Bogomolov on tori revisited.

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1 Introduction.

Let $V \subseteq \mathbb{G}_m^n \subseteq \mathbb{P}^n$ be a geometrically irreducible variety which is not torsion (*i. e.* not a translate of a subtorus by a torsion point). For $\theta > 0$ let $V(\theta)$ be the set of $\boldsymbol{\alpha} \in V(\overline{\mathbb{Q}})$ of Weil's height $h(\boldsymbol{\alpha}) \leq \theta$. By the toric case of Bogomolov conjecture (which is a theorem of Zhang),

$$\hat{\mu}^{\mathsf{ess}}(V) = \inf\{\theta > 0, \quad \overline{V(\theta)} = V\} > 0 \,.$$

If we assume moreover that V is not a translate of a subtorus by a point (eventually of infinite order) we can give a lower bound for $\hat{\mu}^{\mathsf{ess}}(V)$ depending only on $\deg(V)$ (see [Bom-Zan 1995], [Dav-Phi 1999], [Sch 1996]).

Let us define the obstruction index $\omega(V)$ as the minimum degree of an hypersurface containing V. We remark that $\omega(V) \leq n \deg(V)^{1/\operatorname{codim}(V)}$ ([Cha]). Assume that V is not transverse (*i. e.* is not contained in a translate of a subtorus). In [Amo-Dav 2003] we conjecture

$$\hat{\mu}^{\mathsf{ess}}(V) \geq c(n)\omega(V)^{-1}$$

for some $c(n) > 0$ and we prove

$$\hat{\mu}^{\mathsf{ess}}(V) \geq c(n)\omega(V)^{-1}(\log(3\omega(V))^{-\lambda(\operatorname{codim}(V))}$$

where $\lambda(k) = \big(9(3k)^{k+1}\big)^k$.

The aim of this paper is to give a more simple proof of a slightly improved (and explicit) version of this result (theorem 4.1), based on a very simple determinant argument (see section 2). More precisely the proof presented here

- avoid the use of the absolute Siegel's lemma of Zhang (see [Dav-Phi 1999], lemme 4.7)
- don't need any variant of zero's lemma and the subsequent combinatorial arguments (section 4 of [Amo-Dav 2003])

- don't use the weighted obstruction index $\omega(T;V)$ defined in [Amo-Dav 2003], definition 2.3.

Let
$$V^0 = V \setminus \bigcup_{B \subseteq V} B.$$
where the union is on the set of translates B of subgroups of positive dimension contained in V. In [Amo-Dav 2006], theorem 1.5 we deduce from a lower bound for the essential minimum of V, a lower bound for height for all but finitely points of V^0. Here we prove (theorem 5.1) an again slightly improved (and explicit) version of that result. We also correct a mistake which appears in that paper: in *op. cit.*, theorem 1.5, $\delta(V)$ must be defined as the minimum degree δ such that V is, as a set, intersection of hypersurface of degree $\leq \delta$ (see remark 5.2 for details).

The determinant argument allow us to prove also very precise results concerning the normalized height $\hat{h}(V)$ of an hypersurface V (see section 3 for the definition). In this special case we conjecture :

Conjecture 1.1 *Assume one of the following:*

i) V is geometrically irreducible and it is not a translate of a subtorus.

ii) V is defined and irreducible over the rationals and is not torsion.

Then, there exists an absolute constant $c > 0$ such that $\hat{h}(V) \geq c$.

We remark that Lehmer's conjecture implies conjecture ii), *via* an argument of Lawton. We shall prove

Theorem 1.2 *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees $(D_1, \ldots, D_n)$ with discrete stabilizer. Then, if $n \geq 9$ and*
$$\max D_j \leq 3^{2^n}$$
we have
$$\hat{h}(V) \geq \frac{1}{23}\,.$$

This result shows that an eventual example contradicting conjecture i) in n variable must be realized by polynomials of very big degree (or comes from an hypersurface of less variables). This could suggests an even more optimistic conjecture:

Let V be a geometric irreducible hypersurface of $\mathbb{G}_m^n$ with discrete stabilizer. Then $\hat{h}(V) \geq f(n)$, where $f(n) \to +\infty$ for $n \to \infty$.

In section 3 we also provide a counterexample to this last statement.

2 A determinant argument.

The following proposition is the key argument for the proof of the main theorems.

Let $S \subseteq \mathbb{P}_n$ and let $I \subset \mathbb{C}[\mathbf{x}]$ be the ideal defining its Zariski closure. For $\nu \in \mathbb{N}$ we denote by $H(S;\nu)$ the Hilbert function $\dim[\mathbb{C}[\mathbf{x}]/I]_\nu$. Let T be a positive integer and let $I^{(T)}$ be the T-symbolic power of I, *i. e.* the ideal of polynomials vanishing on S with multiplicity $\geq T$. We put $H(S,T;\nu) = \dim[\mathbb{C}[\mathbf{x}]/I^{(T)}]_\nu$.

Similarly, if $S \subseteq (\mathbb{P}_1)^n$ and $\boldsymbol{\nu} = (\nu_1,\dots,\nu_n) \in \mathbb{N}^n$ we denote its multi-homogeneous Hilbert function by

$$H(S;\boldsymbol{\nu}) = \dim([\overline{\mathbb{Q}}[x_1,\dots,x_n]/I]_{\nu_1,\dots,\nu_n})$$

where $I \subset \mathbb{C}[\mathbf{x}]$ is the ideal defining $\overline{S}$. More generally, if T is a positive integer we put $H(S,T;\boldsymbol{\nu}) = \dim([\overline{\mathbb{Q}}[x_1,\dots,x_n]/I^{(T)}]_{\nu_1,\dots,\nu_n})$.

Proposition 2.1 *Let ν, T be positive integers and let p be a prime number. Let also h be a positive real number and S be a subset (eventually infinite) of $\mathbb{G}_m^n$ of points of height $\leq h$. Then*

$$h \geq \left(1 - \frac{H(S,T;\nu)}{H(\ker[p]\cdot S;\nu)}\right)\frac{T\log p}{p\nu} - \frac{n}{2\nu}\log(\nu+1)\ . \tag{2.1}$$

In particular, if

$$H(S,T;\nu) \leq \frac{1}{2}H(\ker[p]\cdot S;\nu) \tag{2.2}$$

and

$$T\log p \geq 2np\log(\nu+1)\ , \tag{2.3}$$

then

$$h \geq \frac{T\log p}{4p\nu} \geq \frac{n\log(\nu+1)}{2\nu}\ .$$

Proof. Let for brevity $S' = \ker[p]S$. We consider the (eventually infinite) matrix

$$(\boldsymbol{\beta}^{\boldsymbol{\lambda}})_{\substack{\beta\in S'\\ |\boldsymbol{\lambda}|\leq\nu}}$$

of rang $L = H(\ker[p]\cdot S;\nu)$. We select $\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_L \in S'$ and $\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_L$ with $|\boldsymbol{\lambda}_j| \leq \nu$ such that the determinant

$$\Delta = \left|\det(\boldsymbol{\beta}_i^{\boldsymbol{\lambda}_j})_{i,j=1,\dots,L}\right|$$

is non-zero. Let $L_0 = H(\ker[p]\cdot S;\nu) - H(S,T;\nu)$. Then, by definition, there exist linearly independent polynomials $G_k = \sum_{j=1}^L g_{kj}\mathbf{x}^{\boldsymbol{\lambda}_j}$ $(k = 1,\dots,L_0)$ vanishing on S with multiplicity $\geq T$. Let K be a sufficiently large field and let v be a non archimedean place of K dividing p. After renumbering the multi-indexes $\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_L$ and after making some linear combinations, we can assume

$$G_k = \sum_{j=1}^{L-k+1} g_{kj}\mathbf{x}^{\boldsymbol{\lambda}_j}$$

and moreover

$$|g_{k,j}|_v \begin{cases} \leq 1, & \text{if } j = 1, \dots, L-k; \\ = 1, & \text{if } j = L-k+1; \end{cases}$$

for $k = 1, \dots, L_0$. By elementary operations on columns we replace the last L_0 columns of Δ by the columns

$${}^{\tau}\big(G_k(\boldsymbol{\beta}_1), \dots, G_k(\boldsymbol{\beta}_L)\big)\ , \qquad k = 1, \dots, L_0\ .$$

