Termination of (Canonical) Context-Sensitive Rewriting*

Salvador Lucas

DSIC, Universidad Politécnica de Valencia
Camino de Vera s/n, E-46022 Valencia, Spain
e.mail: slucas@dsic.upv.es

Abstract. Context-sensitive rewriting (*CSR*) is a restriction of rewriting which forbids reductions on selected arguments of functions. A *replacement map* discriminates, for each symbol of the signature, the argument positions on which replacements are allowed. If the replacement restrictions are less restrictive than those expressed by the so-called *canonical* replacement map, then *CSR* can be used for computing (infinite) normal forms of terms. Termination of such *canonical CSR* is desirable when using *CSR* for these purposes. Existing transformations for proving termination of *CSR* fulfill a number of new properties when used for proving termination of canonical *CSR*.

Keywords: (infinitary) normalization, term rewriting, termination.

1 Introduction

A *replacement map* is a mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ satisfying $\mu(f) \subseteq \{1, \ldots, k\}$, for each k-ary symbol f of the signature $\mathcal{F}$ [Luc98]. We use them to discriminate the argument positions on which replacements are allowed. In this way, we obtain a restriction of rewriting which we call *context-sensitive rewriting* (*CSR* [Luc98]). Terminating TRSs are *μ-terminating* (i.e., no term initiates an infinite sequence of *CSR* under μ). However, *CSR* can *achieve* termination, by pruning (all) infinite rewrite sequences. Several methods have been developed to formally prove μ-termination of TRSs [BLR02,FR99,GM99,GM02,Luc96,SX98,Zan97].

Example 1. Consider the following TRS $\mathcal{R}$:

```
sel(0,x:y)     → x                 first(0,x)       → []
sel(s(x),y:z)  → sel(x,z)          first(s(x),y:z)  → y:first(x,z)
from(x)        → x:from(s(x))
```

Let $\mu(\mathtt{s}) = \mu(\mathtt{:}) = \mu(\mathtt{from}) = \{1\}$ and $\mu(\mathtt{sel}) = \mu(\mathtt{first}) = \{1, 2\}$. The μ-termination of $\mathcal{R}$ can be proved by using Zantema's techniques [Zan97].

Up to now, termination of *CSR* has been studied disregarding other computational issues such as achieving some kind of completeness in computations. This

* Work partially supported by CICYT TIC2001-2705-C03-01, Acciones Integradas HI 2000-0161, HA 2001-0059, HU 2001-0019, and Generalitat Valenciana GV01-424.

problem has been studied in [Luc98]. Since computing (infinite) normal forms is the essential task of most rewriting-based engines (interpreters of rewriting-based (lazy) programming languages, theorem provers, etc.), the application of *CSR* in practice requires undertaking these issues. In Section 3, we define *canonical CSR* as an specialization of *CSR* that only uses replacement maps which are less restrictive than the *canonical* replacement map $\mu_{\mathcal{R}}^{can}$ of the TRS $\mathcal{R}$. This is the least replacement map ensuring that all non-variable subterms of every left-hand side of the rules of $\mathcal{R}$ are replacing [Luc98]. Section 4 discusses the use of canonical *CSR* for computing normal forms and infinite normal forms of terms (with left-linear, possibly non-terminating TRSs). Termination of canonical *CSR* becomes essential for these tasks. Thus, for the purpose of computing (infinite) normal forms, proving termination of canonical *CSR* can be priorized over proofs of termination: a proof of termination of canonical *CSR* can be used to achieve normalizations (with possibly non-terminating, left-linear TRSs) and also to arbitrarily approximate infinite normal forms. In the remainder of the paper we try to analyze whether proving termination of canonical *CSR* for a *terminating* TRS is much more *difficult* than proving its termination.

In Section 5, we review the main (transformational) techniques for proving termination of *CSR* and prove a couple of new interesting properties. Most of them appear when these techniques apply for proving termination of canonical *CSR*. We prove that all transformations introduced by Giesl and Middeldorp [GM99,GM02] are *complete* for proving termination of canonical *CSR*. We also prove that the contractive transformation of [Luc96] is complete for proving termination of canonical *CSR* if the replacing variables of the right-hand sides of each rule of the TRS are also replacing in the left-hand side (conservativeness [Luc96]). We show that, in both cases, Zantema's transformation [Zan97] and Ferreira and Ribeiro's transformation [FR99] remain incomplete. In Section 6, we also analyze the (relative) behavior of the transformations when *simple* termination, rather than termination, is considered. Simple termination covers the use of most usual automatizable orderings for proving termination of rewriting [Der87]. We obtain a new hierarchy of the transformations which is helpful for guiding their practical use. We prove that the transformations for proving termination of *CSR* preserve (simple) termination. Thus, proving termination of canonical *CSR* does not seem to be essentially more difficult than proving termination. Section 7 provides a first experimental evidence supporting this claim.

2 Preliminaries

Throughout the paper, $\mathcal{X}$ denotes a countable set of variables and $\mathcal{F}$ denotes a signature, i.e., a set of function symbols $\{\mathtt{f}, \mathtt{g}, \ldots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. A TRS is a pair $\mathcal{R} = (\mathcal{F}, R)$ where R is a set of rewrite rules. Given TRSs $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{R}' = (\mathcal{F}', R')$, we let $\mathcal{R} \cup \mathcal{R}'$ be the TRS $(\mathcal{F} \cup \mathcal{F}', R \cup R')$. $L(\mathcal{R})$ denotes the set of lhs's of $\mathcal{R}$. Given a signature $\mathcal{F}$, we consider the left-linear TRS $\mathcal{E}mb(\mathcal{F}) = (\mathcal{F}, \{f(x_1, \ldots, x_k) \to x_i \mid f \in \mathcal{F}, i \in \{1, \ldots, ar(f)\}\})$. A TRS $\mathcal{R}$ is simply terminating if $\mathcal{R} \cup \mathcal{E}mb(\mathcal{F})$ is terminating.

3 (Canonical) context-sensitive rewriting

A mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ is a *replacement map* (or $\mathcal{F}$-map) if $\forall f \in \mathcal{F},\ \mu(f) \subseteq \{1, \ldots, ar(f)\}$ [Luc98]. The ordering $\sqsubseteq$ on $M_{\mathcal{F}}$, the set of all $\mathcal{F}$-maps, is: $\mu \sqsubseteq \mu'$ if for all $f \in \mathcal{F}$, $\mu(f) \subseteq \mu'(f)$. Thus, $\mu \sqsubseteq \mu'$ means that μ considers less positions than μ' (for reduction), i.e., μ is more restrictive than μ'. According to $\sqsubseteq$, $\mu_\bot$ (resp. $\mu_\top$) given by $\mu_\bot(f) = \varnothing$ (resp. $\mu_\top(f) = \{1, \ldots, ar(f)\}$) for all $f \in \mathcal{F}$, is the minimum (maximum) of $M_{\mathcal{F}}$. Given a TRS $\mathcal{R} = (\mathcal{F}, R)$, we write $M_{\mathcal{R}}$ rather than $M_{\mathcal{F}}$. The set of *μ-replacing positions* $\mathcal{P}os^\mu(t)$ of $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is: $\mathcal{P}os^\mu(t) = \{\Lambda\}$, if $t \in \mathcal{X}$ and $\mathcal{P}os^\mu(t) = \{\Lambda\} \cup \bigcup_{i \in \mu(root(t))} i.\mathcal{P}os^\mu(t|_i)$, if $t \notin \mathcal{X}$. The set of *replacing* variables $\mathcal{V}ar^\mu(t)$ of t is $\mathcal{V}ar^\mu(t) = \{x \in \mathcal{V}ar(t) \mid \mathcal{P}os_x(t) \cap \mathcal{P}os^\mu(t) \neq \varnothing\}$. In *context-sensitive rewriting* (*CSR* [Luc98]), we (only) rewrite *replacing* redexes: t μ-rewrites to s, written $t \hookrightarrow_\mu s$, if $t \xrightarrow{p}_{\mathcal{R}} s$ and $p \in \mathcal{P}os^\mu(t)$. The $\hookrightarrow_\mu$-normal forms are called μ-normal forms. Note that, except for the trivial case $\mu = \mu_\top$, the μ-normal forms strictly include all normal forms of $\mathcal{R}$. A TRS $\mathcal{R}$ is μ-terminating if $\hookrightarrow_\mu$ is terminating. The canonical replacement map $\mu^{can}_{\mathcal{R}}$ is *the most restrictive replacement map ensuring that the non-variable subterms of the left-hand sides of the rules of $\mathcal{R}$ are replacing.* Note that $\mu^{can}_{\mathcal{R}}$ is easily obtained from $\mathcal{R}$: $\forall f \in \mathcal{F}$, $i \in \{1, \ldots, ar(f)\}$,

