ON THE GROWTH OF THE SPECTRAL MEASURE

A. BOUMENIR
King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Dhahran 31262, SAUDI ARABIA

ABSTRACT. We are concerned with the asymptotics of the spectral measure associated with a self-adjoint operator. By using comparison techniques we shall show that the eigenfunctionals of L_2 are close to the eigenfunctionals L_1 if and only if $d\Gamma_1 \asymp d\Gamma_2$ as $\lambda \to \infty$.

KEY WORDS AND PHRASES: Spectral asymptotics, spectral function, Sturm-Liouville operators.
1991 AMS SUBJECT CLASSIFICATION CODES: 34L20.

1   INTRODUCTION

We would like to obtain a relation between the growth of the spectral measure of a self-adjoint operator and the behaviour of its eigenfunctionals. In this study we shall assume that we have two "close" self-adjoint operators acting in the same separable Hilbert space, H say. Without loss of generality we can assume that both operators have simple spectra. To this end, let us denote by $\varphi(\lambda)$ and $y(\lambda)$ the eigenfunctionals of L_1 and L_2 respectively. Recall that the spectrum of a self-adjoint operator is defined by

$$\forall \lambda \in \sigma_i \quad \exists \ \varphi_{i,n} \in D_{\{L_i\}} \ / \ \|\varphi_{i,n}\| = 1 \ \text{ and } \ \|L_i\varphi_{i,n} - \lambda\varphi_{i,n}\| \xrightarrow{n\to\infty} 0$$

where $i = 1, 2$. In case λ is in the continuous spectrum the sequence is not compact in the Hilbert space H. For this we can assume the existence of a countably normed perfect space Φ, such that

$$\Phi \hookrightarrow H \hookrightarrow \Phi'$$

where the embeddings are compact, for further details see [1] and [2]. For the sake of simplicity we shall assume that the embeddings are given by the identities and so

$$f \in \Phi \quad \psi \in H \quad (f,\psi) \equiv < f, \psi >_{\Phi \times \Phi'}.$$

Since the sequence φ_n is bounded in H, it is then compact in Φ', which implies

$$\varphi_n \xrightarrow{\Phi'} \varphi(\lambda) \in \Phi'$$

and similarly for the operator L_2; Since both operators are acting in the same Hilbert space H, we shall assume that the space Φ' contains both systems of eigenfunctionals; i.e.,

$$\{y(\lambda)\} \subset \Phi' \quad \text{and} \quad \{\varphi(\lambda)\} \subset \Phi'.$$

Recall that the system $\{y(\lambda)\}$ helps define an isometry for L_2

$$\forall f \in \Phi \quad f \longrightarrow \hat{f}^2(\lambda) \equiv < f, y(\lambda) >_{\Phi \times \Phi'}$$

$$f = \int \overline{\hat{f}^2(\lambda)} y(\lambda) d\Gamma_2(\lambda) \quad \text{where} \quad \hat{f}^2(\lambda) \in L^2_{d\Gamma_2(\lambda)}$$

Similarly for $\varphi(\lambda)$;

$$\forall f \in \Phi \quad f \longrightarrow \hat{f}^1(\lambda) \equiv < f, \varphi(\lambda) >_{\Phi \times \Phi'},$$

$$f = \int \overline{\hat{f}^1(\lambda)}\varphi(\lambda)d\Gamma_1(\lambda) \text{ where } \hat{f}^1(\lambda) \in L^2_{d\Gamma_1(\lambda)}$$

These transforms define isometries, and Parseval equality yields

$$\int_{\sigma_1} \hat{f}^1(\lambda)\overline{\hat{\psi}^1(\lambda)}d\Gamma_1(\lambda) = (f,\psi)_H = \int_{\sigma_2} \hat{f}^2(\lambda)\overline{\hat{\psi}^2(\lambda)}d\Gamma_2(\lambda).$$

where the nondecreasing functions $\Gamma_1(\lambda)$ and $\Gamma_2(\lambda)$ are called the spectral measures associated with L_1 and L_2, respectively. It is these functions that we would like to estimate as $\lambda \to \infty$.

