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In this way, Plotkin and Stirling generate intuitionistic modal logics in which modal operators are interpreted similarly to the intuitionistic quantifiers ∀ and ∃, and the two modal operators are not interdefinable. An extension of the work presented in this paper to full substructural modal logics (including negation) will allow better comparisons with [PS86] and others.

In future papers we also plan to extend our research in the following directions:

- accommodating Girard's modalities (the "exponentials" of Linear Logic) into the framework outlined in the present paper;
- investigating the decision problem for modal substructural logics, on the basis of the **LKE** approach; this involves introducing a suitable notion of *completed* or *saturated* **LKE** tree, and adapting existing constraint satisfaction methods for solving the systems of inequalities generated when the applications of the labelled PB rule are carried out with variable labels (see Example 10 in Section 8);
- extending the present approach to cover modal many-valued logics;
- investigating the relation between our methods and the methods based on Belnap's Display Logic ([Bel82, Bel90, Wan94]).

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in particular [Dôs88, Dôs89], but also [Avr88]. The Kripke-style "semantics" of implication described below, and which can be found in a number of papers, is nothing but a generalization of Urquhart's semantics for relevant implication described in [Urq72]. For more semantic investigations into substructural logics which can be related to the contents of Section 3 see also [Wan93], [Ono93], [Sam93], [AD93], [Abr91], [Mac96].

Previous work on the specific topic of this paper (combining modal and substructural logics) has been concentrating on the intuitionistic modal logics. The implication fragments of these logics belong to the wide family of substructural modal systems presented in this paper (i.e. those satisfying *all* the structural rules of Table 2, and are captured by the corresponding **LKE** systems, i.e. the **LKEC** systems obtained by including in the set **C** *all* the structural constraints. We consider in particular the works of Božić and Došen [BD84], and Plotkin and Stirling [PS86] (see also [Sym93] for an overview). The main problem in defining modal connectives on Kripke-style intuitionistic structures is to guarantee the intuitionistic hereditary property for modal formulae. This can be achieved either by defining approriate satisfiaction clauses for the modal operators which embed some "sort of " monotonicity (or hereditary property) feature (e.g. [PS86]) or by imposing conditions between the accessibility intuitionistic relations (e.g. [BD84]). Božić and Došen propose [BD84] three different intuitionistic modal logics, one with only the necessity operators (i.e $HK_\Box$), one with only the possibility operator (i.e. $HK_\Diamond$) and the third with both modalities (i.e. $HK_{\Diamond,\Box}$). In the $HK_\Box$ system, a frame is a possible world structure $(W, \leq, R)$ where the partial ordering $\leq$ and the binary relation R satisfy the following condition.

$$\textit{If } x \leq y \;\; \textit{and} \;\; yRz \textit{ then } \exists z' \textit{ such that } xRz' \textit{ and } z \leq z' \tag{18}$$

A class of models is obtained by adding on a frame $HK_\Box$ a valuation functions V which satisfies the hereditary property for the propositional variables. Standard intuitionistic and modal satisfaction clauses are then considered. In the $HK_\Diamond$, the following condition is added to the same type of frame:

$$\textit{If } x \leq y \textit{ and } xRz \textit{ then } \exists z' \textit{ such that } yRz' \textit{ and } z \leq z' \tag{19}$$

These are Conditions (6) and (5) in this paper.

Plotkin and Stirling ([PS86]) combine the two approaches described above in that they consider the condition 19 of Božić and Došen, which allows to preserve the traditional (classical) satisfaction clause for $\Diamond$, but they define an alternative satisfaction clause for the $\Box$

$$x \models \Box A \Leftrightarrow_{\text{def}} \forall y \; x \leq y \; \forall z \textit{ if } yRz \textit{ then } z \models A \tag{20}$$

where the PB-formula $TA \to B : x$ is never used in $\mathcal{T}_1^*$. It follows that $\mathcal{T}'$ is equivalent to the following tree $\mathcal{T}''$:

$$\begin{array}{c} \vdots \\ X \\ \\ \mathcal{T}_1^* \end{array}$$

Finally, notice that the degree of non-analyticity of the new tree $\mathcal{T}''$ is strictly less than that of $\mathcal{T}$. This completes the proof of case 1.

Let us now consider case 2. Then $\mathcal{T}$ has the following form:

$$\begin{array}{ccc} & \vdots & \\ & X & \\ \swarrow & & \searrow \\ T\Box A : x & & F\Box A : x \\ \mathcal{T}_1 & & \mathcal{T}_2 \end{array}$$

Observe that, being $\Box A$ a formula of maximal complexity, $T\Box A : x$ can be used in $\mathcal{T}_1$ only as premiss of the E$T\Box$ rule with $TA :!x$ as conclusion. Moreover, $F\Box A : x$ can be used in $\mathcal{T}_2$ only as premiss of the E$F\Box$ rule with $FA :!x$ as conclusion. Let $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ be the results of removing, respectively, all the occurrences of $TA :!x$ in $\mathcal{T}_1$ and all the occurrences of $FA :!x$ in $\mathcal{T}_2$. Then it is easy to see that the following tree $\mathcal{T}'$ is a closed tree equivalent to $\mathcal{T}$:

$$\begin{array}{ccc} & \vdots & \\ & X & \\ \swarrow & & \searrow \\ TA :!x & & FA :!x \\ \mathcal{T}_1^* & & \mathcal{T}_2^* \end{array}$$

Notice, again, that the degree of non-analyticity of the new tree $\mathcal{T}'$ is strictly less than that of $\mathcal{T}$. This completes the proof of case 2. The proof of case 3 is similar to that of case 2 and is omitted. $\Box$

11 Discussion and related work

Related work on the semantics of substructural logics sketched in Section 3 (following [DG94]) can be found in a wide number of papers. We mention, only as major premiss of an application of the E$T\rightarrow$ rule (since there is no formula in the tree with complexity greater than $A \rightarrow B$), with some (possibly several) LS-formulae of the form $TA : y$ as minor premisses, to obtain LS-formulae of the form $TB : x \circ y$. Notice also that in $\mathcal{T}_2$, the LS-formula $FA \rightarrow B : x$ can be used only to generate the two LS-formulae $TA : a$, for some new atomic label a, and $FB : x \circ a$.

Let ϕ be a brach containing an LS-formula $TB : x \circ y$ obtained by using the PB-formula $TA \rightarrow B : x$ as major premiss with $TA : y$ as minor premiss. Then ϕ will have the following form:

$$\begin{array}{c} \vdots \\ TA : y \\ \vdots \\ TB : x \circ y \\ \vdots \end{array}$$

Let $\mathcal{T}_2^*$ be the result of removing every occurrence of $TA : a$ from $\mathcal{T}_2$. Now, replace ϕ with the following tree:

$$\begin{array}{ccc} & \vdots & \\ & TA : y & \\ & \vdots & \\ & \diagup \quad \diagdown & \\ TB : x \circ y & & FB : x \circ y \\ \vdots & & \mathcal{T}_2^*[a/y] \end{array}$$

where $\mathcal{T}_2^*[a/y]$ is the result of replacing every occurrence of the atomic label a with y in $\mathcal{T}_2^*$.

It is not difficult to see that the resulting tree is a closed tree equivalent to $\mathcal{T}$. Moreover, the LS-formula $TB : x \circ y$ no longer depends on the PB-formula $TA \rightarrow B : x$. By applying the same procedure for every LS-formula resulting from an application of E$T\rightarrow$ with the PB-formula $TA \rightarrow B : x$ as major premiss, we obtain a tree $\mathcal{T}'$ of the following form

$$\begin{array}{ccc} & \vdots & \\ & X & \\ & \diagup \quad \diagdown & \\ TA \rightarrow B : x & & FA \rightarrow B : x \\ \mathcal{T}_1^* & & \mathcal{T}_2 \end{array}$$

we mean that they are constructed starting from the same set of initial LS-formulae.

Proposition 5 *There is a procedure to turn every closed* **LKE(C)***-tree into an equivalent one which enjoys the subformula property.*

Proof First, notice that all the elimination rules of our systems preserve the subformula property. So, to prove the above proposition we only need to prove that all the applications of the labelled PB rule which violate the subformula property can be eliminated without loss of deductive power. If an **LKE**(**C**)-tree violates the subformula property, some of the formulae introduced by means of the labelled PB rule (let us call them *PB-formulae*) are not subformulae of any formula occurring above in the same branch. We shall call such PB-formulae *non-analytic*. Now, we associate with every **LKE**(**C**)-tree $\mathcal{T}$ the pair (c, n) where c is the maximal complexity[5] of a non-analytic PB-formula in $\mathcal{T}$, while n is the number of non-analytic PB-formulae of maximal complexity. We call this pair *degree of non-analyticity* of $\mathcal{T}$. Obviously, for every tree enjoying the subformula property, both c and n are equal to zero.

We can define an ordering on degrees of non-analyticity as follows: $(c, n) < (c', n')$ if either $c < c'$, or $c = c'$ and $n < n'$. Now, suppose there is a procedure to transform every closed tree with degree of non-analyticity $d = (c, n) \neq (0, 0)$ into an equivalent tree with degree of analyticity d' such that $d < d'$. Then, by iterating this procedure we shall eventually obtain a closed tree equivalent to the initial one, but with the subformula property.

It is not difficult to define such a procedure. Let $\mathcal{T}$ with degree of non-analyticity $d \neq 0$, and let C be a non-analytic PB-formula of *maximal* complexity. We distinguish three cases: (1) $C = A \to B$, (2) $C = \Box A$ and (3) $C = \Diamond A$. Let us consider case (1). Then $\mathcal{T}$ has the following form:

$$
\begin{array}{ccc}
 & \vdots & \\
 & X & \\
 & \swarrow \quad \searrow & \\
TA \to B : x & & FA \to B : x \\
\mathcal{T}_1 & & \mathcal{T}_2
\end{array}
$$

Where X is the parent node of the two sibling nodes containing the PB-formulae. Now, notice that in $\mathcal{T}_1$, the LS-formula $TA \to B : x$ can be used

[5] The complexity of a formula is measured in one of the usual ways, e.g. by the total number of occurrences of propositional letters and logical operators.