Let Δ' the new determinant; then $|\Delta'|_v = |\Delta|_v$. Since G_k vanish on S with multiplicity $\geq T$ and since its coefficients are v-integers, we also have

$$|G_k(\boldsymbol{\beta}_i)|_v \leq p^{-T/(p-1)} \max\{1, |\beta_{i,1}|_v, \dots, |\beta_{i,n}|_v\}^{\nu} \qquad (i = 1, \dots, L;\ k = 1, \dots, L_0)\ .$$

By developping Δ' with respect to the last L_0 columns we obtain

$$|\Delta'|_v = |\Delta|_v \leq p^{-L_0 T/(p-1)} \prod_{i=1}^{L} \max\{1, |\beta_{i,1}|_v, \dots, |\beta_{i,n}|_v\}^{\nu L}\ .$$

By the product's formula (using a trivial lower bound for $v \nmid p$)

$$1 \leq p^{-L_0 T/(p-1)} L^{L/2} e^{\nu h L}$$

and, using $L \leq \binom{\nu+1}{n} \leq (\nu+1)^n$,

$$\log h \geq \frac{L_0}{L} \times \frac{T \log p}{p\nu} - \frac{n}{2\nu} \log(\nu+1)$$

and the statement of proposition 2.1 follows.

□

The following is a multihomogeneous version of proposition 2.1.

Proposition 2.2 *Let $\nu_1, \dots, \nu_n$, T be positive integers and let p be a prime number. Let also $h_1, \dots, h_n$ be a positive real number and S be a subset (eventually infinite) of $\mathbb{G}_m^n$ of points $\boldsymbol{\alpha}$ satisfying $h(\alpha_j) \leq h_j$ for $j = 1, \dots, n$. Then*

$$\nu_1 h_1 + \dots + \nu_n h_n \geq \left(1 - \frac{H(S, T; \boldsymbol{\nu})}{H(\ker[p] \cdot S; \boldsymbol{\nu})}\right) \frac{T \log p}{p} - \frac{n}{2} \log(\nu_{\max} + 1) \qquad (2.4)$$

where $\nu_{\max} = \max\{\nu_1, \dots, \nu_n\}$.

Proof. Let for brevity $S' = \ker[p]S$. We consider the matrix

$$\left(\boldsymbol{\beta}^{\boldsymbol{\lambda}}\right)_{\substack{\boldsymbol{\beta} \in S' \\ |\lambda_1| \leq \nu_1, \dots, |\lambda_n| \leq \nu_1}}$$

of rang $L = H(\ker[p] \cdot S; \boldsymbol{\nu})$. We select $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_L \in S'$ and $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_L$ with $|\lambda_{j,l}| \leq \nu_l$ such that the determinant

$$\Delta = \left| \det(\boldsymbol{\beta}_i^{\boldsymbol{\lambda}_j})_{i,j=1,\ldots,L} \right|$$

is non-zero. Let $L_0 = H(\ker[p] \cdot S; \boldsymbol{\nu}) - H(S, T; \boldsymbol{\nu})$. Then, by definition, there exists linearly independent polynomials $G_k = \sum_{j=1}^{L} g_{kj} \mathbf{x}^{\boldsymbol{\lambda}_j}$ $(k = 1, \ldots, L_0)$ vanishing on S with multiplicity $\geq T$. Let K be a sufficiently large field and let v be a non archimedean place of K dividing p. After renumbering the multi-index $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_L$ and after making some linear combinations, we can assume

$$G_k = \sum_{j=1}^{L-k+1} g_{kj} \mathbf{x}^{\boldsymbol{\lambda}_j}$$

and moreover

$$|g_{k,j}|_v \begin{cases} \leq 1, & \text{if } j = 1, \ldots, L-k; \\ = 1, & \text{if } j = L-k+1; \end{cases}$$

for $k = 1, \ldots, L_0$. By elementary operations on columns we replace the last L_0 columns of Δ by the columns

$${}^{\tau}\big(G_k(\boldsymbol{\beta}_1), \ldots, G_k(\boldsymbol{\beta}_L)\big) \;, \qquad k = 1, \ldots, L_0 \;.$$

Let Δ' the new determinant; then $|\Delta'|_v = |\Delta|_v$. Since G_k vanish on S with multiplicity $\geq T$ and since its coefficients are v-integers, we also have

$$|G_k(\boldsymbol{\beta}_i)|_v \leq p^{-T/(p-1)} \prod_{j=1}^{n} \max\{1, |\beta_{i,j}|_v\}^{\nu_j} \qquad (i = 1, \ldots, L;\; k = 1, \ldots, L_0) \;.$$

By developping Δ' with respect to the last L_0 columns we obtain

$$|\Delta'|_v = |\Delta|_v \leq p^{-L_0 T/(p-1)} \prod_{i=1}^{L} \prod_{j=1}^{n} \max\{1, |\beta_{i,j}|_v\}^{\nu_j L} \;.$$

By the product's formula (using a trivial lower bound for $v \nmid p$)

$$1 \leq p^{-L_0 T/(p-1)} L^{L/2} e^{(\nu_1 h_1 + \cdots + \nu_n h_n) L}$$

and, using $L \leq (\nu_{\max} + 1)^n$,

$$\nu_1 h_1 + \cdots + \nu_n h_n \geq \frac{L_0}{L} \times \frac{T \log p}{p} - \frac{n}{2} \log(\nu_{\max} + 1)$$

and the statement of proposition 2.2 follows.

□

3 Hypersurfaces.

In this section we are interested in the case of a hypersurface V. For these varieties we have a “natural” definition of height (which coincide with the previous one) since we can extend the Mahler measure to polynomials in several variables. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$; we define its Mahler measure as:

$$M(P) = \exp \int_0^1 \cdots \int_0^1 \log \left| f \left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n} \right) \right| dt_1 \ldots dt_n.$$

Let now K be a number field and let V be an hypersurface in $\mathbb{G}_m^n$ defined over K:

$$V = \{ \boldsymbol{\alpha} \in \mathbb{G}_m^n \text{ such that } f(\boldsymbol{\alpha}) = 0 \}$$

for some polynomial $f \in K[\mathbf{x}]$ (irreducible over $\overline{\mathbb{Q}}[\mathbf{x}]$). We define:

$$\hat{h}(V) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log M_v(f),$$

where $M_v(f)$ is the maximum of the v-adic absolute values of the coefficients of f if v is non archimedean, and $M_v(f)$ is the Mahler measure of σf if v is an archimedean place associated with the embedding $\sigma \colon K \hookrightarrow \overline{\mathbb{Q}}$.

We prove:

Proposition 3.1 *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees $D_1, \ldots, D_n$ and assume that V is not a translated of a torus. Let $D_{\max} = \max\{D_1, \ldots, D_n\}$. Then, for any prime number $p \geq 5$,*

$$\hat{h}(V) \geq \frac{\log p}{7p} - \frac{nk' \log p}{p^{k'}} - \frac{n \log(n^2 D_{max})}{2p^{k'}} \,. \tag{3.5}$$

where k' is the codimension of the stabilizer of V.

Proof. Since V is not a translated of a torus, $k' \geq 2$. This implies $n \geq 2$ and $p^{k'} \geq 9$.

