IV. Discrete Version of the Bode Procedure

a) Let us first consider the case where all poles of $H(z)$ are simple. It can be shown that $H_R(z)$ in (3) can be expanded in the following form:

$$H_R(z) = \left(\frac{n_m}{d_m} + \sum_{i=1}^{m} \frac{k_i}{p_i}\right) + \sum_{i=1}^{m} \frac{k_i}{z - p_i} + \sum_{i=1}^{m} \frac{k_i}{z^{-1} - p_i} \quad (8)$$

where k_i can be evaluated from

$$k_i = N(z) \Big/ \frac{d}{dz} D(z)\Big|_{z=p_i}. \quad (9)$$

The required transfer function $H(z)$ is then identified as

$$H(z) = \left(\frac{n_m}{d_m} + \sum_{i=1}^{m} \frac{k_i}{p_i}\right) + \sum_{i=1}^{m} \frac{2k_i}{z - p_i} \quad (10)$$

an expression which only requires the calculation of $\{k_i\}$ in accordance with (9).

b) When multiple order poles are involved, the Bode method is computationally somewhat more involved than the Brune–Gewertz method extended by Mitra and Vaidyanathan [2] along with simplifications thereof recently proposed by Dutta Roy [6]. However, for the sake of completeness, we note the following steps. The "partial fraction expansion" of $H_R(z)$ takes the form

$$\begin{aligned} H_R(z) = \frac{n_m}{d_m} &- \sum_i \frac{k_{ir_i} + k_{i(r_i-1)}(-p_i) + \cdots + k_{i1}(-p_i)^{r_i-1}}{(-p_i)^{r_i}} \\ &+ \sum_i \frac{k_{ir_i} + k_{i(r_i-1)}(z - p_i) + \cdots + k_{i1}(z - p_i)^{r_i-1}}{(z - p_i)^{r_i}} \\ &+ \sum_i \frac{k_{ir_i} + k_{i(r_i-1)}(z^{-1} - p_i) + \cdots + k_{i1}(z^{-1} - p_i)^{r_i-1}}{(z^{-1} - p_i)^{r_i}} \end{aligned} \quad (11)$$

where the pole at p_i is of multiplicity r_i and

$$k_{ix} = \frac{1}{(r_i - x)!} \frac{d^{r_i-x}}{dz^{r_i-x}} \left[\frac{N(z)}{D(z)}(z - p_i)^{r_i}\right]\Bigg|_{z=p_i}. \quad (12)$$

$H(z)$ is deduced therefrom as

$$\begin{aligned} H(z) = \frac{n_m}{d_m} &- \sum_i \frac{k_{ir_i} + k_{i(r_i-1)}(-p_i) + \cdots + k_{i1}(-p_i)^{r_i-1}}{(-p_i)^{r_i}} \\ &+ 2\sum_i \frac{k_{ir_i} + k_{i(r_i-1)}(z - p_i) + \cdots + k_{i1}(z - p_i)^{r_i-1}}{(z - p_i)^{r_i}}. \end{aligned} \quad (13)$$

V. Example

We consider the same example as in [2].

$$H_R(e^{jw}) = \frac{1 + \cos w + \cos 2w}{17 - 8\cos 2w}$$

$$H_R(z) = \frac{1 + \frac{1}{2}(z + z^{-1}) + \frac{1}{2}(z^2 + z^{-2})}{17 - 4(z^2 + z^{-2})} = \frac{N(z)}{D(z)}$$

$$D(z) = 17 - 4(z^2 + z^{-2}) = 17 - 8C_2(u/2)$$

$$= 17 - 8\left[2(u/2)^2 - 1\right]$$

$$Q(u) = 25 - 4u^2.$$

The zeros of $Q(u)$ are $u_1 = 5/2$ abd $u_2 = -5/2$. The two equations $z + z^{-1} = \pm 5/2$ yield the roots $\pm 1/2$ and ± 2 and we select the values $p_1 = 1/2$; $p_2 = -1/2$ for the poles of $H(z)$.

