Cogrowth of Arbitrary Graphs

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Abstract. A "cogrowth set" of a graph G is the set of vertices in the universal cover of G which are mapped by the universal covering map onto a given vertex of G. Roughly speaking, a cogrowth set is large if and only if G is small. In particular, when G is regular, a cogrowth constant (a measure of the size of the cogrowth set) exists and has been shown to be as large as possible if and only if G is amenable.

We present two approaches to the problem of extending this to the non-regular case. First, we show that the result above extends to the case when G is not regular but is the cover of a finite graph. This proof is based on some properties of a family of Laplacians related to the zeta function of the covered graph. An example is given where this result fails when G does not cover a finite graph. Second, for any graph with transient covering tree, we define a new cogrowth constant expressed in terms of harmonic measure and show that G is amenable if and only if this constant is 1.

Finally, we show that if G covers a finite graph, then the radial limit set of a cogrowth set has largest possible Hausdorff dimension if and only if G is amenable.

1. Introduction

The concept of amenability originated with von Neumann who once conjectured, though not in these words, that every non-amenable group was an extension of a free group on two generators. This conjecture turned out to be false but it was not until 1984 that a counterexample was found. Ol'shanskii elaborately constructed a group which was neither a finite extension of F_2 nor was amenable; this last step utilized a criterion for amenability developed by Grigorchuk. Essentially, a finitely generated group is not amenable if the number of reduced words of length n grows at the same rate as the number of reduced words of length less than n. Since every finitely generated group is the quotient of a free group F, it is conceivable that a coset in the quotient of F is big in some well defined way if and only if G is amenable. Grigorchuk's criterion is this: G is amenable if and only if the number of words of length n in a coset grows as fast as the total number of words of length n in F grows.

It was later noticed that this result can be extended to regular graphs (see [8], for example). The concept of amenability was extended to graphs by Gerl: we say that a graph is amenable if and only if

$$\inf_K \frac{|\partial K|}{|K|} = 0 \ ,$$

where the infimum is over all finite non-empty sets of vertices in G, and ∂K is the set of all edges connecting vertices of K to vertices not in K.

A d-regular graph is covered by a d-regular tree T (i.e., there exists a map θ from the vertices of T onto the vertices of G which preserves vertex degree and adjacency). Clearly, the number of vertices of distance n from a fixed vertex o in T is asymptotic to $(d-1)^n$, and we say that the "growth number" of T is $gr(T) = d - 1$. The "coset" $[o] = \theta^{-1}(o)$ also has a growth rate which we call the "cogrowth number" of G and define by

$$cogr(G) = \limsup_{n\to\infty} |S_n(o) \cap [o]|^{\frac{1}{n}} \ , \tag{1.1}$$

where $S_n(x)$ is the metric sphere in T of radius n and center x. It has been shown (see [8]) that G is amenable if and only if

$$cogr(G) = d - 1 \ .$$

Our aim is to extend this to the case when G has bounded vertex degree but is not necessarily regular. First, in the non-regular case, although $cogr(G)$ will still be defined by (1.1), the quantity $d-1$ no longer represents the growth of T; we define the growth number of T to be, in general,

$$gr(T) = \limsup_{n\to\infty} |S_n(o)|^{\frac{1}{n}} \ .$$

We note that our definition of $gr(T)$ differs from that in Lyons [7] (he uses the lim inf), but, under the additional hypothesis that G covers a finite graph, $\lim_{n\to\infty} |S_n(o)|^{\frac{1}{n}}$ exists and thus both definitions agree. A natural conjecture is that $cogr(G) = gr(T)$ if and only if G is amenable. Unfortunately, this is not true in general (see example below); a main difficulty is that if G has arbitrarily long chains (i.e., sequences of adjacent vertices of degree 2) then G is amenable but $cogr(G) < gr(T)$. One criterion that eliminates these possibilities is that G covers a finite graph. Then indeed we get the desired result:

Theorem 1. *Let G be a simple connected graph which covers a finite graph. Then G is amenable if and only if $cogr(G) = gr(T)$.*

This result seems fairly "tight" in that the hypothesis that G covers a finite graph is used several times in seemingly independent places in the proof.