Propositions 3 and 2 imply that **LKE** is complete with respect to modal implication structures. To close the circle we only need to prove that **LKE** is *sound* with respect to this semantics.

Proposition 4 *If* $\vdash_{\mathbf{LKE}(\mathbf{C}')} A$*, then* A *is verified in all* $\mathbf{C}'$*-frames*

The proof is routine and is omitted.

It immediately follows from Propositions 3, 2 and 4 that

Corollary 1 $\vdash_{\mathbf{C}} A$ *if and only if* $\vdash_{\mathbf{LKE}(\mathbf{C}')} A$ *if and only if* A *is verified in all* $\mathbf{C}'$*-frames.*

10 The subformula property

In the classical **KE** system, the application of the bivalence rule PB (which is equivalent to classical cut) can be restricted, without loss of completeness, so as to preserve the subformula property. Is a similar restriction possible also in the **LKE** systems presented here? As remarked in Section 4, the labelled version of PB introduces a new type of non-determinism in the system, since this rule allows for the introduction not only of arbitrary *formulae*, but also of arbitrary *labels*. In this section we show how the first type of non-determinism, concerning the formulae, can be removed without side-effects on the labels . This is a crucial step towards the development of systematic refutation procedures based on our method. However, this step is by no means sufficient. In order to obtain a fully mechanical refutation procedure we need also to "tame" the second type of non-determinism related to the use of labels in the generalized PB rule, i.e. we have to develop a procedure which terminates either with a closed tree or with a tree containgin enough information to construct a countermodel. The solution of this problem crucially depends on what structural constraints are allowed in the algebra of the labels and on the development of suitable algorithms for solving the systems of inequations associated with $\mathbf{LKE}(\mathbf{C})$-trees (as illustrated in Example 10 above). It seems plausible that efficient decision algorithms based on our method will have to be logic-specific, exploiting the computational properties of each given labelling algebra.

Let us say that an $\mathbf{LKE}(\mathbf{C})$-tree enjoys the *subformula property* if for every LS-formula $sA : x$ occuring in it, where s is one of the two signs T or F, the formula A is a subformula of some formula B such that $sB : y$, for some s and some y, occurs above in the same branch. In the context of the next proposition, by saying that two closed $\mathbf{LKE}(\mathbf{C})$-trees are *equivalent*

$$
\begin{array}{cc}
\multicolumn{2}{c}{T\Delta : \delta} \\
\multicolumn{2}{c}{T? : \gamma} \\
\multicolumn{2}{c}{T\Lambda : \lambda} \\
\multicolumn{2}{c}{FB : \delta^\circ \circ \gamma^\circ \circ \lambda^\circ} \\
/ & \backslash \\
TA : \gamma^\circ & FA : \gamma^\circ \\
\mathcal{T}_2[a/\gamma^\circ] & \mathcal{T}_1
\end{array}
$$

where $\mathcal{T}_2[a/\gamma^\circ]$ denotes the result of substituting every occurrence of the atomic label a in $\mathcal{T}_2$ with the label γ°. (The reader can check that such a substitution does not affect correcteness.) The above closed tree shows that $\Delta, ?, \Lambda \vdash_{\mathbf{LKE}} B$.

To complete the proof we need to show that for any given set $\mathbf{C}$ of structural rules and modal axioms, $\vdash_{\mathbf{LKE(C')}}$, where $\mathbf{C}'$ is the set of constraints corresponding to the elements of $\mathbf{C}$, is closed under $\mathbf{C}$. As for the modal axioms, some of them are dealt with in the examples of Section 8 above and the others can be proved in a similar way. For the structural rules, we show only closure under contraction and leave the rest to the reader.

To show closure under contraction, assume $?, A, A, \Delta \vdash_{\mathbf{LKE(C)}} B$. We wish to show $?, A, \Delta \vdash_{\mathbf{LKE(C)}} B$, whenever the constraint corresponding to contraction, i.e. $a \circ a \sqsubseteq a$, belongs to $\mathbf{C}$. By hypothesis there is a closed $\mathbf{LKE}(\mathbf{C})$-tree $\mathcal{T}_1$ with initial LS-formulae

$$T? : \gamma, TA : a, TA : b, T\Delta : \delta, FB : \gamma^\circ \circ a \circ b \circ \delta^\circ$$

We can replace, without affecting correctness, every occurrence of b in this tree with an occurrence of a and the result will still be a closed $\mathbf{LKE}(\mathbf{C})$-tree with initial LS-formulae:

$$T? : \gamma, TA : a, T\Delta : \delta, FB : \gamma^\circ \circ a \circ a \circ \delta^\circ$$

So, if $a \circ a \sqsubseteq a$ belongs to $\mathbf{C}$, , the following tree will also be closed:

$$
\begin{array}{cc}
\multicolumn{2}{c}{T? : \gamma} \\
\multicolumn{2}{c}{TA : a} \\
\multicolumn{2}{c}{T\Delta : \delta} \\
\multicolumn{2}{c}{FB : \gamma^\circ \circ a \circ \delta^\circ} \\
/ & \backslash \\
TB : \gamma^\circ \circ a \circ a \circ \delta^\circ & FB : \gamma^\circ \circ a \circ a \circ \delta^\circ \\
\times & \mathcal{T}_1[b/a]
\end{array}
$$

□

Structural Constraints		Modal Constraints	
commutativity	$x \circ y \sqsubseteq y \circ x$	seriality	$!x \sqsubseteq ?x$
contraction	$x \circ x \sqsubseteq x$	reflexivity	$!x \sqsubseteq x$
expansion	$x \sqsubseteq x \circ x$	transitivity	$!x \sqsubseteq !!x$
monotonicity	$x \sqsubseteq x \circ y$	symmetry	$x \sqsubseteq ?!x$
		Euclideanism	$?x \sqsubseteq ?!x$
		directedness	$!?x \sqsubseteq ?!x$

Let $\mathbf{C}$ be, as above, a set of structural rules and/or modal axioms, and let $\mathbf{C}'$ be the corresponding constraints (according to the correspondence outilined in Table 2 and Table 6). We call $\mathbf{LKE}(\mathbf{C}')$ the basic **LKE** system obtained by augmenting the basic labelling algebra (see page 10) with the additional constraints in $\mathbf{C}'$, and we denote by $\vdash_{\mathbf{LKE}(\mathbf{C}')}$ the deducibility relation associated with this system. Then we can prove the following:

Proposition 3 *If* $\vdash_{\mathbf{C}} A$, *then* $\vdash_{\mathbf{LKE}(\mathbf{C}')} A$.

Proof We need to show that $\vdash_{\mathbf{LKE}(\mathbf{C}')}$, intended as a (finitary) consequence relation, is a **MIL** and is closed under all the structural rules and modal axioms in $\mathbf{C}$. First of all we have to show that $\vdash_{\mathbf{LKE}(\mathbf{C}')}$ is a **MIL**, namely that it is closed under the basic conditions for a consequence relation (identity and surgical cut), under conditions C$\rightarrow_i$ and under conditions (16) and (17). We shall show only that $\vdash_{\mathbf{LKE}(\mathbf{C}')}$ is closed under surgical cut and leave it to the reader to verify that the other conditions are also satisfied.

Suppose $? \vdash_{\mathbf{LKE}} A$ and $\Delta, A, \Lambda \vdash_{\mathbf{LKE}} B$. We wish to show that $\Delta, ?, \Lambda \vdash_{\mathbf{LKE}} B$. Let us write $T? : \gamma$ to denote a sequence of LS-formulae obtained by the sequence of formulae ? by (i) signing them with T and (ii) attaching to each of them a distinct atomic label. We shall also use γ° to denote the $\circ$-concatenation of these atomic labels. Similarly, we shall use the abbreviations $T\Delta : \delta$, δ°, $T\Lambda : \lambda$ and λ°. By hypothesis there is a closed **LKE**-tree $\mathcal{T}_1$ with initial LS-formulae $T? : \gamma, FA : \gamma^\circ$, and there is also a closed **LKE**-tree $\mathcal{T}_2$ with initial LS-formulae $T\Delta : \delta, TA : a, T\Lambda : \lambda, FB : \delta^\circ \circ a \circ \lambda^\circ$. We can assume, without loss of generality, that the sets of the atomic labels occuring in $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint.

Then the following tree is closed:

the righthand branch is closed if $?(!a\circ!c) \sqsubseteq !?(x \circ c)$, because of the pair of LS-formulae TB :$?(!a\circ!c)$ and FB :$!?(x \circ c)$. Hence, this tree is closed for a given class of frames if the system consisting of the two inequations:

$$x \circ b \sqsubseteq a \circ b \qquad\qquad ?(!a\circ!c) \sqsubseteq !?(x \circ c)$$

has a solution in the corresponding algebra of the labels. In this simple case it is easy to work out the solution $x = a$ for the algebra of the labels corresponding to directed regular frames. To verify this is a solution of the inequation generated by the righthand branch, it is sufficient to go through the following steps:

$!a\circ!c \sqsubseteq !(a \circ c)$	by regularity
$?(!a\circ!c) \sqsubseteq ?!(a \circ c)$	by the monotonicity of ?
$?!(a \circ c) \sqsubseteq !?(a \circ c)$	by directedness and duality
$?(!a\circ!c) \sqsubseteq !?(a \circ c)$	by transitivity of $\sqsubseteq$

A general technique for solving systems of inequations generated by **LKE**-trees (with variables) within a given algebra of the labels is crucial for developing decision algorithms based on our method. This problem will be discussed in a subsequent paper. Here we only observe that our approach separates each proof in two components: (i) a logical component, which depends only on the "universal" meaning of the logical operators as defined by the inference rules which are a straightforward generalization of the classical ones, and (ii) an algebraic component, which consists in solving a system of inequations within a given labelling algebra that mirrors the specific properties of the information system in which the inferential process takes place. While algorithms for the first component can be devised as straightforward extensions of known algorithms for classical logic (simply by shifting from formulae to labelled formulae), algorithms for the second component can be devised by generalizing existing algebraic methods for solving similar problems in related areas (e.g., unification, term-rewriting, constraint programming, etc.).