$$i \in \mu^{can}_{\mathcal{R}}(f) \quad \text{iff} \quad \exists l \in L(\mathcal{R}), p \in \mathcal{P}os_{\mathcal{F}}(l), (root(l|_p) = f \wedge p.i \in \mathcal{P}os_{\mathcal{F}}(l))$$

where $\mathcal{P}os_{\mathcal{F}}(l)$ is the set of positions of non-variable subterms of l. Let $CM_{\mathcal{R}} = \{\mu \in M_{\mathcal{R}} \mid \mu^{can}_{\mathcal{R}} \sqsubseteq \mu\}$ be the set of replacement maps which are less or equally restrictive than $\mu^{can}_{\mathcal{R}}$. We say that $\hookrightarrow_\mu$ is *a canonical* context-sensitive rewrite relation if $\mu \in CM_{\mathcal{R}}$.

4 Termination and normalization properties

The most important semantic property of canonical *CSR* is that, for left-linear TRSs, μ-normal forms are head-normal forms[1] [Luc98]. We define a normalization procedure $norm_\mu$ which is based on performing a *layered* μ-normalization of terms (see Figure 1). In Figure 1, the *maximal replacing context* $MRC^\mu(t)$ of t is the maximal prefix of t whose positions are μ-replacing in t. Also, *μ-norm* is a mapping from sets of terms to μ-normal forms of them. The normalization of a term proceeds by first μ-normalizing terms and next recursively normalizing (in this way) the maximal non-replacing subterms of the obtained μ-normal forms.

Theorem 1 (Correctness). *Let $\mathcal{R} = (\mathcal{F}, R)$ be a left-linear TRS, $\mu \in CM_{\mathcal{R}}$, and $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If $s \in norm_\mu(\{t\})$, then s is a normal form of t.*

This procedure is not ensured to be correct or terminating if $\mu \notin CM_{\mathcal{R}}$.

Example 2. Consider $\mathcal{R}$ and μ as in Example 1. Let $t =$ `first(5,from(0))` (numbers n are a shorthand for $\mathtt{s}^n(\mathtt{0})$) and $\mu'(f) = \mu(f)$ for all symbol f except

[1] A head-normal form is a term which does not rewrite to a redex.

```
Procedure norm_μ(T)
   T := μ-norm(T)
   for each t ∈ T
     let t = C[t_1, ..., t_n], where C[ ] = MRC^μ(t)
     for i := 1, ..., n do S_i := norm_μ({t_i})
     T_t := C[S_1, ..., S_n]
   return ⋃_{t∈T} T_t
end procedure norm_μ
```

Given a context $C[\,]$ and sets of terms $T_1, \ldots, T_n$, notation $C[T_1, \ldots, T_n]$ denotes the set $\{C[t_1, \ldots, t_n] \mid t_1 \in T_1, \ldots, t_n \in T_n\}$.

Fig. 1. Algorithm for *normalization via μ-normalization*

$\mu'(\texttt{first}) = \{1\}$. Note that $\mu' \notin CM_{\mathcal{R}}$, but $\mathcal{R}$ is still μ'-terminating. We have

$$\begin{aligned} norm_{\mu'}(\{t\}) &= \texttt{first(5,}\,norm_{\mu'}(\{\texttt{from(0)}\})\texttt{)} \\ &= \texttt{first(5,0:}\,norm_{\mu'}(\{\texttt{from(1)}\})\texttt{)} \\ &= \texttt{first(5,0:1:}\,norm_{\mu'}(\{\texttt{from(2)}\})\texttt{)} \\ &\vdots \end{aligned}$$

Thus, $norm_{\mu'}(t)$ does not terminate and there is no output. Even if $norm_{\mu'}$ terminates, the outcome can be incorrect. Let $s = \texttt{first(1,first(1,[0,1,2]))}$ (`[0,1,2]` is a shorthand for `0:1:2:[]`). Now:

$$\begin{aligned} norm_{\mu'}(\{s\}) &= \texttt{first(1,}\,norm_{\mu'}(\{\texttt{first(1,[0,1,2])}\})\texttt{)} \\ &= \texttt{first(1,0:}\,norm_{\mu'}(\{\texttt{first(0,[1,2])}\})\texttt{)} \\ &= \{\texttt{first(1,[0])}\} \end{aligned}$$

The outcome is *not* a normal form: $\texttt{first(1,[0])} \to^* \texttt{[0]}$.

In the following theorem we say that μ-*norm* is *complete for μ-normalization* if for all $T \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $t \in T$, the μ-normal forms of t are in μ-$norm(T)$.

Theorem 2 (Completeness). *Let $\mathcal{R} = (\mathcal{F}, R)$ be a left-linear TRS, $\mu \in CM_{\mathcal{R}}$, and $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. Assume that μ-norm is complete for μ-normalization. If s is a normal form of t, then $s \in norm_\mu(\{t\})$.*

Correctness and completeness of $norm_\mu$ can easily be achieved if canonical μ-termination of the TRS $\mathcal{R}$ can be ensured: in this case, we let μ-$norm(T)$ to be the set of all μ-normal forms of each $t \in T$.

Example 3. Consider $\mathcal{R}$ and μ as in Example 1. Now we can successfully apply the 'normalization via μ-normalization' procedure to t in Example 2:

$$\begin{aligned} norm_\mu(\{t\}) &= \texttt{0:}\,norm_\mu(\{\texttt{first(4,from(1))}\}) \\ &= \texttt{0:1:}\,norm_\mu(\{\texttt{first(3,from(2))}\}) \\ &\vdots \\ &= \texttt{0:1:2:3:4:}\,norm_\mu(\{\texttt{first(0,from(5))}\}) \\ &= \{\texttt{[0,1,2,3,4]}\} \end{aligned}$$

Lazy programming languages admit giving *infinite values* as the meaning of expressions. Infinite values are limits of converging infinite sequences of *partially defined* values which are more and more defined. In this setting, the following 'termination-like' property has been envisaged:

Top-termination: A TRS is *top-terminating* if no infinitary reduction sequence performs infinitely many rewrites at topmost position Λ [DKP91].

Top-termination is, in general, an undecidable property. We can prove top-termination of a left-linear TRS $\mathcal{R}$ by proving its *canonical μ-termination.*

Theorem 3. *Let $\mathcal{R}$ be a left-linear TRS and $\mu \in CM_{\mathcal{R}}$. If $\mathcal{R}$ is μ-terminating, then $\mathcal{R}$ is top-terminating.*

A TRS is *infinitary normalizing* if every (finite) term t admits a *strongly convergent sequence* (i.e., a sequence that, ultimately[2], reduces deeper and deeper redexes) starting from t and ending into a (possibly infinite) normal form (i.e., a term without redexes). For left-linear TRSs, top-termination is a sufficient condition for infinitary normalization. This fact easily follows from[3] [DKP91]. Still, there are infinitary normalizing TRSs which are not top-terminating.

