AN OVERVIEW OF ABSOLUTE CONTINUITY AND ITS APPLICATIONS

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Abstract. The aim of this paper is to illustrate the usefulness of the notion of absolute continuity in a series of fields such as Functional Analysis, Approximation Theory and PDE.

1. Introduction

The basic idea of *absolute continuity* is to control the behavior of a function $f : X \to \mathbb{R}$ via an estimate of the form

$$|f| \leq \varepsilon q + \delta(\varepsilon)p, \quad \text{for every } \varepsilon > 0, \tag{1.1}$$

where $p, q : X \to \mathbb{R}$ are suitably chosen nonnegative functions. Technically, this means that for every $\varepsilon > 0$, one can find $\delta(\varepsilon) > 0$ such that $|f(x)| \leq \varepsilon q(x) + \delta(\varepsilon)p(x)$, for all $x \in X$. Thus the property of absolute continuity can be seen as a relaxation of the condition of domination

$$|f| \leq p.$$

In this respect (1.1) allows us to interpolate between two extreme cases: $|f| \leq q$ and $|f| \leq p$, one appearing as "too weak" and the other "too special".

Measure Theory offers us the important case of σ-additive measures defined on a σ-algebra $\mathcal{T}$ (of subsets of a set T). In this context, a measure $m : \mathcal{T} \to \mathbb{C}$ is said to be *absolutely continuous* with respect to a positive measure $\mu : \mathcal{T} \to \mathbb{R}$ (abbreviated, $m \ll \mu$) if for every $\varepsilon > 0$ there is a $\eta = \eta(\varepsilon) > 0$ such that for all $A \in \mathcal{T}$ with $\mu(A) \leq \eta$ we have

$$|m(A)| \leq \varepsilon.$$

Since m has finite variation $|m|$ (see [10], Theorem 19.13 (v)), the condition $m \ll \mu$ yields

$$|m(A)| \leq \varepsilon + \frac{|m|(T)}{\eta}\mu(A) \quad \text{for all } A \in \mathcal{T} \text{ and } \varepsilon > 0, \tag{1.2}$$

that represents the case of (1.1) when $X = \mathcal{T}$, $f = m$, $q = 1$, $p = \mu$ and $\delta = |m|(T)/\eta$. In turn, (1.2) yields the absolute continuity of m with respect to μ since for every $A \in \mathcal{T}$ with

$$\mu(A) \leq \frac{\varepsilon\eta(\varepsilon/2)}{2\,|m|(T)}$$

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we have $|m(A)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$.

The main criterion of absolute continuity in the above context is provided by the membership of negligible sets:

$$m \ll \mu \text{ if and only if } \mu(A) = 0 \text{ implies } m(A) = 0. \tag{1.3}$$

See [10], Exercise 19.67, p. 339.

The subject of absolutely continuous functions in Real Analysis can be covered by the above discussion since a function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if and only if it is of the form

$$f(x) = f(a) + m([a, x]),$$

for a suitable Borel measure m which is absolutely continuous with respect to the Lebesgue measure.

The theory of inequalities offers many interesting applications where the concept of absolute continuity is instrumental. In particular this is the case of the famous Hardy-Landau-Littlewood inequalities. See [18].

The aim of this paper is to illustrate the usefulness of the notion of absolute continuity in other areas of mathematics such as Functional Analysis, Approximation Theory and PDE. In particular we show how this notion allows us to derive some quantitative facts from different qualitative properties.

Most of the results we discuss below are not in full generality, but it was our option to emphasize ideas rather than technical results.

2. Absolute Continuity in Functional Analysis

Inspired by the case of Measure Theory, the author initiated in the early 70s an operator theoretical generalization of the concept of absolute continuity, which proved to be useful in understanding the properties of weakly compact operators defined on some special Banach spaces such as $C(K)$ and its relatives; as usually, $C(K)$ represents the Banach space (endowed with the sup norm) of all continuous real-valued functions defined on a compact Hausdorff space K.

The basic fact, which led to the concept of absolutely continuous operator, is as follows:

Theorem 1. (*C. P. Niculescu* [15], [16]). *Suppose that E is a Banach space. A bounded linear operator $T \in L(C(K), E)$ is weakly compact if and only if there exists a positive Borel measure μ on K such that for every $\varepsilon > 0$ one can find a $\delta(\varepsilon) > 0$ such that*

$$\|T(f)\| \leq \varepsilon \|f\| + \delta(\varepsilon) \int_K |f| \, d\mu, \tag{2.1}$$

whenever $f \in C(K)$.

