBMS Issues in Mathematics Education*. Vol. 8., 239-289.

Ervynck, G. (1981). Conceptual difficulties for first year university students in the acquisition of limit of a function. *Proceedings of the Psychology of Mathematics Education, 5* (pp. 330-333), Grenoble, France.

Fernandez, E. (2004). The students' take on the epsilon-delta definition of a limit. *Primus, 14*, 1, 43-54.

Ferrini-Mundy, J., & Graham, K. (1994). Research in calculus learning: Understanding of limits, derivatives, and integrals. In E. Dubinsky & J.J. Kaput (Eds.) *Research issues in undergraduate mathematics learning: Preliminary analyses and results*. Washington D.C.: MAA, 31-45.

Frid, S. (1994). Three approaches to undergraduate calculus instruction: Their nature and potential impact on students' language use and sources of conviction. In E. Dubinsky & A. Schoenfeld & J.J. Kaput (Eds.) *Research in collegiate mathematics education IV* (Vol. 4, pp.69-100). Providence: American mathematical society.

Gass, F. (1992). Limits via graphing technology. *Primus, 2*, 1, 9-15.

Gravemeijer, K. (1998). Developmental research as a research method. In J. Kilpatrick & A.Sierpinska (Eds.), *Mathematics Education as a Research Domain: A Search for Identity (ICMI Study Publication)* (Book 2, pp. 277-297). Dordrecht, The Netherlands: Kluwer.

Gravemeijer, K., Cobb, P., Bowers, J., and Whitenack, J. (2000). Symbolizing, modeling and instructional design. In P. Cobb, E. Yackel and K. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms*, Erlbaum, Mahwah, NJ, pp. 225–273.

Juter, K. (2006). Limits of functions as they developed through time and as students learn them today. *Mathematical thinking and learning, 8,* 4, 407-431.

Larsen, S. (2001). Understanding the formal definition of limit. Unpublished manuscript, Arizona State University.

Larsen, S. (2004). Supporting the guided reinvention of the concepts of group and isomorphism: A developmental research project (Doctoral dissertation, Arizona State University, 2004) Dissertation Abstracts International, B65/02, 781.

Marrongelle, K., & Rasmussen, C. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for research in mathematics education, 37*, 5, pp. 388-420

Monaghan, J. (1991). Problems with the language of limits. *For the learning of mathematics*, *11*, 3, 20-24.

Oehrtman, M. (2003). Strong and weak metaphors for limits, Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th Conference of PME-NA, Vol 3, (pp. 397-404). Honolulu, HI.

Oehrtman, M. (2004). Approximation as a foundation for understanding limit concepts. *Proceedings of the Psychology of Mathematics Education North American Chapter, 23* (pp. 95-102), Snowbird, UT.

Piaget, J. (1971). *Genetic epistemology*. New York: W. W. Norton.

Piaget, J. (1977). *Psychology and epistemology: Towards a theory of knowledge*. New York: Penguin.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*, 2, 114-145.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Tall, D.O. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D.A. Grouws (Ed.) *Handbook of research on mathematics teaching and learning*. New York: Macmillan, 495-511.

Tall, D.O., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics, 12,* 151-169.

Thompson, P. W. & Saldanha, L. (2000). Epistemological analyses of mathematical ideas: A research methodology. *Proceedings of the Twenty-second Annual Meeting of the International Group for the Psychology of Mathematics Education*. Tucson, Arizona.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Boston: Kluwer.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*, Falmer Press, London.

Weber, K., and Larsen, S. (2005). Teaching and learning group theory. To appear in forthcoming MAA Notes volume, *Making the Connection: Research and Teaching in Undergraduate Mathematics*. Mathematical Association of America. Manuscript submitted for publication.

Williams, S. (1991). Models of limit held by college calculus students. *Journal for research in mathematics education, 22*, 3, 219-236.

Williams, S. (2001). Predications of the limit concept: An Application of repertory grids. *Journal for research in mathematics education, 32,* 4, 343-367.

Zaslavsky & Shir (2005). Students' conceptions of a mathematical definition. *Journal for research in mathematics education, 36,* 4, 317-346.