We assume first that $p \nmid [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$, so that $V' = \ker[p]V$ is a union of $p^{k'}$ translate of V, and we prove

$$\hat{h}(V) \geq \frac{\log p}{7p} - \frac{nk' \log p}{p^{k'}} - \frac{n \log(n D_{max})}{2p^{k'}} \,, \tag{3.6}$$

Let $\varepsilon > 0$ and assume $D_{\max} = D_n$. The proposition 2.7 of [Amo-Dav 2000] shows that the set

$$S = \{ (\zeta_1, \ldots, \zeta_{n-1}, \alpha) \in V(\overline{\mathbb{Q}}),\ \zeta_1, \ldots, \zeta_{n-1} \text{ roots of unity},\ h(\alpha) \leq \hat{h}(V)/D_n + \varepsilon \}$$

is Zariski dense in V. We apply proposition 2.2 with $h_1 = \cdots = h_{n-1} = 0$ and $h_n = \hat{h}(V)/D_n + \varepsilon$. We choose, for $j = 1, \ldots, n-1$,

$$\nu_j = n p^{k'} D_j - 1$$

and $\nu_n = p^{k'}D_n - 1$. We remark that $\nu_{\max} = \max\{\nu_1, \ldots, \nu_n\} \le np^{k'}D_{\max} - 1$. We also choose $T = [p^{k'}/2]$. Then

$$\begin{aligned} H(V, T; \boldsymbol{\nu}) &= (\nu_1+1)\cdots(\nu_n+1) - (\nu_1 - TD_1 + 1)\cdots(\nu_n - D_n + 1) \\ &= n^{n-1}p^{k'n} - \frac{1}{2}\left(n - \frac{1}{2}\right)^{n-1} p^{k'n} \end{aligned}$$

and

$$\begin{aligned} H(V'; \boldsymbol{\nu}) &= (\nu_1+1)\cdots(\nu_n+1) - (\nu_1 - p^{k'}D_1 + 1)\cdots(\nu_n - p^{k'}D_n + 1) \\ &= n^{n-1}p^{k'n} \end{aligned}$$

so that

$$1 - \frac{H(V,T;\boldsymbol{\nu})}{H(V';\boldsymbol{\nu})} \ge \frac{1}{2}\left(1 - \frac{1}{2n}\right)^{n-1} \ge \frac{1}{2\sqrt{e}} \, .$$

Inequality (2.4) now gives

$$\begin{aligned} \nu_n h_n &= (p^{k'}D_n - 1)\left(\frac{\hat{h}(V)}{D_n} + \varepsilon\right) \\ &\ge \frac{T \log p}{2\sqrt{e}p} - \frac{n}{2}\log(\nu_{\max} + 1) \\ &\ge \frac{p^{k'} \log p}{4\sqrt{e}p} - \frac{\log p}{2\sqrt{e}p} - \frac{n}{2}\log(np^{k'}D_{\max}) \\ &\ge \frac{p^{k'}\log p}{7p} - nk'\log p - \frac{n}{2}\log(nD_{\max}) \, . \end{aligned}$$

By letting $\varepsilon \mapsto 0$ we obtain the lower bound (3.6).

If $\mathrm{Stab}(V)$ is not connected, by inspection of the proof of proposition 2.4 of [Amo-Dav 2000] we obtain an hypersurface W with connected stabilizer of the same codimension k', multi-degree $(D'_1, \ldots, D'_n)$ with $D'_j \le nD_j$ and normalized height $\hat{h}(W) \le \hat{h}(V)$. Therefore, by (3.6),

$$\hat{h}(V) \ge \hat{h}(W) \ge \frac{\log p}{7p} - \frac{nk'\log p}{p^{k'}} - \frac{n\log(n^2 D_{max})}{2p^{k'}} \, .$$

□

Let now assume $k' = n$, *i. e.* Stab(V) discrete. Choosing $p = 5$ we obtain:

Theorem 3.2 *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees $(D_1, \ldots, D_n)$ with discrete stabilizer. Then, if $n \geq 9$ and*

$$\max D_j \leq 3^{2^n}$$

we have

$$\hat{h}(V) \geq \frac{1}{23} \ .$$

Proof. We apply the proposition above with $p = 5$, assuming $D_{\max} \leq 3^{2^n}$ and $k' = n$. We obtain

$$\begin{aligned} \hat{h}(V) &\geq \frac{\log 5}{35} - \frac{n^2 \log 5}{5^n} - \frac{n \log(n^2 D_{max})}{2 \times 5^n} \\ &\geq \frac{\log 5}{35} - \frac{n^2 \log 5}{5^n} - \frac{2n \log n}{2 \times 5^n} - \frac{n 2^n \log 3}{2 \times 5^n} =: f(n) \ . \end{aligned}$$

An easy computation shows that f is an increasing function and $f(9) > 1/23$.

□

As stated in the introduction, we could conjecture that for any geometric irreducible hypersurface $V \subseteq \mathbb{G}_m^n$ with discrete stabilizer we had $\hat{h}(V) \geq f(n)$ for some function $f(n) \to +\infty$ for $n \to \infty$. This is false, as the the following example prove. Let $F(x_1) = x_1^3 - x_1 - 1$ and define inductively

$$F_n(x_1, \ldots, x_n) = F^*(x_1, \ldots, x_{n-1}) x_n - F(x_1, \ldots, x_{n-1})$$

where F^* indicated the reciprocal polynomial. Since the rational function

$$R(x_1, \ldots, x_{n-1}) = \frac{F(x_1, \ldots, x_{n-1})}{F^*(x_1, \ldots, x_{n-1})}$$

satisfy $|R(z_1, \ldots, z_{n-1})| = 1$ for $|z_1| = \cdots = |z_{n-1}| = 1$, we have for any integer n $M(F_n) = \theta_0$ where θ_0 is the root > 1 of F_1. Moreover, it is easy to see that F_n is irreducible (over $\overline{\mathbb{Q}}$ if $n \geq 2$) and that $V_n = \{F_n = 0\}$ has trivial stabilizer.

We conclude this section with a more a general (and technical) lower bound for the normalized height of an hypersurface:

Theorem 3.3 *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees $(D_1, \ldots, D_n)$ and assume that V is not a translated of a torus. Then,*

$$\hat{h}(V) \geq \frac{1}{56} \times \max\left(\frac{\log(n \log(n^2 D_{\max}))}{k'}, 1 \right) \times \left(\frac{\log(n \log(n^2 D_{\max}))}{28 n k' \log(n^2 D_{\max})} \right)^{1/(k'-1)}$$

where k' is the codimension of the stabilizer of V and $D_{\max} = \max D_j$. In particular,

$$\hat{h}(V) \geq \frac{\log(n \log(n^2 D_{\max}))^2}{6272 n \log(n^2 D_{\max})} \ .$$

Proof. Let

$$N = \left(\frac{28nk' \log(n^2 D_{\max})}{\log(n \log(n^2 D_{\max}))} \right)^{1/(k'-1)} \tag{3.7}$$

and choose a prime number p such that $N \leq p \leq 2N$. By

$$\log x \leq x^{1/2} \qquad (x > 0) \tag{3.8}$$

we have $\log(n \log(n^2 D_{\max})) \leq \log(n(n^2 D_{\max})^{1/2}) \leq \log(n^2 D_{\max})$; hence

$$p^{k'-1} \geq 28nk' \ .$$

We also remark that, again by (3.8),

$$\log p \geq \frac{\log(28n^{1/2} k' \log(n^2 D_{\max})^{1/2})}{k'-1} \geq \frac{\log(n \log(n^2 D_{\max}))}{2k'} \tag{3.9}$$

Therefore,

$$p^{k'-1} \log p \geq 14n \log(n^2 D_{\max}) \ .$$

Thus, by proposition 3.1 we have

$$\begin{aligned}
\hat{h}(V) &\geq \frac{\log p}{7p} - \frac{nk' \log p}{p^{k'}} - \frac{n \log(n^2 D_{max})}{2p^{k'}} \\
&\geq \frac{\log p}{7p} - \frac{\log p}{28p} - \frac{\log p}{28p} \\
&= \frac{\log p}{14p} \ .
\end{aligned}$$

By (3.9) we obtain:

$$\begin{aligned}
\hat{h}(V) &\geq \frac{1}{14} \times \max\left(\frac{\log(n \log(n^2 D_{\max}))}{2k'}, \log 2 \right) \times \frac{1}{2N} \\
&\geq \frac{1}{56} \times \max\left(\frac{\log(n \log(n^2 D_{\max}))}{k'}, 1 \right) \times \left(\frac{\log(n \log(n^2 D_{\max}))}{28nk' \log(n^2 D_{\max})} \right)^{1/(k'-1)} \ .
\end{aligned}$$