Example 4. Consider the TRS $\mathcal{R}$:

$$\mathtt{f(a)} \to \mathtt{f(f(a))}$$
$$\mathtt{f(a)} \to \mathtt{a}$$

This TRS is infinitary normalizing but not top-terminating:

$$\underline{\mathtt{f(a)}} \to \mathtt{f}(\underline{\mathtt{f(a)}}) \to \underline{\mathtt{f(a)}} \to \cdots$$

Hence, from Theorem 3, we conclude that μ-termination criteria (see Section 5 below) can also be used for proving infinitary normalization. Infinitary normalizing TRSs can be thought of as playing the role of normalizing TRSs in the infinitary setting, i.e., ensuring that every term has a meaning (its, possibly infinite, normal form). Our normalization via μ-normalization technique easily generalizes to infinitary normalization (see [Luc02] for details).

5 Termination of *CSR* by transformation

The μ-termination of a TRS $\mathcal{R} = (\mathcal{F}, R)$ is usually proved by demonstrating *termination* of a transformed TRS $\mathcal{R}^{\mu}_{\Theta} = (\mathcal{F}^{\mu}_{\Theta}, R^{\mu}_{\Theta})$ obtained from $\mathcal{R}$ and $\mu \in M_{\mathcal{F}}$ by using a transformation Θ [FR99,GM99,GM02,Luc96,SX98,Zan97].

Definition 1. *A transformation Θ is*

1. correct *(regarding μ-termination) w.r.t. $\mathcal{M} \subseteq M_{\mathcal{F}}$ if, for all $\mu \in \mathcal{M}$, termination of $\mathcal{R}^{\mu}_{\Theta}$ implies μ-termination of $\mathcal{R}$.*

[2] We only consider sequences of length at most ω.

[3] But note that the notion of infinitary normal form used in [DKP91] (a term t such that $t = t'$ whenever $t \to t'$) differs from the usual one.

2. complete *(regarding μ-termination) w.r.t. $\mathcal{M} \subseteq M_{\mathcal{F}}$ if, for all $\mu \in \mathcal{M}$, μ-termination of $\mathcal{R}$ implies the termination of $\mathcal{R}^{\mu}_{\Theta}$.*

If $\mathcal{M} = M_{\mathcal{F}}$ in Definition 1, we just say that Θ is correct (or complete). Obviously, complete transformations Θ preserve termination of $\mathcal{R}$, i.e., if $\mathcal{R}$ is terminating, then $\mathcal{R}^{\mu}_{\Theta}$ is terminating. The simplest (and trivial) correct transformation for proving μ-termination is the identity: $\mathcal{R}^{\mu}_{ID} = \mathcal{R}$ (terminating TRSs are μ-terminating for every replacement map μ). We review the main (nontrivial) correct transformations for proving termination of *CSR* and prove a number of new properties (e.g., completeness) which appear when they are used for proving termination of canonical *CSR*.

5.1 The contractive transformation

With the contractive transformation [Luc96], the non-μ-replacing arguments of all symbols in $\mathcal{F}$ are *removed* and a new, *μ-contracted* signature $\mathcal{F}^{\mu}_{L}$ is obtained (possibly reducing the *arity* of symbols). The function $\tau_{\mu} : \mathcal{T}(\mathcal{F}, \mathcal{X}) \to \mathcal{T}(\mathcal{F}^{\mu}_{L}, \mathcal{X})$ *drops* the non-replacing immediate subterms of a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and constructs a 'μ-contracted' term by joining the (also transformed) replacing arguments below the corresponding operator of $\mathcal{F}^{\mu}_{L}$. A TRS $\mathcal{R} = (\mathcal{F}, R)$ is μ-contracted into $\mathcal{R}^{\mu}_{L} = (\mathcal{F}^{\mu}_{L}, \{\tau_{\mu}(l) \to \tau_{\mu}(r) \mid l \to r \in R\})$.

Example 5. Consider the TRS $\mathcal{R}$ (borrowed from [GM02]):

```
nats  → adx(zeros)     adx(x:y)  → incr(x:adx(y))
zeros → 0:zeros        incr(x:y) → s(x):incr(y)
```

and $\mu(\mathtt{:}) = \mu(\mathtt{incr}) = \mu(\mathtt{adx}) = \mu(\mathtt{s}) = \{1\}$. Then $\mathcal{R}^{\mu}_{L}$ (we use the same symbols with possibly decreased arities):

```
nats  → adx(zeros)     adx(:(x))  → incr(:(x))
zeros → :(0)           incr(:(x)) → :(s(x))
```

is terminating (use an *rpo* with precedence $\mathtt{nats} > \mathtt{adx}, \mathtt{zeros} > \mathtt{incr} > \mathtt{:}, \mathtt{0}, \mathtt{s}$). Hence, $\mathcal{R}$ is μ-terminating.

The contractive transformation only works well with *μ-conservative* TRSs, i.e., satisfying that $\mathcal{V}ar^{\mu}(r) \subseteq \mathcal{V}ar^{\mu}(l)$ for all rule $l \to r$ of $\mathcal{R}$ [Luc96]; otherwise, extra variables will appear in a rule of $\mathcal{R}^{\mu}_{L}$ thus becoming non-terminating. Let

$$CoCM_{\mathcal{R}} = \{\mu \in CM_{\mathcal{R}} \mid \text{for all rules } l \to r \text{ in } \mathcal{R}, \mathcal{V}ar^{\mu}(r) \subseteq \mathcal{V}ar^{\mu}(l)\}$$

That is: $CoCM_{\mathcal{R}}$ contains the replacement maps $\mu \in CM_{\mathcal{R}}$ that make $\mathcal{R}$ μ-conservative. Since $\mu_{\top} \in CoCM_{\mathcal{R}}$ for all TRS $\mathcal{R}$, $CoCM_{\mathcal{R}}$ is not empty. We prove that the contractive transformation becomes complete w.r.t. $CoCM_{\mathcal{R}}$.

Theorem 4 (Completeness). *Let $\mathcal{R}$ be a left-linear TRS and $\mu \in CoCM_{\mathcal{R}}$. If $\mathcal{R}$ is μ-terminating, then $\mathcal{R}^{\mu}_{L}$ is terminating.*

Even though μ-conservativeness can be difficult to achieve, there still are interesting examples fulfilling this requirement (see [BLR02,Luc02]). Theorem 4 does not hold if $\mu \notin CM_{\mathcal{R}}$ (even for μ-conservative TRSs).

Example 6. Consider the terminating (hence μ-terminating) TRS $\mathcal{R}$ [GM02]:

$$\texttt{f(b,x)} \to \texttt{f(c,x)}$$

together with $\mu(\texttt{f}) = \{2\}$. Note that $\mathcal{R}$ is μ-conservative but $\mu \notin CM_{\mathcal{R}}$; $\mathcal{R}_L^\mu$:

$$\texttt{f(x)} \to \texttt{f(x)}$$

is not terminating.

Left-linearity cannot be dropped in Theorem 4.

Example 7. Consider the TRS $\mathcal{R}$:

$$\texttt{f(x,x)} \to \texttt{f(a,b)}$$

with $\mu(\texttt{f}) = \{1\}$. Since $\mathcal{R}$ is terminating, $\mathcal{R}$ is μ-terminating. However, $\mathcal{R}_L^\mu$:

$$\texttt{f(x)} \to \texttt{f(a)}$$

is not terminating. Note that $\mu \in CoCM_{\mathcal{R}}$.