9 Completeness of LKE

The following table summarizes the constraints on the algebra of the labels we have been considering so far.

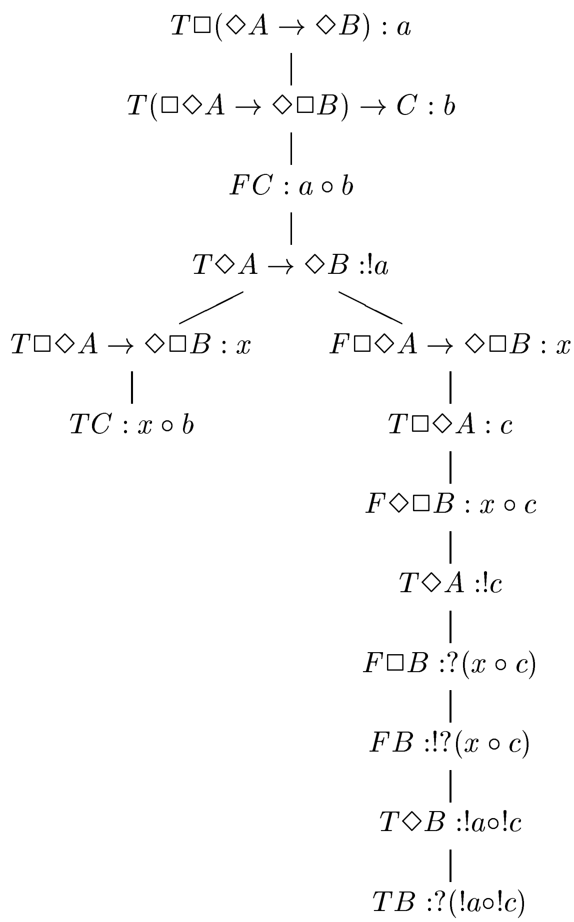

Table 7: Example 10: this tree is closed for all regular and directed frames, under the substitution $x = a$.

$$\frac{\vdash A}{\vdash \Box A}$$

is valid in all monotonic frames.

This is shown by the following closed **LKE**-tree:

$$\begin{array}{ccc} & F\Box A : 1 & \\ & | & \\ & FA :!1 & \\ / & & \backslash \\ TA : 1 & & FA : 1 \\ & & \mathcal{T} \end{array}$$

where $\mathcal{T}$ is a closed **LKE**-tree for $FA : 1$ which exists by hypothesis. Here the lefthand brach is closed because $1 \sqsubseteq 1\circ!1$ by the monotonicity of $\circ$.

Example 10 [Regular modal implication systems] We can define the class of *regular* **MIL**'s, as the class of all **MIL**'s closed under the additional axiom

$$\Box(A \to B) \to (\Box A \to \Box B).$$

(Recall that all **MIL**'s considered in this paper satisfy the other condition that— in the context of classical modal logic— is used to characterized regular modal logics, namely Condition 16.) The class of *regular frames*, corresponding to regular **MIL**'s, is characterized by the following constraint:

$$!x\circ!y \sqsubseteq!(x \circ y)$$

The reader can easily check that the above axiom becomes provable if this constraint is added to the algebra of the labels. The tree in Table 7 shows that the following formula

$$\vdash \Box(\Diamond A \to \Diamond B) \to (\Box\Diamond A \to \Diamond\Box B) \to C) \to C$$

is verified in all frames which are both *regular* and *direct* (we use x, y, z as label-variables and skip the first few nodes in the construction of the tree). Since this example involves an application of the branching rule of generalized bivalence, it can serve also the purpose of illustrating the use of *variables in the labels* hinted at the end of Section 4.

Since the lefthand branch of this tree contains the pair of LS-formulae $TC : x \circ b$ and $FC : a \circ b$, it is closed if $x \circ b \sqsubseteq a \circ b$. On the other hand, This one-branch tree is closed in all expansive-symmetric frames. For, $a \sqsubseteq a \circ a$ by expansion; $(a \circ a) \sqsubseteq ?!(a \circ a)$ by symmetry. These two inequalities imply that $a \sqsubseteq ?!(a \circ a)$.

Example 7 *The formula $\Diamond(A \to A)$ is valid in all monotonic frames.*

$$\begin{array}{c} F\Diamond(A \to A) : 1 \\ | \\ FA \to A :?1 \\ | \\ TA : a \\ | \\ FA :?1 \circ a \end{array}$$

In all monotonic frames, $a \sqsubseteq ?1 \circ a$ and the branch is closed.

Example 8 *The proof rule*

$$\frac{\vdash A \to B}{\vdash \Box A \to \Box B}$$

is valid in all modal quantale frames.

This is shown by the following closed **LKE**-tree:

$$\begin{array}{ccc} & F\Box A \to \Box B : 1 & \\ & | & \\ & T\Box A : a & \\ & | & \\ & F\Box B : a & \\ & | & \\ & TA :!a & \\ & | & \\ & FB :!a & \\ \swarrow & & \searrow \\ TA \to B : 1 & & FA \to B : 1 \\ | & & \\ TB :!a & & \mathcal{T} \end{array}$$

where $\mathcal{T}$ is a closed **LKE**-tree for $FA \to B : 1$ which exists by hypothesis.

Example 9 *The proof rule*

Example 5 *The formula*

$$\Box A \to (B \to \Box\Box A)$$

is valid in all frames which are both monotonic and transitive.

$$F\Box A \to (B \to \Box\Box A) : 1$$
$$|$$
$$T\Box A : a$$
$$|$$
$$FB \to \Box\Box A : a$$
$$|$$
$$TB : b$$
$$|$$
$$F\Box\Box A : a \circ b$$
$$|$$
$$TA :!a$$
$$|$$
$$F\Box A :!(a \circ b)$$
$$|$$
$$FA :!!(a \circ b)$$

This one-branch tree is closed for all monotonic-transitive frames. For, $a \sqsubseteq a \circ b$ by monotonicity; therefore, $!a \sqsubseteq !(a \circ b)$ by (13); moreover, by transitivity, $!(a \circ b) \sqsubseteq !!(a \circ b)$; so, $!a \sqsubseteq !!(a \circ b)$ and the branch is closed.

Example 6 *The formula*

$$A \to (A \to \Box\Diamond A)$$

is valid in all frames which are expansive and symmetric.

$$FA \to (A \to \Box\Diamond A) : 1$$
$$|$$
$$TA : a$$
$$|$$
$$FA \to \Box\Diamond A : a$$
$$|$$
$$F\Box\Diamond A : a \circ a$$
$$|$$
$$F\Diamond A :!(a \circ a)$$
$$|$$
$$FA :?!(a \circ a)$$

$$
\begin{array}{c}
F\Diamond A \to \Box\Diamond A : 1 \\
| \\
T\Diamond A : a \\
| \\
F\Box\Diamond A : 1 \circ a (= a) \\
| \\
TA : ?a \\
| \\
F\Diamond A : !a \\
| \\
FA : ?!a
\end{array}
$$

A similar tree, making use of the dual constraint $?x \sqsubseteq !x$, shows the validity of the dual axiom $\Diamond\Box A \to \Box A$.

Example 4 *The formula*

$$(A \to (A \to B)) \to (\Box A \to B)$$

is valid in all contractive and reflexive frames.

$$
\begin{array}{c}
F(A \to (A \to B)) \to (\Box A \to B) : 1 \\
| \\
TA \to (A \to B) : a \\
| \\
F\Box A \to B : a \\
| \\
T\Box A : b \\
| \\
FB : a \circ b \\
| \\
TA : !b \\
| \\
TA \to B : a \circ !b \\
| \\
TB : a \circ !b \circ !b
\end{array}
$$

In all contractive frames $!b \circ !b \sqsubseteq !b$, so $a \circ !b \circ !b \sqsubseteq a \circ !b$. Moreover, in all reflexive frames $!b \sqsubseteq b$. So, in all contractive and reflexive frames $a \circ !b \circ !b \sqsubseteq a \circ b$.

8 Examples

Example 1 *The axiom T,* $\Box A \to A$ *is valid in all reflexive frames.*

$$
\begin{array}{c}
F\Box A \to A : 1 \\
| \\
T\Box A : a \\
| \\
FA : 1 \circ a (= a) \\
| \\
TA : !a
\end{array}
$$

This one-branch tree is closed in all reflexive frames since $!a \sqsubseteq a$. Observe that, by the duality principle, the constraint $x \sqsubseteq ?x$ also holds in all reflexive frames and, by means of this, the dual of T, $A \to \Diamond A$ can be proved.

Example 2 *The axiom D,* $\Box A \to \Diamond A$ *is valid in all serial frames:*

$$
\begin{array}{c}
F\Box A \to \Diamond A : 1 \\
| \\
T\Box A : a \\
| \\
F\Diamond A : a \\
| \\
TA : !a \\
| \\
FA : ?a
\end{array}
$$

This tree is closed in all serial frames because $!a \sqsubseteq ?a$.

Example 3 *The axiom 5,* $\Diamond A \to \Box\Diamond A$ *is valid in all Euclidean frames.*

scribes a modal implication structure (based on a $\mathbf{C}'$ frame) which does not verify A. $\square$

7 Modal LKE rules

Let us enrich our labelling algebra with the two unary operators ! and ? satisfying conditions 11, 13 12, and the duality principle (see the Duality Lemma above). The valuation clauses (14) and (15) immediately imply simple **LKE**-style rules for the modal operators.

