Monotonically Computable Real Numbers

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Abstract

A real number x is called *k-monotonically computable* (k-mc), for constant $k > 0$, if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x such that the convergence is k-monotonic in the sense that $k \cdot |x - x_n| \geq |x - x_m|$ for any $m > n$ and x is *monotonically computable* (mc) if it is k-mc for some $k > 0$. x is weakly computable if there is a computable sequence $(x_s)_{s\in\mathbb{N}}$ of rational numbers converging to x such that the sum $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}|$ is finite. In this paper we show that all mc real numbers are weakly computable but the converse fails. Furthermore, we show also an infinite hierarchy of mc real numbers.

Key words: Monotonically computable real number, Weakly computable real number, Semi-computable real number, Hierarchy

1 Introduction

It is well known that classical recursion theory or computability theory studies exclusively the effectivity notions of discrete objects like natural numbers or words on some alphabet. The reason for this restriction is that people understand computation as discrete actions. For example, a (classical) Turing machine can accept only a finite string as input and outputs some finite string as well, if it halts. However, the effectiveness of non-discrete objects were also discussed in the very beginning of computability theory. Alan Turing, e.g., defined the notion of computable real numbers in his famous paper [17] about "Turing machines". According to his definition, a computable real number can be described intuitively as one for which we can effectively generate as long a decimal expansion as we wish. Of course, the decimal expansion is only

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one of various possible classical definitions of the real number. A real number can also be defined by a Cauchy sequence of rational numbers, by a binary expansion, by a Dedekind cut, by a sequence nested rational intervals and so on. It is interesting to ask whether we get the same notion of computable real number by the "effectivizations" of different classical approaches used to define real numbers. This question was first mentioned by E. Specker [16], with the restriction that all pertinent functions are primitive recursive. Under this restriction they are not equivalent. However, R. M. Robinson [12] and H. G. Rice [11] have shown that they are equivalent if general computable functions are allowed.

In effective analysis, a computable real number is defined typically by a computable fast-converging Cauchy sequence. Namely, a real number x is *computable* if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x effectively in the sense that $|x - x_n| < 2^{-(n+1)}$ for any $n \in \mathbb{N}$. Here the effectivity of the convergence is essential, because Specker [16] shows that the real number $x_A := \Sigma_{i\in A}2^{-(i+1)}$ is not computable, if $A \subseteq \mathbb{N}$ is a non-recursive r.e. set, although it is a limit of some computable increasing sequence of rational numbers. Roughly speaking, the sequence $(x_n)_{n\in\mathbb{N}}$ converges effectively to x means that we can effectively find as close an approximation x_n of x as we wish. Therefore, we will call the sequence $(x_n)_{n\in\mathbb{N}}$ an effective approximation of x, if it converges effectively to x. Especially, from an effective approximation $(x_n)_{n\in\mathbb{N}}$ of x, we can define a new computable sequence $(y_n)_{n\in\mathbb{N}}$ by $y_n := x_n - 2^{-n}$ which converges also to x such that $|x - y_n| \geq |x - y_m|$ for any $m \geq n$. That is, the later element of $(y_n)_{n\in\mathbb{N}}$ is always a better approximation to x. We will call a sequence $(z_n)_{n\in\mathbb{N}}$ properly monotonically convergent to x if $x = \lim_{n\to\infty} z_n$ and $|x - z_n| \geq |x - z_m|$ holds for all $m \geq n$. A real number x is properly monotonically computable if there is a computable sequence of rational numbers which converges to it properly monotonically. Thus, every computable real number is also properly monotonically computable.

Obviously, any monotone sequence converges properly monotonically, but not vice versa. We call a real number x *left* (*right*) *computable* if there is an increasing (decreasing) computable sequence of rational numbers which converges to x. Left and right computable real numbers are called *semi-computable*. Namely, x is semi-computable if and only if there is a computable monotone sequence of rational numbers which converges to it. Thus all semi-computable real numbers are properly monotonically computable too. Although a properly monotonically convergent sequence is not necessarily monotone, we can show that (see Proposition 3.2) properly monotonically computable real numbers are semi-computable. By the observation of Specker above, the set of semi-computable real numbers, or equivalently of properly monotonically computable real numbers, is a proper superset of the computable real number set, since x_A is not computable but left computable if A is a non-recursive r.e. set. The left computable real numbers are also called computably enumerable by some authors (see [3,2]). To some extent this class of real numbers plays a similar role as recursive enumerable sets in recursion theory. Thus it is also widely discussed in the literature ([13,14,5,3,6,1]).

At a first glance, it seems difficult to understand why a properly monotonically computable real number can be non-computable. For a properly monotonically computable x, there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which approximates to x always better and better. The problem here is that, although x_{n+1} is a better approximation than x_n, the improvement can be very small. This improvement can even be smaller and smaller with the increasing of n. Therefore we cannot decide effectively how accurate our current approximation to x will be. However, an effective estimation of approximation errors is possible, if we know in advance that there is a lower bound for the improvements. More precisely, we can show (see Proposition 3.2) that x is computable if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x and satisfies, for some $0 < k < 1$, the condition

$$\forall n, m \in \mathbb{N}\, (m > n \Longrightarrow k \cdot |x - x_n| \geq |x - x_m|) \tag{1}$$

Here k is a lower bound of the improvement of the approximation.

More generally, we will discuss in this paper also the computable sequences $(x_n)_{n\in\mathbb{N}}$ of rational numbers which satisfy the condition (1) for some $k \geq 1$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x *k-monotonically*, if it converges to x and satisfies condition (1). The sequence $(x_n)_{n\in\mathbb{N}}$ converges monotonically if it converges k-monotonically for some $k > 0$. Notice that, the condition (1) alone does not guarantee the convergence of the sequence. A real number x is *k-monotonically computable* (k-mc, in short) if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x k-monotonically and x is *monotonically computable* (mc, in short) if it is k-mc for some k. Thus properly monotonically computable real numbers above are simply the 1-mc real numbers and k-mc real numbers are computable, if $0 < k < 1$. It is worth notice that the monotone convergence and monotonicity of the sequence are different. In general, any (bounded) monotone sequence converges (1-)monotonically, but a monotonically convergent sequence is not necessarily monotone.

k-monotone convergence was also discussed by C. Calude and P. Hertling in [4]. They show that, if a computable sequence $(x_n)_{n\in\mathbb{N}}$ converges monotonically to a computable real x, then it converges to x computably in the sense that there is a recursive function $e : \mathbb{N} \to \mathbb{N}$ such that $|x - x_m| \leq 2^{-n}$ for any $m \geq e(n)$. That is, any computable sequence which converges monotonically to some computable real number converges also very "fast", although there are other sequences which converge to x slowly (see [4]). We are more interested in the monotonically convergent computable sequences which converge to some non-computable, or even non-semi-computable real numbers. The first natural question is, whether there is a real number x which is not monotonically computable but it is a limit of some computable sequence of rational numbers (i.e., so-called recursively approximable real numbers, or r.a. real numbers in short). We will answer this question affirmatively in section 3. In fact we show a stronger result that the class of all monotonically computable real numbers is properly contained in the class of weakly computable real numbers. Here x is *weakly computable* if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x *weakly effectively*, that is, the sum $\Sigma_{i\in\mathbb{N}}|x_i - x_{i+1}|$ is finite or, equivalently, there are semi-computable real numbers y, z such that $x = y - z$ (see [1,19]).

By definition, if $k_1 \leq k_2$ and x is k_1-mc, then it is k_2-mc too. Then it is also quite natural to ask, whether the classes of k_1-mc and k_2-mc real numbers are different if $k_1 \neq k_2$. Namely, whether the k-mc real number sets form a proper hierarchy of all mc real numbers. A partially positive answer will be shown in section 4. That is, for any k, there is a $k_1 > k$ such that the set of all k-mc real numbers is a proper subset of the set of all k_1-mc real numbers. Thus, although we are not sure whether k_1-mc and k_2-mc are different for any pair of different $k_1, k_2 \geq 1$, there is still an infinite hierarchy of mc real numbers [1].

2 Preliminaries

In this section we recall some notions and notations which are useful for later sections. We assume only very basic notions and results from classical computability theory and CCA (Computability and Complexity in Analysis). The systematical explanation of these topics can be found in [15] and [8,9,18].

Let $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ be sets of the natural, rational and real numbers, respectively. For any sets A and B, $f :\subseteq A \to B$ is a partial function with $\text{dom}(f) \subseteq A$ and $\text{range}(f) \subseteq B$. If f is a total function, i.e., $\text{dom}(f) = A$, then we denote this by $f : A \to B$. The computability notions like computable (or recursive) function, recursive and r.e. (recursively enumerable) set, etc., on $\mathbb{N}$ are well defined and developed in classical computability theory. Let $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ be a pairing function defined by $\langle m, n \rangle := (n + m)(n + m + 1)/2 + m$ and $\pi_1, \pi_2 : \mathbb{N} \to \mathbb{N}$ be its two inverse functions, i.e., $\pi_1 \langle n, m \rangle = n$ and $\pi_2 \langle n, m \rangle = m$ for any $n, m \in \mathbb{N}$. Obviously $\langle \cdot, \cdot \rangle$, π_1 and π_2 are computable. Then we can define a coding $\sigma : \mathbb{N} \to \mathbb{Q}$ of rational numbers using $\mathbb{N}$ by $\sigma(\langle n, m \rangle) := n/(m + 1)$. By this coding, the computability notions on $\mathbb{N}$ can be easily transferred to

[1] A completely positive answer is recently obtained by the first and second author and is reported in MFCS'01 ([10]). Namely, for any $k_2 > k_1 > 1$, the classes of k_2-mc and k_1-mc real numbers are different.

that of $\mathbb{Q}$. For example, a function $f : \mathbb{Q} \to \mathbb{Q}$ is computable if there is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $f(\sigma(n)) = \sigma(g(n))$ for any $n \in \mathbb{N}$, and $A \subseteq \mathbb{Q}$ is recursive if $\{n \in \mathbb{N} : \sigma(n) \in A\}$ is recursive, and so on. More directly, we can also define the Turing machines in such a way that their inputs and outputs can be rational numbers as well as natural numbers. Then the corresponding computability notions for $\mathbb{Q}$ can be developed directly from the Turing machines as usual.