In all that follows $y(\lambda) \sim \varphi(\lambda)$ as $\lambda \to \infty$ means $\forall f \in \Phi$,

$$\hat{f}^1(\lambda) \asymp \hat{f}^2(\lambda) \text{ as } \lambda \to \infty$$

and $d\Gamma_1(\lambda) \sim d\Gamma_2(\lambda)$ as $\lambda \to \infty$ means that $\forall F \in L^1_{d\Gamma_1(\lambda)} \cap L^1_{d\Gamma_2(\lambda)}$

$$\int_\lambda^\infty F(\eta)d\Gamma_1(\eta) \asymp \int_\lambda^\infty F(\eta)d\Gamma_2(\eta) \text{ as } \lambda \to \infty.$$

In this work, we shall try to answer the following problem:

Statement of the Problem: under what conditions

$$y(\lambda) \sim \varphi(\lambda) \text{ as } \lambda \to \infty \iff d\Gamma_1(\lambda) \sim d\Gamma_2(\lambda) \text{ as } \lambda \to \infty.$$

In order to answer the above question, we shall compare the self-adjoint operators L_1 and L_2, see [3]. Recall that a shift operator or transmutation is defined by

$$y(\lambda) = V\varphi(\lambda) \quad \lambda \in \sigma_1;$$

Clearly the definition of V depends on σ_2 and σ_1 and we shall agree to set

$$y(\lambda) = 0 \text{ if } \lambda \notin \sigma_2, \text{ and } \varphi(\lambda) = 0 \text{ if } \lambda \notin \sigma_1$$

$$y(\lambda) = V\varphi(\lambda) \quad \lambda \in \sigma_2 \subset \sigma_1 \subset R.$$

Condition $\sigma_2 \subset \sigma_1$ insures that $V0 = 0$ and so defines an operator on the algebraic span of $\{\varphi(\lambda)\}$. Thus it is clear that in order for V and V^{-1} to exist as linear operator it is necessary that $\sigma_2 \subset \sigma_1$ and $\sigma_1 \subset \sigma_2$

$$\sigma_2 \equiv \sigma_1.$$

It is readily seen that $\{\varphi(\lambda)\}$ form a complete set in the reflexive space (perfect) Φ', and so the space generated by $\{\varphi(\lambda)\}$ is dense in Φ'. Consequently V is densely defined. This in turns allows us to define the adjoint operator $V' : \Phi \to \Phi$.

2   MAIN RESULTS

We shall agree to say $\Gamma_1(\lambda)$ is Abs-$d\Gamma_2$ if there exists $g(\eta) \in L^{1,loc}_{d\Gamma_2}$ such that

$$\Gamma_1(\lambda) = \int_0^\lambda g(\eta)d\Gamma_2(\eta) + \Gamma_1(0)$$

This fact shall be denoted by

$$g(\lambda) \equiv \frac{d\Gamma_1}{d\Gamma_2}(\lambda) \in L^{1,loc}_{d\Gamma_2}$$

In this case the condition $d\Gamma_1(\lambda) \sim d\Gamma_2(\lambda)$ in the statement of the problem can be restated as $g(\lambda) \asymp 1$ as $\lambda \to \infty$. Recall that due to reflexivity of the space Φ, the operator V' is defined in Φ and since $\Phi \hookrightarrow H$, V' is actually defined in H. Let us denote this extension to the space H by $\bar{V}$. Since we are interested in the case where $y(\lambda) \sim \varphi(\lambda)$ we can expect V to be bounded. In this regard we have the following result:

Theorem 1: If the extension $\tilde{V} : H \to H$, is a bounded operator then $\Gamma_1(\lambda)$ is $d\Gamma_2$-ABS continuous.