Elimination rules for $\square$

$$\frac{T\square A : x}{TA : !x}\ \mathrm{E}T\square \qquad\qquad \frac{F\square A : x}{FA : !x}\ \mathrm{E}F\square$$

Elimination rules for $\diamond$

$$\frac{T\diamond A : x}{TA : ?x}\ \mathrm{E}T\diamond \qquad\qquad \frac{F\diamond A : x}{FA : ?x}\ \mathrm{E}F\diamond$$

The rules have a clear intuitive interpretation: if $\square A$ is true at a point x, then it is true at all points accessible from it and, therefore, it is true at the special point $!x$ which verifies all and only the formulae which are true at all points in $S(x)$. Conversely, if $\square A$ is false at x, then it must be false at $!x$.

Similarly, if $\diamond A$ is true at a point x, then it is true at some point accessible from it and, therefore, at the special point $?x$ which verifies all and only the formulae which are true at at least one point in $S(x)$. Conversely, if $\diamond A$ is false at x, then it must be false at $?x$.

By adding these rules to the implication rules we obtain an **LKE**-system for modal substructural implication. Different systems will be characterized by adding to the labelling algebra different sets of structural and modal constraints to be used in checking for branch-closure. So, this extended method preserves the "separation-by-closure" property of the original method. Moreover, it also preserves the atomic closure property (see Section 4 above). Notice that, by virtue of the duality principle, whenever the constraint $\sigma_1 x \sqsubseteq \sigma_2 y$ is derivable in the algebra of the labels, so is its dual $\sigma_2' y \sqsubseteq \sigma_1' x$.

In the next section we shall see some examples of proofs in this modal extension of the **LKE** system.

We now show that the valuation V defined as above satisfies also clauses (1), (3) and (4) and, therefore, the canonical frame together with this valuation is a modal implication structure. We shall call this structure *the canonical model* (generated by the given **MIL** $\vdash$).

Clause (1) For the only-if direction, we assume that $V(A \to B, x) = T$ and that for an arbitrary $y \in M$, $V(A, y) = T$. Then, $A \to B \in x$ and $A \in y$. By the identity property of $\vdash$ and by the deduction theorem, we can easily derive that $A \to B\, A \vdash B$. Hence, by definition of $\circ$, $B \in x \circ y$, which means that $V(B, y) = T$. For the if direction, assume that for all y, $V(A, y) = T \Longrightarrow V(B, x \circ y) = T$. Let $y = \{C \mid A \vdash C\}$. Then, for such a y, $A \in y$ (since $A \vdash A$) and, therefore, $V(A, y) = T$. It then follows that $V(B, x \circ y) = T$. This means that $B \in x \circ y$. By definition of $\circ$, there exist $C \in x$ and $D \in y$ such that $C, D \vdash B$. Since every formula in y is proved by A, by surgical cut we obtain $C, A \vdash B$ and, therefore, $C \vdash A \to B$. Since $C \in x$, we can conclude that $A \to B \in x$ and, hence, $V(A \to B, x) = T$.

Clause (3) For the only-if direction, let $V(\Box A, x) = T$ and let y be an arbitrary point such that xRy. By definition of V, $\Box A \in x$, which implies that $A \in h(x)$. Since xRy, $h(x) \subseteq y$, then $A \in y$. Hence $V(A, y) = T$. For the converse, we assume that for all y such that xRy, $V(A, y) = T$. This implies that $\bigsqcap\{V(A, y) \mid y \in S(x)\} = T$. Moreover, by definition, $\bigsqcap S(x) = h(x)$. Then, by the continuity property of V, $V(A, h(x)) = T$, that is $A \in h(x)$. Hence, $\Box A \in x$ which gives by definition of V that $V(\Box A, x) = T$.

Clause (4) For the only-if direction, we assume that $V(\Diamond A, x) = T$. By definition of V, $\Diamond A \in x$ which implies that $A \in g(x)$. Notice that $xRg(x)$, since $h(x) \subseteq g(x) \subseteq g(x)$. Hence, there exists a y, namely $g(x)$ such that xRy and $V(A, y) = T$. For the converse, we assume that $V(A, y) = T$ for some y in $S(x)$. It follows that $A \in y$ and, *a fortiori*, $A \in g(x)$. Hence, $\Diamond A \in x$ and $V(\Diamond A, x) = T$.

Let $\mathbf{C} = \mathbf{S} \cup \mathbf{M}$, where $\mathbf{S}$ is an arbitrary subset of the structural rules in Table 2 and let $\mathbf{M}$ is an arbitrary subset of the modal axioms in Table 6. We denote by $\vdash_{\mathbf{C}}$ the smallest **MIL** satisfying all the rules and axioms in $\mathbf{C}$. Moreover, let $\mathbf{C}'$ be the set of constraints corresponding to the rules and axioms in $\mathbf{C}$ as outlined in Tables 2 and 6. By a *C'-frame* we mean a frame satisfying all the constraints in $\mathbf{C}'$. Observe that the canonical frame generated by $\vdash_{\mathbf{C}}$ is a $\mathbf{C}'$-frame (the proof is left to the reader). Then, by considering the canonical model, it is easy to show the following:

Proposition 2 *If A is verified in all $\mathbf{C}'$-frames, then $\vdash_{\mathbf{C}} A$.*

Proof Suppose $\not\vdash_{\mathbf{C}} A$. Then the canonical model generated by $\vdash_{\mathbf{C}}$ de-

1. R satisfies the continuity property (2);
2. $x \subseteq y$ implies $h(x) \subseteq h(y)$;
3. $x \subseteq y$ implies $g(x) \subseteq g(y)$;
4. $x \subseteq y$ and xRz implies $\exists z', yRz'$ and $z \subseteq z'$; to see this take $z' = g(y)$; we have that yRz', because $h(y) \subseteq g(y) \subseteq g(y)$; moreover, since xRz, we have that $z \subseteq g(x)$, and since $g(x) \subseteq g(y)$, we can conclude $z \subseteq g(y)$. Therefore $z \subseteq z'$;
5. $x \subseteq y$ and yRz implies $\exists z', xRz'$ and $z' \subseteq z$; to see this take $z' = h(x)$; notice that $h(x) \subseteq h(x) \subseteq g(x)$, so xRz'; moreover, since $h(x) \subseteq h(y)$ and $h(y) \subseteq z$ (because yRz), we have that $z' = h(x) \subseteq h(y) \subseteq z$.
6. $\bigcap S(x) = h(x)$;
7. $\bigcup S(x) = g(x)$;
8. $\Box A \in x$ *iff* $A \in h(x)$;
9. $\Diamond A \in x$ *iff* $A \in g(x)$;

Properties 4 and 5 above show that R satisfies the conditions (5) and (6) and therefore the canononical frame is a modal quantale frame.

A *valuation* V over M can be defined simply as follows:

$$V(A, x) = T \Longleftrightarrow A \in x.$$

Such a function V satisfies all the conditions on the definition of a valuation (Definition 3). As for the heredity property (condition 1), observe that $V(A, x) = T$ implies, by definition of V, that $A \in x$, which together with the condition $x \subseteq y$ gives that $A \in y$ and, therefore, $V(A, y) = T$. As for the continuity property (condition 2), notice that since the relation $\subseteq$ is set-inclusion, the $\bigsqcup$ and $\bigsqcap$ are the set operations $\bigcup$ and $\bigcap$ respectively. So, suppose that for a non-empty family $S \in M$, $V(A, \bigcup S) = T$. By definition of V, $A \in \bigcup S$. Then, there exists a set $x \in S$ such that $A \in x$, so for such an x, $V(A, x) = T$. Hence, $\bigsqcup\{V(A, x) \mid x \in S\} = T$. For the converse, suppose that $\bigsqcup\{V(A, x) \mid x \in S\} = T$. Then, there exists an $x \in S$ such that $V(A, x) = T$, that is $A \in x$. Therefore, $A \in \bigcup\{x \mid x \in S\}$. Hence, $V(A, \bigcup S) = T$. An analogous argument shows that V also satisfies the other continuity property $V(A, \bigsqcap) = \bigsqcap\{V(A, x) \mid x \in S\}$.

6 The canonical model

In the canonical model the points of the underlying quantale frame are decreasing sets of formulae, i.e. if A belongs to any such set and $A \vdash B$, then B also belongs to the set, where $\vdash$ is the deducibility relation of an arbitrary **MIL** (we call these sets "decreasing" because $A \vdash B$ can be interpreted as "the information carried by A is less than or equal to the information carried by B").

For every **MIL** $\vdash$, the *canonical frame* generated by $\vdash$ is then a structure $\langle M, \circ, 1, \subseteq, R \rangle$ such that

- M is the set of all decreasing sets of formulae with respect to the given $\vdash$;
- $\circ$ is a binary operation defined as follows:
$$x \circ y =_{\text{def}} \{C \mid (\exists A \in x)(\exists B \in y) A, B \vdash C\};$$
- 1 is the set $\{A \mid [] \vdash A\}$, where $[]$ denotes the empty list;
- the relation $\subseteq$ is set-inclusion.

Clearly, M is a complete lattice under $\subseteq$ with standard union and intersection as join and meet. Moreover, it is not difficult to verify that $\circ$ is a monoid operation with 1 as identity. Furthermore, it is easy to show that $\circ$ distributes over arbitrary joins. Therefore, the structure $\langle M, \circ, 1, \subseteq \rangle$ is a quantale frame.

Let us now define two functions on the points of the canonical model:

$$h(x) =_{\text{def}} \{A | \Box A \in x\}$$

and

$$g(x) =_{\text{def}} \{A | \Diamond A \in x\}.$$

Notice that, for every x, $h(x)$ and $g(x)$ are themselves decreasing sets of formulae, because of the monotonicity properties of $\Box$ and $\Diamond$ (16) and (17). Therefore they also belong to the canonical model. Notice also that our "seriality" assumption implies that for all x:

$$h(x) \subseteq g(y).$$

Accessibility between points of this canonical model is then defined as follows:

$$xRy \textit{ iff } h(x) \subseteq y \subseteq g(x)$$

It follows from our definitions that

Proof The proof is an easy induction on the length of the chain $\phi_0 \sqsubseteq \cdots \sqsubseteq \phi_n$.

Base: $n = 1$. Then the inequality $\sigma_1 x \sqsubseteq \sigma_2 x$ is an element of $\mathbf{M}^*$. The reader can easily verify that the lemma holds for all primitive inequalities in $\mathbf{M}$.