This prove the first inequality of theorem 3.3. For the second one, we remark that $k' \geq 2$ and $k'(nk')^{1/(k-1)} \leq 4n$. So

$$\begin{aligned}
\hat{h}(V) &\geq \frac{1}{56} \times \max\left(\frac{\log(n \log(n^2 D_{\max}))}{k'}, 1 \right) \times \left(\frac{\log(n \log(n^2 D_{\max}))}{28nk' \log(n^2 D_{\max})} \right)^{1/(k'-1)} \\
&\geq \frac{\log(n \log(n^2 D_{\max}))^2}{56 \times 28 \times 4n \log(n^2 D_{\max})} \\
&= \frac{\log(n \log(n^2 D_{\max}))^2}{6272n \log(n^2 D_{\max})} \ .
\end{aligned}$$

□

4 Essential minimum.

In this section we prove the following theorem, which slightly umprove theorem 1.4 of [Amo-Dav 2003]:

Theorem 4.1 *Let V be a subvariety of $\mathbb{G}_m^n$ of codimension $k < n$. Then either there exists a translate B of a subgroup such that $V \subseteq B \subsetneq \mathbb{G}_m^n$ and*

$$\deg(B)^{1/\operatorname{codim}(B)} \leq \big(250 n^3 \log(2n\omega(V))\big)^{\lambda(k)+1} \omega(V)$$

or

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \big(2400 n^4 \log(2n\omega(V))\big)^{-\lambda(k)} \omega(V)^{-1}$$

where $\lambda(k) = \frac{k+1}{k}\big((k+1)^k - 1\big) - 1 \leq n^n - 3$.

Proposition 2.1 gives the following result:

Proposition 4.2 *Let V be a subvariety of $\mathbb{G}_m^n$ et let $\omega = \omega(V)$. Let also p be a prime, $3 \leq p \leq \omega$ and assume :*

$$\hat{\mu}^{\mathsf{ess}}(V) < \frac{\log p}{10 n p \omega} \ .$$

Then,

$$\omega([p]V) \leq \frac{18 n^2 \omega \log(5n\omega)}{\log p} \ .$$

Proof. Let h such that $\hat{\mu}^{\mathsf{ess}}(V) < h < \frac{\log p}{10np\omega}$ and let

$$S = \{\boldsymbol{\alpha} \in V, \quad h(\boldsymbol{\alpha}) < h\} \ .$$

Thus $H(S, T; \nu) = H(V, T; \nu)$ and $H(\ker[p] \cdot S; \nu) = H(\ker[p] \cdot V; \nu)$. Let us define

$$T = \left[\frac{7np \log(5n\omega)}{\log p}\right]$$

and $\nu = (2n+1)\omega T$. We first show that there exists a a non zero polynomial $F \in \overline{\mathbb{Q}}[x_1, \ldots, x_n]$ of total degree $\leq \nu$, vanishing on $\ker[p]V$. Since $3 \leq p \leq \omega$, we have

$$\nu + 1 \leq 3n\omega \cdot 7np \cdot 5n\omega + 1 \leq (5n\omega)^3$$

and $T \log p \geq 6np \log(5n\omega)$. Thus inequality (2.3) of proposition 2.1, *i. e.* $T \log p \geq 2np \log(\nu + 1)$, is satisfied. We also have

$$\frac{T \log p}{4p\nu} = \frac{\log p}{4p(2n+1)\omega} > h \ .$$

By proposition 2.1, we must have

$$H(\ker[p] \cdot V; \nu) < 2H(V, T; \nu) \leq 2\left(\binom{\nu+n}{n} - \binom{\nu - \omega T + n}{n}\right) \ .$$

We remark that

$$\binom{\nu+n}{n}\binom{\nu-\omega T+n}{n}^{-1} = \prod_{j=1}^{n} \frac{\nu+j}{\nu-\omega T+j} \leq \left(1+\frac{\omega T}{\nu-\omega T}\right)^{n}$$

$$= \left(1+\frac{1}{2n}\right)^{n} \leq \sqrt{e} < 2\ .$$

Thus

$$H(\ker[p]\cdot V;\nu) < \binom{\nu+n}{n}\ ,$$

i. e. there exists a non zero polynomial $F \in \overline{\mathbb{Q}}[x_1,\ldots,x_n]$ vanishing on $\ker[p]V$ of total degree $\leq \nu$. By the zero's lemma of P. Philippon (see [Phi 1986]), there exists a variety Z containing V such that

$$\deg(\ker[p]Z) \leq \nu^{\mathrm{codim}(Z)}\ .$$

Indeed, let W be the algebraic set defined by the equations $F(\boldsymbol{\zeta}\mathbf{x}) = 0$ for $\boldsymbol{\zeta} \in \ker[p]$. Since F vanishes on $\ker[p]V$, there exists a geometrically irriducible component Z of W containing V. Since W is stable by translation by p-torsion points, all $\boldsymbol{\zeta}V$ are components of W for $\boldsymbol{\zeta} \in \ker[p]$. Proposition 3.3 of [Phi 1986] (with $p = 1$, $N_1 = n$ and $D_1 = \nu$) then gives the desired upper bound for $\deg(\ker[p]Z)$.
Since

$$\deg(\ker[p]Z) = \deg([p]^{-1}[p]Z) = p^{\mathrm{codim}(Z)}\deg([p]Z)$$

we obtain

$$\omega([p]V) \leq \deg([p]Z)^{1/\,\mathrm{codim}(Z)} \leq p^{-1}\nu\ .$$

We finally remark that

$$\frac{1}{p}\nu \leq \frac{1}{p}\cdot\frac{5}{2}n\omega\cdot\frac{7np\log(5n\omega)}{\log p} < \frac{18n^2\omega\log(5n\omega)}{\log p}\ .$$

□

In order to prove theorem 4.1 we need, as in [Amo-Dav 2003], a descent argument. In what follows we fix a geometrically irreducible subvariety $V \subsetneq \mathbb{G}_m^n$ of dimension $k < n$ (thus $n \geq 2$) and we let $\omega = \omega(V)$. For $j = 1,\ldots,k$ let $\rho_j = (k+1)^{k-j+1} - 1$ and $P_j = (2\Delta)^{\rho_j}$ where $\Delta = Cn^3\log(2n\omega)$ and $C = 120$.

The following elementary relations will be used several time

Lemma 4.3 *We have:*

i) $\log(2n\omega) > 1$ *and* $\Delta > 960$.

ii) For $j \in \{0, \ldots, k\}$ we have

$$\sum_{l=j+1}^{k} \rho_l = (k+1)\frac{(k+1)^{k-j}-1}{k} - (k-j) \ .$$

Definition 4.4 *Let $\mathcal{W}$ be the set of triples $(s, \mathbf{p}, \mathbf{W})$, where $s \in [0, k]$ is an integer, $\mathbf{p} = (p_1, \ldots, p_s)$ is a s-tuple of prime numbers with $P_i/2 \leq p_i \leq P_i$, and where $\mathbf{W} = (W_0, \ldots, W_s)$ is a $(s+1)$-tuple of strict geometrically irreducible subvarieties $\subsetneq \mathbb{G}_m^n$, satisfying:*

i) $V \subseteq W_0$. Moreover, for $i = 1, \ldots, s$,

$$[p_i]W_{i-1} \subseteq W_i \quad \textit{and} \quad p_i \nmid [\mathrm{Stab}(W_{i-1}) : \mathrm{Stab}(W_{i-1})^0] \ ;$$

ii) For $i = 0, \ldots, s$

$$\deg(W_i)^{1/\operatorname{codim}(W_i)} \leq \Delta^{k-i} p_{i+1} \cdots p_k \omega([p_1 \ldots p_i]V) \ ;$$

iii) For $i = 1, \ldots, s$

$$\omega([p_1 \ldots p_i]V) \leq \Delta \omega([p_1 \ldots p_{i-1}]V) \ .$$

Remark 4.5 Let $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}$ and assume $0 \leq i \leq j \leq s$. Then

$$\omega([p_1 \ldots p_j]V) \leq \Delta^{j-i} \omega([p_1 \ldots p_i]V) \ .$$

We want to prove that there exists $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}$, such that $\dim(W_{i-1}) = \dim(W_i)$ for at least one index i. Let

$$\mathcal{W}_0 = \{(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}, \ \text{such that} \ \dim(W_0) < \dim(W_1) < \cdots < \dim(W_s)\} \ .$$

Proposition 4.6 *Assume*

$$\hat{\mu}^{\mathsf{ess}}(V) < \left(10n\Delta^{k-1} P_1 \cdots P_k \omega\right)^{-1} \ . \tag{4.10}$$

Then $\mathcal{W}_0 \neq \mathcal{W}$.