Proof: It is clear that for $f \in D_{V'}$

$$< f, y(\lambda) >_{\Phi \times \Phi'} \; = \; < f, V\varphi(\lambda) >_{\Phi \times \Phi'}$$
$$= \; < V'f, \varphi(\lambda) >_{\Phi \times \Phi'}$$

In other words

$$\hat{f}^2(\lambda) = \widehat{V'f}^1(\lambda). \tag{2.1}$$

Equation 2.1 obviously holds for $f \in H$. Indeed let $f_n \in D_{V'} \subset H$ such that $f_n \xrightarrow{H} f \in H$. Given that $\tilde{V}$ is a bounded operator in H, we obviously have $\tilde{V} f_n \to \tilde{V} f$. Using the fact that $\forall n, \quad \hat{f}_n^2(\lambda) = \widehat{V'f_n}^1(\lambda)$ and the isometries are bounded operators we have $\hat{f}_n^2 \to \hat{f}^2$ and $\widehat{\tilde{V}f_n}^1 \to \widehat{\tilde{V}f}^1$. Therefore

$$\hat{f}^2(\lambda) = \widehat{\tilde{V}f}^1(\lambda) \quad f \in H. \tag{2.2}$$

From which we deduce that $\forall f \in H$

$$\int \hat{f}^2(\lambda)\overline{\hat{f}^2(\lambda)}d\Gamma_1(\lambda) \; = \; \int \widehat{\tilde{V}'f}^1\overline{\widehat{\tilde{V}'f}^1} \, d\Gamma_1(\lambda)$$
$$= \; (\tilde{V}'f, \tilde{V}'f)$$
$$= \; \|\tilde{V}'f\|^2$$
$$\leq \; c\|f\|^2$$
$$\leq \; c\int | \hat{f}^2(\lambda) |^2 \, d\Gamma_2(\lambda) \qquad \forall f \in H.$$

Thus each $d\Gamma_2$ negligible set is a $d\Gamma_1$ negligible set. Henceforth $\Gamma_1(\lambda)$ to be $d\Gamma_2(\lambda)$-Abs continuous. The above inequality is exactly a sufficient condition for the Radon-Nikodym theorem to hold, see [4].

In all that follows we shall assume that $d\Gamma_1(\lambda)$ is $d\Gamma_2 - Abs$ continuous which is denoted by

$$g(\lambda) \equiv \frac{d\Gamma_1}{d\Gamma_2}(\lambda).$$

We now need to define a function of an operator, namely $g(L_2)$ for the next result:

$$\Phi \xrightarrow{g(L_2)} H$$
$$f \longrightarrow g(L_2)f \equiv \int g(\lambda)\overline{\hat{f}^2(\lambda)}y(\lambda)d\Gamma_2(\lambda).$$

Theorem 2: Assume that V admits closure in Φ' and Γ_1 is Abs-$d\Gamma_2(\lambda)$ then

$$\forall \psi \in D_{V'} \subset \Phi \quad \left(\sqrt{\frac{d\Gamma_2}{d\Gamma_1}(L_2)}\right)' \left(\sqrt{\frac{d\Gamma_2}{d\Gamma_1}(L_2)}\right)\psi = \overline{V}V'\psi \quad \text{in } \Phi'.$$

Proof: From equation 2.1 and the fact that the embeddings are defined by identities, we deduce that $\forall f, \psi \in D_{V'} \subset \Phi$

$$\int \hat{f}^2(\lambda)\overline{\hat{\psi}^2(\lambda)}d\Gamma_1(\lambda) \; = \; \int \widehat{V'f}^1\overline{\widehat{V'\psi}^1}d\Gamma_1(\lambda) \tag{2.3}$$
$$= \; (V'f, V'\psi) \tag{2.4}$$
$$= \; < V'f, V'\psi >_{\Phi \times \Phi'}$$
$$= \; < f, \overline{V}V'\psi >_{\Phi \times \Phi'} .$$