Proof. If T is weakly compact, then the set

$$\mathcal{K} = \{|x' \circ T| : x' \in E', \ \ \|x'\| \leq 1\}$$

is relatively weakly compact in $C(K)'$ (see [19], p. 119); according to the Riesz representation theorem (see [10], p. 177), the functionals on a space $C(K)$ can be viewed as Borel regular measures, so here modulus means variation. By a classical result due to A. Grothendieck [9], the relative weak compactness of $\mathcal{K}$ means that for every bounded sequence of Borel measurable functions $f_n : K \to \mathbb{R}$ which is pointwise convergent to 0 we have

$$\lim_{n\to\infty} \int_K f_n d\nu = 0, \quad \text{uniformly for } \nu \in \mathcal{K}. \tag{2.2}$$

Claim: For every $\varepsilon > 0$ there exist a number $\eta(\varepsilon) > 0$ and a finite subset $\mathcal{K}_\varepsilon \subset \mathcal{K}$ such that every Borel measurable function $f : K \to \mathbb{R}$ with $0 \le f \le 1$ and $\sup_{\nu\in\mathcal{K}_\varepsilon} \int_K f d\nu \le \eta(\varepsilon)$ verifies the inequality

$$\sup_{\nu\in\mathcal{K}} \int_K f d\nu \le \varepsilon.$$

Once the claim is proved, we can easily infer that the measure

$$\mu = \sum_{n=1}^{\infty} \left(\frac{1}{2^n} \sup_{\nu\in\mathcal{K}_{1/n}} \nu \right)$$

verifies a condition of the following form,

$$f \in C(K), \ \ \|f\| \le 1, \ \ \int_K |f|\, d\mu \le \tilde{\eta}(\varepsilon) \Rightarrow \|T(f)\| \le \varepsilon, \tag{2.3}$$

where $\tilde{\eta}(\varepsilon) > 0$ can be obtained from $\eta(\varepsilon)$ by rescaling. Now it is clear that T verifies the inequality (2.1) for $\delta(\varepsilon) = \|T\| / \tilde{\eta}(\varepsilon)$.

The *Claim* can be proved by reductio ad absurdum. In fact, if the contrary is true, then there are a number $\varepsilon_0 > 0$ and two sequences $(f_n)_n$ (of Borel measurable functions on K) and $(\nu_n)_n$ (of elements of $\mathcal{K}$) such that

i) $0 \le f_n \le 1$

ii) $\sup_{1\le k\le n} \int_K f_n d\nu_k \le 2^{-n-1}$

iii) $\int_K f_n d\nu_{n+1} \ge \varepsilon_0$

for all n. Put $g_n = \sup\{f_k : k \ge n\}$ and $g = \inf\{g_n : n \ge 1\}$. Then

$$\sup_{1\le k\le n} \int_K g_n d\nu_k \le 2^{-n},$$

so by (2.2) we infer that

$$\int_K g d\nu_k = \lim_{n\to\infty} \int_K g_n d\nu_k = 0$$

uniformly for $k \in \mathbb{N}$, a fact that contradicts the inequalities *iii)* above. Thus the proof of *Claim* is done.

Suppose now that T verifies the estimate (2.1). We shall show that T maps the weak Cauchy sequences of elements of $C(K)$ into norm convergent sequences in E (whence T is weakly compact by a result due to Grothendieck [9]). In fact, if $(f_n)_n$ is a weak Cauchy sequence in $C(K)$, then by Lebesgue's dominated convergence theorem we get

$$\lim_{m,n\to\infty} \int_K |f_m - f_n|\, d\mu = 0$$

and thus from (2.1) we conclude that $(Tf_n)_n$ is a norm Cauchy sequence. ■

Since the inclusion $L^2(\mu) \subset L^1(\mu)$ is continuous, the inequality (2.1) yields the following one,

$$\|T(f)\| \leq \varepsilon \|f\| + \delta(\varepsilon) \left(\int_K |f|^2 \, d\mu \right)^{1/2}. \tag{2.4}$$