Our second generalization involves the topological boundary ∂T of T. The random walk in T, if transient, converges to a point in ∂T, and the distribution of that random point is "harmonic" measure (called that since every harmonic function on T has an integral representation with respect to harmonic measure). In the regular case, harmonic measure has a particularly simple form:

$$\mu(T_\eta) = (d-1)^{-|\eta|} \ ,$$

where T_η is the set of rays starting at o which go through η. This hints at how to extend $gr(T)$ and $cogr(G)$: let

$$cogr_\mu(G) = \limsup_{n\to\infty}[\sum_{\eta\in S_n(o)\cap[o]} \mu(T_\eta)]^{1/n} \tag{1.2}$$

and

$$gr_\mu(T) = \limsup_{n\to\infty}[\sum_{\eta\in S_n(o)} \mu(T_\eta)]^{1/n} .$$

Note that $gr_\mu(T) = 1$ and, in the regular case, $cogr_\mu(G) = cogr(G)/(d-1)$. We shall prove

Theorem 2. *Let G be a simple connected graph with bounded vertex degree for which the random walk on the covering tree T is transient. Then G is amenable if and only if $cogr_\mu(G) = 1$.*

Finally, we consider measuring the size of the cogrowth set $[o]$ by how big its limit set in ∂T is. That is, let R be the set of rays in ∂T that hit $[o]$ infinitely often. As was proved for the regular case in [N3], we show:

Theorem 3. *If G is the cover of a finite graph then G is amenable if and only if* $\dim(R) = \dim(\partial T)$.

2. First situation

In this section, we shall prove Theorem 1. Given a graph G, which we assume is simple and connected, we additionally assume that it covers a finite graph G_0. That is, there exists a function $\theta_0 : G \to G_0$ such that θ_0 preserves adjacency and vertex degree (i.e., the vertex degree of x in G equals the vertex degree of $\theta_0(x)$ in G_0 and, if $x, y \in G$ are adjacent, then $\theta_0(x)$ and $\theta_0(y)$ are adjacent in G_0). Such a map is a discrete analog of a "local homeomorphism". In general, such a map is called a "cover" (of G_0 by G), and every graph is covered by a graph (if only by itself). The largest such cover of G is necessarily a tree, called the "universal covering tree", and denoted by T. Let $\theta : T \to G$ denote the covering map of T onto G. Given a vertex $x \in G$, let $[x] = \theta^{-1}(\theta(x))$; equivalently, $[x]$ is the equivalence class containing x with respect to the equivalence relation induced by θ. Since G covers a finite graph, it has a bounded vertex degree; let M denote an upper bound for the vertex degrees of G.

Let

$$K_u(x,y) = \sum_{n=0}^{\infty} |S_n(x)\cap[y]|u^n . \tag{2.1}$$

Since $K_u(x,y)$ can also be written as $\sum_{z\in[y]} u^{d(x,z)}$, it is clear that the convergence of K_u is independent of the choice of x and y. By (1.1), it is clear that K_u exists if

$u < 1/cogr(G)$, but diverges if $u > 1/cogr(G)$. Even if K_u exists, the result when applied to a function need not. Consider K_u applied to a constant function:

$$K_u 1(x) = \sum_y K_u(x,y) = \sum_{n=0}^{\infty} |S_n(x)| u^n = \sum_z u^{d(x,z)} \ .$$

By the last equality, it is clear that the convergence of $K_u 1(x)$ is independent of x and, by the definition of $gr(T)$, $K_u 1$ exists if $u < 1/gr(T)$ and diverges if $u > 1/gr(T)$. For convenience, let

$$u_0 = 1/gr(T) \ .$$

Then there is a gap between $cogr(G)$ and $gr(T)$ if and only if $K_{u_0+\epsilon}$ exists for some $\epsilon > 0$.

As a first step in studying the kernels K_u, we first find their inverses. A useful tool for this is the study of the "covering operators".