In order to prove proposition 4.6, we endow the set of finite sequences of integers with the following (total) order $\preccurlyeq$. Let $(v) = (v_i)_{0 \leq i \leq s}$ and $(v') = (v'_j)_{0 \leq j \leq s'}$ two such sequences. Then $(v) \preccurlyeq (v')$ if

$$(v_i)_{0 \leq i \leq \min\{s,s'\}} < (v'_i)_{0 \leq i \leq \min\{s,s'\}}$$

for the lexicographical order, or if $(v_i)_{0 \leq i \leq \min\{s,s'\}} = (v'_i)_{0 \leq i \leq \min\{s,s'\}}$ and $s \geq s'$.

We also need the following technical lemma:

Lemma 4.7 *Let $s \in \mathbb{N}$, $p_1, \ldots, p_s, p_{s+1}$ positive integers, $W_0, \ldots, W_s \subsetneq \mathbb{G}_m^n$ geometrically irreducible subvarieties. Let us assume $V \subseteq W_0$ and $[p_i]W_{i-1} \subseteq W_i$ for $i = 1, \ldots, s$. Then, there exists an integer $s' \in [0, s+1]$ and a geometrically irreducible subvariety $Z_{s'}$ of degree*

$$\deg(Z_{s'}) \leq p_{s'+1} \ldots p_{s+1} \omega([p_1 \ldots p_{s+1}]V) \deg(W_{s'}) \ , \tag{4.11}$$

such that $[p_{s'}]W_{s'-1} \subseteq Z_{s'}$, $\operatorname{codim}(Z_{s'}) = \operatorname{codim}(W_{s'}) + 1$ (with the following convention: $\operatorname{codim}(W_{s+1}) = 0$, $\deg(W_{s+1}) = 1$, $W_{-1} = V$ and $p_0 = 1$) and:

$$(\dim(W_0), \ldots, \dim(W_{s'-1}), \dim(Z_{s'})) \prec (\dim(W_0), \ldots, \dim(W_s)) \ . \tag{4.12}$$

Proof. Let Z_{s+1} be an hypersurface containing $[p_1 \ldots p_{s+1}]V$ of minimal degree $\omega([p_1 \ldots p_{s+1}]V)$. Thus if $s' = s+1$ (4.11) is satisfied. We construct by induction subvarieties $Z_0, \ldots, Z_s$ such that, for $i = 0, \ldots, s$,

i) $Z_i \subseteq W_i$ and $Z_i \neq W_i \Rightarrow \operatorname{codim}(Z_i) = \operatorname{codim}(W_i) + 1$.

ii) $[p_{i+1} \ldots p_{s+1}]Z_i \subseteq Z_{s+1}$.

iii) $[p_{i+1}]Z_i \subseteq Z_{i+1}$.

iv) $\deg(Z_i) \leq p_{i+1} \ldots p_{s+1} \omega([p_1 \ldots p_{s+1}]V) \deg(W_i)$.

We start by the construction of Z_0. If $[p_1 \ldots p_{s+1}]W_0 \subseteq Z_{s+1}$, we set $Z_0 = W_0$. Otherwise we choose for Z_0 a geometrically irreducible component of maximal dimension of $W_0 \cap [p_1 \ldots p_{s+1}]^{-1} Z_{s+1}$ containing V. By Bézout's inequality we have:

$$\deg(Z_0) \leq \deg(W_0) \deg([p_1 \ldots p_{s+1}]^{-1} Z_{s+1}) \leq p_1 \ldots p_{s+1} \omega([p_1 \ldots p_{s+1}]V) \deg(W_0) \ .$$

Let now $i \in [0, s-1]$ be an integer and assume that $Z_0, \ldots, Z_i$ satisfy conditions i)–iv). If

$$[p_{i+2} \ldots p_{s+1}]W_{i+1} \subseteq Z_{s+1} \ ,$$

we set $Z_{i+1} = W_{i+1}$. Otherwise we choose for Z_{i+1} a geometrically irreducible component of maximal dimension of $[p_{i+2} \ldots p_{s+1}]^{-1} Z_{s+1} \cap W_{i+1}$ containing $[p_{i+1}]Z_i$. We can do this, since $[p_{i+1}]W_i \subseteq W_{i+1}$ (by assumption) $Z_i \subseteq W_i$ (by induction i)) and since

$$[p_{i+1} \ldots p_{s+1}]Z_i \subseteq Z_{s+1}$$

(by induction i)). The variety Z_{i+1} verify conditions i)–iii). As before, by Bézout's inequality we have:

$$\deg(Z_{i+1}) \leq p_{i+2} \ldots p_{s+1} \omega([p_1 \ldots p_{s+1}]V) \deg(W_{i+1}) \ .$$

and the variety Z_{i+1} also verify condition iv).

We now choose the integer s'. We define s' as the least integer i such that $Z_i \subsetneq W_i$, if such an integer exists. Otherwise we set $s' = s+1$. We remark that in both cases (4.12) holds.

□

Proof of proposition 4.6. The set $\mathcal{W}_0$ is a finite non-empty set (indeed, let W_0 be an hypersurface of $\mathbb{G}_m^n$ containing V of degree ω; then $(0, \emptyset, (W_0)) \in \mathcal{W}_0$). Thus, there exists a minimal element $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}_0$, *i. e.*

$$(\dim W_i)_{0 \leq i \leq s} \preccurlyeq (\dim W'_i)_{0 \leq i \leq s'} .$$

for all $(s', \mathbf{p}', \mathbf{W}') \in \mathcal{W}_0$. We remark that $s \leq k-1$, since

$$n - k = \dim(V) \leq \dim(W_0) < \dim(W_1) < \cdots < \dim(W_s) \leq n - 1 .$$

We need the following computation:

Lemma 4.8 *There exists a prime p_{s+1} such that $P_{s+1}/2 \leq p_{s+1} \leq P_{s+1}$ and*

$$p_{s+1} \nmid [\mathrm{Stab}(W_s) : \mathrm{Stab}(W_s)^0] .$$

Proof. By Theorems 9 and 10 of [Ros-Sch 1962], $\sum_{p \leq x} \log p \leq 1.02x$ for $x \geq 1$ and $\sum_{p \leq x} \log p \geq 0.84x$ for $x \geq 101$. Thus

$$\begin{aligned} \sum_{P_{s+1}/2 \leq p \leq P_{s+1}} \log p &\geq \big(0.84 - 1.02/2\big) P_{s+1} \\ &> P_{s+1}/4 . \end{aligned}$$

If for any prime p with $P_{s+1}/2 \leq p \leq P_{s+1}$ we had $p \mid [\mathrm{Stab}(W_s) : \mathrm{Stab}(W_s)^0]$, then

$$2 \log \deg(W_s) \geq P_{s+1}/4 ,$$

since $\deg(\mathrm{Stab}(W_s)) \leq \deg(W_s)^2$. By assertion ii) of definition 4.4 and by remark 4.5, we have :

$$\begin{aligned} \log \deg(W_s) &\leq \mathrm{codim}(W_s) \big(k \log(\Delta + \sum_{j=s+1}^{k} \log P_j + \log(\omega)\big) \\ &\leq k \Big(\big(k + \sum_{j=s+1}^{k} \log \rho_j\big) \log(2\Delta) + \log \omega \Big) . \end{aligned}$$