An obvious corollary of Theorem 4 is that the μ-contractive transformation preserves termination of left-linear TRSs $\mathcal{R}$ whenever $\mu \in CoCM_{\mathcal{R}}$. Examples 6 and 7 show the need for these restrictions.

5.2 Zantema's transformation

Zantema's transformation *marks* the *non-replacing arguments* of function symbols (disregarding their positions within the term) [Zan97]. Given $\mathcal{R} = (\mathcal{F}, R)$ and $\mu \in M_{\mathcal{F}}$, $\mathcal{R}_Z^\mu = (\mathcal{F} \cup \mathcal{F}' \cup \{\texttt{activate}\}, R_Z^\mu)$ where R_Z^μ consists of two parts The first part results from R by replacing every function symbol f occurring in a left or right-hand side with f' (a fresh function symbol of the same arity as f which, then, is included in $\mathcal{F}'$) if it occurs in a non-replacing *argument* of the function symbol directly above it. These new function symbols are used to block further reductions at this position. In addition, if a variable x occurs in a non-replacing position in the *lhs* l of a rewrite rule $l \to r$, then all occurrences of x in r are replaced by $\texttt{activate}(x)$. Here, `activate` is a new unary function symbol which is used to activate blocked function symbols again.

The second part of R_Z^μ consists of rewrite rules that are needed for blocking and unblocking function symbols:

$$\begin{aligned} f(x_1, \ldots, x_k) &\to f'(x_1, \ldots, x_k) \\ \texttt{activate}(f'(x_1, \ldots, x_k)) &\to f(x_1, \ldots, x_k) \end{aligned}$$

for every[4] $f' \in \mathcal{F}'$, together with the rule $\texttt{activate}(x) \to x$ Zantema's transformation can succeed when the contractive transformation fails.

Example 8. Let $\mathcal{R}$ and μ be as in Example 1. Since $\mathcal{R}$ is not μ-conservative, $\mathcal{R}_L^\mu$ cannot be used for proving μ-termination of $\mathcal{R}$. However, $\mathcal{R}_Z^\mu$:

[4] Actually, the rules $\texttt{activate}(f'(x_1, \ldots, x_k)) \to f(x_1, \ldots, x_k)$ can be avoided if no symbol `activate` appears in the rules of the first part of the transformation.

```
first(0,x)       → []                        from(x)            → from'(x)
first(s(x),y:z)  → y:first'(x,activate(z))   activate(from'(x)) → from(x)
sel(0,x:y)       → x                         first(x,y)         → first'(x,y)
sel(s(x),y:z)    → sel(x,activate(z))        activate(first'(x,y)) → first(x,y)
from(x)          → x:from'(s(x))             activate(x)        → x
```

is terminating (use *rpo* based on $\texttt{sel} > \texttt{activate} \approx \texttt{first} > \texttt{from}, \texttt{:}, \texttt{first'}, \texttt{[]}$ and $\texttt{from} > \texttt{:}, \texttt{from'}, \texttt{s}$, and giving `sel` the usual lexicographic status).

Zantema's transformation remains incomplete for proving canonical μ-termination even with left-linear, μ-conservative TRSs.

Example 9. Consider $\mathcal{R}$ and μ as in Example 5. Note that $\mu \in CoCM_{\mathcal{R}}$. Remember that $\mathcal{R}_L^\mu$ is (simply) terminating. However, $\mathcal{R}_Z^\mu$:

```
nats      → adx(zeros)          incr(x:y) → s(x):incr'(activate(y))
zeros     → 0:zeros'            adx(x:y)  → incr(x:adx'(activate(y)))
incr(x)   → incr'(x)            zeros     → zeros'
adx(x)    → adx'(x)             activate(incr'(x)) → incr(x)
activate(zeros') → zeros        activate(adx'(x))  → adx(x)
activate(x)      → x
```

is not terminating:

$$\begin{aligned}\texttt{adx}(\underline{\texttt{zeros}}) &\to \underline{\texttt{adx(0:zeros')}} \to \texttt{incr(0:adx'(}\underline{\texttt{activate(zeros')}}\texttt{))}\\ &\to \underline{\texttt{incr(0:adx'(zeros))}} \to \texttt{s(0):incr'(}\underline{\texttt{activate(adx'(zeros))}}\texttt{)}\\ &\to \texttt{s(0):incr'(adx(}\underline{\texttt{zeros}}\texttt{))} \to \cdots\end{aligned}$$

Zantema's transformation preserves termination of TRSs.

Theorem 5. *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If $\mathcal{R}$ is terminating, then $\mathcal{R}_Z^\mu$ is terminating.*

In [FR99], Ferreira and Ribeiro propose a variant of Zantema's transformation which has been proved strictly more powerful than Zantema's one (see [GM02]). Again, $\mathcal{R}_{FR}^\mu$ has two parts. The first part results from the first part of $\mathcal{R}_Z^\mu$ by marking all function symbols (except `activate`) which occur below an already marked symbol. Therefore, all function symbols of non-replacing subterms are marked. The second part consists of the rule $\texttt{activate}(x) \to x$ plus the rules:

$$\begin{aligned} f(x_1,\ldots,x_k) &\to f'(x_1,\ldots,x_k)\\ \texttt{activate}(f'(x_1,\ldots,x_k)) &\to f([x_1]_1^f,\ldots,[x_k]_k^f)\end{aligned}$$

for every $f \in \mathcal{F}$ for which f' appears in the first part of $\mathcal{R}_{FR}^\mu$, where $[t]_i^f = \texttt{activate}(t)$ if $i \in \mu(f)$ and $[t]_i^f = t$ otherwise. They also include rules

$$\texttt{activate}(f(x_1,\ldots,x_k)) \to f([x_1]_1^f,\ldots,[x_k]_k^f)$$

for k-ary symbols f where f' does *not* appear in the first part of $\mathcal{R}_{FR}^\mu$. However, Giesl and Middeldorp have recently shown that these rules are not necessary for obtaining a correct transformation [GM02].

Note that $\mathcal{R}$ and μ of Example 5 also show incompleteness of Ferreira and Ribeiro's transformation $\mathcal{R}_{FR}^\mu$.

5.3 Giesl and Middeldorp's transformations

Giesl and Middeldorp introduced a transformation that explicitly *marks* the replacing positions of a term (by using a new symbol `active`). Given a TRS $\mathcal{R} = (\mathcal{F}, R)$ and $\mu \in M_{\mathcal{F}}$, the TRS $\mathcal{R}^{\mu}_{GM} = (\mathcal{F}^{\mu}_{GM}, R^{\mu}_{GM})$ consists of $\mathcal{F}^{\mu}_{GM} = \mathcal{F} \cup \{\mathtt{active}, \mathtt{mark}\}$ and the rules R^{μ}_{GM} given by (for all $l \to r \in R$ and $f \in \mathcal{F}$):

$$\begin{aligned} \mathtt{active}(l) &\to \mathtt{mark}(r) \\ \mathtt{mark}(f(x_1, \ldots, x_k)) &\to \mathtt{active}(f([x_1]_f, \ldots, [x_k]_f)) \\ \mathtt{active}(x) &\to x \end{aligned}$$

where $[x_i]_f = \mathtt{mark}(x_i)$ if $i \in \mu(f)$ and $[x_i]_f = x_i$ otherwise [GM99]. Let $\mathcal{M} = (\mathcal{F}^{\mu}_{GM}, \{\mathtt{mark}(f(x_1, \ldots, x_k)) \to \mathtt{active}(f([x_1]_f, \ldots, [x_k]_f)) \mid f \in \mathcal{F}\})$; note that $\mathcal{M}$ is a confluent and terminating TRS [GM99]. Giesl and Middeldorp showed that, in general, this transformation is not complete. We prove the *completeness* of this transformation for proving termination of canonical *CSR* when dealing with left-linear TRSs.