We say that a function $\hat{f} : T \to R$ covers $f : G \to R$ if

$$\hat{f} = f \circ \theta \ ,$$

and we say that a kernel (i.e., generalized matrix) $\widehat{M} : T \to R$ covers the kernel $M : G \to R$ if

$$M(\theta(\xi), \theta(\eta)) = \sum_{\rho \in [\eta]} \widehat{M}(\xi, \rho) \ .$$

An example of this last case is afforded by the "adjacency matrices" of T and G: for $x, y \in G$, let $A(x,y)$ be 1 or 0 according to whether x and y are adjacent or not. Similarly, let $\widehat{A}$ denote the adjacency matrix of T. Since θ preserves vertex degree, $\widehat{A}$ covers A.

Another such matrix is Q on G and $\widehat{Q}$ on T defined by

$$Qf(x) = (d(x) - 1) f(x) \ .$$

Clearly, $\widehat{Q}$ covers Q.

It is easy to verify that the covering relation is preserved by matrix multiplication (i.e., $\widehat{MN} = \widehat{M}\widehat{N}$ by which we mean: if $\widehat{M}$ covers M and $\widehat{N}$ covers N, then $\widehat{M}\widehat{N}$ covers MN). Also, if $\widehat{f}$ covers f, then $\widehat{Mf} = \widehat{M}\widehat{f}$. If we define $\widehat{K}_u(x,y) = u^{d(x,y)}$, then $\widehat{K}_u$ covers K_u as defined by (2.1).

Lemma 1. $(I - uA + u^2 Q) K_u = K_u (I - uA + u^2 Q) = (1 - u^2) I$.

Proof. Note that

$$\begin{aligned} \widehat{A}\widehat{K}_u(\xi,\eta) &= \sum_{\rho \sim \xi} u^{d(\rho,\eta)} \\ &= u^{d(\xi,\eta)} [(d(\xi) - 1)u + 1/u - (1/u - u)\hat{I}(\xi,\eta)] \\ &= \widehat{K}_u(\xi,\eta) [d(\xi) u + (1/u - u)(1 - \hat{I}(\xi,\eta)] \ , \end{aligned}$$

and so

$$\widehat{A}\widehat{K}_u = u\widehat{D}\widehat{K}_u + (1/u - u)(\widehat{K}_u - \hat{I}) \ .$$

Hence

$$AK_u = uDK_u + (1/u - u)(K_u - I) \ ,$$

and so

$$(I - uA + u^2Q)K_u = (1 - u^2)I \ .$$

The equality

$$K_u(I - uA + u^2Q) = (1 - u^2)I$$

can be treated similarly or by using the facts that $\widehat{K}_u\widehat{A}$ and $\widehat{K}_u\widehat{Q}$ are the transposes of $\widehat{A}\widehat{K}_u$ and $\widehat{Q}\widehat{K}_u$ respectively. □

We define a generalized *Laplacian* by $\Delta_u \equiv I - uA + u^2Q$. This terminology is motivated by the fact that $\Delta_1 = D - A$ is equivalent (i.e., equal up to multiplication by a bounded function which is also bounded away from 0) to the usual Laplacian on graphs ($\Delta = D^{-1}A - I$). In general, Δ_u is equivalent to the Schrödinger operator $\Delta + q$, where $q(x) = u - u^2 - \frac{1-u^2}{d(x)}$ (which is constant when G is regular). The operator Δ_u has long appeared (though not with this notation) in the literature on zeta functions for graphs. For example, Bass [1] was the first to prove:

$$Z(u) \equiv \prod_C (1 - u^{|C|})^{-1} = \frac{1}{(1-u^2)^r det(\Delta_u)} \ ,$$

where Z, the zeta function of a finite graph, is the product over "prime" cycles C, and r is the Betti number of the graph. See also papers [9, 11, 6] for other proofs of this generalization of Ihara's theorem.

Lemma 1 then states that Δ_u is essentially the inverse of K_u. We say that a function is u-superharmonic if $\Delta_u f \geq 0$. As in the usual case, Harnack's inequality holds.