Computable sequences of rational numbers play a very important role in this paper. We can simply define such sequences as computable total function from natural numbers $\mathbb{N}$ to rational numbers $\mathbb{Q}$. Namely, $(x_n)_{n\in\mathbb{N}}$ is a computable sequence of rational numbers if there is a computable total function $f : \mathbb{N} \to \mathbb{Q}$ such that $x_n = f(n)$ for all n. From time to time, we would like to diagonalize against all computable sequences of rational numbers or some subset of such sequences. In this case, an effective enumeration of all computable sequences of rational numbers would be very useful. Unfortunately, the computable total functions are not effectively enumerable and hence there is also no such effective enumeration of the sequences. Instead we consider the effective enumeration $(\varphi_e)_{e\in\mathbb{N}}$ of all computable partial functions $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$. Of course, all computable sequences of rational numbers (or total computable functions $f : \mathbb{N} \to \mathbb{Q}$, more precisely) appear in this enumeration. Thus it suffices to implement our diagonalization against this enumeration. Concretely, the enumeration $(\varphi_e)_{e\in\mathbb{N}}$ can be defined from some effective enumeration $(M_e)_{e\in\mathbb{N}}$ of Turing machines. Namely, $\varphi_e :\subseteq \mathbb{N} \to \mathbb{Q}$ is the function computed by the e-th Turing machine M_e. Furthermore, let $\varphi_{e,s}$ be the approximation of φ_e computed by M_e until the stage s. Then $(\varphi_{e,s})_{e,s\in\mathbb{N}}$ is an uniformly effective approximation of $(\varphi_e)_{e\in\mathbb{N}}$ which satisfies the following conditions:

$$\begin{aligned}
&\{(e, s, n, r) : \varphi_{e,s}(n) \downarrow= r\} \text{ is decidable, and} \\
&\varphi_{e,s}(n) \downarrow= r \Longrightarrow \forall\, t \geq s\, (\varphi_{e,t}(n) \downarrow= r = \varphi_e(n)),
\end{aligned}$$

where $\varphi_{e,s}(n) \downarrow= r$ means that $\varphi_{e,s}(n)$ is defined and equal to r.

In the last section we have mentioned the notions of computable, left computable, right computable, semi-computable, weakly computable and recursively approximable real numbers. The corresponding classes of these real numbers will be denoted by $\mathbf{C}_e, \mathbf{C}_{lc}, \mathbf{C}_{rc}, \mathbf{C}_{sc}, \mathbf{C}_{wc}$ and $\mathbf{C}_{ra}$, respectively. Some important properties about these classes are summarized in the following theorem.

Theorem 2.1 (Weihrauch and Zheng [19])

(1) The classes $\mathbf{C}_e, \mathbf{C}_{lc}, \mathbf{C}_{rc}, \mathbf{C}_{sc}, \mathbf{C}_{wc}$ and $\mathbf{C}_{ra}$ are all different;
(2) $x \in \mathbf{C}_{lc}$ iff $-x \in \mathbf{C}_{rc}$; $\mathbf{C}_e = \mathbf{C}_{lc} \cap \mathbf{C}_{rc}$ and $\mathbf{C}_{sc} = \mathbf{C}_{lc} \cup \mathbf{C}_{rc}$;
(3) $x \in \mathbf{C}_{wc}$ iff there are $y, z \in \mathbf{C}_{lc}$ such that $x = y - z$. Furthermore, $\mathbf{C}_{wc}$ is

the arithmetic closure of $\mathbf{C}_{lc}$; and

(4) *The classes $\mathbf{C}_e, \mathbf{C}_{wc}$ and $\mathbf{C}_{ra}$ are algebraic fields. That is, they are closed under the arithmetic operations $+, -, \times$ and $\div$.*

Notice that, for any class $\mathbf{C} \subseteq \mathbb{R}$ discussed in this paper, $x \in \mathbf{C}$ if and only if $x \pm n \in \mathbf{C}$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Therefore we can assume, without loss of generality, that any real number and corresponding sequence of rational numbers discussed in this paper is usually in the interval $[0; 1]$ except for being expressed otherwise. For any left computable real number x, there is an increasing computable sequence of rational numbers converging to it by the definition. In fact, a nondecreasing computable sequence $(x_s)_{s\in\mathbb{N}}$ suffices too, because we can define an increasing computable sequence $(y_s)_{s\in\mathbb{N}}$ by $y_s := x_s - 2^{-s}$ which converges obviously also to x. The situation for right computable real numbers is similar.

At last, we fix some further notations: For any alphabet Σ, let Σ^* and Σ^ω be the sets of all finite strings and infinite sequences of Σ, respectively. The set of all strings $w \in \Sigma^*$ of length n is denoted by Σ^n. For $u, v \in \Sigma^*$, denote by uv the concatenation of v after u. If $w \in \Sigma^* \cup \Sigma^\omega$, then $w[n]$ denotes its n-th element. Thus, $w = w[0]w[1]\cdots w[n-1]$, if $|w|$, the length of w, is n, and $w = w[0]w[1]w[2]\cdots$, if $|w| = \infty$. Obviously, $w[n]$ is defined only for $n < |w|$. We will say also that $w[n]$ is undefined and denote by $w[n] = \uparrow$, if $n \geq |w|$. The unique string of length 0 is always denoted by λ (so-called empty string). For any finite string $w \in \{0; 1\}^*$, and number $n \leq |w|$, the restriction $w \upharpoonright n$ is defined by $(w \upharpoonright n)[i] := w[i]$ if $i < n$ and $(w \upharpoonright n)[i] := \uparrow$, otherwise. The generalized restriction $w \lceil n$ is defined by $w \lceil n := w \upharpoonright (n+1)$, if $n < |w|$ and $w \lceil n := w1^{n+1-|w|}$, otherwise. Then the length $|w \upharpoonright n| = n$ and $|w \lceil n| = n + 1$. For $u, v \in \Sigma^* \cup \Sigma^\omega$, if $u = v \upharpoonright n$ for some $n \leq |v|$, then we call u an initial segment of v and denote $u \sqsubseteq v$. We denote also $u \neq v \ \& \ u \sqsubseteq v$ by $u \sqsubset v$. If Σ is linearly ordered by $<$, then we can define a length-lexicographical ordering $<_L$ on $\Sigma^* \cup \Sigma^\omega$ by $u <_L v$ if and only if $u \sqsubset v$ or $\exists n\, (u \upharpoonright n = v \upharpoonright n \ \& \ u[n] < v[n])$. $u \leq_L v$ is defined by $u <_L v$ or $u = v$.

3 Monotone Computability vs Weakly Computability

In this section we will discuss the relationships between monotonically computable real numbers and weakly computable real numbers. It will be shown that the class of all monotonically computable real numbers is a proper subset of the weakly computable real number set. At first we define our notions precisely.

Definition 3.1 Let $k \in \mathbb{R}$, $k > 0$ and $x \in \mathbb{R}$.

(1) A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers *converges k-monotonically* to x if it converges to x and satisfies the condition (1) on page 3 and $(x_n)_{n\in\mathbb{N}}$ *converges monotonically* to x if, for some $k > 0$, it converges to x k-monotonically.
(2) The real number x is called *(k-)monotonically computable* (k-mc and mc, respectively, in short) if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x (k-)monotonically. The classes of all k-mc and mc real numbers are denoted by $\mathbf{C}^k_{mc}$ and $\mathbf{C}_{mc}$, respectively.

About k-mc and mc real numbers we have at first following easy observations.

Proposition 3.2 *Let x be a real number.*

(1) If $k_1 \leq k_2$ and x is a k_1-mc real number, then it is k_2-mc too;
(2) For any $k < 1$, x is k-mc iff it is computable, namely, $\mathbf{C}^k_{mc} = \mathbf{C}_e$;
(3) x is 1-mc iff it is semi-computable. Thus, $\mathbf{C}_{sc} = \mathbf{C}^1_{mc}$.

Proof. 1. This follows immediately from the definition.

2. Let $k < 1$ and x a k-mc real number. There is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x and satisfies the condition (1). Assume w.l.o.g. that $|x_0 - x| \leq 1$. Since $0 < k < 1$, there is a natural number $d > 0$ such that $k^d < 2^{-2}$. Define a computable sequence $(y_n)_{n\in\mathbb{N}}$ of rational numbers by $y_n := x_{d\cdot n}$. Then we have $|y_n - x| = |x_{d\cdot n} - x| \leq k \cdot |x_{d\cdot n-1} - x| \leq \cdots \leq k^{d\cdot n} \cdot |x_0 - x| \leq 2^{-2n} \leq 2^{-(n+1)}$. That is, the computable sequence $(y_n)_{n\in\mathbb{N}}$ converges to x effectively and hence x is a computable real number.

On the other hand, suppose that x is a computable real number and the computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers converges to x effectively, i.e., $|x - x_n| \leq 2^{-(n+1)}$ for any $n \in \mathbb{N}$. Let $y_n := x_{tn} - 2^{-tn}$ for some $t \in \mathbb{N}$ with $t \geq 2$. Then we have $x - 2^{-tn} - 2^{-(tn+1)} \leq y_n \leq x - 2^{-(tn+1)}$, and hence $2^{-(tn+1)} \leq |y_n - x| \leq 2^{-tn} + 2^{-(tn+1)}$ for any $n \in \mathbb{N}$. This implies immediately that, for any $m > n$, $|x - y_m| \leq 2^{-tm} + 2^{-(tm+1)} \leq 2^{-tn-t} + 2^{-(tn+t+1)} = 2^{-(tn+1)} \cdot (3 \cdot 2^{-t}) \leq (3 \cdot 2^{-t}) \cdot |x - y_n|$. That is, the sequence $(y_n)_{n\in\mathbb{N}}$ converges to x k_1-monotonically for $k_1 := 3 \cdot 2^{-t} < 1$. For any $0 < k < 1$, let $t \in \mathbb{N}$ large enough such that $k_1 := 3 \cdot 2^{-t} < k$. Then x is also k-monotonically computable.

3. The inclusion $\mathbf{C}_{sc} \subseteq \mathbf{C}^1_{mc}$ is trivial since any monotone sequence converges always 1-monotonically. We prove now the nontrivial direction.

Let x be an 1-mc real number. Then there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x and satisfies the condition (1). Notice that, if $x_n < x_{n+1}$, then $x_n < x$, otherwise, $|x - x_n| = x_n - x < x_{n+1} - x = |x - x_{n+1}|$ which contradicts the condition (1). Similarly $x_n > x$ holds if $x_n > x_{n+1}$ for any n.

If there are infinitely many n such that $x_n < x_{n+1}$, then we can choose an infinite subsequence $(x_{s(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(\forall n)(x_{s(n)} < x_{s(n)+1})$. Since $x_{s(n)} < x$ for all n, we can define a nondecreasing computable sequence $(y_n)_{n\in\mathbb{N}}$ by $y_n := \max\{x_{s(m)} : m \leq n\}$ which converges obviously also to x, hence it is left computable.

Otherwise, suppose that there are at most finitely many n such that $x_n < x_{n+1}$ holds. This means that $x_n \geq x_{n+1}$ holds for almost all n. If $x_n = x_{n+1}$ holds for almost all n, then $x = \lim_{n\to\infty} x_n$ is a rational number which is, of course, semi-computable. Otherwise, there are infinitely many n such that $x_n > x_{n+1}$. In this case, we can show similarly to the above case that x is right computable. Therefore, in both cases x is a semi-computable real number. □

Notice that the proof of the Proposition 3.2.3 is not uniform in the sense that we don't know whether x is left or right computable, if we know only that a computable sequence converges 1-monotonically to x. In fact we can even show that it is not effectively decidable from an 1-monotonically convergent computable sequence to determine whether its limit is left or right computable.