Step: $n = k + 1$. Let ρ be the prefix of ϕ_k, i.e. $\phi_k = \rho x$. By inductive hypothesis, $\rho' x \sqsubseteq \sigma'_1 x$. Moreover, since $\rho x \sqsubseteq \sigma_2 x$, again by inductive hypothesis we have that $\sigma'_2 x \sqsubseteq \rho' x$. Hence, $\sigma'_2 x \sqsubseteq \sigma'_1 x$. □

Let now $\mathbf{L}'$ be a propositional language containing the two binary operators $\rightarrow_1$ and $\rightarrow_2$ plus the the two modal operators $\Box$ and $\Diamond$. (Recall that the two conditionals collapse in all the systems allowing the structural rule of *Exchange*, in which case we use the symbol $\rightarrow$ without subscripts.) By a *modal implication logic*— or **MIL** for short— we mean a (substructural) implication logic over $\mathbf{L}'$ satisfying the following conditions on the operators $\Box$ and $\Diamond$:

$$\frac{A \vdash B}{\Box A \vdash \Box B} \tag{16}$$

and

$$\frac{A \vdash B}{\Diamond A \vdash \Diamond B} \tag{17}$$

In addition to (16) and (17) above, different **MIL**'s may also satisfy different subsets of the following *modal axioms*:

1. $\Box A \rightarrow \Diamond A$
2. $\Box A \rightarrow A$
3. $\Box A \rightarrow \Box\Box A$
4. $A \rightarrow \Box\Diamond A$
5. $\Diamond A \rightarrow \Box\Diamond A$
6. $\Diamond\Box A \rightarrow \Box\Diamond A$

In this paper we restrict our attention to *serial* **MIL**'s, i.e. those satisfying the modal axiom $\Box A \rightarrow \Diamond A$.

In the next section we shall describe a concrete example of a modal implication structure defined in terms of the deducibility relation $\vdash$ associated with any given (serial) **MIL**. We call this concrete model *the canonical model*.

Modal formulae	Conditions on R	Modal constraints
$\Box A \to A$	$(\forall x) xRx$	$!x \sqsubseteq x$
$\Box\Box A \to \Box A$	$(\forall x)(\forall y)(\forall z)(xRy \textit{ and } yRz \Longrightarrow xRz)$	$!x \sqsubseteq !!x$
$\Box\Diamond A \to A$	$(\forall x)(\forall y)(xRy \Longrightarrow yRx)$	$x \sqsubseteq ?!x$
$\Box\Diamond A \to \Diamond A$	$(\forall x)(\forall y)(\forall z)(xRy \textit{ and } xRz \Longrightarrow yRz)$	$?x \sqsubseteq ?!x$
$\Box\Diamond A \to \Diamond\Box A$	$(\forall x)(\forall y)(\forall z)(xRy \textit{ and } xRz \Longrightarrow (\exists u)(yRu \textit{ and } zRu))$	$?!x \sqsubseteq !?x$

Table 6: Correspondence between modal formulae, conditions on R and constraints on the modal quantale frames.

The condition corresponding to this property is the one stated below:

(Euclideanism) $\quad (\forall x)(?x \sqsubseteq ?!x)$.

Directedness A frame is *directed* if the following condition holds true:

$$(\forall x)(\forall y)(\forall z)(xRy \textit{ and } xRz \Longrightarrow (\exists u)(yRu \textit{ and } zRu))$$

which corresponds to:

(Directedness) $\quad (\forall x)(!?x \sqsubseteq ?!x)$.

Observe that a frame is directed if and only if it verifies the following formula:

$$\Diamond\Box A \to \Box\Diamond A.$$

Our discussion so far amounts to a reformulation of the well-known correspondence theory in our new setting. This correspondence is summarized in table 6.

We shall now state a crucial lemma. Let σ_1 and σ_2 be (possibly empty) strings of ! and ?. The *dual* of σ_i, denoted by σ_i' is the string obtained by interchanging ! and ? in σ_i. Moreover, let $\mathbf{M}^* \subseteq \mathbf{M}$, where $\mathbf{M}$ is the set of modal conditions listed above. We have the following *duality principle*:

Duality Lemma *Assume that $\sigma_1 x \sqsubseteq \sigma_2 x$ holds for all x in all $\mathbf{M}^*$-frames, i.e. there is a chain of inequalities $\phi_0 \sqsubseteq \cdots \sqsubseteq \phi_n$ such that (i) $\phi_0 = \sigma_1 x$, (ii) $\phi_n = \sigma_2 x$, and (iii) each inequality $\phi_i \sqsubseteq \phi_{i+1}$, with $i = 0, \ldots, n-1$, is one of of the primitive inequalities in $\mathbf{M}^*$. Then $\sigma_2' x \sqsubseteq \sigma_1' x$ also holds for all x in all $\mathbf{M}^*$-frames.*

A frame is transitive if and only if it verifies the following formula:

$$\Box A \to \Box\Box A.$$

In our notation, transitivity is expressed by the following condition:

(Transitivity) $(\forall x)(!x \sqsubseteq !!x)$

Assume that $!x \sqsubseteq !!x$ for all x. Now, suppose that the frame is not transitive. Then, there exists a point a, a valuation V and a formula A such that $V(\Box A, a) = T$ but $V(\Box\Box A, a) = F$. This implies by (14) that $V(A, !a) = T$ and $V(A, !!a) = F$. But, since $!a \sqsubseteq !!a$, this is impossible. Hence, if $!x \sqsubseteq !!x$ for all x, the frame is transitive.

Conversely, suppose the frame is transitive. We show that $!x \sqsubseteq !!x$. Let A be any formula which is true at $!x$. Then A is true at all points in $S(x)$. Moreover, $!x$ is accessible from x. Since the frame is transitive, if a point y is accessible from $!x$, it is also accessible from x. So, if A is true at all the points accessible from x it is also true at all the points accessible from $!x$, that is $!x \sqsubseteq !!x$.

Symmetry A frame is said to be *symmetric* if the following condition holds:

$$(\forall x)(\forall y)(xRy \Longrightarrow yRx).$$

A frame is symmetric if and only if it verifies the following formula:

$$A \to \Box\Diamond A.$$

In our notation, symmetry is expressed by the following condition:

(Symmetry) $(\forall x)(x \sqsubseteq ?!x)$

For this property as well as for the next two, We leave it to the reader to verify the equivalence between the correspoding formulations.

Euclideanism A frame is said to be *Euclidean* if the following condition holds true:

$$(\forall x)(\forall y)(\forall z)(xRy \textit{ and } xRz \Longrightarrow yRz).$$

A frame is Euclidean if and only if it verifies the following formula:

$$\Diamond A \to \Box\Diamond A.$$

every point in $S(x)$ is contradicted. Hence, $\Box A$ must be true at x. A similar argument shows the equivalence of (4) and (15).

We can now exploit the new operators to express complex statements about the accessibility relation R concisely, in the form of simple inequalities of the form $\alpha \sqsubseteq \beta$, where α and β are expressions built up from atomic terms by means of the operators $\circ$, ! and ?.

Let us consider some of the most familiar properties of the accessibility relation R.

Seriality A frame is *serial* if for every point x, $S(x) \neq \emptyset$, that is, for every point x there exists a y such that xRy. Observe that a frame is serial if and only if it verifies the formula:

$$\Box A \rightarrow \Diamond A.$$

As mentioned above, in our approach all frames are serial, as a consequence of (2), and seriality corresponds to the assumption that $!x \sqsubseteq ?x$ for all x.

Reflexivity A frame is said to be *reflexive* if xRx for all x. Observe that a frame is reflexive if and only if it verifies the following formula:

$$\Box A \rightarrow A.$$

In our notation reflexivity can be expressed by the following condition:

(Reflexivity) $(\forall x)(!x \sqsubseteq x)$

The equivalence between the two formulations is immediately seen as follows. First, assume that $!x \sqsubseteq x$. Then every formula true at $!x$ is true also at x. Suppose xRx does *not* hold for some x. This means that there exists a point a such that $\neg aRa$. Now, consider a valuation V and a formula A such that $V(A, x) = T$ for all $x \in S(a)$, i.e. $V(\Box A, a) = T$, but $V(A, a) = F$ (which is possible, since $a \notin S(a)$). For this V, we have that $V(A, !a) = T$ and, since $!a \sqsubseteq a$, $V(A, a) = T$. This is a contradiction. Hence, if $!x \sqsubseteq x$ for all x, then xRx for all x. Conversely, if xRx for all x, every formula that is true at all points accessible from x is true also at x, i.e. for all x, $!x \sqsubseteq x$.

Transitivity A frame is said to be *transitive* if the following condition holds:

$$(\forall x)(\forall y)(\forall z)(xRy \textit{ and } yRz \Longrightarrow xRz).$$

So, in our approach, the points $!x$ and $?x$ have a special status: $!x$ is a point in $S(x)$ that verifies all and only the formulae which are true at *all* points in $S(x)$, while $?x$ is a point in $S(x)$ that verifies all and only the formulae which are true at *at least one* point in $S(x)$.

It is not difficult to show that (5) and (6) above are respectively equivalent to the following conditions on ? and ! expressing the fact that the operators ? and ! are both *order-preserving*:

$$x \sqsubseteq y \implies ?x \sqsubseteq ?y \tag{12}$$

$$x \sqsubseteq y \implies !x \sqsubseteq !y \tag{13}$$

Proposition 1 *In every modal quantale frame, (6) holds if and only if (13) holds, and (5) holds if and only if (12) holds.*

Proof We show only that (6) is equivalent to (13) and leave it to the reader to show the other equivalence. First, assume (6), and suppose $x \sqsubseteq y$. By (10), we know that $yR!y$. By (6), $(\exists z')z' \sqsubseteq !y$ and xRz'. Then, by definition of !, $!x \sqsubseteq z'$ and, therefore, $!x \sqsubseteq !y$. Now, assume (13) and suppose (i) $x \sqsubseteq y$, (ii) yRz. Then $!y \sqsubseteq z$ and, by (13), $!x \sqsubseteq z$. Since $xR!x$, there exists z' such that xRz' and $z' \sqsubseteq z$. □

Given our definitions of the operators ! and ?, the valuation clauses for □ and ◇ can be reformulated more concisely as below:

$$V(\Box A, x) = T \textit{ iff } V(A, !x) = T \tag{14}$$

and

$$V(\Diamond A, x) = T \textit{ iff } V(A, ?x) = T \tag{15}$$

Let us now show the equivalence of the valuation clauses (14) and (15) with (3) and (4) respectively. First, we assume that the clause in (3) holds and show that (14) holds too.