Lemma 2. *If f is non-negative and u-superharmonic for some $u > 0$, then there exists $C > 0$ such that $f(y) \leq Cf(x)$ for all pairs of adjacent vertices x, y.*

Proof.

$$\begin{aligned} f(y) &\leq \sum_{z \sim x} f(z) = Af(x) \\ &\leq (1 + u^2q(x))f(x)/u \leq f(x)(1 + u^2(M-1))/u \ . \end{aligned}$$

□

Lemma 3. *If $f > 0$ and $\Delta_u f \geq \lambda f$, then $\forall \sigma < \lambda : \exists \epsilon > 0 : \Delta_{u+\epsilon} f \geq (\lambda - \sigma)f$.*

Proof. By Harnack's inequality and bounded vertex degree, choose $\epsilon > 0$ such that $Af(x) \leq \frac{\sigma}{\epsilon} f$. Then

$$-\epsilon Af + 2\epsilon uQf + \epsilon^2 Qf \geq -\sigma f \ ,$$

and so by hypothesis,

$$\Delta_{u+\epsilon} f = f - uAf + u^2 Qf - \epsilon Af + 2\epsilon uQf + \epsilon^2 Qf \geq (\lambda - \sigma) f \ .$$

□

A necessary and sufficient condition for $K_{u_0+\epsilon}$ to exist (equivalently, for $cogr(G) < gr(T)$) follows.

Proposition 1. *Suppose G is a simple connected graph which covers a finite graph. Then $\Delta_{u_0} f \geq \lambda f$ for some positive function f and some $\lambda > 0$ if and only if $K_{u_0+\epsilon}$ exists for some $\epsilon > 0$.*

Proof. The idea of the proof here is that $K_{u_0+\epsilon}$ is an analogue of the resolvent kernel G^ϵ for the usual Laplacian and we merely follow the proof of the analogous theorem in the usual case. One difficulty, that arises here is that there is no "resolvent equation". However, it turns out that equation (2.2) below is sufficient for our purposes.

Suppose $f > 0$ and $\Delta_{u_0} f \geq \lambda f$ for some $\lambda > 0$. By Lemma 3, for sufficiently small ϵ, there exists $\sigma < \lambda$ such that $\Delta_{u_0+\epsilon} f \geq (\lambda - \sigma) f$. Then $\widehat{\Delta}_{u_0+\epsilon} \hat{f} \geq (\lambda - \sigma)\hat{f}$. On T, $\widehat{K}_{u_0+\epsilon}$ exists, and

$$[1 - (u_0 + \epsilon)^2]\hat{f} = \widehat{K}_{u_0+\epsilon}\widehat{\Delta}_{u_0+\epsilon}\hat{f} \geq (\lambda - \sigma)\widehat{K}_{u_0+\epsilon}\hat{f} \ ,$$

from which it follows that

$$K_{u_0+\epsilon} f(x) \leq \frac{1 - (u_0 + \epsilon)^2}{\lambda - \sigma} f(x) \ ,$$

and therefore $K_{u_0+\epsilon}$ exists.

To prove the other way, we note that $K_u 1$ takes on only finitely many values (since K_u covers a corresponding kernel on a finite graph), and thus $K_u 1$ is bounded if $u < u_0$.

Suppose that, for some C,

$$\widehat{K}_{u_0}\widehat{K}_{u_0+\epsilon} \leq C\widehat{K}_{u_0+\epsilon} \ . \tag{2.2}$$

Then $g(x) \equiv K_{u_0+\epsilon}(x, x_0)$ satisfies $K_{u_0} g \leq Cg$ and $g \geq 0$. Choose λ such that $(1 - u_0^2)g \geq \lambda K_{u_0} g$. By Lemma 1, $\Delta_{u_0} K_{u_0} g \geq \lambda K_{u_0} g$. Letting $f = K_{u_0} g$, we find $f > 0$ and $\Delta_{u_0} f \geq \lambda f$.

It remains to prove (2.2) under the hypothesis that $K_u 1(x)$ is bounded for $u < u_0$. Fix $\xi, \eta \in T$ and suppose $\gamma = (\xi = \gamma_0, \gamma_1, \ldots, \gamma_n = \eta)$ is the path connecting ξ to η in T. Define $T(i) = \{\rho : d(\rho, \gamma_i) = d(\rho, \gamma)\}$. For convenience, let $s = u_0$ and $t = u_0 + \epsilon$. Then