From (9)

$$H(z) = \frac{n_m}{d_m} + \frac{N\left(\frac{1}{2}\right)/D'\left(\frac{1}{2}\right)}{\frac{1}{2}} + \frac{N\left(-\frac{1}{2}\right)/D'\left(-\frac{1}{2}\right)}{-\frac{1}{2}} + \frac{N\left(\frac{1}{2}\right)/D'\left(\frac{1}{2}\right)}{z - \frac{1}{2}} + \frac{N\left(-\frac{1}{2}\right)/D'\left(-\frac{1}{2}\right)}{z + \frac{1}{2}}.$$

Now

$$N\left(\frac{1}{2}\right)\Big/D'\left(\frac{1}{2}\right) = \frac{1 + \frac{1}{2}(z + z^{-1}) + \frac{1}{2}(z^2 + z^{-2})}{-8z + 8z^{-3}}\Bigg|_{z=1/2} = 7/96.$$

Likewise

$$N\left(-\tfrac{1}{2}\right)/D'\left(-\tfrac{1}{2}\right) = -1/32.$$

Hence

$$H(z) = -1/8 + \frac{7/96}{\frac{1}{2}} + \frac{-1/32}{-\frac{1}{2}} + \frac{7/48}{z - \frac{1}{2}} + \frac{-1/16}{z + \frac{1}{2}} = \frac{1/12(z^2 + z + 1)}{z^2 - \frac{1}{4}}.$$

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A Prony Method for Noisy Data: Choosing the Signal Components and Selecting the Order in Exponential Signal Models

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Prony's method is a simple procedure for determining the values of parameters of a linear combination of exponential functions. Until recently, even the modern variants of this method have performed poorly in the presence of noise. We have discovered improvements to Prony's method which are based on low-rank approximations to data matrices or estimated correlation matrices [6]–[8], [15]–[21], [34]. Here we present a different, often simpler procedure for estimation of the signal parameters in the presence of noise. This procedure has received only limited dissemination [35]. It is very close in form and assumptions to Prony's method. However, in preliminary tests, the performance of the method is close to that of the best available, more complicated, approaches which are based on maximum likelihood or on the use of eigenvector or singular value decompositions.

I. Introduction

Nearly two hundred years ago, Prony developed a simple procedure for determining the values of parameters of a linear combina-

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tion of exponential functions from uniformly spaced samples [1]. Today "Prony's method" is usually taken to mean the least squares extension of the method as presented, for example, by Hildebrand [2, pp. 378–382].

A short record of a data sequence $y(n)$, $n = 1,2,\cdots,N$, is assumed to be composed of uniformly spaced samples of a sum of exponential signals $x(n)$ and measurement noise $w(n)$. That is,

$$y(n) = x(n) + w(n), \qquad \text{for } n = 1,2,\cdots,N \tag{1}$$

where

$$x(n) = \sum_{k=1}^{M} a(k)(C(k))^n \tag{2}$$

$$N > 2M \tag{3}$$

$$C(k) = \exp(s(k)). \tag{4}$$

The values of the signal parameters $a(k)$ and $s(k)$ for $K = 1,2,\cdots,M$ are unknown complex numbers. Often the value of M is also unknown. However, let us initially assume that the value of M is known.

Following the derivation of Hildebrand [2, formula 9.4.6., p. 379] we note that the signal $x(n)$ satisfies a linear, homogeneous difference equation with constant coefficients

$$\sum_{k=0}^{M} b(k)x(n-k) = 0, \qquad \text{for } M < n \leqslant N \tag{5}$$

where

$$b(0) = 1. \tag{6}$$

The roots of the prediction-error-filter polynomial $B(z)$ provide the values of the exponent parameters $C(k)$, and hence $s(k)$

$$B(z) = \sum_{k=0}^{M} b(k)z^{-k} = \prod_{k=1}^{M}\left[1 - C(k)z^{-1}\right]. \tag{7}$$

Hildebrand explicitly considers noisy data and specifies Prony's method by the following three steps:

1. Using the method of least squares, minimize the approximation error

$$A = \sum_{n=M+1}^{N}\left|\sum_{k=1}^{M}\hat{b}(k)y(n-k) + y(n)\right|^2 \tag{8}$$

by best choice of the coefficients $\hat{b}(k)$ [3]. For $N > 2M$ and for noisy data, the solution will be unique with high probability. However, if the resulting set of normal equations is singular, then the pseudo-inverse of the coefficient matrix can be used to choose the minimum norm solution.

2. After the M values of $\hat{b}(k)$ are determined, the roots of the prediction-error-filter polynomial $\hat{B}(Z)$ are found, using $\hat{b}(0) = 1$.

$$\hat{B}(z) = \sum_{k=0}^{M}\hat{b}(k)z^{-k} = \prod_{k=1}^{M}\left(1 - \hat{C}(k)z^{-1}\right). \tag{9}$$

The corresponding exponent values $\hat{s}(k)$ can then be found from (4).

3. Having determined the values $\hat{C}(k)$ for $k = 1,2,\cdots,M$, the error in approximating the observed data by a linear combination of exponential signal components then becomes linear in the M values of $a(k)$, the complex signal scale factors

$$e(n) = y(n) - \sum_{k=1}^{M} a(k)\left[\hat{C}(k)\right]^n, \qquad \text{for } n = 1,2,\cdots,N. \tag{10}$$

The M estimates $\hat{a}(k)$ can be determined by minimizing the summed, magnitude-squared error

$$E = \sum_{n=1}^{N}\left|y(n) - \sum_{k=1}^{M} a(k)\left[\hat{C}(k)\right]^n\right|^2. \tag{11}$$

It is well known that the errors in signal parameters which are estimated by Prony's method can be discouragingly large [2], [3]. For insight into this phenomenon we recommend calculation and study of the Cramer–Rao (CR) bounds for the variance of the error in the estimated parameters and comparison of the threshold of estimation of the Prony method with that of the maximum-likelihood method [6]–[9]. By the threshold of estimation, we mean the value of signal-to-noise ratio (SNR) at which the variance of an estimation error of an unbiased estimate begins to depart very rapidly from the corresponding CR-bound value.

As another example of the application of Prony's method, consider the problem of estimating the parameters of a zero-mean, autoregressive, moving average (ARMA) stationary random sequence from estimates of its covariance values. Various investigators have recognized that, after a finite number of lags, the true underlying covariance values satisfy a linear, homogeneous, difference equation with constant coefficients [10]–[13]. That is, after a finite number of lags, the estimated covariance sequence can be represented as a linear combination of exponentials (i.e., the true, underlying covariance sequence which satisfies the homogeneous difference equation) plus measurement noise. This measurement noise may be only the error sequence in estimating the covariance values from a finite observation of the ARMA sequence. Part of it might also be due to additive noise in the observation of the ARMA sequence. Some of these ideas have been restated by Cadzow [13], [14].

II. Prony Methods for Noisy Data

In previous and related work, we have shown how one can extend the threshold of estimation of Prony's method to much lower values of SNR and how one can improve parameter estimation at values of SNR above this threshold [6]–[9], [15]–[21]. The major source for these improvements is the use of information about the rank M of a matrix of signal covariance values or a matrix of samples of the signal. If there is no prior information about this rank, it is estimated from the data using singular value decomposition (SVD). The most important computational step is a preprocessing step, before application of Prony's method. A prediction order L which is larger than the value of M is chosen. The measured data matrix or the matrix of estimated covariance values is replaced by a matrix of the prescribed rank M which is the best least squares approximation to the given matrix. Other investigators have presented closely related approaches [22]–[28] and [36].

In this work we advocate a simpler procedure which appears to provide the same desirable attributes to nearly the same extent. This procedure consists of the following two steps:

1. Use Prony's method on the given data, but with a prediction order L which is larger than the maximum number of exponentials which are expected in the signal. The result is a set of L exponentials which are candidates for signal components.

2. Out of the L exponential functions which are provided by the high-order Prony calculation, determine the best subset of size M. A best subset of M exponentials is one for which a linear combination of the M exponentials best approximates the observed data using a least squares criterion. One can check all of the $\binom{L}{M}$ possible subsets of size M of the L exponential functions to find the best combination.

A simpler approach to step 2 is to use the procedure of Hocking and Leslie [29] as we have previously suggested [30]. In the procedure of Hocking and Leslie, a best subset can usually be found without searching over all possible subsets. Hocking and Leslie accomplish this by first solving a related, but different, problem. They search for the basis functions (exponentials, in our application) which contribute most to the summed magnitude-squared errors by their deletion. This provides an initial importance ordering of the exponentials. Hocking and Leslie prove that the fitting errors associated with these single-deletion sets provide convenient threshold values of the error for recognizing the global optimality of a particular combination of M exponentials being tested.

If M is not known *a priori*, an estimate of M, $(\hat{M})$, can be found as follows. Choose $\hat{M} = 1$, and find the best subset of size unity that best fits the data. Call the corresponding minimum error E_1.

Then, choose $\hat{M} = 2$ and find the best subset of size two and the corresponding minimum error E_2. Repeat the procedure until the rate of decrease of the error with increasing values of $\hat{M}$ is small, consistent with the modeling of broad-band noise. The integer i at which E_i shows the significant drop in rate of decrease is taken as $\hat{M}$. We now give a simulation example.

III. Simulation Results

If the data are known to be composed of undamped sinusoids, as we assume in this example, forward and backward prediction equations can be used simultaneously to obtain extra prediction equations for Hildebrand's least squares form of the Prony method [31]–[33].

A sequence $y(n)$ consisting of two complex sinusoids and white, complex Gaussian noise $w(n)$ was generated using the formula below

$$y(n) = a_1 \exp[j(\omega_1 n + \phi_1)] + a_2 \exp[j(\omega_2 n + \phi_2)] + w(n), \quad n = 0, 1, 2 \cdots, 24.$$

Here $a_1 = a_2 = 1$, $\omega_1 = 2\pi(0.52)$, $\omega_2 = 2\pi(0.5)$, and $j = \sqrt{-1}$. The variance of the real or imaginary part of $w(n)$ is σ^2. SNR is defined as $10 \log_{10} (a_1^2/2\sigma^2)$. The coefficients of the polynomial $B(z)$ were found by solving the forward–backward linear prediction equations as in [34] in the least squares sense. L was chosen to be 12 ($N/2$). The 12 zeros of $G(z)$ were found and the best subset of 2 out of the 12 which minimized E in (11) was computed. The frequency estimates of the two sinewaves, $\hat{f}_1$ and $\hat{f}_2$, are the angles of the two chosen exponents (divided by 2π). This simulation was repeated 500 times and the root mean square (rms) value of the frequency estimation error was computed at SNR values in the range of 30 to 7 dB. They are given in Table 1 along with the appropriate CR bounds and SVD-method values which were taken from [34]. Comparing these figures with those in [34], we note that the SVD-based methods are slightly better in performance. This difference is due to the signal enhancement achieved by SVD.

Table 1 Mean Square Error of the Frequency (f_1) Estimation Error versus SNR. (CRB stands for the Cramer–Rao bound which is the lower bound on the standard deviation of the frequency estimation error for an unbiased estimator. The bias in the frequency estimates was insignificant except at SNR = 7 dB. Below 7 dB, the mean square error is large due to the presence of outliers. For the proposed subset selection method, the prediction order is twelve ($L = 12$). For comparison, two values of the error for the SVD method are provided [34].)

SNR (dB)	$\sqrt{\text{Mean Square Error}}$	CRB
30	0.427×10^{-3}	0.311×10^{-3}
	(SVD) $0.403 \times 10^{-3} (L = 12)$	
20	0.130×10^{-2}	0.984×10^{-3}
15	0.238×10^{-2}	0.175×10^{-2}
12	0.373×10^{-2}	0.276×10^{-2}
	(SVD) $0.313 \times 10^{-2} (L = 12)$	
10	0.417×10^{-2}	0.311×10^{-2}
7	0.601×10^{-2}	0.490×10^{-2}

Fig. 1 shows the minimum subset error $E_{\hat{M}}$ for different choices of $\hat{M}$ at different SNR values. The value E_0 at $\hat{M} = 0$ is the data "energy"

$$\sum_{n=1}^{N} |y(n)|^2.$$

Note the clear drop in the rate of change of E at $\hat{M} = 2$.

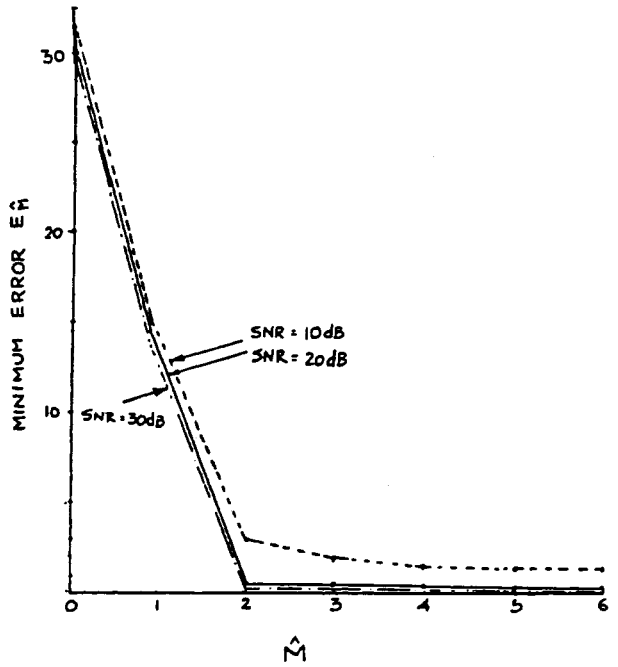

Fig. 1. Minimum subset error versus hypothesized subset size (the estimated number of exponential signal components).

IV. Discussion and Conclusions

Ideally, to fit exponentials to a data sequence $y(n)$, one has to minimize the error

$$\sum_{n=1}^{N} \left| y(n) - \sum_{k=1}^{M} \hat{a}_k e^{\hat{s}_k n} \right|^2$$

with respect to $\hat{a}_k$'s and $\hat{s}_k$'s simultaneously. This is a difficult problem even if the value of M is known. Instead, we find the exponents $\hat{s}(k)$ separately as is often done. However, we have made use of the fact that, if the data are composed of exponentials and noise, overestimating the degree L ($> M$) of the polynomial $B(z)$ improves the accuracy of the M signal-zero locations. Subsequently, we select the M out of the L exponentials that best explain the data. The new procedure extends the threshold of the forward–backward covariance method [31]–[33] and is only slightly inferior to SVD based methods [7], [34].

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Sensitivity Analysis of the Ampacities of Self-Contained, Oil-Paper-Insulated, Naturally Cooled, and Forced Cooled 300-kV Underground Cables

C. S. INDULKAR

The effects of the various parameters on the ampacity of 300-kV oil-paper-insulated, naturally cooled, and forced cooled underground cable systems are considered, and sensitivity values are tabulated.

INTRODUCTION

The highest safe operating temperature of the oil-paper insulation, the cable dimensions, and the insulating and thermal properties of the materials of cable construction set a limit to the current that the cable conductor may be allowed to carry. Ampacity computations are in general tedious, and it is therefore preferable to have a sensitivity analysis carried out at the design stage in order to determine the sensitivity of the cable ampacity to the various parameters. In this letter, such a sensitivity analysis has been carried out for a typical 300-kV cable and the results are tabulated.

AMPACITY [1]

The current-carrying capacity of a self-contained oil-paper-insulated cable is given by

$$I = \sqrt{\frac{T_c - T_a - P_d(0.5R_i + R_j + kR_e)}{R_{ac}(R_i + R_j + kR_e) + R_s'(R_j + kR_e)}} \tag{1}$$

where

T_c conductor temperature
T_a soil surface temperature
P_d dielectric loss
R_i thermal resistance of internal insulation
R_j thermal resistance of external insulation
R_e soil thermal resistance
k thermal proximity factor
R_{ac} ac resistance of cable conductor
R_s' resistance of sheath.

Furthermore,

$$P_d = \frac{w\epsilon V^2 \cos\phi}{2\ln\left(\frac{r_2}{r_1}\right)9 \times 10^{11}} \tag{2}$$

$$R_i = \frac{\rho_i}{2\pi}\ln\frac{r_2}{r_1} \tag{3}$$

$$R_j = \frac{\rho_j}{2\pi}\ln\frac{r_4}{r_3} \tag{4}$$

$$R_e = \frac{\rho_e}{2\pi}\ln\frac{2D}{r_4} \tag{5}$$

and

$$k = 1 + \frac{\ln\frac{(4D^2 + S^2)}{S^2}}{\ln\left(\frac{2D}{r_4}\right)} \tag{6}$$

where

ρ_i thermal resistivity of oil-paper insulation

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