According to the Banach-Saks theorem, every bounded sequence in a Hilbert space has a Cesàro converging subsequence. Thus from Theorem 1 we infer the following interesting property of weakly compact operators defined on a $C(K)$ space:

Corollary 1. *If $T \in L(C(K), E)$ is weakly compact, then T maps every bounded sequence into a sequence with Cesàro converging subsequences.*

Another direct consequence of Theorem 1 is as follows:

Corollary 2. *Suppose that $T \in L(C(K), E)$ is an weakly compact operator and $(f_n)_n$ is a bounded sequence of functions in $C(K)$ which converges pointwise to a function $f \in C(K)$. Then $\|T(f_n) - T(f)\| \to 0$.*

For further developments related to Theorem 1 see our papers [14], [15], [16], [17], and the monograph of J. Diestel, H. Jarchow and A. Tonge [7], Ch. 15.

The property of absolute continuity is also instrumental in establishing the Radon-Riesz property for L^p-spaces with $1 \leq p < \infty$. See Corollary 3 below, which is a consequence of following result due to H. Brezis and E. H. Lieb [4], about the "missing term" in Fatou's Lemma:

Theorem 2. . *Let $(f_n)_n$ be a sequence of functions in a space $L^p(\mu)$ with $p \in [1, \infty)$, which verifies the following conditions:*

i) $\sup \|f_n\| < \infty$;

ii) $f_n \to f$ *almost everywhere.*

Then $f \in L^p(\mu)$ *and* $\lim_{n\to\infty} \left(\|f_n\|^p - \|f_n - f\|^p\right) = \|f\|^p$.

Corollary 3. *Assume that $(f_n)_n$ is a sequence of functions in a space $L^p(\mu)$ ($p \in [1, \infty)$) such that:*

i) $\|f_n\| \to \|f\|$;

ii) $f_n \to f$ *almost everywhere.*

Then $\|f_n - f\| \to 0$.

Proof of Theorem 2. We start by noticing the following inequality (that illustrates a property of absolute continuity): For each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\left| |a + b|^p - |a|^p \right| \leq \varepsilon |a|^p + \delta |b|^p \tag{2.5}$$

for all $a, b \in \mathbb{R}$.

This is clear for $p = 1$. For $p > 1$ we shall use the convexity of the function $|x|^p$. Indeed,

$$\begin{aligned} |a + b|^p \leq (|a| + |b|)^p &= \left((1 - \lambda) \frac{|a|}{(1 - \lambda)} + \lambda \frac{|b|}{\lambda} \right)^p \\ &\leq |a|^p + \left((1 - \lambda)^{1-p} - 1 \right) |a|^p + \lambda^{1-p} |b|^p \end{aligned}$$

for all $a, b \in \mathbb{R}$ and $\lambda \in (0, 1)$. For $\lambda = 1 - (1 + \varepsilon)^{-1/(p-1)}$, this inequality yields (2.5).

The membership of f to the space $L^p(\mu)$ is motivated by Fatou's lemma. According to (2.5),

$$\begin{aligned} g_{n,\varepsilon} &= \left(\left| |f_n|^p - |f_n - f|^p - |f|^p \right| - \varepsilon \, |f_n - f|^p \right)^+ \\ &\leq (1+\delta) \, |f|^p \end{aligned}$$

so that by the dominated convergence theorem we get

$$\lim_{n \to \infty} \int g_{n,\varepsilon} d\mu = 0.$$

Taking into account the inequality

$$\left| |f_n|^p - |f_n - f|^p - |f|^p \right| \leq g_{n,\varepsilon} + \varepsilon \, |f_n - f|^p \,,$$

we infer that

$$\limsup_{n \to \infty} \int \left| |f_n|^p - |f_n - f|^p - |f|^p \right| d\mu \leq \varepsilon \sup_{n \in \mathbb{N}} \|f_n - f\|^p \,,$$

whence $\lim_{n \to \infty} \left(\|f_n\|^p - \|f_n - f\|^p \right) = \|f\|^p$. ■

In what follows we shall concentrate on the connection between absolute continuity and the Arzelà-Ascoli criterion of compactness. Roughly speaking, this criterion asserts that in a function space, the property of being relatively compact means the boundedness plus a certain kind of equi-membership.

If M is a metric space, then an estimate of the form

$$|f(s) - f(t)| \leq C d(s,t) \quad \text{for all } s,t \in M$$

is characteristic for the Lipschitz functions $f : M \to \mathbb{R}$. The following relaxation in terms of absolute continuity

$$|f(s) - f(t)| \leq \varepsilon + \delta(\varepsilon) d(s,t) \quad \text{for all } s,t \in M \tag{2.6}$$

represents precisely the condition of uniform continuity. Indeed, a function $f : M \to \mathbb{R}$ is uniformly continuous if and only if there is a nonnegative function $\omega : [0, \infty) \to \mathbb{R}$ such that $\omega(0) = 0$, ω is continuous at $x = 0$ and

$$|f(s) - f(t)| \leq \omega \left(d(s,t) \right) \quad \text{for all } s,t \in M.$$

As a consequence of (2.6) we easily infer the well known fact that every uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ verifies an estimate of the form

$$|f(x)| \leq a \, |x| + b.$$

A characterization of the metric spaces on which every continuous function is also uniformly continuous appeared in [11].

In the special case when M is also compact, the role of the distance function in (2.6) can be taken by any separating function for M. Recall that a *separating function* is a nonnegative continuous function $\gamma : M \times M \to \mathbb{R}$ such that

$$\gamma(s,t) = 0 \text{ implies } s = t.$$

If M is a compact subset of $\mathbb{R}^N$, and $f_1, ..., f_m \in C(M)$ is a family of functions which separates the points of M (in particular this is the case of the coordinate functions $\mathrm{pr}_1, ..., \mathrm{pr}_N$), then

$$\gamma(s,t) = \sum_{k=1}^{m} \left(f_k(s) - f_k(t) \right)^2 \tag{2.7}$$

is a separating function.

More generally, all separating prametrics in General Topology (see [2]) are also separating functions.

The separating functions play an important role in Approximation Theory. This will be detailed in the next section.

Lemma 1. *If K is a compact metric space, and $\gamma : K \times K \to \mathbb{R}$ is a separating function, then any real-valued continuous function f defined on K verifies an estimate of the following form*

$$|f(s) - f(t)| \leq \varepsilon + \delta(\varepsilon)\gamma(s,t) \quad \text{for all } s,t \in K.$$

Proof. In fact, if the estimate above doesn't work, then for a suitable $\varepsilon_0 > 0$ one can find two sequences $(s_n)_n$ and $(t_n)_n$ of elements of K such that

$$|f(s_n) - f(t_n)| \geq \varepsilon_0 + 2^n \gamma(s_n, t_n) \tag{2.8}$$

for all n. Without loss of generality we may assume (by passing to subsequences) that both sequences $(s_n)_n$ and $(t_n)_n$ are convergent, respectively to s and t. Since f is bounded, the inequality (2.8) forces $s = t$. Indeed,

$$\frac{|f(s_n) - f(t_n)|}{2^n} \geq \gamma(s_n, t_n) \to \gamma(s,t) \geq 0.$$

On the other hand, from (2.8) we can infer that $|f(s) - f(t)| \geq \varepsilon_0$ (and thus $s \neq t$). This contradiction shows that the conclusion of Lemma 1 is true. ∎

Lemma 1 is the topological counterpart of a well known result in Measure Theory (precisely, of the equivalence (1.3)).

Now, a careful inspection of the proof of the Arzelà-Ascoli criterion of compactness in a space $C(K)$ shows that this criterion can be reformulated in terms of absolute continuity as follows:

Theorem 3. *If K is a compact metric space, then a bounded subset $\mathcal{A}$ of the Banach space $C(K)$ is relatively compact if and only if for every $\varepsilon > 0$ there is a number $\delta = \delta(\varepsilon) > 0$ such that*

$$|f(s) - f(t)| \leq \varepsilon + \delta(\varepsilon) d(s,t) \quad \text{for all } s,t \in K \text{ and } f \in \mathcal{A}.$$

Here the role of the distance function can be taken by any separating function for K.

We leave the details to the reader, as an exercise.

The above discussion can be easily extended to the case of functions with values in a complete metric space. Besides, the result of Theorem 3 remains valid for many other spaces, for example, for the space $C^r([a,b])$, of all functions $f : [a,b] \to \mathbb{R}$ which are r-times continuously differentiable, endowed with the norm

$$\|f\|_r = \sum_{k=0}^{r} \sup_{a \leq t \leq b} \left| f^{(k)}(t) \right|.$$

In fact, $C^r([a,b])$ is isomorphic to a subspace of $C([a,b] \times \{0, \ldots, r\})$. This remark can be used to prove that the canonical inclusion

$$j : C^{r+1}([a,b]) \to C^r([a,b]) \tag{2.9}$$

is compact.

A variant of Theorem 3 in the case of functions defined on a noncompact domain is as follows:

Theorem 4. *Given a bounded open subset Ω of $\mathbb{R}^N$, we denote by $BC(\Omega)$ the Banach space of all continuous bounded functions $f:\Omega\to\mathbb{R}$, endowed with the sup norm. A bounded subset $\mathcal{A}$ of $BC(\Omega)$ is relatively compact if and only if for every $\varepsilon>0$ there is a number $\delta=\delta(\varepsilon)>0$ such that*

$$|f(s)-f(t)|\leq\varepsilon+\delta(\varepsilon)d(s,t)\quad \text{for all } s,t\in\Omega \text{ and } f\in\mathcal{A}.$$

3. Absolute Continuity and Approximation Theory

We start with the beautiful result of P. P. Korovkin [12], which put in a new perspective the whole subject of approximation in the case of continuous functions. In order to state this result we need a preparation.

Suppose that E is a Banach lattice. A linear operator $T:E\to E$ is called *positive* if

$$x\geq 0 \text{ implies } T(x)\geq 0.$$

Such an operator is always bounded. See [19], p. 84. For $E=C(K)$ this fact can be checked easily since

$$-\|f\|\cdot 1\leq f\leq\|f\|\cdot 1 \text{ implies } -\|f\|\cdot T(1)\leq T(f)\leq\|f\|\cdot T(1)$$

and thus $\|T(f)\|\leq\|T(1)\|\cdot\|f\|$.

Theorem 5. (*P.P. Korovkin* [12]). *Consider the functions $e_0(x)=1$, $e_1(x)=x$, $e_2(x)=x^2$ in $C([0,1])$, and suppose there is given a sequence*

$$T_n:C([0,1])\to C([0,1])\quad(n\in\mathbb{N})$$

of positive linear operators such that $T_n(f)\to f$ uniformly on $[0,1]$ for $f\in\{e_0,e_1,e_2\}$. Then

$$T_n(f)\to f\quad \text{uniformly on } [0,1]$$

for every $f\in C([0,1])$.

The proof is both simple and instructive, so we shall include here the details. The main ingredient is the fact that every function $f\in C([0,1])$ verifies an estimate of the form

$$|f(s)-f(t)|\leq\varepsilon+\delta(\varepsilon)|s-t|^2.$$

See Lemma 1. Then

$$|f-f(t)e_0|\leq\varepsilon e_0+\delta(\varepsilon)\left(e_2-2te_1+t^2e_0\right)$$

which implies that $|T_n(f)(s)-f(t)T_n(e_0)(s)|$ is bounded above by

$$\varepsilon T_n(e_0)(s)+\delta(\varepsilon)[T_n(e_2)(s)-2tT_n(e_1)(s)+t^2T_n(e_0)(s)]$$

for every $s\in[0,1]$. Therefore

$$\begin{aligned}|T_n(f)(t)-f(t)|&\leq|T_n(f)(t)-f(t)T_n(e_0)(t)|+|f(t)|\cdot|T_n(e_0)(t)-1|\\&\leq\varepsilon T_n(e_0)(t)+\delta(\varepsilon)[T_n(e_2)(t)-2tT_n(e_1)(t)+t^2T_n(e_0)(t)]\\&\qquad\qquad+\|f\|\cdot|T_n(e_0)(t)-1|\end{aligned}$$

whence we conclude that $T_n(f)\to f$ uniformly on $[0,1]$.

The above argument (based on Lemma 1) is actually strong enough to cover a much more general result:

Theorem 6. *Suppose that K is a compact metric space and γ is a separating function for M. If $T_n : C(K) \to C(K)$ $(n \in \mathbb{N})$ is a sequence of positive linear operators such that $T_n(1) \to 1$ uniformly and*

$$T_n(\gamma(\cdot, t))(t) \to 0 \quad \textit{uniformly in } t, \tag{3.1}$$

then $T_n(f) \to f$, uniformly for each $f \in C(K)$.

Proof. In fact, taking into account Lemma 1, we have

$$\begin{aligned} |T_n(f)(t) - f(t)| &\leq |T_n(f)(t) - f(t)T_n(1)(t)| + |f(t)| \cdot |T_n(1)(t) - 1| \\ &\leq T_n\left(|f - f(t)|\right)(t) + \|f\| \cdot |T_n(1)(t) - 1| \\ &\leq T_n\left(\varepsilon + \delta(\varepsilon)\gamma(\cdot, t)\right)(t) + \|f\| \cdot |T_n(1)(t) - 1| \\ &\leq \varepsilon T_n(1)(t) + \delta(\varepsilon)T_n(\gamma(\cdot, t))(t) \\ &\qquad + \|f\| \cdot |T_n(1)(t) - 1| \end{aligned}$$

and the conclusion follows from our hypothesis. ■

Theorem 6 is a variant of a recent result by H. E. Lomeli and C. L. Garcia [13] (based on a slightly different concept of separating function).

In order to understand how Theorem 6 extends the Theorem of Korovkin, let us consider the case were M is a compact subset of $\mathbb{R}^N$ and

$$\gamma(s, t) = \sum_{k=1}^{m} \left(f_k(s) - f_k(t)\right)^2$$

is the separating function (associated to a family of functions $f_1, ..., f_m \in C(M)$ which separates the points of M). In this case the condition (3.1) of uniform convergence can be obtained by imposing that

$$T_n(f) \to f \quad \text{uniformly for } t \in M,$$

for each of the functions $f \in \left\{1, f_1, ..., f_m, f_1^2, ..., f_m^2\right\}$. For $M = [0, 1]$ the identity separates the points of M, a fact that leads to the Theorem of Korovkin.

Corollary 4. (*Weierstrass Approximation Theorem*). *If f belongs to $C([a, b])$, then there exists a sequence of polynomials that converges to f uniformly on $[a, b]$.*

Proof. We can restrict to the case where $[a, b] = [0, 1]$ (by performing the linear change of variable $t = (x - a)/(b - a)$). Then we apply Theorem 6 for $M = [0, 1]$, $\gamma(s, t) = (s - t)^2$ and T_n the nth Bernstein operator,

$$T_n(f)(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} f\left(k/n\right).$$

In fact,

$$T_n(\gamma(\cdot, t))(t) = \frac{t(1 - t)}{n}$$

for all $t \in [0, 1]$. This computation is part of Bernstein's classical proof of the Weierstrass Approximation Theorem. See [6], pp. 290-292. ■

Corollary 5. (*Féjer Approximation Theorem*). *The Cesàro averages of the Fourier partial sums of a continuous function f of period 2π converge uniformly to f.*

Proof. We have to consider the Féjer kernels

$$K_n(t) = \begin{cases} \frac{1}{2n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 & \text{if } t \neq 2k\pi,\ k \in \mathbb{Z} \\ \frac{n}{2} & \text{if } t = 2k\pi,\ k \in \mathbb{Z}. \end{cases}$$

A direct computation shows that

$$K_n(t) = \frac{1}{2} + \frac{1}{n}\sum_{m=1}^{n-1}\sum_{k=1}^{m}\cos kt.$$

The result of Corollary 5 follows from Theorem 6, applied to $M = \mathbb{R} \bmod 2\pi$, $\gamma(s,t) = 1 - \cos(s-t)$ and the sequence of operators

$$T_n(f)(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t-s)f(s)ds. \quad \blacksquare$$

Since Lemma 1 does not work for all metric spaces, we cannot use arbitrary separating functions in the case of noncompact metric spaces. However we can still formulate a Korovkin type criterion of convergence for operators acting on the Banach lattice $BUC(M)$ (of all uniformly continuous bounded functions on the metric space M, endowed with the sup norm).

Theorem 7. *Suppose that M is a metric space and*

$$T_n : BUC(M) \to BUC(M) \quad (n \in \mathbb{N})$$

is a sequence of positive linear operators such that $T_n(1) \to 1$ uniformly and

$$T_n(d(\cdot,t)^\alpha)(t) \to 0 \quad \textit{uniformly in } t, \tag{3.2}$$

for a positive real number α. Then $T_n(f) \to f$, uniformly for each $f \in BUC(M)$.

The usual technique of mollification for approximating the continuous functions by smooth functions can be derived as a consequence of Theorem 7. In the next theorem, a *mollifier* is meant as any continuous function $\varphi : \mathbb{R}^N \to \mathbb{R}$ such that

$$\varphi(x) \leq C(1 + \|x\|)^{-p} \text{ for some } C > 0 \text{ and } p > N$$

and

$$\int_{\mathbb{R}^N} \varphi(x)dx = 1.$$

The standard mollifier is the function $\varphi(x) = (2\pi)^{-N/2}\, e^{-\|x\|^2/2}$.

Theorem 8. *If $\varphi : \mathbb{R}^N \to \mathbb{R}$ is a mollifier and $f \in BUC(\mathbb{R}^N)$, then*

$$n^N \int_{\mathbb{R}^N} \varphi\left(n(y-x)\right) f(y)dy \to f(x)$$

uniformly on $\mathbb{R}^N$.

Proof. We apply Theorem 7 for $M = \mathbb{R}^N$, $\alpha \in (p - N)$ arbitrarily fixed, and the sequence of operators

$$T_n(f)(x) = n^N \int_{\mathbb{R}^N} \varphi\left(n(y-x)\right) f(y)ds.$$

In order to prove that the condition (3.2) is fulfilled we need the following estimate:

$$\|y-x\|^{\alpha}\varphi\left(n(y-x)\right) \leq \frac{n^{N}\|y-x\|^{\alpha}}{C(1+n\|y-x\|)^{p}} \leq \frac{n^{N-\alpha}}{C(1+n\|y-x\|)^{p-\alpha}}.$$

Then

$$0 \leq T_n(\|\cdot - x\|^{\alpha})(x) = n^N \int_{\mathbb{R}^N} \varphi\left(n(y-x)\right)\|y-x\|^{\alpha}\,ds \leq C'\frac{1}{n^{\alpha}}\int_{\mathbb{R}^N}\frac{ds}{(1+n\|y-x\|)^{p-\alpha}},$$

where C' is a constant and the integral in the right hand side is convergent because $p-\alpha > N$. Consequently $T_n(\|\cdot - x\|^{\alpha})(x) \to 0$ uniformly, as $n \to \infty$, and the proof is complete. ■

The technique of mollification works outside the framework of continuous functions. It would be interesting to enlarge the theory above to encompass some spaces of differentiable functions (for example, the Sobolev spaces). A nice account of the most significant developments in the Korovkin theory (including Bauer's approach [3] in terms of Choquet boundary) can be found in the monograph [1].

4. Absolute Continuity and PDE

There are many instances when the concept of absolute continuity appears in PDE (see [8]) but we shall restrict here to the remarkable theorem of F. Rellich concerning the compact embedding of Sobolev spaces.

Theorem 9. *If Ω is a bounded open subset of $\mathbb{R}^N$ then the canonical injection*

$$i : \mathring{H}^{m+1}(\Omega) \to \mathring{H}^{m}(\Omega)$$

is compact.

Recall that $\mathring{H}^{m}(\Omega)$ is the closure of $C_c^{\infty}(\Omega)$ into $H^m(\Omega)$, the Sobolev space of all functions $f : \Omega \to \mathbb{R}$ that have weak derivatives $D^{\alpha}f \in L^2\left(\mathbb{R}^N\right)$ of all orders α with $|\alpha| \leq m$. The natural norm on $H^m(\Omega)$ (and thus on $\mathring{H}^{m}(\Omega)$) is

$$||f||_{H^m} = \left(\sum_{|\alpha|\leq m}\int_{\Omega}|D^{\alpha}f(x)|^2\,dx\right)^{1/2}.$$

Before to enter the details of Theorem 9, we shall discuss an easy (though important) application, related to a property of absolute continuity of compact operators.

Lemma 2. *(Ehrling's Lemma). Assume that E, F, G are Banach spaces. If $T \in L(E,F)$ is a compact linear operator and $S \in L(F,G)$ is an one-to-one bounded linear operator, then for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that*

$$\|Tx\| \leq \varepsilon\|x\| + \delta(\varepsilon)\|S(Tx)\| \quad \text{for all } x \in E.$$

The proof is similar to the proof of Lemma 1, and we shall omit the details.