$$\begin{aligned}
\widehat{K}_{u_0}\widehat{K}_{u_0+\epsilon}(\xi,\eta) &= \sum_{\rho} s^{d(\xi,\rho)}t^{d(\rho,\eta)} \\
&= \sum_{i=0}^{n}\sum_{\rho\in T(i)} s^{d(\rho,\gamma_i)+i}t^{d(\rho,\gamma_i)+n-i} \\
&= t^n\sum_{i=0}^{n}(s/t)^i \sum_{\rho\in T(i)}(st)^{d(\rho,\gamma_i)} \\
&= t^n\sum_{i=0}^{n}(s/t)^i \sum_{k=0}^{\infty}(st)^k|S_k(\gamma_i)\cap T(i)| \\
&\le t^n\sum_{i=0}^{n}(s/t)^i \sum_{k=0}^{\infty}(st)^k|S_k(\gamma_i)| \\
&\le t^n\frac{1}{1-\frac{s}{t}}\sup_{\xi}\widehat{K}_{st}1(\xi)\ .
\end{aligned}$$

The result follows, since $t^n = \widehat{K}_t(\xi,\eta)$ and $st < s = u_0$. □

Essential to the proof of Theorem 1 will be the fact that there exists a positive u_0-harmonic function on G. The proof of this, basically an application of the Perron–Frobenius theorem, appears in a paper on zeta functions on graphs by Kotani and Sunada [6].

Lemma 4. *There exists $h > 0$ such that $\Delta_{u_0}h = 0$ on G.*

Proof. Let G_0 be a finite graph covered by G, and let $\theta_0 : G \to G_0$ be the covering map. By theorem 1.6 of [6], there exists a positive valued function h_0 on G_0 such that $\Delta_{u_0}h_0 = 0$ (the α in [6] is our $1/gr(T)$ where T is the covering tree of G_0 and thus of G). The "lift" $h \equiv h_0 \circ \theta_0$ is positive u_0-harmonic on G. □

We define the usual inner product on G:

$$\langle f,g\rangle = \sum_{x\in G} f(x)g(x)\ .$$

Let $\mathcal{E} = \{[x,y] : x \sim y\}$ denote the set of directed edges in G, and for functions $u, v : \mathcal{E} \to R$, define

$$\langle u,v\rangle = \frac{1}{2}\sum_{[x,y]\in\mathcal{E}} u([x,y])v([x,y])\ .$$

We remark that Δ_u is self-adjoint since A is. We write $f \asymp g$ if and only if there exists $C > 0$ such that $\frac{1}{c} < f(x)/g(x) < C$ for all x.

The following proposition is a standard fact about self-adjoint operators and appears, in a slightly less general form, in [3].

Proposition 2. *Let $\lambda \geq 0$. Then there exists $h > 0$ such that $\Delta_u h \geq \lambda h$ if and only if*

$$\inf_f \langle f, \Delta_u f\rangle / \langle f, f\rangle \geq \lambda \ .$$

Proof. Suppose that $\Delta_{u_0} h \geq \lambda h$ for some $h > 0$. Define

$$\nabla f([x,y]) = \alpha(x,y) f(y) - \alpha(y,x) f(x) \ ,$$

where $\alpha(x,y) = \sqrt{u_0 h(x)/h(y)}$, and $[x,y]$ is a directed edge in G. Then, the usual inner product gives, for square summable f:

$$\begin{aligned} \langle \nabla f, \nabla g\rangle &= \frac{1}{2} \sum_{[x,y]} [\alpha(x,y) f(y) - \alpha(y,x) f(x)][\alpha(x,y) g(y) - \alpha(y,x) g(x)] \\ &= \sum_x f(x) g(x) \sum_{y \sim x} \alpha(y,x)^2 - \sum_x f(x) \sum_{y \sim x} \alpha(x,y)\alpha(y,x) g(y) \ . \end{aligned}$$

Since $\sum_{y\sim x} \alpha(y,x)^2 \leq 1 - u_0^2 + u_0^2 d(x) - \lambda$,

$$0 \leq \langle \nabla f, \nabla f\rangle \leq \langle f, \Delta_{u_0} f\rangle - \lambda \langle f, f\rangle \ ,$$

and therefore

$$\inf_f \langle f, \Delta_{u_0} f\rangle / \langle f, f\rangle \geq \lambda \ .$$