However the left handside of equation 2.3 can rewritten as

$$\int \hat{f}^2(\lambda)\overline{\hat{\psi}^2(\lambda)}d\Gamma_1(\lambda) = \int \hat{f}^2(\lambda)\overline{\hat{\psi}^2(\lambda)}g(\lambda)d\Gamma_2(\lambda)$$

$$= \int \sqrt{g(\lambda)}\hat{f}^2(\lambda)\overline{\sqrt{g(\lambda)}\hat{\psi}^2}d\Gamma_2(\lambda)$$

$$= \int \widehat{\sqrt{g(L_2)}f}^2 \overline{\widehat{\sqrt{g(L_2)}\psi}^2} \, d\Gamma_2(\lambda)$$

$$= (\sqrt{g(L_2)}f, \sqrt{g(L_2)}\psi) \qquad (2.5)$$

$$= <f, \sqrt{g(L_2)}'\sqrt{g(L_2)}\psi>_{\Phi\times\Phi'} .$$

Observe that if we set $f = \psi$ in equations 2.4 and 2.5 then we would obtain

$$\|\sqrt{g(L_2)}f\| = \|V'f\| \qquad (2.6)$$

from which we deduce that $D_{V'} \subset D_{\sqrt{g(L_2)}} \subset \Phi$, from we obtain

$$\forall \psi \in D_{V'} \quad \sqrt{g(L_2)}'\sqrt{g(L_2)}\psi = \overline{V}V'\psi. \qquad (2.7)$$

Remark: Observe that both operators $\sqrt{g(L_2)}'\sqrt{g(L_2)}$ and $\overline{V}V'$ are mappings from $\Phi \longrightarrow \Phi'$.

It is easy to see that if we restrict equation 2.7 to

$$f \in D_{g(L_2)} \equiv \{f \in \Phi \, / \, g(\lambda)\hat{f}^2(\lambda) \in L^2_{d\Gamma_2}\}$$

then it reduces to

$$\forall f \in D_{V'} \cap D_{g(L_2)} \quad \frac{d\Gamma_1}{d\Gamma_2}(L_2) = g(L_2) = \overline{V}V' \quad \text{in} \quad \Phi' \qquad (2.8)$$

The next result describes the domain of $\tilde{V}'$.

Theorem 3: $\tilde{V}$ is densely defined if and only if $L^2_{d\Gamma_1(\lambda)} \cap L^2_{d\Gamma_2(\lambda)}$ is dense in $L^2_{d\Gamma_2(\lambda)}$.

Proof: From equation 2.2 it is readily seen that

$$f \in D_{\tilde{V}} \Leftrightarrow \hat{f}^2(\lambda) \in L^2_{d\Gamma_1(\lambda)} \cap L^2_{d\Gamma_2(\lambda)}$$

Then use the fact that $f \longrightarrow \hat{f}^2$ is an isometry between H and $L^2_{d\Gamma_2(\lambda)}$.

This work is based on the following result.

Theorem 4: Assume that

• V admits closure in Φ'

• Γ_1 is Abs- $d\Gamma_2(\lambda)$

• $\overline{V}^{-1}$ exists

• $\overline{V} : \Phi \longrightarrow \Phi$ is a bounded operator

then

$$g(\lambda)\varphi(\lambda) - y(\lambda) = (V' - 1)y(\lambda) \quad \text{in} \quad \Phi'.$$

Proof: Notice that conditions of Theorem 2 hold and so it follows that

$$\sqrt{g(L_2)}'\sqrt{g(L_2)} = \overline{V}V' \quad \text{in} \quad \Phi' . \qquad (2.9)$$

By the above condition we have that $\sqrt{g(L_2)}'\sqrt{g(L_2)}f \in \Phi$ if $f \in D_{V'} \subset \Phi$. However since it is assumed that $\overline{V}^{-1}$ exists, then equation 2.8 yields

$$\overline{V}^{-1}\left(\sqrt{g(L_2)}\right)'\sqrt{g(L_2)} = V' \quad \text{in} \quad \Phi' \qquad (2.10)$$