Theorem 6 (Completeness). *Let $\mathcal{R}$ be a left-linear TRS and $\mu \in CM_{\mathcal{R}}$. If $\mathcal{R}$ is μ-terminating, then $\mathcal{R}^{\mu}_{GM}$ is terminating.*

Example 1 of [GM99] shows that this result does not hold if $\mu \notin CM_{\mathcal{R}}$. Giesl and Middeldorp also proposed two refinements of this transformation that are intended to provide better results in practice (being still incomplete).

1. The first modification ($\mathcal{R}^{\mu}_{mGM}$), see [GM02] Definition 3, consists in replacing the single symbol `active` by fresh symbols f_a for every $f \in \mathcal{F}$. Thus, $\mathcal{F}^{\mu}_{mGM} = \mathcal{F} \cup \{f_a \mid f \in \mathcal{F}\} \cup \{\mathtt{mark}\}$. The pattern $\mathtt{active}(f(\cdots))$ is also replaced everywhere by $f_a(\cdots)$. The rule $\mathtt{active}(x) \to x$ is expanded into all rules of the form $f_a(x_1, \ldots, x_k) \to f(x_1, \ldots, x_k)$ and, hence, removed. Now, let $\mathcal{M}' = (\mathcal{F}^{\mu}_{mGM}, \{\mathtt{mark}(f(x_1, \ldots, x_k)) \to f_a([x_1]_f, \ldots, [x_k]_f) \mid f \in \mathcal{F}\})$.
2. The second modification ($\mathcal{R}^{\mu}_{nGM}$) consists of normalizing the *rhs*'s of rules $f_a(l_1, \ldots, l_k) \to \mathtt{mark}(r)$ in $\mathcal{R}^{\mu}_{mGM}$ (where $f(l_1, \ldots, l_k) \to r \in R$) w.r.t. $\mathcal{M}'$, see Section 7 of [GM02].

Since $\mathcal{R}^{\mu}_{GM}$, $\mathcal{R}^{\mu}_{mGM}$ and $\mathcal{R}^{\mu}_{nGM}$ are equivalent regarding termination [GM02], they are also complete under the conditions of Theorem 6. Also, $\mathcal{R}^{\mu}_{GM}$ and $\mathcal{R}^{\mu}_{mGM}$ are equivalent regarding simple termination.

Theorem 7. *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. Then, $\mathcal{R}^{\mu}_{GM}$ is simply terminating if and only if $\mathcal{R}^{\mu}_{mGM}$ is simply terminating.*

Theorem 7 does not hold for $\mathcal{R}^{\mu}_{nGM}$.

Example 10. Consider the following non-simply terminating TRS $\mathcal{R}$:

```
f(0,1,x) → c(f(x,x,x))
```

Let $\mu(\mathtt{f}) = \{1, 2\}$ and $\mu(\mathtt{c}) = \varnothing$. Then, with $\mathcal{R}^{\mu}_{mGM}$:

$$
\begin{array}{lll}
\mathtt{f}_a(\mathtt{0},\mathtt{1},\mathtt{x}) & \to \mathtt{mark}(\mathtt{c}(\mathtt{f}(\mathtt{x},\mathtt{x},\mathtt{x}))) & \mathtt{f}_a(\mathtt{x},\mathtt{y},\mathtt{z}) \to \mathtt{f}(\mathtt{x},\mathtt{y},\mathtt{z}) \\
\mathtt{mark}(\mathtt{f}(\mathtt{x},\mathtt{y},\mathtt{z})) & \to \mathtt{f}_a(\mathtt{mark}(\mathtt{x}),\mathtt{mark}(\mathtt{y}),\mathtt{z}) & \mathtt{0}_a \to \mathtt{0} \\
\mathtt{mark}(\mathtt{0}) & \to \mathtt{0}_a & \mathtt{1}_a \to \mathtt{1} \\
\mathtt{mark}(\mathtt{1}) & \to \mathtt{1}_a & \mathtt{c}_a(\mathtt{x}) \to \mathtt{c}(\mathtt{x}) \\
\mathtt{mark}(\mathtt{c}(\mathtt{x})) & \to \mathtt{c}_a(\mathtt{x}) &
\end{array}
$$

we have the following infinite derivation (in $\mathcal{R}^{\mu}_{mGM} \cup \mathcal{E}mb(\mathcal{F}^{\mu}_{mGM})$):

$$
\begin{array}{l}
\underline{\mathtt{f}_a(\mathtt{0},\mathtt{1},\mathtt{f}(\mathtt{0},\mathtt{1},\mathtt{x}))} \to_{\mathcal{R}^{\mu}_{mGM}} \mathtt{mark}(\mathtt{c}(\mathtt{f}(\mathtt{f}(\mathtt{0},\mathtt{1},\mathtt{x}),\mathtt{f}(\mathtt{0},\mathtt{1},\mathtt{x}),\mathtt{f}(\mathtt{0},\mathtt{1},\mathtt{x})))) \\
\quad \to^{+}_{\mathcal{E}mb(\mathcal{F}^{\mu}_{mGM})} \underline{\mathtt{mark}(\mathtt{f}(\mathtt{0},\mathtt{1},\mathtt{f}(\mathtt{0},\mathtt{1},\mathtt{x})))} \to_{\mathcal{R}^{\mu}_{mGM}} \mathtt{f}_a(\mathtt{mark}(\mathtt{0}),\mathtt{mark}(\mathtt{1}),\mathtt{f}(\mathtt{0},\mathtt{1},\mathtt{x})) \\
\quad \to^{+}_{\mathcal{E}mb(\mathcal{F}^{\mu}_{mGM})} \mathtt{f}_a(\mathtt{0},\mathtt{1},\mathtt{f}(\mathtt{0},\mathtt{1},\mathtt{x})) \to \cdots
\end{array}
$$

Thus, $\mathcal{R}^{\mu}_{mGM}$ is not simply terminating. However, $\mathcal{R}^{\mu}_{nGM}$:

$$
\begin{array}{lll}
\mathtt{f}_a(\mathtt{0},\mathtt{1},\mathtt{x}) & \to \mathtt{c}_a(\mathtt{f}(\mathtt{x},\mathtt{x},\mathtt{x})) & \mathtt{f}_a(\mathtt{x},\mathtt{y},\mathtt{z}) \to \mathtt{f}(\mathtt{x},\mathtt{y},\mathtt{z}) \\
\mathtt{mark}(\mathtt{f}(\mathtt{x},\mathtt{y},\mathtt{z})) & \to \mathtt{f}_a(\mathtt{mark}(\mathtt{x}),\mathtt{mark}(\mathtt{y}),\mathtt{z}) & \mathtt{0}_a \to \mathtt{0} \\
\mathtt{mark}(\mathtt{0}) & \to \mathtt{0}_a & \mathtt{1}_a \to \mathtt{1} \\
\mathtt{mark}(\mathtt{1}) & \to \mathtt{1}_a & \mathtt{c}_a(\mathtt{x}) \to \mathtt{c}(\mathtt{x}) \\
\mathtt{mark}(\mathtt{c}(\mathtt{x})) & \to \mathtt{c}_a(\mathtt{x}) &
\end{array}
$$

is *rpo*-terminating (use the precedence $\mathtt{mark} > \mathtt{f}_a, \mathtt{0}_a, \mathtt{1}_a > \mathtt{c}_a > \mathtt{f}, \mathtt{0}, \mathtt{1} > \mathtt{c}$).

Thus, $\mathcal{R}^{\mu}_{mGM}$ (or $\mathcal{R}^{\mu}_{GM}$) and $\mathcal{R}^{\mu}_{nGM}$ are *not* equivalent regarding simple termination. However, we have the following.