Now we will discuss the relationships between monotone computability and weak computability. We show at first that any monotonically computable real number is in fact weakly computable. Since $\mathbf{C}^k_{mc} \subseteq \mathbf{C}_{sc}$, for $k \leq 1$, we consider only the case $k > 1$.

Let's introduce a few new notations at first. Given a computable sequence $(x_i)_{i\in\mathbb{N}}$ of rational numbers, we will denote by V the set of all pairs (x_i, x_j) with $i < j$ and by V_t the set $\{(x_i, x_j) \in V : i < j \leq t\}$. Namely, V consists of all successor pairs of the sequence $(x_i)_{i\in\mathbb{N}}$ and V_t is its initial part up to x_t. For given $(x_i, x_j) \in V$ and $k > 1$ let

$$\overrightarrow{I}_k(x_i, x_j) := \begin{cases} [x_i;\ x_i + (x_j - x_i)/(k+1)] & \text{if } x_i < x_j, \\ \emptyset & \text{otherwise;} \end{cases}$$

$$\overleftarrow{I}_k(x_i, x_j) := \begin{cases} [x_i - (x_i - x_j)/(k+1);\ x_i] & \text{if } x_i > x_j, \\ \emptyset & \text{otherwise;} \end{cases}$$

$$I_k(x_i, x_{i+1}) := \overrightarrow{I}_k(x_i, x_{i+1}) \cup \overleftarrow{I}_k(x_i, x_{i+1});$$

$$J_k(x_i;\ x_j) := \begin{cases} [x_i; x_i + (x_i - x_j)/(k-1)] & \text{if } x_i \geq x_j, \\ [x_i - (x_j - x_i)/(k-1);\ x_i] & \text{otherwise ;} \end{cases}$$

and, for any $A \subseteq V$,

$$I_k(A) := \bigcup\{I_k(x,y) : (x,y) \in A\};$$
$$J_k(A) := \bigcup\{J_k(x,y) : (x,y) \in A\}.$$

Notice that, if the sequence $(x_n)_{n\in\mathbb{N}}$ converges k-monotonically to x, then $x \notin I_k(x_n, x_m) \cup J_k(x_n, x_m)$ for any $n < m$, because of condition (1). Therefore we can "speed up" the sequence by removing all redundant elements $x_s \in I_k(x_n, x_m) \cup J_k(x_n, x_m)$ for $s > m$. In this way we obtain a so-called k-reduced sequence. More precisely we have the following definition.

Definition 3.3 A k-monotonically convergent sequence $(x_i)_{i\in\mathbb{N}}$ of real numbers is called *k-reduced*, if $x_t \notin I_k(V_t) \cup J_k(V_t)$ for all $t \in \mathbb{N}$.

The following lemma follows easily from the definition.

Lemma 3.4 *A real number x is k-monotonically computable if and only if there is a k-reduced computable sequence $(x_i)_{i\in\mathbb{N}}$ of rational numbers which converges to x k-monotonically.*

Proof. We prove the non-trivial direction. Suppose that x is a k-mc real number and the computable sequence $(x_i)_{i\in\mathbb{N}}$ of rational numbers converges k-monotonically to x, i.e., the condition (1) is satisfied. Define a function $s : \mathbb{N} \to \mathbb{N}$ inductively by $s(0) := 0$ and $s(n+1) := \min\{s > s(n) : x_s \notin I_k(V_{s(n)}) \cup J_k(V_{s(n)})\}$. By condition (1), $x \notin I_k(V_t) \cup J_k(V_t)$ and hence there are infinitely many s such that $x_s \notin I_k(V_t) \cup J_k(V_t)$ for any $t \in \mathbb{N}$. That is, the function s is a well defined total function and the computable subsequence $(x_{s(i)})_{i\in\mathbb{N}}$ is k-reduced and converges to x k-monotonically too. □

Next, we would like to show that any k-reduced sequence $(x_n)_{n\in\mathbb{N}}$ converges in fact weakly effectively, namely, $\sum_{n\in\mathbb{N}} |x_n - x_{n+1}| \leq c$ for some constant c. Let's divide this sum into two parts by

$$\sum_{i\in\mathbb{N}} |x_i - x_{i+1}| = \sum_{(x,y)\in\overleftarrow{V}} (x-y) + \sum_{(x,y)\in\overrightarrow{V}} (y-x), \tag{2}$$

where the sets $\overrightarrow{V}$ and $\overleftarrow{V}$ are defined, respectively, by $\overrightarrow{V} := \bigcup_{t\in\mathbb{N}} \overrightarrow{V}_t$ and $\overleftarrow{V} := \bigcup_{t\in\mathbb{N}} \overleftarrow{V}_t$ for

$$\overrightarrow{V}_t := \{(x_i, x_{i+1}) : x_i \leq x_{i+1} \ \&\ i < t\}$$
$$\overleftarrow{V}_t := \{(x_i, x_{i+1}) : x_i > x_{i+1} \ \&\ i < t\},$$

which correspond to the increasing and decreasing, respectively, immediate successor pairs (x_i, x_{i+1}) of the sequence $(x_n)_{n\in\mathbb{N}}$. We will show that both parts in right side of (2) are finite. By the symmetry, we need only to consider the sum $\sum_{(x,y)\in\overrightarrow{V}}(y-x)$. Let μ be the Lebesgue-measure on interval $[0;1]$ and $\dot{-}$ the arithmetical difference defined by $x \dot{-} y := x - y$ if $x \geq y$, and $x \dot{-} y := 0$, otherwise. Then, we have

$$\sum_{(x,y)\in\overrightarrow{V}}(y-x) = \sum_{t\in\mathbb{N}}(x_{t+1} \dot{-} x_t) = (k+1)\sum_{t\in\mathbb{N}}\mu(\overrightarrow{I}_k (x_t, x_{t+1}))$$

where the second equality follows directly from the definition of $\overrightarrow{I}_k$. Since $\mu(I_k(\overrightarrow{V}_{t+1})) - \mu(I_k(\overrightarrow{V}_t)) = \mu(\overrightarrow{I}_k (x_t, x_{t+1})) - \mu(I_k(\overrightarrow{V}_t) \cap \overrightarrow{I}_k (x_t, x_{t+1}))$ for any $t \in \mathbb{N}$, we have furthermore that

$$\begin{aligned}
&\sum_{t\in\mathbb{N}}\mu(\overrightarrow{I}_k (x_t, x_{t+1}))\\
&= \sum_{t\in\mathbb{N}}(\mu(I_k(\overrightarrow{V}_{t+1})) - \mu(I_k(\overrightarrow{V}_t))) + \sum_{t\in\mathbb{N}}\mu(I_k(\overrightarrow{V}_t) \cap \overrightarrow{I}_k (x_t, x_{t+1}))\\
&= \mu(I_k(\overrightarrow{V})) - \mu(I_k(\overrightarrow{V}_0)) + \sum_{t\in\mathbb{N}}\mu(I_k(\overrightarrow{V}_t) \cap \overrightarrow{I}_k (x_t, x_{t+1}))\\
&\leq 1 + \sum_{t\in\mathbb{N}}\mu(I_k(\overrightarrow{V}_t) \cap \overrightarrow{I}_k (x_t, x_{t+1})),
\end{aligned}$$

since $I_k(\overrightarrow{V}) \subseteq [0;1]$ and $\overrightarrow{V}_0 = \emptyset$. From the discussions above, it is clear that, in order to show the sum $\sum_{s\in\mathbb{N}}|x_s - x_{s+1}|$ is finite, it suffices to prove that the sum $\sum_{t\in\mathbb{N}}\mu(I_k(\overrightarrow{V}_t) \cap \overrightarrow{I}_k (x_t, x_{t+1}))$ is finite, namely, the intervals $\overrightarrow{I}_k (x_t, x_{t+1})$ do not overlap too much. To this end, we prove at first the following technical lemma which asserts that, in any interval $\overrightarrow{I}_k (x_t, x_{t+1})$, there is a "not small" part which is not overlapped with any earlier intervals $\overrightarrow{I}_k (x_s, x_{s+1})$ for $s < t$.

Lemma 3.5 *Let $k \geq 2$, $(x_n)_{n\in\mathbb{N}}$ be a k-reduced computable sequence of rational numbers which converges to x k-monotonically and $t \geq 1$ such that $x_t < x_{t+1}$. For any $i \leq t$, if $I_k(\overrightarrow{V}_i) \cap I_k(x_t, x_{t+1}) \neq \emptyset$, then there are sequences $(B^i_j)_{j<m_i}$ and $(C^i_j)_{j<m_i}$ of rational intervals (for some $m_i \leq i+1$), which satisfy, for all $j < m_i$, the following conditions.*

(I) $B^i_j \cap B^i_{j_1} = \emptyset$*, if* $j_1 \neq j$*;*
(II) $C^i_j \subseteq B^i_j \subseteq I_k(x_t, x_{t+1})$*;*
(III) $I_k(\overrightarrow{V}_i) \cap I_k(x_t, x_{t+1}) \subseteq B^i \setminus C^i$*;*
(IV) $\mu(C^i) \geq \mu(B^i)/(k+1)k$*.*

where $B^i := \bigcup_{s<m_i} B^i_s$ *and* $C^i := \bigcup_{s<m_i} C^i_s$*.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a k-reduced computable sequence of rational numbers which converges k-monotonically to x, $t \geq 1$ and $x_t < x_{t+1}$. For any $i \leq t$, we will define an $m_i \in \mathbb{N}$, two finite sequences $r^i_0 < r^i_1 \cdots < r^i_{m_i-1}$ and $s^i_0 < s^i_1 < \cdots < s^i_{m_i-1}$ of natural numbers and a finite sequence $(l^i_j)_{j<m_i}$ of rational numbers and furthermore

$$B^i_j := \left[a^i_j;\ x_{r^i_j} + l^i_j\right] \text{ and } C^i_j := \left[b^i_j;\ x_{r^i_j}\right] \text{ where} \tag{3}$$

$$a^i_j := x_{r^i_j} - |x_{r^i_j} - x_{s^i_j}|/(k-1) \tag{4}$$

$$b^i_j := x_{r^i_j} - |x_{r^i_j} - x_{s^i_j}|/(k-1)(k+1). \tag{5}$$

Then the interval sequences $(B^i_j)_{j<m_i}$ and $(C^i_j)_{j<m_i}$ satisfy conditions (I) – (IV). This is achieved by choosing the sequences $(r^i_j)_{j<m_i}$, $(s^i_j)_{j<m_i}$ and $(l^i_j)_{j<m_i}$ in such a way that they satisfy, for all $j < m_i$, the following conditions

(i) $r^i_j < s^i_j$ and $x_{r^i_j} < x_{s^i_j}$;

(ii) $x_{r^i_j}, x_{r^i_j} + l^i_j \in I_k(x_t, x_{t+1})$;

(iii) $l^i_j \leq (x_{s^i_j} - x_{r^i_j})$;

(iv) $x_{r^i_j} + l^i_j < a^{i+1}_j := x_{r^i_{j+1}} - \frac{1}{k-1}(x_{s^i_{j+1}} - x_{r^i_{j+1}})$;

(v) $I_k(\overrightarrow{V}_i) \cap [x_{r^i_0} - \frac{1}{k-1}(x_{s^i_0} - x_{r^i_0});\ x_{r^i_0}] = \emptyset$, and

(vi) $I_k(\overrightarrow{V}_i) \cap I_k(x_t, x_{t+1}) \subseteq B^i \setminus C^i$.