In order to proceed further we need to extend the operator V' to Φ'. For this observe that since $\overline{V} : \Phi \longrightarrow \Phi$ is a bounded operator, $\overline{V'} = V'$ is a bounded operator in Φ'. Hence V' is defined for all elements in Φ', and in particular for $y(\lambda)$, thus

$$V^{-1}\sqrt{g(L_2)}'\sqrt{g(L_2)}y(\lambda) = V'y(\lambda).$$

We now need to compute $\sqrt{g(L_2)}'\sqrt{g(L_2)}y(\lambda)$. Let $f \in D_{V'} \subset \Phi$ then

$$
\begin{aligned}
< f, \sqrt{g(L_2)}'\sqrt{g(L_2)}y(\lambda) >_{\Phi\times\Phi'} &= < \sqrt{g(L_2)}f, \sqrt{g(L_2)}y(\lambda) >_{\Phi\times\Phi'} \\
&= < \sqrt{g(L_2)}'\sqrt{g(L_2)}f, y(\lambda) >_{\Phi\times\Phi'} \\
&= \overline{\sqrt{g(\lambda)}\sqrt{g(\lambda)}\hat{f}^2(\lambda)} \\
&= g(\lambda)\hat{f}^2(\lambda) \\
&= < f, g(\lambda)y(\lambda) >
\end{aligned}
$$

where we have used the fact that $\sqrt{g(L_2)}'\sqrt{g(L_2)}f = \overline{V}V'f \in \Phi$. Hence

$$\sqrt{g(L_2)}'\sqrt{g(L_2)}y(\lambda) = g(\lambda)y(\lambda) \quad \text{in} \quad \Phi' \ d\Gamma_2 a.e.$$

where $g(\lambda) \equiv \frac{d\Gamma_1}{d\Gamma_2}(\lambda)$ is a real function. Hence we have

$$g(\lambda)\overline{V}^{-1}y(\lambda) = V'y(\lambda).$$

Since by definition we have $\overline{V}^{-1}y(\lambda) = \varphi(\lambda)$ we obtain

$$g(\lambda)\varphi(\lambda) - y(\lambda) = (V' - 1)y(\lambda) \quad \text{in} \quad \Phi'.$$

We easily deduce the following result:

Corollary 1: Let conditions of Theorem 4 hold then

$$g(\lambda)\varphi(\lambda) - y(\lambda) \xrightarrow{\Phi'} 0 \quad \Leftrightarrow \quad (V' - 1)y(\lambda) \xrightarrow{\Phi'} 0$$

Corollary 2: Let conditions of Theorem 4 hold and $(V' - 1)y(\lambda) \xrightarrow{\Phi'} 0 \quad \lambda \to \infty$ then

$$g(\lambda) \sim 1 \quad \text{as} \quad \lambda \to \infty \Leftrightarrow \varphi(\lambda) - y(\lambda) \xrightarrow{\Phi'} 0 \quad \text{as} \quad \lambda \to \infty.$$

Proof: By hypothesis and Corollary 1 we have $\forall f \in \Phi$

$$g(\lambda)\hat{f}^1(\lambda) - \hat{f}^2(\lambda) \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Thus if $g(\lambda) \to 1$ then $\hat{f}^1(\lambda) - \hat{f}^2(\lambda) \to 0$ which means that $\varphi(\lambda) - y(\lambda) \xrightarrow{\Phi'} 0$ as $\lambda \to \infty$. Conversely $\hat{f}^1(\lambda) - \hat{f}^2(\lambda) \to 0$ together with $y(\lambda) - g(\lambda)\varphi(\lambda) \xrightarrow{\Phi'} 0$ implies that

$$g(\lambda)\hat{f}^1(\lambda) - \hat{f}^2(\lambda) \to 0$$

i.e. $g(\lambda) \to 1$ as $\lambda \to \infty$.

Corollary 2 suggests to write $V = 1 + K$. In this case Theorem 2 would read

$$g(\lambda)\varphi(\lambda) - y(\lambda) \xrightarrow{\Phi'} 0 \quad \Leftrightarrow \quad K'y(\lambda) \xrightarrow{\Phi'} 0 \quad \text{as} \quad \lambda \to \infty.$$

The question we would like to answer now is under what condition would

$$K'y(\lambda) \xrightarrow{\Phi'} 0 \quad \text{as} \quad \lambda \to \infty.$$

First we need to observe that the above convergence holds in Φ'. Indeed by construction the function $y(\lambda)$ is in Φ' and so the operator K' originally was defined in Φ must be extended to Φ'. This is easily achieved if the operator K, i.e. V, is bounded in $\Phi \longrightarrow \Phi$.