For the only-if direction, let us assume $V(\Box A, x) = T$; then by (3), $V(A, y) = T$ for all $y \in S(x)$. Now, by the definition of valuation, A remains true at the greatest lower bound of $S(x)$, that is at $!x$. For if-direction let us assume $V(\Box A, x) = F$. It follows that there exists a point $y \in S(x)$ such that A is false at y. Since, $!x \sqsubseteq y$, A must be false at $!x$.

Now we show that (14) implies (3). For the only-if direction, suppose $V(\Box A, x) = T$. Then, by (14), $V(A, !x) = T$. So, A is true at every point y such that $!x \sqsubseteq y$ and, therefore, at every point in $S(x)$. For the if-direction, assume that A is true at every point in $S(x)$. Suppose, that $\Box A$ is false at x. Then, by (14), $V(A, !x) = F$. So the assumption that A is true at

Observe that these definitions are essentially different from the classical definitions. In classical modal logics a formula is verified in a modal structure (or "model" as structures are often called in the literature) if it is verified in *all* the worlds beloging to that structure. Our definition is equivalent to the classical one only in the case of *monotonic* information frames. For, suppose a formula A is verified at the identity point 1 of a structure based on a monotonic frame, i.e. $V(A,1) = T$. Then, by the monotonicity of the frame $1 \sqsubseteq 1 \circ a (= a)$ for every point a in the frame. So, by the hereditary property of valuations $V(A,a) = T$ for all the points in the frame.

Our final aim is to incorporate the semantics of the modal operators into the proof-theoretical framework of Section 4. For this purpose we shall simplify this semantics by replacing the binary relation R with a pair of unary operators corresponding to the modalities. (These operators are reminiscent of the ones used in the algebraic semantics of classical modal logics, for which see [BS84].) Such a reformulation will allow us, in section 7, to formulate an algebra of the labels which is a simple extension of the one characterizing substructural implication systems obtained by adding to the structural constraints a (possibly empty) set of *modal constraints*, i.e. inequations of the form $\alpha \sqsubseteq \beta$, where α and β are terms of the extended algebra including the new unary operators.

Let us denote by $S(x)$ *the sphere* of x, i.e. the set of all pieces of information accessible from x. We shall also use $\bigsqcup$ and $\bigsqcap$ to denote the usual lattice join and meet. We now define the two unary operators ! and ? as follows:

$$!x =_{\text{def}} \bigsqcap S(x) \tag{7}$$

and

$$?x =_{\text{def}} \bigsqcup S(x) \tag{8}$$

Now observe that (2) implies that the following property of R is satisfied in every frame:

$$(\forall x)(\exists y) xRy. \tag{9}$$

In other words, for all x, $S(x) \neq \emptyset$. In fact, for every point x, both $!x$ and $?x$ are accessible from x, that is

$$(\forall x) xR!x \textit{ and } xR?x \tag{10}$$

In the theory of classical modal logics a property like (9) is known as *seriality*. So, in our approach *all* frames are serial. Using the ! and ? operators, this assumption is expressed by the following inequation which holds for all modal quantale frames:

$$!x \sqsubseteq ?x \tag{11}$$

and for $\diamond$:

$$V(\diamond A, x) = T \textit{ iff } V(A, y) = T \text{ for some } y, \text{ such that } xRy \tag{4}$$

In order to preserve the "hereditary" property of valuations (see point 1 of Definition 3), R must also satisfy the following conditions (see [BD84] and [PS86]):

$$x \sqsubseteq y \textit{ and } xRz \Longrightarrow (\exists z')(yRz' \textit{ and } z \sqsubseteq z') \tag{5}$$

and

$$x \sqsubseteq y \textit{ and } yRz \Longrightarrow (\exists z')(xRz' \textit{ and } z' \sqsubseteq z). \tag{6}$$

Definition 5 A *modal quantale frame* is a pair $(\mathcal{F}, R)$ where $\mathcal{F}$ is a quantale frame and R is a binary accessibility relation defined on the domain of $\mathcal{F}$ and satisfying conditions (2), (5) and (6). $\square$

If R is defined as above, any atomic valuation V over a quantale frame can be extended over a modal quantale frame by means of the valuation clauses for $\square$ and $\diamond$ preserving the "hereditary" property of valuations.

Definition 6 A *modal implication structure* is a triple $(\mathcal{F}, R, V)$, where $(\mathcal{F}, R)$ is a modal quantale frame and V is a valuation satisfying (1), (3) and (4). $\square$

Let M be a modal implication structure $(\mathcal{F}, R, V)$ and let $\equiv$ be the relation that holds between two points x and y when they verify exactly the same formulae, i.e. $x \equiv y$ means that $\forall A, x \Vdash A$ if and only if $y \Vdash A$. We can consider the quotient structure $\mathcal{F}/\equiv$ and define a valuation V' on it as follows:

$$V'(A, [x]) = T \textit{ iff } V(A, x) = T$$

where $[x]$ denotes the equivalence class of x under $\equiv$. Similary, we can define $[x]R'[y]$ *iff* xRy. Clearly, the structure $(\mathcal{F}/\equiv, R', V')$ is a modal implication structure, and we can restrict our attention, without loss of generality, to modal implication structures with the additional property that any two points verifying exactly the same formulae are identical.

Definition 7 We say that a formula A is *verified* in a *modal implication structure* M if it is verified at the identity point 1 of M. We also say that A is *verified* in a *frame* $\mathcal{F}$ if is verified in all modal implication structures based on $\mathcal{F}$. $\square$

$$
\begin{array}{l}
F(A \to (A \to B)) \to (A \to B) : 1 \\
TA \to (A \to B) : a \\
FA \to B : 1 \circ a (= a) \\
TA : b \\
FB : a \circ b \\
TA \to B : a \circ b \\
TB : a \circ b \circ b
\end{array}
$$

Table 5: The contraction axiom is valid for the class of contractive frames (i.e. those satisfying the condition $x \sqsubseteq x \circ x$, for all x). For, $b \circ b \sqsubseteq b$ and, since $\circ$ is order-preserving, $a \circ b \circ b \sqsubseteq a \circ b$. Therefore, this one-branch tree is closed. The axiom is not valid in Linear Logic which is characterised by a class of frames which are not contractive.

as in the canonical procedure for the classical system outlined in Section 2). As for (b), it can be shown that similar restrictions hold for the labels and the best strategy is to apply the PB rule with a *variable* as label. In this way the closure of a branch depends on the solution of an inequation in the given algebra of the labels and the closure of the whole tree on the simultaneous solution of a system of inequations. This a well defined algebraic problem which can be addressed via unification-like techniques. For a detailed and systematic presentation of the **LKE** system (including the other "multiplicative" operators different from $\to$), and more examples of **LKE**-proofs, the reader is referred to [DG94].

5 The modal operators

We introduce in our quantale frames a binary relation R between their points, called the *accessibility relation*. We assume that this relation is "closed" under arbitrary $\bigsqcup$ and $\bigsqcap$, namely:

$$(\forall y \in S) xRy \Longrightarrow xR \bigsqcup S \textit{ and } (\forall y \in S) xRy \Longrightarrow xR \bigsqcap S. \tag{2}$$

We then introduce into our language the two unary operators $\Box$ and $\Diamond$ intended as the usual modalities. Valuations are extended to these modal operators in the obvious way. For $\Box$ we have the following clause:

$$V(\Box A, x) = T \textit{ iff } V(A, y) = T \text{ for all } y \text{, such that } xRy \tag{3}$$

the labelling algebra obtained by adding the constraints in **S** to the basic labelling algebra.

A proof of the validity of a formula A for the class of **S**-frames consists in a refutation of the assumption that A is false at the identity element 1 of some implication structure based on an **S**-frame. Such a refutation is represented by a closed **LKE**(**S**) tree starting with the LS-formula $FA : 1$, where the constraints in **S** may be used together with the basic labelling algebra in order to close a branch. Whenever such a closed tree can be constructed, we say that A is a **LKE**(**S**)-theorem.

We shall also denote by $\vdash_{\mathbf{LKE(S)}}$ the (finitary) deducibility relation of **LKE**(**S**) defined as follows (where [] represents the empty list of formulae):

1. $[] \vdash_{\mathbf{LKE(S)}} A$ *iff* A is an **LKE**(**S**)-theorem;

2. $A_1, \ldots, A_n \vdash_{\mathbf{LKE(S)}} B$ *iff* $[] \vdash_{\mathbf{LKE(S)}} A_1 \rightarrow (A_2 \rightarrow \cdots \rightarrow (A_n \rightarrow B) \cdots)$.

We shall also write simply$\vdash_{\mathbf{LKE(S)}}$ instead of $[] \vdash_{\mathbf{LKE(S)}} A$. This definition implies that $A_1, \ldots, A_n \vdash_{\mathbf{LKE(S)}} B$ if and only if there is a closed **LKE**(**S**)-tree starting with the sequence of LS-formulae

$$TA_1 : a_1, \ldots, TA_n : a_n, FB : a_1 \circ \cdots \circ a_n$$

where $a_1, \ldots, a_n$ are all *distinct atomic* labels.