Theorem 8. *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If $\mathcal{R}^{\mu}_{GM}$ is simply terminating, then $\mathcal{R}^{\mu}_{nGM}$ is simply terminating.*

Unfortunately, the transformations also have some (practical) drawbacks.

Theorem 9. *Let $\mathcal{R}$ be a left-linear TRS and $\mu \in CM_{\mathcal{R}}$. If $\mathcal{R}$ is not simply terminating, then $\mathcal{R}^{\mu}_{GM}$ and $\mathcal{R}^{\mu}_{mGM}$ are not simply terminating.*

The theorem does not hold if $\mu \notin CM_{\mathcal{R}}$.

Example 11. Consider the following TRS:

$$
\begin{array}{ll}
\mathtt{f}(\mathtt{f}(\mathtt{x})) & \to \mathtt{f}(\mathtt{g}(\mathtt{h}(\mathtt{x}))) \\
\mathtt{h}(\mathtt{x}) & \to \mathtt{f}(\mathtt{x})
\end{array}
$$

This TRS is not simply terminating:

$$
\underline{\mathtt{f}(\mathtt{f}(\mathtt{x}))} \to \mathtt{f}(\underline{\mathtt{g}(\mathtt{h}(\mathtt{x}))}) \to_{\mathcal{E}mb(\mathcal{F})} \mathtt{f}(\underline{\mathtt{h}(\mathtt{x})}) \to \mathtt{f}(\mathtt{f}(\mathtt{x})) \to \cdots
$$

Let $\mu(\mathtt{f}) = \mu(\mathtt{g}) = \mu(\mathtt{h}) = \varnothing$. Note that $\mu \notin CM_{\mathcal{R}}$. Now, $\mathcal{R}^{\mu}_{GM}$:

$$
\begin{array}{lll}
\mathtt{active}(\mathtt{f}(\mathtt{f}(\mathtt{x}))) & \to \mathtt{mark}(\mathtt{f}(\mathtt{g}(\mathtt{h}(\mathtt{x})))) & \mathtt{mark}(\mathtt{g}(\mathtt{x})) \to \mathtt{active}(\mathtt{g}(\mathtt{x})) \\
\mathtt{active}(\mathtt{h}(\mathtt{x})) & \to \mathtt{mark}(\mathtt{f}(\mathtt{x})) & \mathtt{mark}(\mathtt{h}(\mathtt{x})) \to \mathtt{active}(\mathtt{h}(\mathtt{x})) \\
\mathtt{mark}(\mathtt{f}(\mathtt{x})) & \to \mathtt{active}(\mathtt{f}(\mathtt{x})) & \mathtt{active}(\mathtt{x}) \to \mathtt{x}
\end{array}
$$

is simply terminating; e.g., the reduction sequence that corresponds to the previous 'dangerous' sequence is not longer possible in $\mathcal{R}^{\mu}_{GM} \cup \mathcal{E}mb(\mathcal{F}^{\mu}_{GM})$.

Theorem 9 does not hold for $\mathcal{R}^{\mu}_{nGM}$ (see Example 10; note that $\mu \in CM_{\mathcal{R}}$). Still, $\mathcal{R}^{\mu}_{nGM}$ can also be non-simply terminating if $\mathcal{R}$ is not simply terminating.

Example 12. Consider the non-simply terminating TRS $\mathcal{R}$:

$$\mathtt{f(f(x))} \to \mathtt{f(g(f(x)))}$$

together with $\mu(\mathtt{f}) = \{1\}$ and $\mu(\mathtt{g}) = \varnothing$. Note that $\mu \in CM_{\mathcal{R}}$. Then, $\mathcal{R}^{\mu}_{nGM}$:

$$\begin{array}{lll} \mathtt{f}_a\mathtt{(f(x))} \to \mathtt{f}_a\mathtt{(g}_a\mathtt{(f(x)))} & & \mathtt{f}_a\mathtt{(x)} \to \mathtt{f(x)} \\ \mathtt{mark(f(x))} \to \mathtt{f}_a\mathtt{(mark(x))} & & \mathtt{g}_a\mathtt{(x)} \to \mathtt{g(x)} \\ \mathtt{mark(g(x))} \to \mathtt{g}_a\mathtt{(x)} & & \end{array}$$

is not simply terminating:

$$\underline{\mathtt{f}_a\mathtt{(f(x))}} \to_{\mathcal{R}^{\mu}_{nGM}} \mathtt{f}_a\mathtt{(}\underline{\mathtt{g}_a\mathtt{(f(x))}} \to_{\mathcal{E}mb(\mathcal{F}^{\mu}_{nGM})} \mathtt{f}_a\mathtt{(f(x))} \to \cdots$$

As one can notice by re-examining the examples of Giesl and Middeldorp's papers [GM99,GM02], the success of their transformations usually depends on using Arts and Giesl's dependency pairs method [AG00]. Our results in this section clarify this situation by showing that, in fact, simplification orderings (alone) should *not* be used to orientate the rules of $\mathcal{R}^{\mu}_{GM}$ or $\mathcal{R}^{\mu}_{mGM}$ in proofs of canonical μ-termination. Rather, some kind of auxiliary technique *must* be additionally used to make them suitable. This is critically true when a proof of canonical μ-termination of a nonterminating (hence non-simply terminating) TRS is attempted. Automatic tools for proving termination that do not provide for adequate preprocessings (based on, e.g., applying argument filtering transformations, exploiting modularity, using dependency pairs, etc.) should not be used together with $\mathcal{R}^{\mu}_{GM}$ or $\mathcal{R}^{\mu}_{mGM}$. For instance, when using C*i*ME 2.0 system[5] for proving termination of $\mathcal{R}^{\mu}_{GM}$ or $\mathcal{R}^{\mu}_{mGM}$, the use of dependency pairs should be activated before.

6 Comparing the transformations

Figure 2 shows how different transformations for proving termination of *CSR* compare. The leftmost diagram of Figure 2 is obtained from that of [GM02] plus Theorem 5. An arrow from Θ to Θ' means that whenever $\mathcal{R}^{\mu}_{\Theta}$ is terminating, then $\mathcal{R}^{\mu}_{\Theta'}$ is also terminating (for μ selected from the subset of replacement maps written below the diagram). Dashed arrows (as well as minimal paths containing them) require left-linearity of $\mathcal{R}$. $^*GM$ stands for the three transformations introduced in the previous section (that are equivalent regarding termination of the transformed TRS); C stands for the *complete* Giesl and Middeldorp's transformation (not described here, see [GM99,GM02]).

Comparisons regarding simple termination are also interesting from a more practical point of view. Apart from Theorems 7 and 8, we have the following.

Theorem 10. *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If $\mathcal{R}$ is simply terminating, then $\mathcal{R}^{\mu}_{Z}$, $\mathcal{R}^{\mu}_{FR}$, and $\mathcal{R}^{\mu}_{GM}$ are simply terminating.*

We conjecture that simple termination of $\mathcal{R}^{\mu}_{Z}$ implies that of $\mathcal{R}^{\mu}_{FR}$ (the opposite is not true). Simple termination can be achieved after applying some transformations.