Theorem 5: Let

- $V : \Phi \longrightarrow \Phi$ be a bounded operator.

- $K \equiv \overline{V} - 1$, be such that $\Phi \xrightarrow{L_2 K} H$ is densely defined in Φ

then

$$K'y(\lambda) \xrightarrow{\Phi'} 0 \quad \text{as} \quad \lambda \to \infty.$$

Proof: Recall that for each λ, there exists a bounded sequence $\varphi_{n,\lambda} \in D_{L_2}$ such that

$$\varphi_{n,\lambda} \in D_{L_2}, \quad \parallel \varphi_{n,\lambda} \parallel = 1, \quad \text{and} \quad \parallel L_2\varphi_{n,\lambda} - \lambda\varphi_{n,\lambda} \parallel \longrightarrow 0$$

The last condition can be written as

$$\lambda\varphi_{n,\lambda} = L_2\varphi_{n,\lambda} + \epsilon(n,\lambda)$$

where $\epsilon(n,\lambda) \to 0$ in H as $n \to \infty$. This allows us to obtain the following limit

$$
\begin{aligned}
< f, K'y(\lambda) >_{\Phi \times \Phi'} &= < Kf, y(\lambda) >_{\Phi \times \Phi'} \\
&= \lim_{n \to \infty} < Kf, \varphi_{n,\lambda} >_{\Phi \times \Phi'} \\
&= \frac{1}{\lambda} \lim_{n \to \infty} (\lambda\varphi_{n,\lambda}, Kf) \\
&= \frac{1}{\lambda} \lim_{n \to \infty} (L_2\varphi_{n,\lambda} + \epsilon(n,\lambda), Kf) \\
&= \frac{1}{\lambda} \lim_{n \to \infty} (L_2\varphi_{n,\lambda}, Kf) + \frac{1}{\lambda} \lim_{n \to \infty} (\epsilon(n,\lambda), Kf) \\
&= \frac{1}{\lambda} \lim_{n \to \infty} (\varphi_{n,\lambda}, L_2Kf) + \frac{1}{\lambda} \lim_{n \to \infty} (\epsilon(n,\lambda), Kf) \\
&\leq \frac{1}{\lambda} \parallel \varphi_{n,\lambda} \parallel \parallel L_2Kf \parallel + \frac{1}{\lambda} \lim_{n \to \infty} \parallel (\epsilon(n,\lambda) \parallel \parallel Kf \parallel
\end{aligned}
$$

So as $\lambda \to \infty$ we shall obtain $< f, K'y(\lambda) >_{\Phi \times \Phi'} \longrightarrow 0$. This last limit means that

$$K'y(\lambda) \xrightarrow{\Phi'} 0 \quad \text{as} \quad \lambda \to \infty.$$

Recall that in order for the conclusion to hold we need L_2K to be at least densely defined in Φ.

Remark: The condition $V : \Phi \longrightarrow \Phi$ bounded can be replaced by densely defined. This forces us to use Baire's Theorem to obtain the density of $\Phi \cap D_V \cap D_{L_2K}$ in Φ.

Theorem 6: Let the conditions of Theorem 2 hold, and

- $V : \Phi \longrightarrow \Phi$ be a bounded operator

- $(g(L_2) - 1)^{-1}K$ be a bounded operator in Φ

then

$$(g(\lambda) - 1)y(\lambda) \xrightarrow{\Phi'} 0 \quad \Rightarrow \quad K'y(\lambda) \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Proof:

$$
\begin{aligned}
< f, K'y(\lambda) >_{\Phi \times \Phi'} &= < Kf, y(\lambda) >_{\Phi \times \Phi'} \\
&= \widehat{Kf}^2(\lambda) \\
&= (g(\lambda) - 1)(g(\lambda) - 1)^{-1}\widehat{Kf}^2(\lambda) \quad \cdot \\
&= (g(\lambda) - 1)\{(g(L_2)-1)^{-1}\widehat{Kf}^2\} \\
&= (g(\lambda) - 1) < (g(L_2) - 1)^{-1}Kf), y(\lambda) >_{\Phi \times \Phi'} \\
&= < (g(L_2) - 1)^{-1}Kf), (g(\lambda) - 1)y(\lambda) >_{\Phi \times \Phi'}
\end{aligned}
$$

Since the $[g(\lambda) - 1]y(\lambda) \xrightarrow{\Phi'} 0$ we obtain $< f, K'y(\lambda) >_{\Phi \times \Phi'} \to 0 \quad \forall f \in \Phi$ and so $K'y(\lambda) \to 0$ as $\lambda \to \infty$.