In this approach the whole family of substructural implication logics is, therefore, characterised by *the same* tree-expansion rules, and different members of the family are identified by the different labelling algebras that can be employed to check branch-closure. This is what we call the *separation-by-closure* property of the **LKE** system. Another remarkable property of **LKE** is the *atomic closure property*: if there is a closed tree starting from a given set of initial LS-formulae, then there is also one starting from the same set of LS-formulae such that the closure rule is applied only with *atomic* LS-formulae, i.e. LS-formulae of the form $sP : x$ (where $s = T$ or F) whith P atomic.

An example of an **LKE**-proof is given in Table 5. This example does not make any use of the branching rule of "generalized bivalence". Indeed, this rule introduces a good deal of "non-determinism" into the system, in that it allows for the use of (a) arbitrary formulae and (b) arbitrary labels in each rule application. However, this "non-determinism" can be tamed to some extent. As for (a), it can be shown that the applications of PB can be restricted to *analytic* ones, i.e. involving only subformulae of formulae previously occurring in the branch (and even further to *canonical* applications case of the closure rule given in the table is the simple rule:

$$\frac{\begin{array}{c}TA : x\\ FA : y\end{array}}{\times}\quad \text{provided } x \sqsubseteq y$$

Indeed this special case is sufficient for completeness except for the implication systems which do not allow weakening but do allow expansion (see [DG94] for the details).

In these rules the declarative units are not just signed formulae as in the classical **KE** system (or in the system of analytic tableaux) but *labelled signed formulae*, or *LS-formulae* for short[3]. The points of the quantale frames are turned into "labels", while signs play the usual role, so that $TA : x$ is interpreted as "A is true at point x" and $FA : x$ is interpreted as "A is *not true* at point x". Different classes of frames correspond to different *labelling algebras*, i.e. different sets of rules that can be used in manipulating the labelling terms to verify whether or not the condition for the application of the closure rule is satisfied.

The *labelling language* consists of a denumerable set of atomic labels, denoted by a, b, c, etc. (possibly with subscripts), a distinguished atomic label 1, and two binary operators $\circ$ and $\sqcap$[4]. Complex labels are built up from the atomic ones by means of the two binary operators. The *basic labelling algebra*, which applies to all frames, consists of the following elements:

- the usual axioms expressing the fact that $\circ$ is a monoid operation with 1 as identity;

- the usual axioms expressing the fact that $\sqcap$ is a semilattice meet.

The partial ordering $\sqsubseteq$ can be defined, as usual, by putting $x \sqsubseteq y$ if and only if $x \sqcap y = x$.

This basic labelling algebra augmented with a class of structural constraints **S**, provides a specific labelling algebra sufficient to charaterize the notion of validity in **S**-frames. We shall denote by **LKE**(**S**), where **S** is an arbitrary set of structural constraints, the **LKE** system equipped with

[3] A similar approach is used by Fitting in his "prefixed" tableaux for classical modal logics [Fit83].

[4] This operator is used only in connection with the general closure rule given in the table, and is not required whenever the simpler special case described above is sufficient for completeness, namely for all classes of frames which either are monotonic or non-expansive.

$$\frac{\begin{array}{l} TA \to B : x \\ TA : y \end{array}}{TB : x \circ y} \qquad\qquad \frac{\begin{array}{l} TA \to B : x \\ FB : x \circ y \end{array}}{FA : y}$$

$$\frac{FA \to B : x}{\begin{array}{l} TA : a \\ FB : x \circ a \end{array}} \ \text{where } a \text{ is a } \textit{new} \text{ atomic label}$$

$$\frac{}{TA : x \mid FA : x} \qquad\qquad \frac{\begin{array}{l} TA : x_1 \\ \vdots \\ TA : x_n \\ FA : y \end{array}}{\times} \ \text{if } x_1 \sqcap \cdots \sqcap x_n \sqsubseteq y$$

Table 4: The **LKE** rules for substructural implication.

is the implication fragment of Girard's Linear Logic [Gir87, Avr88]; if it satisfies both commutativity and contraction, then the system of relevant implication [AB75, Dun86] is obtained; finally, if $\circ$ satisfies all the structural conditions on $\circ$, the resulting logic is intuitionistic implication. See Table 3 for an overview.

4 Implicational LKE

An inferential characterization of substructural implication logics can be obtained by turning the "semantics" described above into the rules of a labelled deductive system (in the sense of [Gab94]). In [DG94] D'Agostino and Gabbay presented a labelled refutation system consisting of a generalization of the classical system **KE** investigated in [DM94]. The rules of this labelled refutation system, that we call **LKE**, are tree-expansion rules which are immediately justified by (and are indeed equivalent to) our previous definitions (see [BDR97] for a detailed discussion of the route which leads from sequent-based presentation of substructural consequence relations to labelled refutation systems, via algebraic and Kripke-style semantics). The rules for the implication fragment are listed in Table 4.

Notice that the only branching rule can be seen as a labelled generalization of the classical principle of bivalence. Observe also that a simple special

Commutativity	$x \circ y \sqsubseteq y \circ x$	$\dfrac{?\,, A, B, \Delta \vdash C}{?\,, B, A, \Delta \vdash C}$
Contraction	$x \circ x \sqsubseteq x$	$\dfrac{?\,, A, A, \Delta \vdash C}{?\,, A, \Delta \vdash C}$
Expansion	$x \sqsubseteq x \circ x$	$\dfrac{?\,, A, \Delta \vdash C}{?\,, A, A, \Delta \vdash C}$
Monotonicity	$x \sqsubseteq x \circ y$	$\dfrac{?\,, \Delta \vdash C}{?\,, A, \Delta \vdash C}$

Table 2: Correspondence between structural constraints and structural rules.

	$x \circ y \sqsubseteq y \circ x$	$x \circ x \sqsubseteq x$	$x \sqsubseteq x \circ x$	$x \sqsubseteq x \circ y$
Lambek's implications				
Linear implication	•			
Relevant implication	•	•		
Mingle implication[2]	•	•	•	
BCK implication	•	•		
Intuitionistic implication	•	•	•	•

Table 3: Correspondence between implication systems and sets of structural constraints.

1. *(Heredity)* For all formulae A, if $V(A,x) = T$ and $x \sqsubseteq y$, then $V(A,y) = T$.

2. *(Continuity)* For each given A and every non-empty $S \subseteq Q$

   - $V(A, \bigsqcup S) = \bigsqcup\{V(A,x)|x \in S\}$ and
   - $V(A, \bigsqcap S) = \bigsqcap\{V(A,x)|x \in S\}$.

□

As usual, when $V(A,x) = T$ we shall also say "A is true at x" or "A is verified at a", or "a verifies A" and sometimes we shall write "$x \Vdash A$". Similarly, when $V(A,x) = F$ we shall also say "A is false at x" or "A is not verified at a", or "a does not verify A" and sometimes write "$x \nVdash A$".

Definition 4 An *implication structure* is a pair $(\mathcal{F}, V)$ where $\mathcal{F}$ is a quantale frame and V is a valuation over $\mathcal{F}$ satisfying the following condition:

$$V(A \to B, x) = T \iff \forall y, V(A,y) = F \textit{ or } V(B, x \circ y) = T. \tag{1}$$

□

The valuation clause for $\to$ is clearly a generalization of Urquhart's semantics of relevant implication [Urq72]. The only difference is that the underlying algebraic structure is a quantale rather than a semilattice.

We say that an implication formula is *verified* in an implication structure if it is true at its identity point 1. Given a set **S** of structural constraints we call **S**-*frame* a frame closed under all the constraints in **S**. An implication formula is *verified* in an **S**-frame if its verified in all the implication structures based on it and is *valid for* **S** if it is verified in all **S**-frames. We also say that a finite sequence $A_1, \ldots, A_n$ of formulae *implies* a formula A *for* **S** if the implication formula $A_1 \to (A_2 \to \cdots \to (A_n \to A) \cdots)$ is valid for **S**.

In [DG94] different substructural logics are seen to correspond to different classes of quantale frames in the expected way, each structural rule being associated with a structural constraint on $\sqsubseteq$. Indeed, one deals only with the implication fragments, full quantale frames are not necessary and semilattice ordered monoids are sufficient. However, as we shall see in the sequel, the extra structure of quantale frames allows us to represent modalities in a very natural way.

The correspondence between structural rules and structural constraints is summarized in Table 2.

Some of the resulting logics are well-known implication systems. For instance, if $\circ$ satisfies only the commutativity condition, the resulting logic

Definition 1 An *information frame* or *quantale frame* is a structure $\mathcal{F} = (Q, \circ, 1, \sqsubseteq)$ such that:

1. Q is a non-empty set of elements called *pieces of information* or *information tokens*;

2. $\sqsubseteq$ is a partial ordering which makes Q into a complete lattice; $x \sqsubseteq y$ can be interpreted as "y contains at least the same information as x";

3. $\circ$ is a binary operation on Q which is

   (a) associative: $x \circ (y \circ z) = (x \circ y) \circ z$;

   (b) distributive over $\bigsqcup$: for every non-empty family $\{z_i\} \subseteq Q$, $\bigsqcup\{z_i \circ x\} = \bigsqcup\{z_i\} \circ x$ and $\bigsqcup\{x \circ z_i\} = x \circ \bigsqcup\{z_i\}$;

4. $1 \in Q$ and for every $x \in Q$, $x \circ 1 = 1 \circ x = x$;

□

Observe that the properties of $\circ$ imply that this operation is order-preserving, i.e.

$$x_1 \sqsubseteq x_2 \Longrightarrow x_1 \circ y \sqsubseteq x_2 \circ y \textit{ and } y \circ x_1 \sqsubseteq y \circ x_2.$$

We can define classes of quantale frames which satisfy additional conditions on the ordering $\sqsubseteq$:

Definition 2 We say that a quantale frame is:

commutative	if	$x \circ y \sqsubseteq y \circ x$
contractive	if	$x \circ x \sqsubseteq x$
expansive	if	$x \sqsubseteq x \circ x$
monotonic	if	$x \sqsubseteq x \circ y$

□

We shall call these conditions *structural constraints.*

Information tokens may verify formulae of the given language **L**. This verification is governed by the notion of valuation.