It is easy to see that the sequences $(B^i_j)_{j<m_i}$ and $(C^i_j)_{j<m_i}$ defined by (3) satisfy the conditions (I) – (IV), if the sequences $(r^i_j)_{j<m_i}$, $(s^i_j)_{j<m_i}$ and $(l^i_j)_{j<m_i}$ satisfy all items (i) – (vi). In fact, condition (I) follows from the item (iv); condition (II) follows from item (ii) and the definitions (3) – (5). Here the second inclusion, $B^i_j \subseteq I_k(x_t, x_{t+1})$, requires that $x_t < x_{r^i_j} - |x_{r^i_j} - x_{s^i_j}|/(k-1)$, which is true because $x_t \notin J_k(x_{r^i_j}, x_{s^i_j})$, since the sequence $(x_s)_{s\in\mathbb{N}}$ is k-reduced. The condition (III) is the same as item (vi). For the condition (IV), we have the following estimations that

$$\begin{aligned}
\mu(B^i) &= \sum_{j<m_i} \left((x_{s^i_j} - x_{r^i_j})/(k-1) + l^i_j\right) \\
&\leq \sum_{j<m_i} \left((x_{s^i_j} - x_{r^i_j})/(k-1) + (x_{s^i_j} - x_{r^i_j})\right) \\
&= \frac{k}{(k-1)} \sum_{j<m_i} \left(x_{s^i_j} - x_{r^i_j}\right).
\end{aligned}$$

and hence

$$\mu(C^i) = \frac{1}{(k+1)(k-1)} \sum_{j<m_i} \left(x_{s^i_j} - x_{r^i_j}\right) \geq \frac{1}{k(k+1)}\mu(B^i).$$

That is, the condition (IV) is satisfied too.

Now we define the numbers $m_i \le i+1$ and the sequences $(r^i_j)_{j<m_i}$, $(s^i_j)_{j<m_i}$ and $(l^i_j)_{j<m_i}$ for $i \le t$ inductively as follows.

For $i = 0$, we simply let $m_0 := 0$, namely, all three sequences are empty sequences. And in general, we define always $m_i := 0$, if $I_k(\overrightarrow{V}_i) \cap I_k(x_t, x_{t+1}) = \emptyset$, for any $i \le t$.

For $0 < i < t$, suppose that m_i and the sequences $(r^i_j)_{j<m_i}$, $(s^i_j)_{j<m_i}$ and $(l^i_j)_{j<m_i}$ are defined and $I_k(\overrightarrow{V}_{i+1}) \cap I_k(x_t, x_{t+1}) \ne \emptyset$.

If $m_i = 0$, that is, (x_i, x_{i+1}) is the first pair with $x_i < x_{i+1}$ such that $I_k(x_i, x_{i+1}) \cap I_k(x_t, x_{t+1}) \ne \emptyset$. This means that $x_i \in I_k(x_t, x_{t+1})$ and $x_t \notin I_k(x_i, x_{i+1})$ since $(x_n)_{n\in\mathbb{N}}$ is k-reduced. In this case we define $m_{i+1} := 1$ and

$$\begin{aligned} &r^{i+1}_0 := i, \quad s^{i+1}_0 := i+1, \quad \text{and} \\ &l^{i+1}_0 := \min\{(x_{i+1} - x_i)/(k+1),\ x_t + (x_{t+1} - x_t)/(k+1) - x_i\}. \end{aligned}$$

This means in fact that

$$\begin{aligned} &C^{i+1}_0 := [x_i - (x_{i+1} - x_i)/(k-1)(k+1);\ x_i] \quad \text{and} \\ &B^{i+1}_0 := J_k(x_i, x_{i+1}) \cup (I_k(x_i, x_{i+1}) \cap I_k(x_t, x_{t+1})). \end{aligned}$$

Notice that $J_k(x_i, x_{i+1}) \subseteq I_k(x_t, x_{t+1})$ holds because $x_i \in I_k(x_t, x_{t+1})$ and $x_t \notin J_k(x_i, x_{i+1})$. Then it is easy to see that C^{i+1}_0 and B^{i+1}_0 satisfy the conditions (I) – (IV).

If $m_i > 0$, i.e., the sequences $(r^i_j)_{j<m_i}$, $(s^i_j)_{j<m_i}$ and $(l^i_j)_{j<m_i}$ are not empty and satisfy the conditions (i) – (vi). If $x_i > x_{i+1}$ or $I_k(x_i, x_{i+1}) \cap I_k(\overrightarrow{V}_{t+1}) \subseteq (B^i \setminus C^i)$, then we just define $m_{i+1} := m_i$, $r^{i+1}_j := r^i_j$, $s^{i+1}_j := s^i_j$ and $l^{i+1}_j := l^i_j$ for all $j < m_i$ and we are done. Otherwise we distinguish the following cases:

Case 1: There exists a $j < m_i$ such that $x_{i+1} \in [x_{r^i_j};\ x_{r^i_j} + l^i_j]$. We show at first the claim that $I_k(x_i, x_{i+1}) \cap C^i_j = \emptyset$. For the case of $x_i \ge x_{r^i_j}$, it is obvious. Suppose now that $x_i < x_{r^i_j}$. Since the sequence $(x_i)_{i\in\mathbb{N}}$ is k-reduced, we have at first $x_i \notin J_k(x_{r^i_j}, x_{s^i_j})$. That is, $x_i < a^i_j < b^i_j$ and hence $x_i = a^i_j - \sigma$ for some $\sigma > 0$. On the other hand we have

$$\begin{aligned} b^i_j - x_i &= b^i_j - a^i_j + \sigma = \frac{k}{(k-1)(k+1)}(x_{s^i_j} - x_{r^i_j}) + \sigma \\ &\ge \frac{1}{k+1}\left(\frac{k}{k-1}(x_{s^i_j} - x_{r^i_j}) + \sigma\right) = \frac{1}{k+1}\left(x_{s^i_j} - a^i_j + \sigma\right) \\ &= \frac{1}{k+1}\left(x_{s^i_j} - x_i\right) \ge \frac{1}{k+1}\left(x_{i+1} - x_i\right). \end{aligned}$$

This means that $I_k(x_i, x_{i+1}) \cap C_j^i = \emptyset$. We define in this case $m_{i+1} := m_i - j$ and let $r_0^{i+1} := (r_0^i$, if $x_{r_0^i} < x_i$; i, otherwise) and $l_0^{i+1} := b_j^i - x_{r_0^{i+1}}$. Then the sequences $r_0^{i+1} < r_j^i < r_{j+1}^i < \cdots < r_{m_i}^i$, $i+1 < s_j^i < \cdots < s_{m_i}^i$ and $l_0^{i+1}, l_j^i, \cdots, l_{m_i}^i$ will work fine for $i+1$. Here the item (vi) follows especially from the fact that $I_k(x_i, x_{i+1}) \cap C_j^i = \emptyset$.

Case 2: There exists a $j < m_i$ such that $x_{i+1} > x_{r_j^i} + l_j^i$ but no $e > j$ such that $x_{i+1} > x_{r_e^i}$. In this case we define also $m_{i+1} := m_i - j$ and simply let r_0^{i+1} be r_0^i, if $x_{r_0^i} < x_i$, and i otherwise. Furthermore let l_0^{i+1} be $x_{i+1} - x_{r_0^{i+1}}$. Then the sequences $r_0^{i+1} < r_{j+1}^i < \cdots < r_{m_i}^i$, $i+1 < s_{j+1}^i < \cdots < s_{m_i}^i$ and $l_0^{i+1}, l_{j+1}^i, \cdots, l_{m_i}^i$ will fulfill conditions (i) – (vi).

Case 3: $x_{i+1} < x_{r_0^i}$. Then we can choose the sequences $i < r_0^{i+1} < r_1^i < \cdots < r_{m_i}^i$, $i+1 < s_0^i < \cdots < s_{m_i}^i$ and $(x_{i+1} - x_i), l_0^i, l_1^i, \cdots, l_{m_i}^i$. They satisfy the conditions (i) – (vi) too.

This completes the proof of the lemma. □

Lemma 3.6 *Let $k \geq 2$. If $(x_n)_{n\in\mathbb{N}}$ is a k-reduced computable sequence of rational numbers which converges k-monotonically to x, then $\sum_{i\in\mathbb{N}} |x_i - x_{i+1}|$ is finite, hence x is weakly computable.*

Proof. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a k-reduced computable sequence of rational numbers which converges to x k-monotonically. As shown before, it suffices to prove that the sum $\sum_{t\in\mathbb{N}} \mu(I_k(\overrightarrow{V}_t) \cap \overrightarrow{I}_k\ (x_t, x_{t+1}))$ is finite. By Lemma 3.5, there are sequences $(B_j^i)_{j<m_i}$ and $(C_j^i)_{j<m_i}$ of rational intervals (for some $m_i \leq i$), which satisfy, for all $j < m_i$, the conditions (I) – (IV) of Lemma 3.5. This implies, for any $(x_t, x_{t+1}) \in \overrightarrow{V}_{t+1}$, that

$$\begin{aligned}
&\mu(I_k(\overrightarrow{V}_{t+1})) - \mu(I_k(\overrightarrow{V}_t)) \\
&= \mu(I_k(x_t, x_{t+1})) - \mu(I_k(\overrightarrow{V}_t) \cap I_k(x_t, x_{t+1})) \\
&\geq \mu(I_k(x_t, x_{t+1})) - \mu(B^t \backslash C^t) && \text{(by Lemma 3.5.III)} \\
&= \mu(I_k(x_t, x_{t+1})) - (\mu(B^t) - \mu(C^t)) && \text{(by Lemma 3.5.II)} \\
&\geq \mu(I_k(x_t, x_{t+1})) - \mu(B^t) + \frac{\mu(B^t)}{k(k+1)} && \text{(by Lemma 3.5.IV)} \\
&= \mu(I_k(x_t, x_{t+1})) - \mu(B^t)\left(1 - \frac{1}{k(k+1)}\right) \\
&\geq \mu(I_k(x_t, x_{t+1}))\left(\frac{1}{k(k+1)}\right) = \frac{x_{t+1} - x_t}{k(k+1)^2} && \text{(since } B^t \subseteq I_k(x_t, x_{t+1}))
\end{aligned}$$

This concludes, for $x_t < x_{t+1}$, that

$$\mu(I_k(\vec{V}_t) \cap \vec{I}_k\ (x_t, x_{t+1})) \leq \mu(\vec{I}_k\ (x_t, x_{t+1})) = \frac{x_{t+1} - x_t}{k+1}$$
$$\leq k(k+1)\left(\mu(I_k(\vec{V}_{t+1})) - \mu(I_k(\vec{V}_t))\right).$$

This implies immediately that the sum $\sum_{t\in\mathbb{N}} \mu(I_k(\vec{V}_t) \cap \vec{I}_k\ (x_t, x_{t+1})) \leq k(k+1)(\mu(I_k(\vec{V}) - \mu(I_0(\vec{V}))) \leq k(k+1)$ is finite.