Using the inequality $\log x < x^{1/3}$ $(x > 100)$ with $x = 2\Delta$ (see lemma 4.3 i)) we obtain

$$\log \deg(W_s) \leq k \big(k + 1 + \sum_{j=s+1}^{k} \log \rho_j\big) (2Cn^3)^{1/3} \log(2n\omega) .$$

Since $s \leq k - 1$, we have, using lemma 4.3 ii),

$$\begin{aligned} k \big(k + 1 + \sum_{j=s+1}^{k} \log \rho_j\big) &= k(k+1) + (k+1)^{k-s+1} - (k+1) - k(k-s) \\ &= (k+1)^{k-s+1} + ks - 1 \\ &\leq 2(k+1)^{2(k-s)} . \end{aligned}$$

Thus, by setting $a = (k+1)^{(k-s)} \geq 2$,

$$2 \log \deg(W_s) \leq 4a^2 (2Cn^3)^{1/3} \log(2n\omega)$$

and

$$\frac{P_{s+1}/4}{2\log\deg(W_s)} \geq \frac{\left(2Cn^3 \log(2n\omega)\right)^{a-1}}{16a^2(2Cn^3)^{1/3}\log(2n\omega)}$$

$$\geq \frac{\left(16C\right)^{a-4/3}}{16a^2} =: f(a) \; .$$

An easy computation shows that $f(a) \geq f(2) > 1$. Contradiction.

□

By the previous lemma, there exists a prime number $p_{s+1} \in [P_{s+1}/2, P_{s+1}]$ such that $p_{s+1} \nmid [\mathrm{Stab}(W_s) : \mathrm{Stab}(W_s)^0]$. We want to apply proposition 4.2 to the variety $V' = [p_1 \dots p_s]V$ choosing $p = p_{s+1}$. We have

$$\hat{\mu}^{\mathsf{ess}}(V') \leq p_1 \dots p_s \hat{\mu}^{\mathsf{ess}}(V)$$

and, by iii) of definition 4.4

$$\omega(V') \leq \Delta^s \omega(V) \; .$$

Thus, by assumption (4.10),

$$\omega(V')\hat{\mu}^{\mathsf{ess}}(V') \leq \Delta^s p_1 \cdots p_s \omega \hat{\mu}^{\mathsf{ess}}(V)$$

$$< (10nP_{s+1})^{-1}$$

$$\leq \frac{\log p_{s+1}}{10np_{s+1}} \; .$$

Proposition 4.2 shows that:

$$\omega([p_{s+1}]V') \leq \frac{18n^2 \log(5n\omega(V'))}{\log p_{s+1}} \omega([p_1 \dots p_s]V)$$

$$\leq 18n^2 \log(5n\omega(V'))\omega(V') \; .$$

Since $s \leq k-1 \leq n$, we have, using remark 4.5,

$$5n\omega(V') \leq 5n\Delta^s\omega \leq \left((C\sqrt{5}/32)(2n\omega)^5\right)^n \; .$$

Thus

$$\Delta - 18n^2\log(5n\omega(V')) \geq Cn^3 \log(2n\omega) - 18n^3 \log\left((C\sqrt{5}/32)(2n\omega)^5\right)$$

$$\geq n^3\left((C - 18 \times 5)\log(4) - 18\log(C\sqrt{5}/32)\right) > 0$$

and

$$\omega([p_1 \dots p_{s+1}]V) = \omega([p_{s+1}]V') \le \Delta\omega(V') = \Delta\omega([p_1 \dots p_s]V) \ .$$

We apply now lemme 4.7. We obtain an integer s' such that $0 \le s' \le s+1 \le k$ and a subvariety $Z_{s'}$ satisfying the properties described in this lemma. We want to show that

$$(s', (p_1, \dots, p_{s'}), (W_0, \dots, W_{s'-1}, Z_{s'})) \in \mathcal{W} \ .$$

All conditions i)–iii) of definition 4.4 are trivially verified, except eventually for the upper bound of $\deg(Z_{s'})$. Using inequality (4.11) of lemma 4.7, the upper bound for the degree of $W_{s'}$ (point ii) of definition 4.4), remark 4.5 and the relation $\mathrm{codim}(Z_{s'}) = \mathrm{codim}(W_{s'+1}) + 1$, we get:

$$\begin{aligned}
\deg(Z_{s'}) &\le p_{s'+1} \dots p_{s+1}\omega([p_1 \dots p_{s+1}]V) \deg(W_{s'}) \\
&\le p_{s'+1} \dots p_{s+1}\Delta^{s-s'+1}\omega([p_1 \dots p_{s'}]V) \deg(W_{s'}) \\
&\le \Delta^{k-s'} p_{s'+1} \cdots p_k \omega([p_1 \dots p_{s'}]V) \deg(W_{s'}) \\
&\le \left(\Delta^{k-s'} p_{s'+1} \cdots p_k \omega([p_1 \dots p_{s'}]V)\right)^{1+\mathrm{codim}(W_{s'+1})} \\
&\le \left(\Delta^{k-s'} p_{s'+1} \cdots p_k \omega([p_1 \dots p_{s'}]V)\right)^{\mathrm{codim}(Z_{s'})} \ .
\end{aligned}$$

Thus $(s', (p_1, \dots, p_{s'}), (W_0, \dots, W_{s'-1}, Z_{s'})) \in \mathcal{W}$. Since

$$(\dim(W_0), \dots, \dim(W_{s'-1}), \dim(Z_{s'})) \prec (\dim(W_0), \dots, \dim(W_s))$$

by relation (4.12) of lemma 4.7 and since $(s, \mathbf{p}, \mathbf{W})$ is a minimal element of $\mathcal{W}_0$, we deduce that:

$$(s', (p_1, \dots, p_{s'}), (W_0, \dots, W_{s'-1}, Z_{s'})) \notin \mathcal{W}_0 \ .$$

□

4.1 Proof of theorem 4.1

Let V be a geometrically irreducible subvariety of $\mathbb{G}_m^n$ of codimension $k < n$ which satisfy the assumption of proposition 4.6. By this proposition, there exists $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W} \setminus \mathcal{W}_0$. Thus there exists an index i such that

$$\mathrm{codim}(W_{i-1}) = \mathrm{codim}(W_i) = r, \qquad [p_i]W_{i-1} \subseteq W_i, \qquad [p_1 \dots p_{i-1}]V \subseteq W_i \ ;$$

and $p_i \nmid [\mathrm{Stab}(W_{i-1}) : \mathrm{Stab}(W_{i-1})^0]$.

Assume first that W_i is a translate of a subtorus. Then the same is true for the connected component B of $[p_1 \dots p_i]^{-1} W_i$ containing V and we have, using ii) of definition 4.4 and remark 4.5,

$$
\begin{aligned}
(\deg B)^{1/\operatorname{codim}(B)} &\leq (p_1 \cdots p_i)^{1/r} \Delta^k p_{i+1} \cdots p_k \\
&\leq \Delta^k P_1 \cdots P_k \\
&\leq (2\Delta)^{\lambda(k)+1}
\end{aligned}
$$

where

$$
\lambda(k) + 1 = k + \sum_{j=1}^{k} \rho_j = \frac{k+1}{k}\big((k+1)^k - 1\big) .
$$

Assume now that W_i is not a translate of a subtorus. Thus

$$
p_i \deg(W_{i-1}) \leq \deg(W_i) \ .
$$

Since $W_{i-1} \supseteq [p_1 \dots p_{i-1}]V$, we have, using ii) and iii) of definition 4.4,

$$
\begin{aligned}
\omega([p_1 \dots p_{i-1}]V) &\leq \big(\deg(W_{i-1})\big)^{1/r} \\
&\leq p_i^{-1/r} \big(\deg(W_i)\big)^{1/r} \\
&\leq p_i^{-1/r} \Delta^{k-i} p_{i+1} \cdots p_k \omega([p_1 \dots p_i]V) \\
&\leq p_i^{-1/r} \Delta^{k-i} p_{i+1} \cdots p_k \times \Delta\omega([p_1 \dots p_{i-1}]V) \ .
\end{aligned}
$$