Let $K \subset G$ be finite, and define Δ_K by

$$\Delta_K f = \Delta_{u_0}(\chi_K f) \ .$$

Furthermore, let $\langle u, v\rangle_K = \sum_{x\in K} u(x) v(x)$. It is then easy to see that

$$\langle f, \Delta_K g\rangle_K = \langle \chi_K f, \Delta_{u_0}(\chi_K g)\rangle_K \ ,$$

and thus Δ_K is self-adjoint and finite dimensional. Let $\mathcal{C}(K)$ be the space of functions supported on K, and

$$\lambda_K = \inf_{f \in \mathcal{C}(K)} \frac{\langle f, \Delta_K f\rangle_K}{\langle f, f\rangle_K} \ .$$

Suppose $\inf_f \langle f, \Delta_u f\rangle / \langle f, f\rangle \geq \lambda$. Then $\lambda_K \geq \lambda \geq 0$ and Δ_K is positive. Since K is finite, there exists an eigenvector f such that $\Delta_K f = \lambda_K f$. We argue that f can be assumed to be positive on K as follows. Let h be as in Lemma 4, and define ∇ as above using this h. Then

$$\begin{aligned} \langle f, \Delta_K f\rangle_K &= \langle \chi_K f, \Delta_{u_0}(\chi_K f)\rangle \\ &= \langle \nabla(\chi_K f), \nabla(\chi_K f)\rangle \\ &\geq \langle \nabla(\chi_K |f|), \nabla(\chi_K |f|)\rangle \\ &= \langle |f|, \Delta_K |f|\rangle_K \ , \end{aligned}$$

where equality holds if and only if f does not change sign. However, since $\langle |f|, |f|\rangle = \langle f, f\rangle$ and $\langle |f|, \Delta_K |f|\rangle_K \geq \lambda_K$, equality indeed holds.

Let $o \in K_1 \subset K_2 \subset \dots$, where $\bigcup K_i = G$, and define h_n to be a positive solution on K_n of $\Delta_{K_n} h_n \geq \lambda h_n$ normalized so that $h_n(o) = 1$. By Lemma 1, there exists a pointwise convergent subsequence, and the pointwise limit, h, is positive and satisfies $\Delta_{u_0} h \geq \lambda h$. □

By combining Propositions 1 and 2, we see that there is a gap between $cogr(G)$ and $gr(T)$ if and only if

$$\inf_f \frac{\langle f, \Delta_{u_0} f\rangle}{\langle f, f\rangle} > 0 \ .$$

Theorem 1 then follows if we show the equivalence of this condition with a similar condition equivalent to the non-amenability of G (see [3]), namely,

$$\inf_f \frac{\langle f, \Delta_1 f\rangle}{\langle f, f\rangle} > 0 \ .$$

With h as in Lemma 4, we indeed have this equivalence since, as in the proof of Proposition 2,

$$\begin{aligned}
\langle f, \Delta_{u_0} f\rangle &= \frac{1}{2}\sum_{[x,y]} \Big(\sqrt{u_o \frac{h(x)}{h(y)}} f(y) - \sqrt{u_o \frac{h(y)}{h(x)}} f(x)\Big)^2 \\
&= \frac{u_0}{2} \sum_{[x,y]} h(x)h(y)\Big(\frac{f(x)}{h(x)} - \frac{f(y)}{h(y)}\Big)^2 \\
&\asymp \sum_{[x,y]} \Big(\frac{f(x)}{h(x)} - \frac{f(y)}{h(y)}\Big)^2 \\
&= \langle \frac{f}{h}, \Delta_1 \frac{f}{h}\rangle \ ,
\end{aligned}$$

and therefore Theorem 1 is proven.

Example 1. We give an example (actually a class of examples) of an amenable graph with $cogr(G) < gr(T)$.

Let G_0 be a non-amenable regular graph which is not a tree. Its cover is a regular tree, call it T_d. Attach an infinite chain to a vertex $o \in G_0$; call the resulting graph G. Then G is amenable. Its cover, T, is T_d with an infinite chain attached to each point in $[o]$. Then $|S_n(o) \cap [o]|$ is the same in T and T_d, and so $cogr(G_0) = cogr(G)$. However, $|S_n(o)|$ is bigger in T than in T_d, so $gr(G_0) \leq gr(G)$. Hence, since G_0 is non-amenable and regular, $cogr(G) = cogr(G_0) < gr(G_0) \leq gr(G)$.