Thus, the computable sequence $(x_n)_{n\in\mathbb{N}}$ converges to x weakly effectively, hence x is a weakly computable real number. $\square$

From Lemma 3.4 and Lemma 3.6, the next theorem follows immediately.

Theorem 3.7 *Any mc-real number is wc-computable. That is,* $\mathbf{C}_{mc} \subseteq \mathbf{C}_{wc}$.

Our next result shows that not every weakly computable real number is monotonically computable. Thus the class $\mathbf{C}_{mc}$ is a proper subset of $\mathbf{C}_{wc}$.

Theorem 3.8 *There is a weakly computable real number which is not monotonically computable, hence,* $\mathbf{C}_{mc} \subsetneq \mathbf{C}_{wc}$.

Proof. Let $(\varphi_i)_{i\in\mathbb{N}}$ be an effective enumeration of all computable (partial) functions $\varphi_i :\subseteq \mathbb{N} \to \mathbb{Q}$. $(\varphi_{i,s})_{s\in\mathbb{N}}$ is the uniformly effective approximation of φ_i. We will construct effectively a computable sequence $(x_s)_{s\in\mathbb{N}}$ of rational numbers which converges to x such that $\sum_{n\in\mathbb{N}} |x_n - x_{n+1}| \leq c$ for some $c \in \mathbb{N}$ and x satisfies, for all $e := \langle i, j\rangle \in \mathbb{N}$, the following requirements

$$R_e: \quad \left\{\begin{array}{l} \varphi_i \text{ is total, } \lim\limits_{n\to\infty} \varphi_i(n) = y_i \text{ exists and} \\ \forall n \forall m \geq n\, (j \cdot |y_i - \varphi_i(n)| \geq |y_i - \varphi_i(m)|) \end{array}\right\} \Longrightarrow x \neq y_i. \tag{6}$$

The strategy for satisfying a single requirement R_e is simple. We need only to fix arbitrarily a nonempty interval (a, b) as base interval and wait for some $t_1, t_2 \in \mathbb{N}$ with $t_1 < t_2$ and some $s \in \mathbb{N}$ such that both $\varphi_{i,s}(t_1) := \varphi_i(t_1)$ and $\varphi_{i,s}(t_2) := \varphi_i(t_2)$ are defined, $\varphi_i(t_1) \neq \varphi_i(t_2)$ and $\varphi_i(t_1), \varphi_i(t_2) \in (a, b)$. Let

$$\delta := \min\left\{\frac{|\varphi_i(t_1) - \varphi_i(t_2)|}{j+1}, \frac{|a - \varphi_i(t_1)|}{2}, \frac{|b - \varphi_i(t_1)|}{2}\right\}$$

and define a subinterval $(a'; b') := (\varphi_i(t_1) - \delta;\ \varphi_i(t_1) + \delta)$. In this case, it is easy to see that $j \cdot |x - \varphi_i(t_1)| < |x - \varphi_i(t_2)|$ holds for any $x \in (a'; b')$. In other words, any $x \in (a'; b')$ meets the requirement $R_{\langle i,j\rangle}$. We will call the interval $(a'; b')$ a witness interval of $R_{\langle i,j\rangle}$. Otherwise, if there are no such s, t_1 and t_2, then the limit $y_i := \lim_{n\to\infty} \varphi_i(n)$, if it exists, cannot be in the interval (a, b). Thus the interval $(a;b)$ is itself a witness interval of the requirement $R_{\langle i,j\rangle}$. Notice that, although this strategy always succeeds, we have no effective way to decide which of the above two possible approaches will be eventually applied.

To satisfy all requirements simultaneously, we will construct a nested interval sequences $((a_e;b_e))_{e\in\mathbb{N}}$ such that $(a_{e+1};b_{e+1})\subset(a_e;b_e)$ and $(a_e;b_e)$ is a witness interval of R_e. In this case, any real number $x\in\bigcap_{e\in\mathbb{N}}(a_e;b_e)$ satisfies all requirements R_e.

Unfortunately, it is not uniformly effectively to define witness intervals for different requirements. However, they can be effectively approximated in the sense that there is a computable sequence $((a_{e,s};b_{e,s}))_{s\in\mathbb{N}}$ which converges to a witness interval $(a_e;b_e)$ of R_e. In the following we will construct such an approximation in stages. At any stage s, we will define finitely many approximations $(a_{e,s};b_{e,s})$ for all $e\leq d_s$, where d_s will be defined in the construction and satisfies $\lim_{s\to\infty}d_s=\infty$. These intervals are also nested in the sense that $(a_{e+1,s};b_{e+1,s})\subset(a_{e,s};b_{e,s})$ and the interval $(a_{e,s};b_{e,s})$ is a correct witness interval for R_e with respect to the approximation sequences $(\varphi_{i,j}(n))_{n\in\mathbb{N}}$ $(e=\langle i,j\rangle)$ instead of $(\varphi_i(n))_{n\in\mathbb{N}}$. Of course we have to correct continuously our approximations according to the behaviors of $(\varphi_{i,j}(n))_{n\in\mathbb{N}}$. Then a priority argument is necessary. We say that R_e has higher priority than R_{e_1} if $e<e_1$. If we define a new witness interval according to above strategy for R_e at some stage, then all old witness intervals for requirements $R_{e'}$ may be destroyed if $e'>e$ ($R_{e'}$ is injured at this stage). We set the current witness intervals for these requirements $R_{e'}$ as undefined and redefine them at some later stages again. On the other hand, whenever some new witness interval I for R_e is defined by the above strategy, then I really witnesses the requirement R_e and we need not do anything more for R_e unless it is destroyed. In this case we will say that R_e is in the state "satisfied" to avoid any further unnecessary action. This makes sure also that any requirement R_e can be injured at most 2^e times.

At any stage s, we define x_s to be the middle point of the smallest witness interval defined at stage s. This guarantees that x_s locates in all currently defined witness intervals. To make sure that the sequence $(x_s)_{s\in\mathbb{N}}$ converges weakly effectively to x, we choose the interval $(a_{e,s};b_{e,s})$ small enough so that its length is not longer than, say, 2^{-2e} so that the approximation of x cannot have big jumps at all.

Here is the formal construction.

Stage $s=0$: Define $(a_{0,s};b_{0,s}):=(0;1)$, $x_0:=1/2$ and $d_0:=0$. Any requirement R_e is in the state "unsatisfied". Stage 0 is called an 0-stage.

Stage $s+1$: Given d_s and $(a_{e,s};b_{e,s})$ for all $e\leq d_s$. We say that a requirement R_e $(e=\langle i,j\rangle)$ requires attention if it is in the state "unsatisfied", $e\leq d_s$ and there are $t_1,t_2\in\mathbb{N}$ with $t_1<t_2$ such that both $\varphi_{i,s}(t_1)\downarrow=\varphi_i(t_1)$ and

$\varphi_{i,s}(t_2)\downarrow = \varphi_i(t_2)$ are defined and

$$\varphi_i(t_1) \neq \varphi_i(t_2) \ \& \ \varphi_i(t_1), \varphi_i(t_2) \in (a_{e,s}; b_{e,s}). \tag{7}$$

If there are no requirement which requires attention at this stage, then define

$$d_{s+1} := d_s + 1 \tag{8}$$

$$(a_{e,s+1}; b_{e,s+1}) := \begin{cases} (a_{e,s}; b_{e,s}) & \text{if } e \leq d_s \\ (a_{e,s} + \eta; a_{e,s} + 2\eta) & \text{if } e = d_s + 1 \\ \text{undefined} & \text{otherwise} \end{cases} \tag{9}$$

where $\eta := (b_{e,s} - a_{e,s}) \cdot 2^{-2e}$ for $e := d_{s+1}$. In this case, the stage $s+1$ is a default d_{s+1}-stage.

Otherwise, choose a minimal natural number $e := \langle i, j\rangle \leq d_s$ such that R_e requires attention. Let $t_1 < t_2$ be the numbers satisfying the condition (7). We define

$$d_{s+1} := e \tag{10}$$

$$(a_{e',s+1}; b_{e',s+1}) := \begin{cases} (a_{e',s}; b_{e',s}) & \text{if } e' < e \\ (\varphi_i(t_1) - \delta;\ \varphi_i(t_1) + \delta) & \text{if } e' = e \\ \text{undefined} & \text{otherwise} \end{cases} \tag{11}$$

where

$$\delta := \min\left\{ \frac{|\varphi_i(t_1) - \varphi_i(t_2)|}{j+1}, \frac{\varphi_i(t_1) - a_{e,s}}{2}, \frac{b_{e,s} - \varphi_i(t_1)}{2}, \frac{b_{e,s} - a_{e,s}}{2^{2(e+1)}} \right\} \tag{12}$$

and set the state of R_e to be "satisfied" and all states for $R_{e'}$ with $e' > e$ to be "unsatisfied". We say that R_e *receives attention* and all requirements $R_{e'}$ for $e < e' \leq d_s$ are injured at this stage, if $R_{e'}$ is in the state "satisfied" at stage s. The stage $s+1$ is called an e-stage in this situation. In both cases we define furthermore that $x_{s+1} := (a_{d_{s+1}} + b_{d_{s+1}})/2$.

We will show that our construction succeeds by the following sublemmas.

Sublemma 3.8.1 *For any $e \in \mathbb{N}$, the requirement R_e receives attention at most 2^e times, hence there are at most $2^{(e+1)}$ e-stages in the above construction.*

Proof. We prove the sublemma by induction on $e \in \mathbb{N}$. For $e = 0$, if R_0 receives attention at some stage s, then it is in the state "satisfied". Since there are no requirements which have higher priority than R_0, it will never be injured, and hence R_0 is always in the state "satisfied". This means that R_0 will never require attention after stage s again. That is, R_0 receives attention at most once. Including the default 0-stage 0, there are at most $2 = 2^{0+1}$ 0-stages.