Since $r \leq k$ and $P_i/2 \leq p_i \leq P_i$, we get :

$$
\begin{aligned}
p_i^{-1/r} \Delta^{k-i} p_{i+1} \cdots p_k \Delta &\leq P_i^{-1/k} 2^{1/k} \Delta^{k-i+1} P_{i+1} \cdots P_k \\
&< P_i^{-1/k} (2\Delta)^{k-i+1} P_{i+1} \cdots P_k \\
&= (2\Delta)^b
\end{aligned}
$$

where (see lemma 4.3 ii))

$$
\begin{aligned}
b &= -\frac{\rho_i}{k} + k - i + 1 + \sum_{j=i+1}^{k} \rho_j \\
&= -\frac{(k+1)^{k-i+1} - 1}{k} + (k - i + 1) + (k+1)\frac{(k+1)^{k-i} - 1}{k} - (k - i) \\
&= 0 \ .
\end{aligned}
$$

This is a contradiction. Hence

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \left(10n\Delta^{k-1}P_1\cdots P_k\omega(V)\right)^{-1} .$$

We finally remark that

$$10n\Delta^{k-1}P_1\cdots P_k \leq \left(20n\Delta\right)^{\lambda(k)} .$$

Theorem 4.1 is proved.

□

5 Petit points.

Given an algebraic set $V \subseteq \mathbb{G}_m^n$ we define, following [Bom-Zan 1995] and [Sch 1996],

$$V^0 = V \setminus \bigcup_{B\subseteq V} B.$$

where the union is on the set of translates B of subgroups of positive dimension contained in V. In this section we prove a slightly improved version of theorem 1.5 of [Amo-Dav 2006]:

Theorem 5.1 *Let $V \subsetneq \mathbb{G}_m^n$ be an algebraic set defined by equations of degree $\leq \delta$. Then, for all but finitely many $\boldsymbol{\alpha} \in V^0$ we have*

$$\hat{h}(\boldsymbol{\alpha}) \geq \theta := \left(2400n^3\log(2n\delta)\right)^{-n^n+3}\delta^{-1} .$$

More precisely, the set of $\alpha \in V$ of height $< \theta$ is contained in a finite union $B_1 \cup \cdots \cup B_m$ of translate of subtori such that

$$\deg(B_j) \leq \left(250n^3\log(2n\delta)\right)^{(2n)^n}\delta^{2^{\operatorname{codim}(B_j)}-1}$$

Proof.
It is enough to prove the following statement:

Let $V \subsetneq \mathbb{G}_m^n$ be an algebraic set defined by equations of degree $\leq \delta$ and let Z be a geometrically irreducible subvariety of V of positive dimension, satifying

$$\hat{\mu}^{\mathsf{ess}}(Z) \leq \left(2400n^3\log(2n\delta)\right)^{-n^n+3}\delta^{-1} . \tag{5.13}$$

Then, there exists a translate B of a subtorus of codimension r such that $Z \subseteq B \subseteq V$ and

$$\deg(B) \leq \left(250n^3\log(2n\delta)\right)^{(2n)^n}\delta^{2^r-1} .$$

We prove this last statement by induction on n. If $n = 2$ it is easily implied by theorem 4.1. Assume $n \geq 3$ and that the conclusion holds for all algebraic set defined by equations of degree $\leq \delta'$ in $\mathbb{G}_m^{n-1}$. Assume further that there exists a positive integer δ, an algebraic set $V \subsetneq \mathbb{G}_m^n$ defined by equations of degree $\leq \delta$ and a geometrically irreducible subvariety Z of V which satisfies (5.13). Let $k = \mathrm{codim}(Z)$. In particular, since $\omega(Z) \leq \delta$ and $\lambda(k) \leq n^n - 3$, theorem 4.1 gives a translate $B = \boldsymbol{\alpha} H$ of codimension k' containing Z, and such that

$$(\deg(B))^{1/k'} \leq \big(250 n^3 \log(2n\delta)\big)^{n^n-2} \delta \ . \tag{5.14}$$

We can assume $\boldsymbol{\alpha} \in Z$ and $\hat{h}(\boldsymbol{\alpha}) \leq 2\hat{\mu}^{\mathsf{ess}}(Z)$; thus we have :

$$\hat{\mu}^{\mathsf{ess}}(\boldsymbol{\alpha}^{-1}Z) \leq \hat{h}(\boldsymbol{\alpha}^{-1}) + \hat{\mu}^{\mathsf{ess}}(Z) \leq n\hat{h}(\boldsymbol{\alpha}) + \hat{\mu}^{\mathsf{ess}}(Z) \leq 3n\hat{\mu}^{\mathsf{ess}}(Z) \ . \tag{5.15}$$

We now fix a $\mathbb{Z}$-base $\mathbf{a}_1, \dots \mathbf{a}_{k'}$ of the $\mathbb{Z}$-module

$$\Lambda := \Big\{ \boldsymbol{\lambda} \in \mathbb{Z}^n, \text{ t.q. } \forall \mathbf{x} \in H, \ \mathbf{x}^{\boldsymbol{\lambda}} = 1 \Big\} \subseteq \mathbb{Z}^n$$

and we consider the $n \times k'$matrix $A = (a_{i,j})$. Let $E = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Then (see for instance [Ber-Phi 1988]) the degree of H is the maximum of the absolute values of the $k' \times k'$ subdeterminants of A, and $\mathrm{Vol}(E/\Lambda)$ is their quadratic mean. Thus

$$\mathrm{Vol}(E/\Lambda) \leq \binom{n}{k'}^{1/2} \deg(B) \leq n^{k'} \deg(B) \ .$$

Let us consider the cube $[-1/2, 1/2]^n \subset \mathbb{R}^n$; by a theorem of Vaaler (see [Vaaler 1979])

$$\mathrm{Vol}(C \cap E) \geq 1 \ .$$

Thus, by Minkowski's theorem on convex bodies, there exists a non-zero $\boldsymbol{\lambda} \in \Lambda$ such that:

$$\max_{1 \leq i \leq n} \{|\lambda_i|\} \leq n \deg(B)^{1/k'} \ .$$

Since H is connected, we can assume $\lambda_1, \dots, \lambda_n$ coprime and also $\lambda_n = D$. Then the equation

$$\boldsymbol{x}^{\boldsymbol{\lambda}} = 1$$

defines a subtorus $H' \supseteq H$ of codimenion 1 and degree

$$D \leq n \deg(B)^{1/k'} \leq (2n)^{-2} \big(250 n^3 \log(2n\delta)\big)^{n^n} \delta \ . \tag{5.16}$$

If $\boldsymbol{\alpha} H' \subseteq V$ we are done. Asume the contrary. We consider the isogeny $\mathbb{G}_m^{n-1} \twoheadrightarrow H'$ defined by

$$\varphi(\mathbf{x}) = \Big(x_1^{\lambda_n}, \dots, x_{n-1}^{\lambda_n}, x_1^{-\lambda_1} \cdots x_{n-1}^{-\lambda_{n-1}} \Big) \ .$$

We remark that, for any $\boldsymbol{\beta} \in \mathbb{G}_m^{n-1}$,

$$h\big(\varphi(\boldsymbol{\beta})\big) \geq h\big(\boldsymbol{\beta}^{\lambda_n}\big) = \lambda_n h\big(\boldsymbol{\beta}\big) = Dh(\boldsymbol{\beta}) \ . \tag{5.17}$$

Let

$$V' = \varphi^{-1}\left(\boldsymbol{\alpha}^{-1} V \cap H\right) \subseteq \mathbb{G}_m^{n-1}$$

Since $\boldsymbol{\alpha} H' \not\subseteq V$ we have $V' \subsetneq \mathbb{G}_m^{n-1}$. Moreover, let $F_j(\mathbf{x})$ $(j = 1, \ldots, N)$ be equations defining V; then V' is defined by the equations

$$F_j\left(x_1^{\lambda_n}, \ldots, x_{n-1}^{\lambda_n}, x_1^{-\lambda_1} \ldots x_{n-1}^{-\lambda_{n-1}}\right) = 0$$

of degree

$$\leq \delta' = \max\{\lambda_n, |\lambda_1 + \cdots + \lambda_{n-1}|\}\delta \leq nD\delta \ .$$