Definition 3 Given a quantale frame $\mathcal{F}$, let V be a function $\mathbf{F} \times Q \mapsto \{T, F\}$ where: (i) $\mathbf{F}$ is the set of formulae of the language, (ii) Q is the set of information tokens in the frame $\mathcal{F}$, (iii) T, F are truth values on which we assume the usual order $F < T$; we say that V is a *valuation over* $\mathcal{F}$ when it satisfies the following conditions:

logics.

3 Information Frames

In this section we briefly review the treatment of substructural implication systems in [DG94] to which we refer the reader for further details. (See Section 11 for related work on this topic.)

In the context of this paper, by a *consequence relation* over a given logical language **L** we intend a relation $\vdash$ between *sequences* of **L**-formulae and **L**-formulae satisfying the following two conditions:

(Identity) $$A \vdash A$$

(Surgical Cut) $$? \vdash A \textit{ and } \Delta, A, \Lambda \vdash B \Longrightarrow \Delta, ?, \Lambda \vdash B$$

Let **L** be a propositional language containing two binary operators $\rightarrow_1$ and $\rightarrow_2$. By an *implication logic* we mean a consequence relation over **L** characterised by the following universal conditions on the operators $\rightarrow_1$ and $\rightarrow_2$:

$C_{\rightarrow_1}$ $$?, A \vdash B \Longleftrightarrow ? \vdash A \rightarrow_1 B$$

$C_{\rightarrow_2}$ $$A, ? \vdash B \Longleftrightarrow ? \vdash A \rightarrow_2 B.$$

Let us now consider the following *structural rules*:

$$\frac{?, A, B, \Delta \vdash C}{?, B, A, \Delta \vdash C}\,Exchange \qquad \frac{?, A, A, \Delta \vdash C}{?, A, \Delta \vdash C}\,Contraction$$

$$\frac{?, A, \Delta \vdash C}{?, A, A, \Delta \vdash C}\,Expansion \qquad \frac{?, \Delta \vdash C}{?, A, \Delta \vdash C}\,Weakening$$

Then we can consider the family of implication logics closed under different collections of these structural rules. We shall call the logical systems belonging to this family *substructural implication logics*.

Notice that in all the systems allowing *Exchange* the two implications $\rightarrow_1$ and $\rightarrow_2$ collapse. In this context we shall use the symbol $\rightarrow$ without subscripts. Notice also that every system closed under *Weakening* is closed also under *Expansion*.

Substructural logics in general, and implication systems in particular, can be characterized in terms of information structures that we call "quantale frames" or "information frames" procedures is *the canonical procedure*, consisting in giving priority to the linear elimination rules over the cut rule, so that the cut rule is applied only when no elimination rule is further applicable, and the choice of the cut formulae is restricted to pairs $A,\neg A$ such that A is an immediate subformula of a formula of type β (in the Smullyan notation) occurring above in the branch.

Disjunction Rules

$$\frac{\begin{array}{l} A \vee B \\ \neg A \end{array}}{B}\, \mathrm{E}\vee 1 \qquad \frac{\begin{array}{l} A \vee B \\ \neg B \end{array}}{A}\, \mathrm{E}\vee 2 \qquad \frac{\neg(A \vee B)}{\begin{array}{l} \neg A \\ \neg B \end{array}}\, \mathrm{E}\neg\vee$$

Conjunction Rules

$$\frac{\begin{array}{l} \neg(A \wedge B) \\ A \end{array}}{\neg B}\, \mathrm{E}\neg\wedge 1 \qquad \frac{\begin{array}{l} \neg(A \wedge B) \\ B \end{array}}{\neg A}\, \mathrm{E}\neg\wedge 2 \qquad \frac{A \wedge B}{\begin{array}{l} A \\ B \end{array}}\, \mathrm{E}\wedge$$

Implication Rules

$$\frac{\begin{array}{l} A \to B \\ A \end{array}}{B}\, \mathrm{E}\to 1 \qquad \frac{\begin{array}{l} A \to B \\ \neg B \end{array}}{\neg A}\, \mathrm{E}\to 2 \qquad \frac{\neg(A \to B)}{\begin{array}{l} A \\ \neg B \end{array}}\, \mathrm{E}\neg\to$$

Negation Rule

$$\frac{\neg\neg A}{A}\, \mathrm{E}\neg\neg$$

Principle of Bivalence

$$\frac{}{A \mid \neg A}\, \mathrm{PB}$$

Table 1: **KE**-rules for unsigned formulae.

In [DG94] the system **KE** is extended into a labelled deductive system (in the sense of [Gab94]) which provides a unifying proof framework for (the “multiplicative” fragment of) classical and intuitionistic substructural as "pieces of information" that may or may not "verify" a given formula. In this paper we shall restrict our attention to the fragment containing the operators $\Box$, $\Diamond$ and $\rightarrow$. The interplay between modalities and the other operators will be discussed in a subsequent work. Owing to space constraints, proper comparisons with related work will also have to be postponed to another occasion.

2 The classical KE system

The system **KE**, like the tableau method and resolution, is a refutation system for classical logic. Unlike resolution, however, **KE** is not restricted to clausal form and, unlike the tableau method, it includes a cut rule which *cannot*, in general, be eliminated. This classical cut rule is called PB (from Principle of Bivalence) and has the following forms, depending on whether we deal with signed or unsigned formulae:

$$\frac{}{TA \mid FA} \qquad\qquad \frac{}{A \mid \neg A}$$

The formula A introduced by an application of this rule is called *PB-formula* or *cut formula*[1]. Once such a cut rule has been allowed, the branching elimination rules typical of the tableau method become unnecessarily strong and can be replaced by weaker *non-branching* rules (with two premises).

The rules of the **KE** system (for unsigned formulae) are illustrated in Table 1. The two-premiss elimination rules correspond to familiar principles of inference: *modus ponens, modus tollens, disjunctive syllogism* its dual. The one-premiss elimination rules are the same as the tableau rules. A **KE**-refutation of a set of formulae ? is, as usual, a closed tree of formulae constructed according to the rules of **KE** starting from formulae in ?.

A crucial property of **KE** is the *analytic cut property*: the applications of the cut rule can be restricted to subformulae of the formulae occurring above in the branch without loss of completeness This property allows for systematic and efficient refutation procedures. Indeed, results in [DM94] imply that any refutation procedure which can be formulated in terms of the tableau rules can be efficiently (linearly) simulated by means of the **KE** rules, but there are efficient and systematic **KE**-procedures which cannot be polynomially simulated by means of the tableau rules. One of these

[1] To see that PB plays the same role as the cut rule in the classical sequent calculus, think of it as a rule that allows one to construct a closed **KE**-tree for Γ, given a closed **KE**-tree for Γ, A and a closed **KE**-tree for $\Gamma, \neg A$; for a discussion of this point see [DM94].

motivations what they may, we must develop the logical capability of adding modalities into a system and have an understanding of the landscape of options available.

This paper addressed the problem of grafting modalities onto substructural implication logics. The term "substructural logics" is now of widespread use to indicate a family of logics weaker than classical logic (including intuitionistic, relevant and linear logic), which result from dropping some or all of the traditional structural rules of the classical sequent calculus (see [Dôs93] for an introduction), and are proving useful in several applications of logic to Computer Science and AI (see, for instance, [Gar89] for an application of relevance logic to modular reasoning systems, and [Ale94] for an overview of computational applications of Linear Logic). The introduction of modalities into such logics adds a new dimension which can explicitly and naturally account for "accessibility" relations involved in the processes that are being modelled.

Formally we have a system with the conditional operator $\rightarrow$ and we want to add unary operators $\Box$ and $\Diamond$ which behave like modalities. We would like to to this in a way which is compatible with the spirit of the system, i.e. to define modalities from within the system in some natural way. Substructural logics can be characterized by structures of "objects" which can be understood as "pieces of information" or, sometimes, as "resources". A natural way of defining modalities within such semantics consists therefore in adding an accessibility relation between pieces of information. This is the approach we shall adopt in this paper. By analogy with classical modal logics the intuitive idea is that the verification of a proposition of the form "$\Box A$" or "$\Diamond A$" by means of a given piece of information (or resource) x, depends on what is verified by other pieces of information (or resources) "accessibile" from x. Unlike classical modal logics, however, the verification of non-modal formulae, such as a conditional formula, by a given piece of information x also depends on what is or is not verified elsewhere in the structure (as with the intuitionistic conditional). We shall not pursue any particular intuitive interpretation, because it may change from one logic to the other, and shall leave it to the reader to envisage his or her favourite application contexts.

Previous work on this topic has been concentrating on intuitionistic modal logics ([FS77, BD84, PS86, AP93, Sym93]; see in particular [Sym93] for a comprehensive overview). Here we start developing a more general approach which fits naturally into the framework for substructural logics presented in [DG94]. The latter consists of a labelled generalization of the classical refutation system **KE** [DM94], where the labels can be interpreted

Grafting Modalities onto substructural implication systems*

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Published in *Studia Logica*, **VII** 1997

Abstract

We investigate the semantics of the logical systems obtained by introducing the modalities $\Box$ and $\Diamond$ into the family of substructural implication logics (including relevant, linear and intuitionistic implication). Then, in the spirit of the LDS (Labelled Deductive Systems) methodology, we "import" this semantics into the classical proof system **KE**. This leads to the formulation of a uniform labelled refutation system for the new logics which is a natural extension of a system for substructural implication developed by the first two authors in a previous paper.

Keywords: Kripke semantics, Labelled Deductive Systems, KE system.

1 Introduction

The notion of modality is central in pure and applied logic. Many systems presented to formalise some application area require the addition of modality to the language for a variety of reasons: to cater for changes of the system in time, or perhaps for the dependency of the system on the context, or even to bring metalevel notions into the object level. Be the reasons and

*We wish to thank Krysia Broda, Marco Mondadori, Guido Governatori and Sara Negri for stimulating discussions and helpful suggestions. This research was supported by EPSRC, grant GR-J14745 ("Labelled Tableaux for Algorithmic Deduction").