Suppose by induction hypothesis that, for any $i < e$, R_i receives attention at most 2^i times and there are at most 2^{i+1} i-stages. Let

$$\begin{aligned} I_i &:= \{s \in \mathbb{N} : R_i \text{ is injured at stage } s\}, \text{ and} \\ A_i &:= \{s \in \mathbb{N} : R_i \text{ receives attention at stage } s\}. \end{aligned}$$

Then we have $|A_i| \leq 2^i$ for any $i < e$. By the construction, R_e can be injured at stage s only if some R_i $(i < e)$ receives attention at this stage. Therefore $|I_e| \leq \sum_{i<e} |A_i| \leq \sum_{i<e} 2^i = 2^e - 1$. On the other hand, if R_e receives attention at some stage s, then R_e is in the state "satisfied" whenever it is not yet injured. Namely, R_e does not require and hence receive attention after stage s unless it is injured. This implies that R_e receives attention at most $|I_e| + 1 = 2^e$ times. In addition, there may be a default e-stage before each of the 2^e e-stages. Thus, there are at most 2^{e+1} e-stages totally. (sublemma) □

Sublemma 3.8.2 *For any $e \in \mathbb{N}$, the limit $\lim_{s\to\infty}(a_{e,s}; b_{e,s}) := (a_e; b_e)$ exists and the interval $(a_e; b_e)$ is a witness interval of R_e in the sense that any real number $x \in (a_e; b_e)$ satisfies R_e.*

Proof. By the construction, the interval $(a_{e,s}; b_{e,s})$ can be changed at stage s if and only if s is an i-stage for some $i \leq e$. Therefore, it follows immediately from Sublemma 3.8.1, that $\lim_{s\to\infty}(a_{e,s}; b_{e,s}) := (a_e; b_e)$ exists.

Choose a minimal s_0 such that $(a_{e,s_0}; b_{e,s_0}) = (a_{e,s}; b_{e,s}) = (a_e; b_e)$ for all $s \geq s_0$. Then no $R_{e'}$, for $e' \leq e$, will receive attention after stage s_0. Let $e = \langle i, j \rangle$. If R_e is in the state "unsatisfied" at stage s_0, then there are no $t_1 < t_2$ which satisfies the condition (7) for $s \geq s_0$, otherwise, R_e will require and hence receives attention at this stage which contradicts the choice of s_0. This implies that, the limit $\lim_{n\to\infty} \varphi_i(n)$, if it exists, will not be in the interval $(a_e; b_e)$ which is hence a correct witness interval of R_e. Otherwise, if R_e is in the state "satisfied", then, by the minimality of s_0, R_e requires and receives attention at stage s_0. In this case, the interval $(a_{e,s_0}; b_{e,s_0})$ is defined according to (11). Namely $(a_{e,s_0}; b_{e,s_0}) := (\varphi_{i,s_0-1}(t_1) - \delta; \varphi_{i,s_0-1}(t_2) + \delta)$ for some $t_1 < t_2$ and δ defined as the minimal one of four values from (12). Especially, we have $\delta \leq |\varphi_{i,s_0-1}(t_1) - \varphi_{i,s_0-1}(t_2)|/(j+1)$. Suppose now that φ_i is total and the sequence $(\varphi_i(n))_{n\in\mathbb{N}}$ converges j-monotonically to y_i. Then we have especially

$j \cdot |y_i - \varphi_i(t_1)| \geq |y_i - \varphi_i(t_2)|$. This implies further that $|y_i - \varphi_i(t_1)| \geq \delta$, hence $y_i \notin (a_e; b_e)$. Thus, $(a_e; b_e)$ is also a witness interval in this case. (sublemma) □

Sublemma 3.8.3 *For all $s \in \mathbb{N}$ and $e \leq d_s$, the following hold*

$$b_{e,s} - a_{e,s} \leq 2^{-2e} \ \&\ b_e - a_e \leq 2^{-2e}; \text{ and} \tag{13}$$

$$(a_{e+1,s}; b_{e+1,s}) \subset (a_{e,s}; b_{e,s}) \ \&\ (a_{e+1}; b_{e+1}) \subset (a_e; b_e). \tag{14}$$

Proof. This follows directly from Sublemma 3.8.2, definition (9) and (11). (sublemma) □

Sublemma 3.8.4 *The sequence $(x_s)_{s\in\mathbb{N}}$ converges weakly effectively to a real number x and x satisfies all the requirements R_e.*

Proof. It suffices to show that the sequence $(x_s)_{s\in\mathbb{N}}$ satisfies the condition $\sum_{n\in\mathbb{N}} |x_n - x_{n+1}| \leq c$ for some constant $c \in \mathbb{R}$. Let

$$S_e := \{s \in \mathbb{N} : s \text{ is an } e\text{-stage}\}.$$

By Sublemma 3.8.1 we have $|S_e| \leq 2^{e+1}$. All stages are divided into different e-stages, i.e., $\mathbb{N} = \bigcup_{e\in\mathbb{N}} S_e$. From the construction it is easy to see that, if $s+1$ is an e-stage, then $d_{s+1} = e$ and hence $x_{s+1} := (a_{e,s+1} + b_{e,s+1})/2 \in (a_{e,s+1}; b_{e,s+1}) \subseteq (a_{e,s}; b_{e,s})$ by (14). On the other hand, it follows again from (14) of Sublemma 3.8.3 that $x_s := (a_{d_s,s} + b_{d_s,s})/2 \in (a_{e,s}; b_{e,s})$ since $e \leq d_s$. By (13), this implies that $|x_s - x_{s+1}| \leq 2^{-2e}$, if $s+1$ is an e-stage. Therefore, we have

$$\begin{aligned}
\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| &= \sum_{e\in\mathbb{N}} \sum_{s+1\in S_e} |x_s - x_{s+1}| \\
&\leq \sum_{e\in\mathbb{N}} \sum_{s+1\in S_e} 2^{-2e} \leq \sum_{e\in\mathbb{N}} |S_e| \cdot 2^{-2e} \\
&\leq \sum_{e\in\mathbb{N}} 2^{(e+1)} \cdot 2^{-2e} = \sum_{e\in\mathbb{N}} 2^{-e+1} = 4
\end{aligned}$$

That is, the computable sequence $(x_s)_{s\in\mathbb{N}}$ converges weakly effectively to some weakly computable real number x.

By a simple induction we can show that, for any $e \in \mathbb{N}$, $x \in (a_e; b_e)$. Thus x satisfies all the requirements R_e by Sublemma 3.8.2. (sublemma) □

From Sublemma 3.8.4, the real number x is weakly computable but not monotonically computable. This completes the proof. □

Corollary 3.9 *The class $\mathbf{C}^c_{mc}$ is not closed under addition and subtraction, and hence it is not an algebraic field, if $c \geq 1$.*

Proof. By Theorem 2.1, the class $\mathbf{C}_{wc}$ is the closure of $\mathbf{C}_{sc}$ under $+$ and $-$. Especially, for any $x \in \mathbf{C}_{wc}$, there are $y, z \in \mathbf{C}_{sc}$ such that $x = y + z$. By Theorem 3.8, we can choose an $x \in \mathbf{C}_{wc} \setminus \mathbf{C}_{mc}$. In this case, $y, z \in \mathbf{C}_{sc} \subseteq \mathbf{C}^c_{mc}$ but $y + z \notin \mathbf{C}^c_{mc}$ for any $c \geq 1$. Therefore $\mathbf{C}^c_{mc}$ is not closed under $+$ and $-$. $\square$

4 A Hierarchy of Monotonically Computable Real Numbers

In Sections 2 and 3 we have shown that, if $k < 1$, then the k-monotone computability (i.e., the classical computability) is different from the 1-monotone computability (or equivalently, the semi-computability). On the other hand, for any $k_1, k_2 < 1$, k_1- and k_2-monotone computability are the same which is simply equal to the computability. One question remains open: whether k_1- and k_2-monotone computability are also the same for different $k_1, k_2 > 1$? Or whether all classes $\mathbf{C}^k_{mc}$ of k-mc real numbers collapse to $\mathbf{C}^1_{mc}$? In this section we will show that is not the case. In fact we show that, for any k, there is a $k' > k$ such that the class of k'-mc real numbers form a proper superset of the class of all k-mc real numbers. Therefore, there is an increasing sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers such that $(\mathbf{C}^{n_i}_{mc})_{i\in\mathbb{N}}$ is a proper hierarchy of the class $\mathbf{C}_{mc}$.

Theorem 4.1 *For any $k \in \mathbb{N}^+$, there is a $k' > k$ and a k'-monotonically computable real number which is not k-computable. Hence $\mathbf{C}^k_{mc} \subsetneq \mathbf{C}^{k'}_{mc}$.*

Proof. Let $k \geq 1$ be any natural number, $(\varphi_i)_{i\in\mathbb{N}}$ be an effective enumeration of all computable functions $\varphi_i :\subseteq \mathbb{N} \to \mathbb{Q}$ and $(\varphi_{i,s})_{s\in\mathbb{N}}$ the uniformly effective approximation of φ_i. We will construct a computable sequence $(x_s)_{s\in\mathbb{N}}$ of rational numbers which satisfies, for some $k' > k$ and any $e \in \mathbb{N}$ the following requirements:

$N : (x_s)_{s\in\mathbb{N}}$ converges k'-monotonically to some x, and
$R_i : (\varphi_i(n))_{n\in\mathbb{N}}$ converges k-monotonically to $y_i \Longrightarrow x \neq y_i$.

The strategy for satisfying a single requirement R_i is as follows. We fix an interval, say, $(0; 1)$, as our base interval and try to find out a so-called witness interval $(a; b) \subseteq (0; 1)$ such that any $x \in (a; b)$ satisfies the requirement R_i. Let $(1/(k+4); 2/(k+4))$ be our first candidate of the witness interval. If no element of the sequence $(\varphi_i(n))_{n\in\mathbb{N}}$ appears in this interval, then it is automatically a correct witness interval, since the limit $\lim_{n\to\infty} \varphi_i(n)$, if exists, will not be in this interval. Otherwise, if there are some $s_1, n_1 \in \mathbb{N}$ such that $\varphi_{i,s_1}(n_1) \in (1/(k+4); 2/(k+4))$, then we choose $((k+2)/(k+4); (k+3)/(k+4))$ as our new candidate of witness interval. Again, if there is no $n_2 > n_1$ such that $\varphi_i(n_2)$ comes into this interval, then any element x from this interval witnesses the requirement R_i. Otherwise, if $\varphi_{i,s_2}(n_2) \in ((k+2)/(k+4); (k+3)/(k+4))$ for some $s_2 > s_1$ and some $n_2 > n_1$, then the old interval $(1/(k+4); 2/(k+4))$ turns out to be again a correct witness interval, since $k|x - \varphi_i(n_1)| < k/(k+4) < |x - \varphi_i(n_2)|$, for any $x \in (1/(k+4); 2/(k+4))$ and hence $(\varphi_i(n))_{n\in\mathbb{N}}$ does not converge to x k-monotonically.