Remark 1. We conjecture that if G has bounded vertex degree and $cogr(G) = gr(T)$, then G is amenable. A counterexample would, of course, not be a cover of a finite graph and not be regular. Furthermore, from Theorem 2, $cogr_\mu(G) < 1$.

3. Another cogrowth constant

For this, we assume that the universal covering tree T of G is transient (that is the random walk on T is transient). This is not a very strong condition since it is satisfied by most graphs G; for example, any G which contains two or more cycles or on which the random walk is transient. The only graphs for which this fails are recurrent graphs with at most one cycle. We shall not consider these graphs here.

Fix $o \in T$, and let ∂T denote the set of all geodesic rays starting at o. Given $\eta \in T$, let T_η denote the set of all paths in ∂T which go through the vertex η. This set is called a "cone", and the set of all cones forms a topology base on ∂T. The simple random walk X_n on T, starting at o, converges a.s., in this topology, to a point X_∞ in ∂T. The distribution of X_∞ is called harmonic measure, and we write:

$$\mu(E) = P_o(X_\infty \in E) .$$

Recall the resolvent kernels for the Laplacian $\Delta = D^{-1}A - I$ on T, denoted G_T^ϵ, are defined by

$$G_T^\epsilon(\xi, \eta) = P_\xi(X_n = \eta)/(1-\epsilon)^{n+1}$$

and satisfy

$$(\Delta + \epsilon I)G_T^\epsilon = -I .$$

Then, the resolvent kernels cover the resolvent kernel G^ϵ on G. We base our proof of Theorem 2 on the fact that G^ϵ exists for some $\epsilon > 0$ if and only if G is not amenable (for example, see [8], and references therein).

As a first step, we prove that Green's function G_T ($=G_T^0$) on T and harmonic measure of cones are comparable.

Lemma 5. *For $\eta \in [o]$, $G_T(o, \eta) \asymp \mu(T_\eta)$.*

Proof. By the Markov property (twice),

$$\mu(T_\eta) = P_o(X_\infty \in T_\eta) = P_o(\exists n : X_n = \eta)P_\eta(X_\infty \in T_\eta) = G_T(o, \eta)\frac{P_\eta(X_\infty \in T_\eta)}{G_T(\eta, \eta)} .$$

The denominator of the fraction above is constant for $\eta \in [o]$ while the numerator takes on at most $d(\eta)$ values. □

Proof of Theorem 2. Suppose that G is not amenable (i.e., G^ϵ exists for some $\epsilon > 0$). Let $c = 1/(1-\epsilon)$. Note that

$$\sum_n p_T^{(n)}(o, \eta)c^{n+1} = \sum_{n \geq |\eta|} p_T^{(n)}(o, \eta)c^{n+1} \geq \sum_{n \geq |\eta|} p_T^{(n)}(o, \eta)c^{|\eta|+1} = G_T(o, \eta)c^{|\eta|+1} ,$$

and so, by Lemma 5,

$$\begin{aligned}\sum_n \Big(\sum_{\eta \in S_n \cap [o]} \mu(T_\eta) \Big) c^n &\asymp \sum_n \Big(\sum_{\eta \in S_n \cap [o]} G_T(o,\eta) \Big) c^n \\ &\leq \sum_{\eta \in [o]} \sum_n p^{(n)}(o,\eta) c^{n+1} \\ &= \sum_{\eta \in [o]} G_T^\epsilon(o,\eta) \\ &= G^\epsilon(o,o) < \infty \ ,\end{aligned}$$

and thus, by (1.2), $cogr_\mu(G) < 1$.

Conversely, suppose $cogr_\mu(G) < 1$. Then there exists some $c > 1$ such that

$$\sum_n \sum_{\eta \in S_n \cap [o]} \mu(T_\eta) c^n < \infty \ ,$$

which, by Lemma 5, implies

$$\sum_{\eta \in [o]} G_T(o,\eta) c^{|\eta|} < \infty \ .$$

Choose $\alpha \in (0,1)$ such that $G_T(o,\eta) \geq k_1 c^{-|\eta|/(1-\alpha)}$ for all $\eta \in [o]$. Then $G_T(o,\eta)^{1-\alpha} \geq k_1^{1-\alpha} c^{-|\eta|}$, and so

$$G_T(o,\eta)^\alpha \leq k_1^{\alpha-1} c^{|\eta|} G_T(o,\eta) \ .$$

Now, choose $\epsilon > 0$ such that both G_T^ϵ and $G_T^{\epsilon'}$ exist (and are bounded), where $\epsilon' = 1 - (1-\epsilon)^{1/(1-\alpha)}$. By Hölder's inequality,

$$G_T^\epsilon(o,\eta) \leq G_T(o,\eta)^\alpha G^{\epsilon'}(o,\eta)^{1-\alpha} \ ,$$

and so $G_T^\epsilon(o,\eta) \leq k c^{|\eta|} G_T(o,\eta)$ for some k and all $\eta \in [o]$. Therefore, there exists some $\epsilon > 0$ such that

$$G^\epsilon(o,o) = \sum_{\eta \in [o]} G_T^\epsilon(o,\eta) \leq k \sum_{\eta \in [o]} G_T(o,\eta) c^{|\eta|} < \infty \ .$$

□

4. The radial limit set and its Hausdorff dimension

We shall now show that a cover of a finite graph is amenable if and only if the radial limit set of $[o]$ has the highest possible Hausdorff dimension. Most of the terminology below appears in the seminal paper by Lyons [7]. The proof is based on the proof of the analogous fact for regular graphs which appeared in [10].

Proof of Theorem 3. We note that since G is a cover of a finite graph, T is also, and so it is "quasispherical". A consequence is that $\lim_{n\to\infty} |S_n|^{\frac{1}{n}}$ exists.

Fix k, and let $R' = \{\gamma \in \partial T : \gamma_{nk} \in [o] \text{ for any } n\}$. Then $R' = \partial T'$, where T' is periodic and generated by a finite tree, namely T_o, the tree of radius k in T with $\partial T_o = S_k \cap [o]$. Then

$$\begin{aligned}\dim(R') &= \log(br(T')) = \log(gr(T'))\\ &= \liminf_{n\to\infty} \frac{1}{n} \log |S'_n| = \liminf_{n\to\infty} \frac{1}{nk} \log |S'_{nk}|\\ &\geq \frac{1}{k} \log |S'_k| = \log(|S_k \cap [o]|)^{\frac{1}{k}} ,\end{aligned}$$

and thus, since $R' \subset R \subset \partial T$,

$$|S_k \cap [o]|^{\frac{1}{k}} \leq e^{\dim(R')} \leq e^{\dim(R)} \leq e^{\dim(\partial T)} = br(T) \leq gr(T) = \lim_{n\to\infty} |S_n|^{\frac{1}{n}} .$$

Therefore, by Theorem 1, if G is amenable, then $\dim(R) = \dim(\partial T)$.

Conversely, suppose $\dim(R) = \delta \equiv \dim(\partial T)$. If there exists $\alpha < \delta$ such that $\sum_{\eta\in[o]} e^{-\alpha|\eta|} < \infty$, then

$$\inf_F \sum_{\eta\in[o]-F} e^{-\alpha|\eta|} = 0 ,$$

where $F \subset [o]$ is finite, and, since R is the radial limit set of any set of the form $[o] - F$, $\dim(R) \leq \alpha < \delta$. Hence, if $\dim(R) = \delta$, then, for $\alpha < \delta$,

$$\sum_n e^{-\alpha n} |S_n \cap [o]| = \sum_{\eta\in[o]} e^{-\alpha|\eta|} = \infty ,$$

and thus $\limsup_{n\to\infty} |S_n \cap [0]|^{\frac{1}{n}} \geq e^{\alpha}$ for all $\alpha < \delta$. Therefore, by (1.2),

$$cogr(G) \geq e^{\dim(\partial T)} = br(T) = gr(T) = \liminf_{n\to\infty} |S_n|^{\frac{1}{n}} ,$$

and, by Theorem 1, G is amenable. □

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