[5] Available at `http://cime.lri.fr`

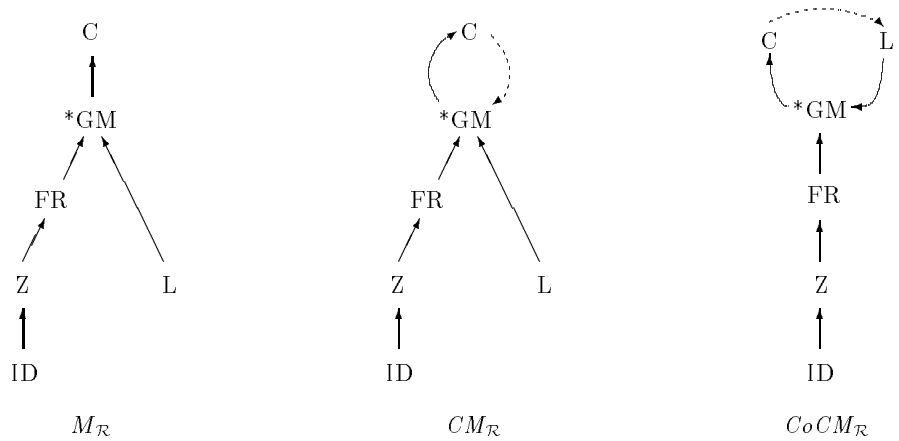

Fig. 2. Comparing transformations regarding termination

Example 13. Consider $\mathcal{R}$ and μ as in Example 12. Recall that $\mathcal{R}$ (and $\mathcal{R}^{\mu}_{GM}$, $\mathcal{R}^{\mu}_{mGM}$, $\mathcal{R}^{\mu}_{nGM}$) is not simply terminating. However, $\mathcal{R}^{\mu}_{L}$:

$$\mathtt{f(f(x))} \to \mathtt{f(g)}$$

is *rpo*-terminating (use precedence $\mathtt{f} > \mathtt{g}$). Also, $\mathcal{R}^{\mu}_{Z}$:

$$\begin{array}{lll} \mathtt{f(f(x))} \to \mathtt{f(g(f'(x)))} & & \mathtt{activate(f'(x))} \to \mathtt{f(x)} \\ \mathtt{f(x)} \to \mathtt{f'(x)} & & \mathtt{activate(x)} \to \mathtt{x} \end{array}$$

is *rpo*-terminating (use the precedence $\mathtt{activate} > \mathtt{f} > \mathtt{g}, \mathtt{f'}$). $\mathcal{R}^{\mu}_{FR}$ also is *rpo*-terminating.

Simple termination of $\mathcal{R}^{\mu}_{nGM}$ does *not* imply simple termination of $\mathcal{R}^{\mu}_{Z}$ or $\mathcal{R}^{\mu}_{FR}$.

Example 14. Consider $\mathcal{R}$ and μ as in Example 5. Note that $\mathcal{R}^{\mu}_{Z}$ (and $\mathcal{R}^{\mu}_{FR}$) is not terminating (see Example 9). However, $\mathcal{R}^{\mu}_{nGM}$:

$$\begin{array}{lll@{\qquad}lll}
\mathtt{nats}_a & \to & \mathtt{adx}_a\mathtt{(zeros}_a\mathtt{)} & \mathtt{mark(0)} & \to & \mathtt{0}_a \\
\mathtt{adx}_a\mathtt{(x:y)} & \to & \mathtt{incr}_a\mathtt{(mark(x):}_a\mathtt{adx(y))} & \mathtt{mark(s(x))} & \to & \mathtt{s}_a\mathtt{(mark(x))} \\
\mathtt{zeros}_a & \to & \mathtt{0}_a\mathtt{:}_a\mathtt{zeros} & \mathtt{nats}_a & \to & \mathtt{nats} \\
\mathtt{incr}_a\mathtt{(x:y)} & \to & \mathtt{s}_a\mathtt{(mark(x)):}_a\mathtt{incr(y)} & \mathtt{adx}_a\mathtt{(x)} & \to & \mathtt{adx(x)} \\
\mathtt{mark(nats)} & \to & \mathtt{nats}_a & \mathtt{zeros}_a & \to & \mathtt{zeros} \\
\mathtt{mark(adx(x))} & \to & \mathtt{adx}_a\mathtt{(mark(x))} & \mathtt{x:}_a\mathtt{y} & \to & \mathtt{x:y} \\
\mathtt{mark(zeros)} & \to & \mathtt{zeros}_a & \mathtt{incr}_a\mathtt{(x)} & \to & \mathtt{incr(x)} \\
\mathtt{mark(x:y)} & \to & \mathtt{mark(x):}_a\mathtt{y} & \mathtt{0}_a & \to & \mathtt{0} \\
\mathtt{mark(incr(x))} & \to & \mathtt{incr}_a\mathtt{(mark(x))} & \mathtt{s}_a\mathtt{(x)} & \to & \mathtt{s(x)}
\end{array}$$

is simply terminating.

For replacement maps $\mu \in CM_{\mathcal{R}}$, an immediate consequence of Theorems 9 and 10 is the equivalence of simple termination of $\mathcal{R}$ and $\mathcal{R}^{\mu}_{GM}$ for left-linear TRSs $\mathcal{R}$. With conservativeness, we have:

Theorem 11. *Let $\mathcal{R}$ be a left-linear TRS and $\mu \in CoCM_{\mathcal{R}}$. If $\mathcal{R}^{\mu}_{Z}$, $\mathcal{R}^{\mu}_{FR}$, or $\mathcal{R}^{\mu}_{nGM}$ are simply terminating, then $\mathcal{R}^{\mu}_{L}$ is simply terminating.*

Figure 3 shows the hierarchy of transformations regarding *simple* termination.

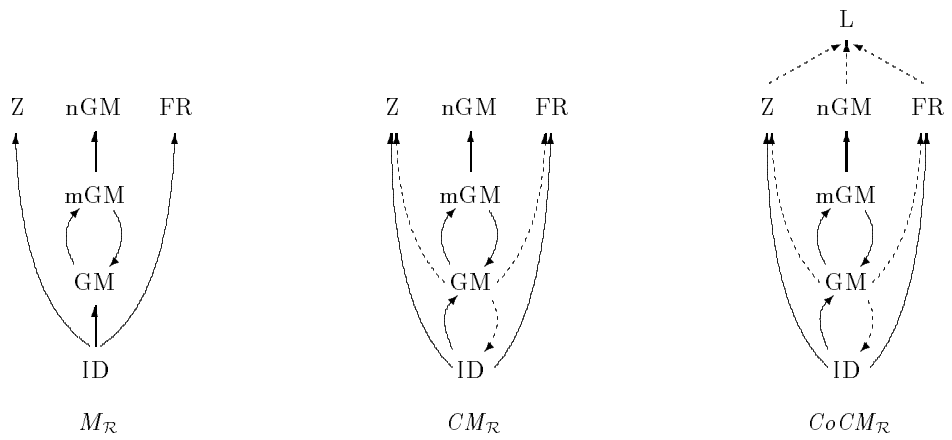

Fig. 3. Comparing transformations regarding simple termination

7 Experiments on proving termination of canonical *CSR*

Consider a left-linear TRS $\mathcal{R}$ for which we want to establish some interesting termination property. We claim that, for the purpose of computing (infinite) normal forms, proving termination of canonical *CSR* could be priorized over proofs of termination: a proof of termination of canonical *CSR* can be used to achieve normalizations (even with non-terminating TRSs) and also to arbitrarily approximate infinite normal forms (see Section 4). Of course, if (it is clear that) $\mathcal{R}$ is not terminating, then only the proof of termination of canonical *CSR* should be attempted. Our results suggest that proving μ-termination of a *terminating* TRS $\mathcal{R}$ is not much harder than directly proving termination of $\mathcal{R}$ (e.g., simple termination is preserved by the transformations, see Theorems 5 and 10). Moreover, proving termination of canonical *CSR* can be easier than attempting a direct proof of termination (see Examples 12 and 13). Table 1 shows the result of a preliminary experience aimed at providing some experimental evidence supporting our claim. We have selected two rather different collections of termination examples: Dershowitz's *33 examples of termination* [Der95] (first group of four examples) and Arts and Giesl's *collection of examples for termination using dependency pairs* [AG01] (most of them non-simply terminating; see the second group of four examples). In our experiments, given a TRS $\mathcal{R}$, we systematically try to prove termination of $\mathcal{R}_{\Theta}^{\mu_{\mathcal{R}}^{can}}$ for $\Theta \in \{ID, L, Z, FR, GM, mGM, nGM\}$ (note that proving termination of $\mathcal{R}_{ID}^{\mu_{\mathcal{R}}^{can}}$ is just proving termination of $\mathcal{R}$). We have used MU-TERM 1.0 for obtaining the transformed[6] TRSs. We have attempted the proofs of termination using C*i*ME 2.0 (MU-TERM 1.0 has also been used for giving the appropriate format) in both its standard version (Std, where polynomial orderings are used for proving simple termination of the TRS), and activating the use of dependency pairs (DP, thus dealing with non-simply

[6] MU-TERM 1.0 is available from `http://www.dsic.upv.es/users/elp/slucas/muterm`.