To satisfy all requirements R_i simultaneously, we need an interval tree. For any $\delta \in \mathbb{N}$, let $\Sigma_\delta := \{0, 1, \cdots, \delta - 1\}$ and $\mathbb{I}$ the set of all rational subintervals of $[0; 1]$. We define a δ-interval tree I on $[0; 1]$ as a function $I : \Sigma_\delta^* \to \mathbb{I}$ by $I(w) := [a_w; b_w]$, for all $w \in \Sigma_\delta^*$, where $a_w := \sum_{i<|w|} w(i) \cdot \delta^{-(i+1)}$ and $b_w := a_w + \delta^{-|w|}$. Now we fix some δ-interval tree I. For any $w \in \Sigma_\delta^*$ of length i, the intervals $I(w)$ are reserved exclusively as base intervals to implement the above strategy for satisfying the requirement R_i. Suppose that $I(w)$ is a current base interval for R_i. We will try to find some subinterval $I(wa) \subseteq I(w)$, for some $a \in \Sigma_\delta$, as a "witness interval" for R_i such that any real number of this interval witnesses the requirement R_i. More precisely, the limit $\lim_{n\to\infty} \varphi_i(n)$, if exists, will not be in the interval $I(wa)$, if the sequence $(\varphi_i(n))_{n\in\mathbb{N}}$ converges k-monotonically. In this way, if $I(w_1)$ and $I(w_2)$ are witness intervals for R_{i_1} and R_{i_2}, respectively, and $i_1 > i_2$, then $I(w_{i_2}) \subset I(w_{i_1})$. Thus, the sequence of all witness intervals for all requirements form a nested interval sequence whose common point x satisfies all requirements R_i for $i \in \mathbb{N}$. At any stage s, we choose some x_s from the smallest witness interval defined at stage s. Then the limit $\lim_{s\to\infty} x_s$ is a common point of final witness intervals of all requirements and hence satisfies all R_e.

To satisfy the requirement N, we have to make some further efforts. Notice that, in the above strategy, it is possible that $x_{s_1}, x_{s_3} \in (1/(k+4); 2/(k+4))$ and $x_{s_2} \in ((k+2)/(k+4); (k+3)/(k+4))$ for some $s_1 < s_2 < s_3$. In this case, we cannot guarantee that $k'|x - x_{s_1}| \geq |x - x_{s_2}|$ for some constant k'. To solve this problem, we divide the base interval $I(w)$ for R_i into $k+8$ instead of $k+4$ subintervals $I(wa)$ for $a < k+8$. Every such subinterval is again divided into $k+8$ subsubintervals $I(wab)$ for $b < k+8$, and so on. That is, we fix a δ-interval tree I for $\delta := k+8$. At any stage, we consider the witness intervals for R_e and R_{e+1} simultaneously. As the default witness intervals we consider $I(w1)$ and $I(w11)$ for R_i and R_{i+1}, respectively. If it is necessary, we will change the witness interval of R_i from $I(w1)$ to $I(w(\delta - 4))$ and again back to $I(w1)$. But in this case, we force that the new default witness interval of R_{i+1} to be $I(w13)$. Later on, R_{i+1} can change its witness intervals from $I(w13)$ to $I(w1(\delta - 2))$ and back to $I(w13)$ again, if it is necessary. In this way, we can make sure that the constructed limit x will not be too close to its early approximation x_{s_1} after some big jump to some x_{s_2}, so that x is

k'-monotonically computable for some proper k'.

Of course, the choice of base and witness intervals has to be corrected continuously according to the behaviors of the sequences $(\varphi_{i,s}(n))_{n\in\mathbb{N}}$ for different $s \in \mathbb{N}$. The choice of the witness intervals for the requirements $R_0, R_1, \cdots, R_{i-1}$ corresponds to a string $w \in \{1, 3, \delta-4, \delta-2\}^*$ of length i. Namely, for any $i \leq |w|$, the interval $I(w \upharpoonright (i+1))$ is the base interval for R_{i+1} and at the same time the witness interval of requirement R_i. We denote by w_s our choice of this string at stage s, which seems correct at least for the s-th approximation sequences $(\varphi_{i,s}(n))_{n\in\mathbb{N}}$ instead of $(\varphi_i(n))_{n\in\mathbb{N}}$ for all $i < |w_s|$. As the limit, $w := \lim_{s\to\infty} w_s \in \Sigma^\omega$ describes a correct sequence of witness intervals $(I(w \upharpoonright i))_{i\in\mathbb{N}}$ for all requirements $(R_i)_{i\in\mathbb{N}}$. Correspondingly, the sequence $(x_s)_{s\in\mathbb{N}}$ defined by $x_s := a_{w_s 1}$ will converges to $x_w := \sum_{i\in\mathbb{N}} w[i]$ which satisfies the theorem. For technical reasons, if some $\varphi_{i,s}(n)$ comes into the old witness interval of R_i, the number n is recorded by $c_s(i) := n$. If there is still no such n until stage s or the action for R_i is destroyed by the action for some R_j of higher priority, i.e., $j < i$, then we will also denote this by $c_s(i) = -1$.

The formal construction of $(w_s)_{s\in\mathbb{N}}$:

Stage $s = 0$: Define simply $w_0 := 1$ and $c_0(e) := -1$ for all $e \in \mathbb{N}$. Namely, we choose the interval $(0; 1)$ as the base interval of R_0 and choose $(1/\delta; 2/\delta)$ as the witness interval of R_0 which is also the base interval of R_1.

Stage $s + 1$: Given the string $w_s \in \{1, 3, \delta-4, \delta-2\}^*$ and the function c_s. For any $i < |w_s|$, the interval $(a_{w_s\upharpoonright i}, b_{w_s\upharpoonright i})$ is the current base interval of R_i and the interval $(a_{w_s\upharpoonright(i+1)}, b_{w_s\upharpoonright(i+1)})$ is the current witness interval of R_i. We say that a requirement R_i *requires attention* if $i + 1 < |w_s|$ and there exists a number $m > c_s(i)$ such that

$$(w_s[i+1] \neq 3 \ \&\ w_s[i+1] \neq \delta-2) \ \&\ \varphi_{i,s}(m) \in (a_{w_s\upharpoonright(i+1)}, b_{w_s\upharpoonright(i+1)}). \quad (15)$$

If there is no i such that R_i requires attention at stage $s + 1$, then we define simply $w_{s+1} := w_s 1, c_{s+1} := c_s$ and go to the next stage. Namely, we introduce a new interval $I(w_s 1)$ as the default witness interval for the requirement $R_{|w_s|}$, if no requirement requires attention.

Otherwise, choose a minimal i such that R_i requires attention at this stage and let m be corresponding number which satisfies condition (15). We define new w_{s+1} and c_{s+1} by

$$w_{s+1} := \begin{cases} (w_s \upharpoonright i)(\delta-4)1 & \text{if } w_s[i] = 1 \\ (w_s \upharpoonright i)(\delta-2)1 & \text{if } w_s[i] = 3 \\ (w_s \upharpoonright i)13 & \text{if } w_s[i] = \delta-4 \\ (w_s \upharpoonright i)33 & \text{if } w_s[i] = \delta-2 \end{cases} \tag{16}$$

$$c_{s+1}(j) := \begin{cases} c_s(j) & \text{if } j < i; \\ m & \text{if } j = i; \\ -1 & \text{otherwise.} \end{cases} \tag{17}$$

Notice that, we have always $w_{s+1} \neq w_s$ and the length $|w_{s+1}| = i + 2$ in this case. At this stage, we define a new witness interval for R_i as well as for R_{i+1}. All (possible) old witness intervals for requirements $R_{i'}$ for $i' > i$ are cancelled, i.e., they are *injured* at this stage. We say that the requirement R_i *receives attention* in this case. This ends the construction.

We show now that our construction succeeds by the following sublemmas.

Sublemma 4.1.1 *For any $i \in \mathbb{N}$, the requirement R_i requires and hence receives attention at most finitely often.*

Proof. We prove the sublemma by induction on $i \in \mathbb{N}$. Assume by induction hypothesis that R_j requires and receives attention at most finitely often for any $j < i$. Then there is a minimal s_0 such that no R_j $(j < i)$ requires and receives attention after stage s_0. By the minimality, we have either $s_0 = 0$ or R_j receives attention at stage s_0 for some $j < i$. Therefore, there is an $s_1 = s_0 + t$ for $t = i + 2 - |w_{s_0}|$ such that $w_{s_1} = w_{s_0}1^t$ and no requirement requires and receives attention between stages s_0 and s_1. Especially, we have $t = 0$ if and only if R_{i-1} receives attention at stage s_0. Furthermore, it is easy to see that $w_s \upharpoonright i = w_{s_1} \upharpoonright i$ for any $s \geq s_1$ since R_j $(j < i)$ will never be injured after stage s_0. Namely, R_i has always the same base interval $I(w_{s_1} \upharpoonright i)$ after stage s_1.

Obviously we have also that $w_{s_1}[i] \in \{1, 3\}$ and $c_{s_1}(i) = -1$. We consider now only the case of $w_{s_1}[i] = 1$. The case of $w_{s_1}[i] = 3$ can be discussed completely similarly.

If there is no $s \geq s_1$ such that R_i requires attention at stage s, then we are done, because R_i requires and receives attention only before stage s_0, hence at most finitely often.

Otherwise, suppose that R_i requires and hence receives its first attention after stage s_1 at stage $s_2 + 1$. Namely the condition (15) is satisfied for $s = s_2$ and for some $m_1 \in \mathbb{N}$. In this case, we define $w_{s_2+1} := (w_{s_2} \upharpoonright i)(\delta - 4)1$ and

$c_{s_2+1}(i) = m_1$ according to (16), hence $w_{s_2+1}[i] = \delta - 4$.

Now if there is no $s \geq s_2$ such that R_i requires attention at stage s, then we are done again. Otherwise, suppose that R_i requires and hence receives its first attention after stage $s_2 + 1$ at stage $s_3 + 1$. That is the condition (15) is satisfied for $s = s_3$ and for some $m_2 > m_1$. Since no R_j $(j \leq i)$ receives attention between stages s_2+1 and s_3+1, we have $w_{s_3}[i] = w_{s_2+1}[i] = \delta-4$. By (16), we define $c_{s_3+1}(i) = m_2$ and $w_{s_3+1} := (w_{s_2} \upharpoonright i)13$, hence $w_{s_3+1}[i+1] = 3$. By the construction, the value of $w_s[i + 1]$ can be changed only from 3 to $\delta - 2$ or from 1 to $\delta - 4$ and vice versa, whenever R_{i+1} receives attention and no R_j for $j \leq i$ receives attention in between. It follows that we have always $w_s[i+1] = 3$ or $w_s[i+1] = \delta-2$ for any $s > s_3$. Therefore R_i will never require attention after stage $s_3 + 1$ because of the condition (15). This concludes that R_i requires and receives attention after stage s_0 at most two times, hence at most finitely often totally. (sublemma) □

Sublemma 4.1.2 *The limit* $w := \lim_{s\to\infty} w_s \in \{1, 3, (\delta - 4), (\delta - 2)\}^\omega$ *exists.*

Proof. It suffices to show that, for any $i \in \mathbb{N}$, there is an s_0 such that $|w_{s_0}| > i$ and $w_{s_0}[i] = w_s[i] \in \{1, 3, (\delta - 2), (\delta - 2)\}$ holds for all $s \geq s_0$.