Let Z' be a geometrically irreducible component of $\varphi^{-1}\left(\boldsymbol{\alpha}^{-1} Z \cap H\right) \subseteq V'$. We have, by (5.17) and (5.15),

$$D\hat{\mu}^{\mathsf{ess}}(Z') \leq \hat{\mu}^{\mathsf{ess}}(\varphi(Z')) = \hat{\mu}^{\mathsf{ess}}(\boldsymbol{\alpha}^{-1} Z) \leq 3n\hat{\mu}^{\mathsf{ess}}(Z) \ .$$

Using the upper bound for $\hat{\mu}^{\mathsf{ess}}(Z)$ and the inequality $\delta' \leq nD\delta$, we deduce

$$\hat{\mu}^{\mathsf{ess}}(Z') \leq 3nD^{-1}\big(2400n^3 \log(2n\delta)\big)^{-n^n+3}\delta^{-1}$$

$$\leq 3n^2\big(2400n^3 \log(2n\delta)\big)^{-n^n+3}\delta'^{-1}$$

Using the inequalities $\delta' \leq nD\delta$, (5.16) and $\log x < x$ we get

$$2n\delta' \leq 2n^2 D\delta \leq (250n^3 \cdot 2n\delta)^{n^n}\delta \leq (2n\delta)^{(250n^3)^{n-1}} \ . \tag{5.18}$$

Thus

$$\big(2400(n-1)^3 \log(2(n-1)\delta')\big)^{(n-1)^{n-1}-3} \leq (3n^2)^{-1}(2400n^3)^a \log(2n\delta)^{n^n-3}$$

where

$$a = 1 + n\big((n-1)^{n-1} - 3\big) \leq n^n - 3 \ .$$

Therefore

$$\hat{\mu}^{\mathsf{ess}}(Z') \leq \big(2400(n-1)^3 \log(2(n-1)\delta')\big)^{-(n-1)^{n-1}+3}\delta'^{-1}$$

By induction there exists a translate $B' \subseteq V'$ of a subtorus of codimension r' such that $Z' \subseteq B'$ and

$$\deg(B') \leq \big(250n^3 \log(2n\delta')\big)^{(2n)^{n-1}}\delta'^{2^{r'}-1} \ .$$

Let for brevity $K = 250n^3 \log(2n\delta)$. From the inequalities (5.18) and $\delta' \leq nD\delta$ we get

$$\deg(B') \leq K^{2^{n-1}n^n}(nD\delta)^{2^{r'}-1} \ .$$

Then $Z = \boldsymbol{\alpha}\varphi(Z') \subseteq \varphi(B') \subseteq V$, $r = \operatorname{codim}\varphi(B') = r' + 1$ and

$$\begin{aligned}\deg\varphi(B') &\leq D\deg(B') \\ &\leq K^{2^{n-1}n^n} n^{2^{r'}-1} D^{2^{r'}} \delta^{2^{r'}-1} \\ &\leq K^{2^{n-1}n^n + 2^{r'}n^n} \delta^{2^{r'+1}-1} \\ &\leq K^{(2n)^n} \delta^{2^r-1}\end{aligned}$$

where we have used the upper bound (5.16) for D.

□

Remark 5.2 In [Amo-Dav 2006], theorem 1.5 we assume that V is geometrically irreducible (which is not necessary) and that V is incompletely defined by forms of degree $\leq \delta$, *i. e.* it is a component of a complete intersection of hypersurfaces of degree $\leq \delta$. Unfortunately, there is a mistake in the proof: at page 561, point (a), we cannot ensure that V' is incompletely defined by forms of degree $\leq nD\delta$. The problem is the following: if V is incompletely defined by forms of degree $\leq \delta$, Z is an hpersurface of degree $\leq \delta$ which not contains V, then an irreducible component of $V \cap Z$ is not a *priori* incompletely defined by forms of degree $\leq \delta$.

References

[Amo-Dav 1999] F. Amoroso et S. David. "Le problème de Lehmer en dimension supérieure", *J. Reine Angew. Math.* **513**, 145-179 (1999).

[Amo-Dav 2000] F. Amoroso et S. David. "Minoration de la hauteur normalisée des hypersurfaces", *Acta Arith.* **92** (2000), **4**, 340-366.

[Amo-Dav 2001] F. Amoroso et S. David. "Densité des points à cordonnées multiplicativement indépendantes", *Ramanujan J.* **5**, 237-246 (2001).

[Amo-Dav 2003] F. Amoroso et S. David. "Minoration de la hauteur normalisée dans un tore", *Journal de l'Institut de Mathématiques de Jussieu*, **2** (2003), no. 3, 335-381.

[Amo-Dav 2006] F. Amoroso et S. David. "Points de petite hauteur sur une sous-varit d'un tore", *Compos. Math.* **142** (2006), 551-562.

[Amo-Zan 2000] F. Amoroso and U. Zannier. "A relative Dobrowolski's lower bound over abelian extensions", *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **29** (2000), no 3, 711-727.

[Ber-Phi 1988] D. Bertrand et P. Philippon. "Sous-groupes algébriques de groupes algébriques commutatifs. Ill". *J. Math.*, t. **32**, pages 263–280, 1988.

[Bom-Zan 1995] E. Bombieri and U. Zannier. "Algebraic points on subvarieties of $\mathbb{G}_m^n$". *Internat. Math. Res. Notices*, **7** (1995), 333-347.

[Cha] M. Chardin – "Une majoration de la fonction de Hilbert et ses conséquences pour l'interpolation algébrique", Bulletin de la Société Mathématique de France, **117**, 305–318, (1988).

[Cha-Phi] M. Chardin et P. Philippon – "Régularité et interpolation", J. Algebr. Geom., **8**, no. 3, 471–481, (1999) ; *erratum*, ibidem, **11**, 599–600, (2002).

[Dav-Phi 1999] S. David et P. Philippon. "Minorations des hauteurs normalisées des sous-variétés des tores". *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **xxviii** (1999), no. 3, 489-543; Errata, *ibidem* **xxix** (2000), no 3, 729-731.

[Leh 1933] D. H. Lehmer. "Factorization of certain cyclotomic functions", *Ann. of Math.* **34** (1933), 461-479.

[Lau 1984] M. Laurent. "Equations diophantiennes exponentielles". *Invent. Math.* **78** (1984), 299-327.

[Law 1983] W. Lawton. "A Problem of Boyd concerning Geometric Means of Polynomial". *J. of Number Theory*, **16** (1983), 356-362.

[Phi 1986] P. Philippon. "Lemmes de zéros dans les groupes algébriques commutatifs". *Bull. Soc. Math. France*, **114** (1986), 353-383.

[Pon 2005] C. Pontreau "Geometric lower bounds for the normalized height of hypersurfaces". *International Journal of Number Theory*, to appear.

[Ros-Sch 1962] J. B. Rosser and L. Schoenfeld. "Approximate formulas for some functions of prime numbers". *Ill. J. Math.*, **6** (1962), 64–94.

[Sch 1991] W. M. Schmidt. Diophantine approximation and Diophantine equations. Springer Lecture Notes in Mathematics, t. **1467**, Springer-Verlag, Berlin–Heidelberg–New-York, viii & 217 pages, 1991.

[Sch 1996] W. M. Schmidt. "Heights of points on subvarieties of $\mathbb{G}_m^n$". In "Number Theory 93-94", S. David editor, London Math. Soc. Ser., volume 235, Cambridge University Press, 1996.

[Vaaler 1979] J. D. Vaaler. "A geometric inequality with applications to linear forms". *Pacific J. Math.*, t. **83** n°2, pages 543–553, 1979.

[Zha 1995] S. Zhang. "Positive line bundles on arithmetic varieties". *J. Amer. Math. Soc.*, **8** (1995), no. 1, 187-221.