	ID		L		Z		FR		GM		mGM		nGM	
Ref. Example	Std	DP	Std	DP	Std	DP	Std	DP	Std	DP	Std	DP	Std	DP
5. Non Simp.	N	0.06	0.03	0.00	0.04	0.00	0.08	0.05	N	4153	N	0.14	N	0.16
7. Dutch Flag	0.09	0.06	0.05	0.02	0.22	0.12	?	0.20	?	0.33	?	0.2	?	0.17
8. Diff.	N	N	0.02	0.00	3.14	1.43	?	?	?	0.70	?	0.59	?	0.47
33. Hydra	N	N	NC	NC	N	?	?	?	N	N	N	N	N	N
3.1. Division v.1	N	2152	NC	NC	?	1.11	?	?	?	?	?	?	?	?
3.5. Remainder	N	2.65	NC	NC	?	9.08	=Z	=Z	?	?	?	?	?	?
3.7. Logarithm	105.0	17.9	=ID	=ID	6469	117.6	=Z	=Z	?	?	?	?	?	?
3.10. Min. sort	N	?	NC	NC	N	?	=Z	=Z	N	N	N	?	N	?

Table 1. Experiments on termination vs. $\mu_{\mathcal{R}}^{can}$-termination with C*i*ME 2.0

terminating TRSs). In this way, we proceeded as a (typical) user of automatic sofware tools having no expertise in proofs of termination.

A positive answer for a given experiment is reported in our figures by indicating the time (in seconds) used by C*i*ME to find out the proof. C*i*ME's negative answers are reported with 'N'. Finally, '?' means that the search for a proof was interrupted without getting any answer (usually after several hours, often days, of execution). Other marks such as 'NC' or '=ID' (in column L), or '=Z' (in column FR) mean that the TRS $\mathcal{R}$ is not $\mu_{\mathcal{R}}^{can}$-conservative ('NC'), or $\mathcal{R}_L^{\mu_{\mathcal{R}}^{can}} = \mathcal{R}$ ('=ID'), or that $\mathcal{R}_{FR}^{\mu_{\mathcal{R}}^{can}} = \mathcal{R}_Z^{\mu_{\mathcal{R}}^{can}}$ ('=Z'). In Table 1, we have selected a representative sample of the (still growing) complete experiment[7]. We can see that, whenever a TRS $\mathcal{R}$ is proved terminating, $\mu_{\mathcal{R}}^{can}$-termination of $\mathcal{R}$ can also be proved using some (non-trivial) transformation. This usually amounts to have some overhead in the proof, but we have also obtained improvements (e.g., with Dutch Flag and $\mathcal{R}_L^{\mu_{\mathcal{R}}^{can}}$, or Division v.1 and $\mathcal{R}_Z^{\mu_{\mathcal{R}}^{can}}$). In several cases we can prove (with C*i*ME) $\mu_{\mathcal{R}}^{can}$-termination even when we *cannot* prove termination of the (terminating) TRS (e.g., with Non Simp. or Diff.). On the other hand, hard problems like Hydra were not managed in any way (with the current conditions for the experiment). We believe that these results are quite encouraging regarding our hypothesis.

8 Conclusions

We have proven that, for left-linear TRSs $\mathcal{R}$, Giesl and Middeldorp's transformations are complete for proving termination of canonical *CSR*; if conservativeness is additionally fulfilled, then the contractive transformation is also complete. We have compared the different transformations regarding *simple* termination. Although undecidable, simple termination copes with the use of most usual orderings for proving termination of rewriting automatically (e.g., *rpo*, *kbo*, and *poly* [Der87]). Giesl and Middeldorp noticed that, even though their transformations subsume the others (see the leftmost diagram of Figure 2), the latter

[7] See `http://www.dsic.upv.es/users/elp/slucas/experiments` for more details.

'may still be useful for the purpose of automation' [GM99]. This is true, indeed, and Figure 3 provides a new (quite different, compared to Figure 2) hierarchy of the transformations which is helpful for guiding their practical use in combination with simplification orderings. Their interaction with other techniques for automatically proving termination (e.g., DP-simple termination [AG00,AG01]) should be further investigated.

We argued that termination of canonical *CSR* is a computational property which can be more interesting to analyze than standard termination. We have partially supported this claim both with theoretical and experimental results.

Acknowledgements. I thank Cristina Borralleras and Albert Rubio for many interesting discussions about the topics of this paper. I also thank Jürgen Giesl and the anonymous referees for their useful remarks.

References

[AG00] T. Arts and J. Giesl. Termination of Term Rewriting Using Dependency Pairs *Theoretical Computer Science*, 236:133-178, 2000.

[AG01] T. Arts and J. Giesl. A collection of examples for termination of term rewriting using dependency pairs. TR AIB-2001-09, RWTH Aachen, 2001.

[BLR02] C. Borralleras, S. Lucas, and A. Rubio. Recursive Path Orderings can be Context-Sensitive. *Proc. of CADE'02*, Springer LNAI to appear, 2002.

[Der87] N. Dershowitz. Termination of rewriting. *JSC*, 3:69-115, 1987.

[Der95] N. Dershowitz. 33 Examples of Termination. LNCS 909:16-26, Springer-Verlag, Berlin, 1995.

[DKP91] N. Dershowitz, S. Kaplan, and D. Plaisted. Rewrite, rewrite, rewrite, rewrite, rewrite. *Theoretical Computer Science* 83:71-96, 1991.

[FR99] M.C.F. Ferreira and A.L. Ribeiro. Context-Sensitive AC-Rewriting. *Proc. of RTA'99*, LNCS 1631:286-300, Springer-Verlag, Berlin, 1999.

[GM99] J. Giesl and A. Middeldorp. Transforming Context-Sensitive Rewrite Systems. *Proc. of RTA'99*, LNCS 1631:271-285, Springer-Verlag, Berlin, 1999.

[GM02] J. Giesl and A. Middeldorp. Transformation Techniques for Context-Sensitive Rewrite Systems. Technical Report AIB-2002-02, Aachen, 2002.

[Luc96] S. Lucas. Termination of context-sensitive rewriting by rewriting. *Proc. of ICALP'96*, LNCS 1099:122-133, Springer-Verlag, Berlin, 1996.

[Luc98] S. Lucas. Context-sensitive computations in functional and functional logic programs. *Journal of Functional and Logic Programming*, 1998(1):1-61, January 1998.

[Luc02] S. Lucas. Context-sensitive rewriting strategies. *Information and Computation*, to appear.

[SX98] J. Steinbach and H. Xi. Freezing – Termination Proofs for Classical, Context-Sensitive and Innermost Rewriting. Institut für Informatik, T.U. München, January 1998.

[Zan97] H. Zantema. Termination of Context-Sensitive Rewriting. *Proc. of RTA'97*, LNCS 1232:172-186, Springer-Verlag, Berlin, 1997.