By the construction, $w_s \in \{1, 3, (\delta-2), (\delta-2)\}$ holds for any $s \in \mathbb{N}$ obviously. Fix an $i \in \mathbb{N}$. By Sublemma 4.1.1, there is an s_1 such that no R_j $(j \leq i)$ receives attention after stage s_1. It is clear, that if $w_s[i]$ is already defined, then $w_t[i]$ must also be defined and $w_t[i] = w_s[i]$ for any $t \geq s$, if no requirement R_j $(j \leq i)$ receives attention between stages s and t. Thus, if $|w_{s_1}| > i$, then we are done by simply setting $s_0 = s_1$. Otherwise, suppose that $|w_{s_1}| \leq i$. Then after stage s_1 we can take only the default action in the construction until stage $s_2 := s_1 + t$ for $t = (i + 2) - |w_{s_1}|$, because for any $j \in \mathbb{N}$, the requirement R_j does not require attention for $j \leq i$ by the choice of s_1 and for $j > i$ by the fact that $j + 1 \geq |w_s|$. This implies that $w_{s_2} = w_{s_1}1^t$ by the construction. In this case $s_0 := s_2$ satisfies $|w_{s_0}| > i$ and $w_{s_0}[i] = w_s[i]$ for all $s \geq s_0$. (sublemma) □

Sublemma 4.1.3 *The sequence* $(x_s)_{s\in\mathbb{N}}$ *converges to* x_w k'*-monotonically, where* $k' := (k + 8)^2$*, i.e.,* x_w *is* k'*-monotonically computable.*

Proof. By the definitions of x_w and x_s, it follows from the Sublemma 4.1.2 immediately that the sequence $(x_s)_{s\in\mathbb{N}}$ converges to x_w. Now we will show that this convergence is also k'-monotone in the sense of (1) for $k' := (k+8)^2$. Namely, we have to show that

$$s < t \Longrightarrow k' \cdot |x_w - x_s| \geq |x_w - x_t|. \tag{18}$$

for any $s, t \in \mathbb{N}$.

Given any $s, t \in \mathbb{N}$ with $s < t$. We have always $x_s \neq x_t$. Then there exists an unique $m \in \mathbb{N}$ and a string $u \in \Sigma_\delta^*$ of length m such that $x_t, x_s \in I(u)$ and $\delta^{-(m+1)} < |x_s - x_t| \leq \delta^{-m}$. By the construction, the numbers x_s, x_t can only be located in the intervals of $I(u1), I(u3), I(u(\delta-4))$ or $I(u(\delta-2))$. If $x_w \notin I(u)$, then $|x_w - x_s| \geq \delta^{-(m+1)}$ and $|x_t - x_s| \leq \delta^{-m}$. Remember that $\delta = (k+8)$ and $k' = (k+8)^2$. This implies that

$$\begin{aligned}
k'|x_w - x_s| &= (k+8)^2|x_w - x_s| \geq |x_w - x_s| + (k+8)|x_w - x_s| \\
&\geq |x_w - x_s| + (k+8)\delta^{-(m+1)} = |x_w - x_s| + \delta^{-m} \\
&\geq |x_w - x_s| + |x_s - x_t| \geq |x_w - x_t|.
\end{aligned}$$

That is, the condition (18) is satisfied.

Suppose now that $x_w \in I(u)$. Notice that x_w belongs to $I(ua)$ for some $a \in \{1, 3, \delta-4, \delta-2\}$. If x_s and x_w do not locate in the same subinterval $I(ua)$ for any $a \in \{1, 3, \delta-4, \delta-2\}$, then $k'|x_w - x_s| \geq k'\delta^{-(m+1)} \geq \delta^{-m} \geq |x_w - x_t|$. Otherwise suppose now that x_w, x_s belong to a single interval $I(ua)$ for some $a \in \{1, 3, \delta-4, \delta-2\}$ and consider the following cases.

Case 1: $x_w, x_s \in I(u1)$. It follows from $x_s \in I(u1)$ that $w_s[m] = 1$ and hence $|w_s| \geq m+1$ and $u \sqsubseteq w_s$. Therefore, the interval $I(u)$ is the base interval of R_m and $I(u1)$ is the witness interval of R_m at stage s. Since $x_w \in I(u1)$, R_m does not change its base interval after stage s any more, otherwise, the limit x_w should be in the interval $I(u3)$ or $I(u(\delta-2))$. This means that $x_{s'} \in I(u)$ for any $s' \geq s$. On the other hand, R_m has to change its witness interval from $I(u1)$ to $I(u(\delta-4))$ and back to $I(u1)$ again after stage s, since there is a $t > s$ such that $x_t \notin I(u1)$. It follows that $x_s \in I(u11)$ or $x_s \in I(u1(\delta-4))$ and $x_w \in I(u13)$ or $x_w \in I(u1(\delta-2))$. Therefore, $k'|x_w - x_s| \geq k'\delta^{-(m+2)} = \delta^{-m} \geq |x_w - x_t|$.

Case 2: $x_w, x_s \in I(u3)$. Notice that the base interval for R_m can be changed only if some requirement R_e with $e < m$ receives attention. If, for some $e < m-1$, R_e receives attention after stage s, then R_m will never come back to its old base interval $I(u)$ hereafter and this contradicts the fact that $x_w \in I(u)$. And R_{m-1} will never require attention after stage s since $w_s[m] = 3$. This means that R_m has always $I(u)$ as its base interval after stage s. By assumption, there is a $t > s$ with $x_t \notin I(u3)$. Then it is only possible that $x_s \in I(u31)$ or $x_s \in I(u3(\delta-4))$, $x_t \in I(u(\delta-2))$, and $x_w \in I(u33)$ or $x_w \in I(u3(\delta-2))$ by the construction. This implies also that $k'|x_w - x_s| \geq k'\delta^{-(m+2)} = \delta^{-m} \geq |x_w - x_t|$.

Case 3. $x_w, x_s \in I(u(\delta-4))$ Because $x_s \in I(u(\delta-4))$, the interval $I(u(\delta-4))$ is the witness interval of R_m at stage s. Since there is a $t > s$ such that $x_t \notin I(u(\delta-4))$, R_m will change its witness interval from $I(u(\delta-4))$ to $I(u1)$ and never back to $I(u(\delta-4))$ again. This contradicts the fact that $x_w \in I(u(\delta-4))$. This concludes that this case can not happen in fact. Similarly, the case of $x_w, x_s \in I(u(\delta-2))$ is also impossible. (sublemma) $\square$

Sublemma 4.1.4 *x_w satisfies all requirements R_e for $e \in \mathbb{N}$, thus x_w is not k-monotonically computable.*

Proof. By Sublemma 4.1.1 and 4.1.2, there is, for any given $e \in \mathbb{N}$, an $s_0 \in \mathbb{N}$ such that, for any $s \geq s_0$, $w_{s_0} \upharpoonright (e+2) = w_s \upharpoonright (e+2)$ and no requirement R_j $(j \leq e)$ receives attention at stage s.

If $w_{s_0}[e]w_{s_0}[e+1] = 11$, then $x_w \in I((w_{s_0} \upharpoonright e)11)$ and $\varphi_{e,s}(m) \notin I((w_{s_0} \upharpoonright e)1)$ for any $s \geq s_0$ and $m \in \mathbb{N}$. This implies that $\lim_{n\to\infty} \varphi_e(n) \neq x_w$, if the limit exists, and hence R_e is satisfied. The same argument holds also for the cases of $w_{s_0}[e]w_{s_0}[e+1] \in \{31, (\delta-2)1, (\delta-4)1\}$.

Suppose that $w_{s_0}[e]w_{s_0}[e+1] = 13$. By the construction, this means that there are $s_1 < s_2 < s_0$ such that

(1) no R_j $(j < e)$ receives attention after stage s_1;
(2) the requirement R_e receives attention at stage s_1. Namely, for some $m_1 \in \mathbb{N}$, $\varphi_{e,s_1}(m_1) \in I((w_{s_1-1} \upharpoonright e)1) = I((w_{s_0} \upharpoonright e)1)$, $c_{s_1}(e) = m_1$ and $w_{s_1} = (w_{s_0} \upharpoonright e)(\delta-4)1$;
(3) the requirement R_e receives attention at stage s_2. Hence there is some $m_2 > m_1$ such that $\varphi_{e,s_2}(m_2) \in I((w_{s_2-1} \upharpoonright e)(\delta-4) = I((w_{s_0} \upharpoonright e)(\delta-4)$ and $w_{s_2} = (w_{s_0} \upharpoonright e)13$.

If φ_e is a total function and the sequence $(\varphi_e(n))_{n\in\mathbb{N}}$ converges k-monotonically to y_e, then we have $k|y_e - \varphi_e(m_1)| \geq |y_e - \varphi_e(m_2)|$ by the condition (1). Since $\varphi_e(m_1) \in I((w_{s_0} \upharpoonright e)1)$ and $\varphi_e(m_2) \in I((w_{s_0} \upharpoonright e)(\delta-4))$, it follows that $y_e \notin I((w_{s_0} \upharpoonright e)1)$, otherwise $k \cdot |y_e - \varphi_e(m_1)| \leq k \cdot \delta^{-(m+1)} < (\delta-6)\delta^{-(m+1)} \leq |y_e - \varphi_e(m_2)|$ which contradicts to the hypothesis. It is obviously from the definition of x_w that $x_w \in I(w \upharpoonright m)$ for any $m \in \mathbb{N}$. Especially we have $x_w \in I((w_{s_0} \upharpoonright e)1)$ in this case. This implies that $x_w \neq y_e$.

Suppose now that $w_{s_0}[e]w_{s_0}[e+1] = 1(\delta-4)$. Then there was a stage $s_2 < s_0$ where $w_{s_2}[2]w_{s_2}[e+1] = 13$ and we can argue as above that $w_w \in I((w_{s_0} \upharpoonright e)1)$ whereas $y_e \notin I((w_{s_0} \upharpoonright e)1)$.

Similarly we can show that the requirement R_e is also satisfied in the cases of $w_{s_0}[e]w_{s_0}[e+1] \in \{33, 1(\delta-2), 3(\delta-2), 3(\delta-4)\}$. Because the cases of $w_{s_0}[e]w_{s_0}[e+1] \in \{\delta-2, \delta-4\} \times \{3, \delta-2, \delta-4\}$ are impossible, we conclude that R_e is always satisfied in any case. (sublemma) □

From Sublemma 4.1.3 and 4.1.4, we know that the real number x_w is k'-monotonically computable for $k' := (k+8)$ and it is not k-monotonically computable. This completes the proof of the theorem. □

From the proof of Theorem 4.1 we know that, for any $k \in \mathbb{N}$ and $k_1 := (k+8)^2$, the class $\mathbf{C}_{wc}^{k}$ is contained properly in the class $\mathbf{C}_{wc}^{k_1}$. Then the following corollary follows immediately.

Corollary 4.2 *There is an increasing sequence* $(n_s)_{s\in\mathbb{N}}$ *of natural numbers such that* $(\mathbf{C}_{mc}^{n_s})$ *is a proper hierarchy of* $\mathbf{C}_{mc}$.

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