Corollary 3: Assume that conditions of Theorem 4, hold and

- $y(\lambda)$ are bounded functionals for large λ

- $(g(L_2) - 1)^{-1}K$ be a bounded operator in Φ

then

$$g(\lambda) - 1 \xrightarrow{\lambda \to \infty} 0 \quad \Rightarrow \quad y(\lambda) - \varphi(\lambda) \xrightarrow{\Phi'} 0 \quad \text{as} \quad \lambda \to \infty$$

Proof: It suffices to see that $(g(\lambda) - 1)y(\lambda) \xrightarrow{\Phi'} 0$, and since Theorem 6, is applicable

$$K'y(\lambda) \to 0 \quad \text{as} \quad \lambda \to \infty.$$

From Theorem 4, we deduce that

$$g(\lambda)\varphi(\lambda) - y(\lambda) \xrightarrow{\Phi'} 0.$$

It remains to see that since $g(\lambda) \asymp 1$ as $\lambda \to \infty \Rightarrow \varphi(\lambda) \xrightarrow{\Phi'} y(\lambda)$ as $\lambda \to \infty.$

3 EXAMPLES

Below we shall consider two simple examples to illustrate the above results.

Let L_1 and L_2 be two self-adjoint differential operators in $L^2[0, \infty)$ defined by

$$\begin{cases} L_1 f \equiv -f''(x) + q(x)f(x) \\ nf(0) - f'(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} L_2 f \equiv -f''(x) \\ nf(0) - f'(0) = 0. \end{cases}$$

where $|n| < \infty$. Let the eigenfunctionals associated with L_1 and L_2 be defined by

$$\begin{cases} L_1\varphi(x, \lambda) \equiv \lambda\varphi(x, \lambda) \\ \varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = n \end{cases} \quad \text{and} \quad \begin{cases} L_2 y(x, \lambda) = \lambda y(x, \lambda) \\ y(0, \lambda) = 1, \ y'(0, \lambda) = n \end{cases}$$

where $y(x, \lambda) = \cos(\sqrt{\lambda}x) + n\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$. It is clear that

$$\varphi(x, \lambda) = y(x, \lambda) + \int_0^x \frac{\sin(\sqrt{\lambda}(x - t))}{\sqrt{\lambda}}q(t)\varphi(t, \lambda)dt.$$

By the Riemman-Lebesgue theorem we have

$$\varphi(x, \lambda) - y(x, \lambda) \longrightarrow 0 \ as \ \lambda \longrightarrow \infty.$$

It is also known that the following representation holds

$$\varphi(x, \lambda) = y(x, \lambda) + \int_0^x K(x, t)y(t, \lambda)dt.$$

Then formally

$$(V' - 1)y(x, \lambda) = \int_x^\infty K(t, x)y(t, \lambda)dt$$

Therefore if $(V' - 1)y(x, \lambda) \xrightarrow{\Phi'} 0$ then

$$\frac{d\Gamma_1(\lambda)}{d\lambda} \asymp \frac{1}{\pi}\frac{\sqrt{\lambda}}{\lambda + n^2} \quad \text{as} \quad \lambda \to \infty.$$

Remark: It is known that if $q'(x) \in L^{1,loc}[0, \infty)$ then for each fixed x $K_{tt}(x, t) \in L^{1,loc}[0, \infty)$ and hence $L_2 K$ is densely defined. Therefore Theorem 5 is applicable.