By combining Ehrling's Lemma with Theorem 9 we get the estimate

$$||f||_{H^{m-1}} \leq \varepsilon \, ||f||_{H^m} + \delta(\varepsilon) \, ||f||_{L^2} \quad \text{for all } f \in \mathring{H}^m(\Omega),$$

which yields

$$||f||_{H^{m-1}} \leq \frac{1}{2} \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha f(x)|^2 \, dx \right)^{1/2} + \frac{1}{2} \, ||f||_{H^{m-1}} + \delta(1/2) \, ||f||_{L^2}$$

that is,

$$||f||_{H^{m-1}} \leq \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha f(x)|^2 \, dx \right)^{1/2} + 2\delta(1/2) \, ||f||_{L^2} .$$

Therefore the norm $||\cdot||_{H^m}$ is equivalent to the norm

$$|f|_{H^m} = \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha f(x)|^2 \, dx \right)^{1/2} + ||f||_{L^2} .$$

The above renorming argument is typical for many Banach spaces of differentiable functions. See [8].

The usual proof of Theorem 9 (and its generalization to the case of Sobolev spaces $\mathring{W}^{m,p}(\Omega)$) is obtained via the mollification technique described in Theorem 8. However it is possible to provide an alternative argument based on Fourier transform.

Indeed, $\mathring{H}^m(\Omega)$ can be viewed as a subspace of $\mathring{H}^m(\mathbb{R}^N)$. The later space has a very simple description in terms of Fourier transform:

$$\mathring{H}^m(\mathbb{R}^N) = \left\{ f \in L^2\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} \left(1 + ||\xi||^2\right)^m \left|\widehat{f}(\xi)\right|^2 d\xi < \infty \right\}$$

Moreover, $||\cdot||_{H^m}$ on $\mathring{H}^m(\mathbb{R}^N)$ is equivalent to the norm $||\,|\cdot|\,||_{H^m}$, where

$$||\,|f|\,||_{H^m} = \left(\int_{\mathbb{R}^N} \left(1 + ||\xi||^2\right)^m \left|\widehat{f}(\xi)\right|^2 d\xi \right)^{1/2} .$$

This gives us a constant $C\,(m) > 0$ such that $||\cdot||_{H^m} \leq C(m)||\,|\cdot|\,||_{H^m}$.

Let $\varepsilon > 0$. Then there is number $A > 0$ such that $1 + ||\xi||^2 \geq C(m-1)/\varepsilon$ for $||\xi|| \geq A$. Consequently, for every sequence $(f_k)_k$ of functions in the unit ball of $\mathring{H}^m(\mathbb{R}^N)$ we have

$$\begin{aligned} ||f_j - f_k||^2_{H^{m-1}} &\leq C(m-1) \int_{\mathbb{R}^N} \left(1 + ||\xi||^2\right)^{m-1} \left|\widehat{f}_j(\xi) - \widehat{f}_k(\xi)\right|^2 d\xi \\ &\leq \varepsilon \int_{||\xi|| > A} \left(1 + ||\xi||^2\right)^m \left|\widehat{f}_j(\xi) - \widehat{f}_k(\xi)\right|^2 d\xi \\ &\quad + \delta(\varepsilon) \int_{||\xi|| \leq A} \left|\widehat{f}_j(\xi) - \widehat{f}_k(\xi)\right|^2 d\xi. \end{aligned} \tag{4.1}$$

The Fourier transform of every function in $\mathring{H}^m(\Omega)$ is holomorphic on $\mathbb{C}^N$, and the Cauchy-Schwarz inequality shows that for every compact subset $K \subset \mathbb{C}^N$ there is a constant $M = M(K) > 0$ such that

$$\sup_{\xi \in K} \left| \widehat{f}(\xi) \right| \leq M \, ||f||_{H^m}$$

for all functions $f \in \mathring{H}^m(\Omega)$. Therefore the functions $(\widehat{f_k})_k$ are uniformly bounded on the compact subsets of $\mathbb{C}^N$. Because they are holomorphic, a compactness principle due to P. Montel assures us that a subsequence should be uniformly convergent on each compact subset of $\mathbb{C}^N$. See [5], p. 209. Taking into account the estimate (4.1), that subsequence should also verify $\limsup_{j,k \to \infty} ||f_j - f_k||^2_{H^{m-1}} = 0$. The proof of Theorem 9 is done.

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