The next example deals with the generalized Sturm Liouville operator. Let

$$\begin{cases} L_1 f \equiv -\frac{1}{w(x)}f''(x) + q(x)f(x) \\ f'(0) = 0. \end{cases} \quad \text{and} \quad \begin{cases} L_2 f \equiv \frac{-1}{x^\alpha}f'(x) \\ f'(0) = 0. \end{cases}$$

where $w(x) \asymp x^\alpha$ as $x \to 0$ and $\alpha > 0$. In this case the operator L_2 corresponds to a string whose length and mass are infinite, and is known to be self-adjoint in the space $L^2_{x^\alpha dx}$, see [5, p. 151] and [9].

We shall see that the behaviour of $w(x) \to 0$ dictates the behaviour of the spectral function at infinity. Although this result is known, see [6], we shall provide a different treatment as it is stated in [7]. For simplicity let the eigenfunctionals associated with L_1 and L_2 be defined by

$$\begin{cases} L_1\varphi(x,\lambda) \equiv \lambda\varphi(x,\lambda) \\ \varphi(0,\lambda) = 1, \ \varphi'(0,\lambda) = 0 \end{cases} \quad \text{and} \quad \begin{cases} L_2 y(x,\lambda) = \lambda y(x,\lambda) \\ y(0,\lambda) = 1, \ y'(0,\lambda) = 0. \end{cases}$$

It is clear that

$$\varphi(x,\lambda) = y(x,\lambda) + \int_0^x R(x,t,\lambda)q(t)\varphi(t,\lambda)dt.$$

where $R(x,t,\lambda)$ is the Greens' function and it is shown, by the semi-classical approximation, see [8], that $R(x,t,\lambda) \longrightarrow 0$ as $\lambda \to \infty$. Therefore we have that $\varphi(x,\lambda) - y(x,\lambda) \longrightarrow 0$ as $\lambda \to \infty$. The solution $y(x,\lambda)$ are known explicitly,

$$y(x,\lambda) = \sqrt{x}AJ_{-\nu}\left(\left(\frac{2\sqrt{\lambda}}{\alpha+2}\right)x^{\frac{\alpha+2}{2}}\right).$$

where $\nu = \frac{1}{\alpha+2}$ and $A = \{\frac{2\sqrt{\lambda}}{\alpha+2}\}^{\frac{1}{\alpha+2}}\frac{1}{\Gamma(1-\nu)}$.

Therefore provided $(V'-1)y(x,\lambda) \xrightarrow{\Phi'} 0$, we shall have

$$\Gamma_1(\lambda) \asymp \Gamma_2(\lambda) \quad \text{as} \quad \lambda \to \infty.$$

where, see [3], $\Gamma_2(\lambda) = c\lambda^{\frac{\alpha+1}{\alpha+2}}$ for $\lambda > 0$.

REFERENCES

[1] BEREZANSKI, Ju.M., Expansions in Eigenfunctions of Selfadjoint Operators, Amer. Math. Transl. Mono, Vol. 17, 1968.

[2] GELFAND, I.M. and SHILOV, G. *Generalized Functions*, Academic Press, 1963.

[3] BOUMENIR, A., A comparison theorem for self-adjoint operators, *Proc. Amer. Math. Soc,* 11, 1 (1991), 161-175.

[4] DIEUDONNEE, J., *Treatise in Analysis*, Vol. 2, Academic Press, 1976.

[5] DYM, H. and McKEAN, H.P., *Gaussian Processes, Function Theoritic and the Inverse Spectral Problem*, Academic Press, 1976

[6] BENNEWITZ, C.S., Spectral asymptotics for Sturm-Liouville equations, *Proc. London. Math. Soc.* 3, **59** (1989) 294-338.

[7] KAC, I.S., Power-Asymptotic estimates for spectral functions of generalized boundary value problems of second order, *Sov. Math. Dokl.*, Vol. 13 (1972), No. 2, 453-455.

[8] NIKIFOROV, A. and OUVAROV, V., *Fonctions speciales de la physique, Mathematique*, M.I.R Publishers, 1983, p. 178.

[9] KREIN, M.G. and KAC, I.S., Spectral function of the string, *Amer. Math. Soc. Transl.*, 2, Vol. 103 (1970), 19-103.

Special Issue on
Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning

- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management

- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk