RICH FAMILIES, W-SPACES AND THE PRODUCT OF BAIRE SPACES

PEIJIE LIN AND WARREN B. MOORS*

ABSTRACT. In this paper we prove a theorem more general than the following. Suppose that X is a Baire space and Y is the product of hereditarily Baire metric spaces then $X \times Y$ is a Baire space.

2000 *Mathematics Subject Classification*: Primary 54B10, 54C35; Secondary 54E52.

Keywords and Phrases: Baire space; barely Baire space; W-space; rich family.

1. INTRODUCTION

A topological space X is said to be a *Baire* space if for each sequence $(O_n : n \in \mathbb{N})$ of dense open subsets of X, $\bigcap_{n\in\mathbb{N}} O_n$ is dense in X and a Baire space Y is called *barely Baire* if there exists a Baire space Z such that $Y \times Z$ is not Baire. It is well known that there exist metrizable barely Baire spaces, (see [5]). On the other hand it has recently been shown that the product of a Baire space X with a hereditarily Baire metric space Y is Baire, [7]. In that same paper the author claims in a "Remark" that the hypothesis on Y can be reduced to: "Y is the product of hereditarily Baire metric spaces". In this paper we substantiate this claim.

The main result of this paper relies upon two notions. The first, which is that of a W-space [6], is recalled in Section 2. The second, which is that of a "rich family" is considered in Section 3. In Section 4, we shall prove our main theorem which states that the product of a Baire space with a W-space that possesses a rich family of Baire subspaces is Baire.

2. W-SPACES

In this paper all topological spaces are assumed to be Hausdorff and nonempty. Furthermore, if X is a topological space and $a \in X$ then we shall always denote by $\mathcal{N}(a)$ the set of all neighbourhoods of a.

For any point a in a topological space X we can consider the following two player topological game, called the $G(a)$-*game*. This game is played between the players α and β and although it may seem unfair, β will always

*The second author is supported by the Marsden Fund research grant, UOA0422, administered by the Royal Society of New Zealand.

be granted the priviledge of the first move. To define this game we must first specify the rules and then also specify the definition of a win.

The moves of the player α are simple. He/she must always select a neighbourhood of the point a. However, the moves of the player β depend upon the previous move of α. Specifically, for his/her first move β may select any point $x_1 \in X$. For α's first move, as mentioned earlier, α must select a neighbourhood O_1 of a. Now, for β's second move he/she must select a point $x_2 \in O_1$. For α's second move he/she is entitled to select any neighbourhood O_2 of a. In general, if α has chosen $O_n \in \mathcal{N}(a)$ as his/her n^{th} move of the $G(a)$-game then β is obliged to choose a point $x_{n+1} \in O_n$. The response of α is then simply to choose any neighbourhood O_{n+1} of a. Continuing in this fashion indefinitely, the players α and β produce a sequence $((x_n, O_n) : n \in \mathbb{N})$ of ordered pairs with $x_{n+1} \in O_n \in \mathcal{N}(a)$ for all $n \in \mathbb{N}$, called a *play* of the $G(a)$-game. A *partial play* $((x_k, O_k) : 1 \leq k \leq n)$ of the $G(a)$-game consists of the first n moves of a play of the $G(a)$-game. We shall declare α the *winner* of a play $((x_n, O_n) : n \in \mathbb{N})$ of the $G(a)$-game if $a \in \overline{\{x_n : n \in \mathbb{N}\}}$, otherwise, β is the winner. That is, β is declared the winner of the play $((x_n, O_n) : n \in \mathbb{N})$ if, and only if, $a \notin \overline{\{x_n : n \in \mathbb{N}\}}$.

A *strategy* for the player α is a rule that specifies his/her moves in every possible situation that can occur. More precisely, a strategy for α is an inductively defined sequence of functions $t := (t_n : n \in \mathbb{N})$. The domain of t_1 is X^1 and for each $(x_1) \in X^1$, $t_1(x_1) \in \mathcal{N}(a)$, i.e., $((x_1, t_1(x_1)))$ is a partial play. Inductively, if $t_1, t_2, \ldots, t_n$ have been defined then the domain of t_{n+1} is defined to be,

$$\{(x_1, x_2, \ldots, x_{n+1}) \in X^{n+1} : (x_1, x_2, \ldots, x_n) \in \mathrm{Dom}(t_n) \\ \text{and } x_{n+1} \in t_n(x_1, x_2, \ldots, x_n)\}.$$

For each $(x_1, x_2, \ldots, x_{n+1}) \in \mathrm{Dom}(t_{n+1})$, $t_{n+1}(x_1, x_2, \ldots, x_{n+1}) \in \mathcal{N}(a)$. Equivalently, for each $(x_1, x_2, \ldots, x_{n+1}) \in \mathrm{Dom}(t_{n+1})$, $((x_k, t_k(x_1, \ldots, x_k)) : 1 \leq k \leq n+1)$ is a partial play.

A *partial t-play* is a finite sequence $(x_1, x_2, \ldots, x_n) \in X^n$ such that $(x_1, x_2, \ldots, x_n) \in \mathrm{Dom}(t_n)$ or, eqivalently, if $x_{k+1} \in t_k(x_1, x_2, \ldots, x_k)$ for all $1 \leq k < n$. A *t-play* is an infinite sequence $(x_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(x_1, x_2, \ldots x_n)$ is a partial t-play.

A strategy $t := (t_n : n \in \mathbb{N})$ for the player α is said to be a *winning strategy* if each play of the form $((x_n, t_n(x_1, x_2, \ldots, x_n)) : n \in \mathbb{N})$ is won by α, or equivalently, if $a \in \overline{\{x_n : n \in \mathbb{N}\}}$ for each t-play $(x_n : n \in \mathbb{N})$.

A topological space X is called a *W-space* if α has a winning strategy in the $G(a)$-game for each $a \in X$, [6].

In the remainder of this section we shall recall some relevant fact concerning W-spaces.

Theorem 2.1. [6, Theorem 3.3] *Every first countable space is a W-space.*

There are of course many W-spaces that are not first countable, see Example 2.7.

A topological space X is said to have *countable tightness* if for each nonempty subset A of X and each $p \in \overline{A}$, there exists a countable subset $C \subseteq A$ such that $p \in \overline{C}$.

Proposition 2.2. [6, Corollary 3.4] *Every W-space has countable tightness.*

Proposition 2.3. [6, Theorem 3.1] *If X is a W-space and $\varnothing \neq A \subseteq X$ then A is a W-space.*

Lemma 2.4. [6, Theorem 3.9] *Suppose that X is a W-space and $a \in X$, then the player α possesses a strategy $s := (s_n : n \in \mathbb{N})$ in the $G(a)$-game such that every s-play converges to a.*

For the remainder of this paper whenever we shall consider a W-space X with $a \in X$ we shall assume that the player α is employing a strategy t, in the $G(a)$-game, in which every t-play converges to a.

Let $\{X_s : s \in S\}$ be a nonempty family of topological spaces and let $a \in \Pi_{s\in S}X_s$. The Σ-*product* of this family with *base point* a, denoted by $\Sigma_{s\in S}X_s(a)$, is the set of all $x \in \Pi_{s\in S}X_s$ such that $x(s) \neq a(s)$ for at most countably many $s \in S$. For each $x \in \Sigma_{s\in S}X_s(a)$, the *support of* x is defined by $\mathrm{supp}(x) := \{s \in S : x(s) \neq a(s)\}$.

Theorem 2.5. [6, Theorem 4.6] *Suppose that $\{X_s : s \in S\}$ is a nonempty family of W-spaces. If $a \in \Pi_{s\in S}X_s$ then $\Sigma_{s\in S}X_s(a)$ is a W-space.*

Corollary 2.6. [6, Theorem 4.1] *If $\{X_n : n \in \mathbb{N}\}$ are W-spaces, then so is $\Pi_{n\in\mathbb{N}}X_n$.*

Example 2.7. Suppose that S is a nonempty set. For each $s \in S$, let $X_s := [0, 1]$ and define $a : S \to [0, 1]$ by, $a(s) := 0$ for all $s \in S$. Then by Theorem 2.5, $X := \Sigma_{s\in S}X_s(a)$ is a W-space. However, X is not first countable whenever S is uncountable.

3. Rich familes

Let X be a topological space, and let $\mathcal{F}$ be a family of nonempty, closed and separable subsets of X. Then $\mathcal{F}$ is *rich* if the following two conditions are satisfied:

(i) for every separable subspace Y of X, there exists an $F \in \mathcal{F}$ such that $Y \subseteq F$;
(ii) for every increasing sequence $(F_n : n \in \mathbb{N})$ in $\mathcal{F}$, $\overline{\bigcup_{n\in\mathbb{N}} F_n} \in \mathcal{F}$.

For any topological space X, the collection of all rich families of subsets forms a partially ordered set, under the binary relation of set inclusion. This partially ordered set has a greatest element, $\mathcal{S}_X := \{S \in 2^X : S$ is a nonempty, closed and separable subset of $X\}$. On the other hand, if X is a separable space, then the partially ordered set has a least element, namely $\{X\}$.

Next we present an important property of rich families. For a proof of this see [2, Proposition 1.1].

Proposition 3.1. *Suppose that X is a topological space. If $\{\mathcal{F}_n : n \in \mathbb{N}\}$ are rich families then so is $\bigcap_{n\in\mathbb{N}} \mathcal{F}_n$.*

Suppose that X is a topological space and S is a separable subset, it can be easily verified that the family $\mathcal{F}_S := \{F \in \mathcal{S}_X : S \subseteq F\}$ is rich. Hence, whenever X is an infinite set and $\mathcal{F}$ is a rich family of subsets of X, then we can always assume, by possibly passing to a sub-family, that all the members of $\mathcal{F}$ are infinite. Indeed, if X has a countably infinite subset A, then by Proposition 3.1, $\mathcal{F} \cap \mathcal{F}_A \subseteq \mathcal{F}$ is a rich family whose members are all infinite.

Proposition 3.2. *If X is a topological space with countable tightness (e.g. if X is a W-space) and E is a dense subset of X then*

$$\mathcal{F} := \{F \in \mathcal{S}_X :\ E \cap F \text{ is dense in } F\}$$

is a rich family.

Proof: Let Y be a separable subspace of X, then Y has a countable dense subset $D := \{d_n : n \in \mathbb{N}\}$. Since X has countable tightness, for each $n \in \mathbb{N}$, there is a countable subset $C_n \subseteq E$ such that $d_n \in \overline{C_n}$. Let $F := \overline{\bigcup_{n\in\mathbb{N}} C_n}$, then $Y = \overline{D} \subseteq F \in \mathcal{S}_X$ and

$$F = \overline{\bigcup_{n\in\mathbb{N}} C_n} \subseteq \overline{E \cap F} \subseteq F.$$

Therefore, $F \in \mathcal{F}$. Now suppose that $(F_n : n \in \mathbb{N})$ is an increasing sequence in $\mathcal{F}$. Then $F' := \overline{\bigcup_{n\in\mathbb{N}} F_n} \in \mathcal{S}_X$ and $F' \cap E$ is dense in F'. Therefore, $F' \in \mathcal{F}$. $\square$

Theorem 3.3. *Suppose that X is a topological space with countable tightness (in particular if X is a W-space) that possesses a rich family $\mathcal{F}$ of Baire subspaces then X is also a Baire space.*

Proof: Let $\{O_n : n \in \mathbb{N}\}$ be dense open subsets of X. For each $n \in \mathbb{N}$, let $\mathcal{F}_n := \{F \in \mathcal{S}_X : O_n \cap F \text{ is dense in } F\}$, then $\mathcal{F}_n$ is a rich family by Proposition 3.2. Let $\mathcal{F}^* = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n \cap \mathcal{F}$, then $\mathcal{F}^*$ is also a rich family by Proposition 3.1. For each $F \in \mathcal{F}^*$, $\bigcap_{n\in\mathbb{N}}(O_n \cap F)$ is dense in F since F is a Baire space. Let $x \in X$, then there is $F \in \mathcal{F}^*$ such that $x \in F$. Then $x \in \overline{\bigcap_{n\in\mathbb{N}}(O_n \cap F)} \subseteq \overline{\bigcap_{n\in\mathbb{N}} O_n}$. Therefore, $\overline{\bigcap_{n\in\mathbb{N}} O_n} = X$. $\square$

Suppose that $\{X_s : s \in S\}$ is a nonempty family of topological spaces and $a \in \Pi_{s\in S} X_s$. A *cube E in $\Sigma_{s\in S} X_s(a)$* is any nonempty product $\Pi_{s\in S} E_s \subseteq \Sigma_{s\in S} X_s(a)$. The set $C_E := \{s \in S : E_s \neq \{a(s)\}\}$ is at most countable and E is homeomorphic to $\Pi_{s\in C_E} E_s$. If for each $s \in S$, $\mathcal{F}_s$ is a rich family of subsets of X_s then the Σ*-product of the rich families, with the base point*

$a \in \Pi_{s\in S} X_s$, denoted by $\Sigma_{s\in S}\mathcal{F}_s(a)$, is the set of all cubes $E := \Pi_{s\in S} E_s$ in $\Sigma_{s\in S} X_s(a)$ such that $E_s \in \mathcal{F}_s$ for each $s \in C_E$.

Lemma 3.4. *Let $\{X_s : s \in S\}$ be a nonempty family of topological spaces. For each $s \in S$, let $(E^s_n : n \in \mathbb{N})$ be an increasing sequence of nonempty subsets of X_s. Then $\overline{\bigcup_{n\in\mathbb{N}}(\Pi_{s\in S} E^s_n)} = \overline{\Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E^s_n)}$.*

Proof: It is easy to see that $\overline{\bigcup_{n\in\mathbb{N}}(\Pi_{s\in S} E^s_n)} \subseteq \overline{\Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E^s_n)}$ since for all $n \in \mathbb{N}$, $\Pi_{s\in S} E^s_n \subseteq \overline{\Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E^s_n)}$.

Let $x \in \overline{\Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E^s_n)}$ and let $U := \Pi_{s\in S} U_s$ be a basic neighbourhood of x. Then there exists $y \in U \cap \Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E^s_n)$. Let M be the finite set $\{s \in S : U_s \neq X_s\}$, and let $N_s := \min\{n \in \mathbb{N} : y(s) \in E^s_n\}$ for all $s \in M$. Let $N := \max\{N_s : s \in M\}$, then $y(s) \in E^s_N$ for all $s \in M$. Let $a \in \Pi_{s\in S} E^s_N$ and let $y' \in U$ be defined by $y'(s) := y(s)$ for all $s \in M$ and $y'(s) := a(s)$ for all $s \in S \setminus M$. Since $y' \in \Pi_{s\in S} E^s_N$, $U \cap \bigcup_{n\in\mathbb{N}}(\Pi_{s\in S} E^s_n) \neq \varnothing$. Therefore, $x \in \overline{\bigcup_{n\in\mathbb{N}}(\Pi_{s\in S} E^s_n)}$. □

Theorem 3.5. *Suppose that $\{X_s : s \in S\}$ is a nonempty family of topological spaces and $a \in \Pi_{s\in S} X_s$. If for each $s \in S$, $\mathcal{F}_s$ is a rich family of subsets of X_s, then $\Sigma_{s\in S}\mathcal{F}_s(a)$ is a rich family of subsets of $\Sigma_{s\in S} X_s(a)$.*

Proof: Let Y be a separable subspace of $\Sigma_{s\in S} X_s(a)$, then it has a countable dense subset D. Let $C := \bigcup_{d\in D} \mathrm{supp}(d)$, then C is a countable set. For each $s \in C$, let P_s be the projection of D onto X_s, then P_s is countable and hence there is some $E_s \in \mathcal{F}_s$ such that $\overline{P_s} \subseteq E_s$. For each $s \in S \setminus C$, let $E_s := \{a(s)\}$. Let $F := \Pi_{s\in S} E_s$, then $F \in \Sigma_{s\in S}\mathcal{F}_s(a)$ and $Y \subseteq F$.

Let $(E_n : n \in \mathbb{N})$ be an increasing sequence in $\Sigma_{s\in S}\mathcal{F}_s(a)$. For each cube $E_n \in \Sigma_{s\in S}\mathcal{F}_s(a)$, let $E_n := \Pi_{s\in S} E^s_n$. Then by Lemma 3.4

$$\overline{\bigcup_{n\in\mathbb{N}} E_n} = \overline{\bigcup_{n\in\mathbb{N}} (\Pi_{s\in S} E^s_n)} = \overline{\Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E^s_n)} = \Pi_{s\in S}(\overline{\bigcup_{s\in S} E^s_n}).$$

It now follows that $\overline{\bigcup_{n\in\mathbb{N}} E_n} \in \Sigma_{s\in S}\mathcal{F}_s(a)$. □

4. Baire spaces and Σ-products

A subset R of a topological space X is *residual* in X if there exist dense open subsets $\{O_n : n \in \mathbb{N}\}$ of X such that $\bigcap_{n\in\mathbb{N}} O_n \subseteq R$.

For any subset R of a topological space X we can consider the following two player topological game, called the *$BM(R)$-game.* This game is played between two players α and β and, as with the $G(a)$-game, the player β is always granted the priviledge of the first move. To define this game we must first specify the rules and then specify the definition of a win.

The player β's first move is to select a nonempty open subset B_1 of X. For α's first move he/she must also select a nonempty open subset A_1 of B_1. Now, for β's second move he/she must select a nonempty open subset B_2 of A_1. For α's second move he/she must select a nonempty open subset A_2

of B_2. In general, if α has chosen A_n as his/her n^{th} move of the $BM(R)$-game then β is obliged to select a nonempty open subset B_{n+1} of A_n. The response of α is then simply to select any nonempty open subset A_{n+1} of B_{n+1}. Continuing in this fashion indefinitely the players α and β produce a sequence $((B_n, A_n) : n \in \mathbb{N})$ of ordered pairs of nonempty open subsets of X such that $B_{n+1} \subseteq A_n \subseteq B_n$ for all $n \in \mathbb{N}$, called a *play* of the $BM(R)$-game. A *partial play* $((B_k, A_k) : 1 \leq k \leq n)$ of the $BM(R)$-game consists of the first n moves of a play of the $BM(R)$-game. We shall declare α the *winner* of a play $((B_n, A_n) : n \in \mathbb{N})$ of the $BM(R)$-game if $\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} B_n \subseteq R$, otherwise, β is declared the winner. That is, β is the winner if, and only if, $\bigcap_{n\in\mathbb{N}} B_n \not\subseteq R$.

A *strategy* for the player α is an inductively defined sequence of functions $t := (t_n : n \in \mathbb{N})$. The domain of t_1 is the family of all nonempty open subsets of X and for each $B_1 \in \mathrm{Dom}(t_1)$, $t_1(B_1)$ must be a nonempty open subset of B_1 or, equivalently, for each $B_1 \in \mathrm{Dom}(t_1)$, $t_1(B_1)$ is defined so that $((B_1, t_1(B_1)))$ is a partial play of the $BM(R)$-game. Inductively, if $t_1, t_2, \ldots, t_n$ have been defined then the domain of t_{n+1} is defined to be:

$$\{(B_1, B_2, \ldots, B_{n+1}) : (B_1, B_2, \ldots, B_n) \in \mathrm{Dom}(t_n) \text{ and } \\ B_{n+1} \text{ is a nonempty open subset of } t_n(B_1, B_2, \ldots, B_n)\}.$$

For each $(B_1, B_2, \ldots, B_{n+1}) \in \mathrm{Dom}(t_{n+1})$, $t_{n+1}(B_1, B_2, \ldots, B_{n+1})$ must be a nonempty open subset of B_{n+1}. Alternatively, but equivalently, for each $(B_1, B_2, \ldots, B_{n+1}) \in \mathrm{Dom}(t_{n+1})$, $t_{n+1}(B_1, B_2, \ldots, B_{n+1})$ is defined so that $((B_k, t_k(B_1, B_2, \ldots, B_k)) : 1 \leq k \leq n+1)$ is a partial play. A *partial t-play* is a finite sequence $(B_1, B_2, \ldots, B_n)$ such that $(B_1, B_2, \ldots, B_n) \in \mathrm{Dom}(t_n)$ or, equivalently, B_{k+1} is a nonempty open subset of $t_k(B_1, B_2, \ldots, B_k)$ for all $1 \leq k < n$. A *t-play* is an infinite sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(B_1, B_2, \ldots, B_n)$ is a partial t-play.

A strategy $t := (t_n : n \in \mathbb{N})$ for the player α is said to be a *winning strategy* if each play of the form $((B_n, t_n(B_1, B_2, \ldots, B_n)) : n \in \mathbb{N})$ is won by α, or equivalently, if $\bigcap_{n\in\mathbb{N}} B_n \subseteq R$ for each t-play $(B_n : n \in \mathbb{N})$. For more information on the $BM(R)$-game see [3].

Our interest in the $BM(R)$-game is revealed in the next lemma.

Lemma 4.1 ([9]). *Let R be a subset of a topological space X. Then R is residual in X if, and only if, the player α has a winning strategy in the $BM(R)$-game played on X.*

The next simple result plays a key role in the proof of our main theorem (Theorem 4.3).

Lemma 4.2. *Let X and Y be topological spaces and let O be a dense open subset of $X \times Y$. Given nonempty open subsets V_1, V_2,...,V_m of Y and a nonempty open subset U of X, there exists a nonempty open subset $W \subseteq U$ and elements $y_i \in V_i$, $1 \leq i \leq m$, such that $W \times \{y_1, ..., y_m\} \subseteq O$.*

Proof: The result will be shown inductively on m.

Base Step: $m = 1$. Since $U \times V_1$ is nonempty and open in $X \times Y$ and O is dense and open in $X \times Y$, $(U \times V_1) \cap O$ is a nonempty open subset of $X \times Y$. Therefore, there is a nonempty open subset $W \subseteq U$ and an element $y_1 \in V_1$ such that $W \times \{y_1\} \subseteq (U \times V_1) \cap O \subseteq O$.

Inductive Step: Suppose that the result holds for $m = k$ and consider the case when $m = k + 1$. According to the inductive hypothesis, there exists a nonempty open subset $W' \subseteq U$ and elements $y_i \in V_i$, $1 \leq i \leq k$, such that $W' \times \{y_1, ..., y_k\} \subseteq O$. By repeating the base step, there is a nonempty open subset $W \subseteq W'$ and an element $y_{k+1} \in V_{k+1}$ such that $W \times \{y_{k+1}\} \subseteq O$. Clearly, $W \times \{y_1, ..., y_{k+1}\} \subseteq O$. □

Theorem 4.3. *Suppose that Y is a W-space and X is a topological space. If Z is a separable subset of Y and $\{O_n : n \in \mathbb{N}\}$ are dense open subsets of $X \times Y$ then for each rich family $\mathcal{F}$ of Y the subset*

$$R := \{x \in X : \text{ there exists a } F_x \in \mathcal{F} \text{ containing } Z \text{ such that } \{y \in F_x : (x, y) \in O_n\} \text{ is dense in } F_x \text{ for all } n \in \mathbb{N}\}$$

is residual in X.

Proof: We are going to apply the $BM(R)$-game and Lemma 4.1 to show that R is residual in X. We shall only consider the case when Y is infinite as the case when Y is finite (and hence has the discrete topology) follows from Lemma 4.2. Thus we can assume that all the members of $\mathcal{F}$ are infinite. Moreover, without loss of generality, we can also assume that all the sets $\{O_n : n \in \mathbb{N}\}$ are decreasing. For each $a \in Y$, let $t^a := (t^a_n : n \in \mathbb{N})$ be a winning strategy for the player α in the $G(a)$-game.

We shall inductively define a strategy $s := (s_n : n \in \mathbb{N})$ for the player α in the $BM(R)$-game played on X, but first let us choose $y \in Y$, set $z_{(i,j,0)} := y$ for all $(i, j) \in \mathbb{N}^2$, set $Z_0 := \{z_{(1,1,0)}\}$ and let $\mathscr{F}_0$ be any countable subset of Y such that $Z \subseteq \overline{\mathscr{F}_0} \in \mathcal{F}$.

Base Step: Suppose that (B_1) is a partial s-play. We shall define the following:

(i) a countable set $\mathscr{F}_1 := \{f_{(1,n)} : n \in \mathbb{N}\}$ such that $Z_0 \cup \mathscr{F}_0 \subseteq \overline{\mathscr{F}_1} \in \mathcal{F}$;
(ii) $s_1(B_1)$ and $z_{(1,1,1)}$ so that:
  (a) $s_1(B_1)$ is a nonempty open subset of B_1;
  (b) $z_{(1,1,1)} \in t_1^{f_{(1,1)}}(z_{(1,1,0)})$, i.e., $(z_{(1,1,0)}, z_{(1,1,1)}) \in \mathrm{Dom}(t_2^{f_{(1,1)}})$;
  (c) $s_1(B_1) \times \{z_{(1,1,1)}\} \subseteq O_1$.
  Note that this is possible by Lemma 4.2.

Finally, define $Z_1 := \{z_{(1,1,1)}\}$.

Inductive Hypothesis: Suppose that $(B_1, ..., B_k)$ is a partial s-play, and for each $1 \leq n \leq k$, the following terms have been defined, $\mathscr{F}_n = \{f_{(n,j)} : j \in \mathbb{N}\}$, $Z_n = \{z_{(i,j,l)} : (i, j, l) \in \mathbb{N}^3 \text{ and } i + j + l \leq n + 2\}$ and s_n so that:

(i) $(\mathscr{F}_{n-1} \cup Z_{n-1}) \subseteq \overline{\mathscr{F}_n} \in \mathcal{F}$;

(ii) $(z_{(i,j,0)}, ..., z_{(i,j,l)}) \in \mathrm{Dom}(t_{l+1}^{f_{(i,j)}})$ for all $i+j+l = n+2$ and

$$s_n(B_1, ..., B_n) \times \{z_{(i,j,l)} : i+j+l = n+2\} \subseteq O_n.$$

Inductive Step: Suppose that $(B_1, ..., B_{k+1})$ is a partial s-play, that is, $(B_1, ..., B_k) \in \mathrm{Dom}(s_k)$ and B_{k+1} is a nonempty open subset of $s_k(B_1, ..., B_k)$. Then:

(i) $Z_k \cup \mathscr{F}_k$ is countable, hence it is contained in some $F \in \mathcal{F}$. Define $\mathscr{F}_{k+1} := \{f_{(k+1,n)} : n \in \mathbb{N}\}$ to be a countable dense subset of F;

(ii) by the inductive hypothesis, $(z_{(i,j,0)}, ..., z_{(i,j,l)}) \in \mathrm{Dom}(t_{l+1}^{f_{(i,j)}})$ for all $i+j+l = k+2$. By re-indexing and noting $(z_{(i,j,0)}) \in \mathrm{Dom}(t_1^{f_{(i,j)}})$ for all $i+j = (k+1)+2$, we get that $(z_{(i,j,0)}, ..., z_{(i,j,l-1)}) \in \mathrm{Dom}(t_l^{f_{(i,j)}})$ for all $i+j+l = (k+1)+2$.

Next, we define $s_{k+1}(B_1, ..., B_{k+1})$ and $z_{(i,j,l)}$ for all $i+j+l = (k+1)+2$ so that:

(a) $s_{k+1}(B_1, ..., B_{k+1})$ is a nonempty open subset of B_{k+1};

(b) $z_{(i,j,l)} \in t_l^{f_{(i,j)}}(z_{(i,j,0)}, ..., z_{(i,j,l-1)})$ for all $i+j+l = (k+1)+2$, i.e., $(z_{(i,j,0)}, ..., z_{(i,j,l)}) \in \mathrm{Dom}(t_{l+1}^{f_{(i,j)}})$ for all $i+j+l = (k+1)+2$;

(c) $s_{k+1}(B_1, ..., B_{k+1}) \times \{z_{(i,j,l)} : i+j+l = (k+1)+2\} \subseteq O_{k+1}$.

Note that this is possible by Lemma 4.2.

Finally, define $Z_{k+1} := \{z_{(i,j,l)} : i+j+l \leq (k+1)+2\}$. This completes the inductive definition of s.

Consider an s-play $(B_n : n \in \mathbb{N})$ of the $BM(R)$-game played on X. For any $x \in \bigcap_{n\in\mathbb{N}} B_n$, let $F_x := \overline{\bigcup_{n\in\mathbb{N}} \mathscr{F}_n} \in \mathcal{F}$. Clearly, $Z \subseteq F_x$. Let $N \in \mathbb{N}$, we will show that the set $\{y \in F_x : (x,y) \in O_N\}$ is dense in F_x. For any open subset U of Y that intersects F_x, there is $f_{(i,j)} \in U \cap (\bigcup_{n\in\mathbb{N}} \mathscr{F}_n)$. Since $t^{f_{(i,j)}}$ is a winning strategy for the player α in the $G(f_{(i,j)})$-game, there is $m > N$ such that $z_{(i,j,m)} \in U \cap F_x$. Moreover, according to the definition of the the strategy s, $(x, z_{(i,j,m)}) \in O_{i+j+m-2} \subseteq O_m \subseteq O_N$. Therefore, $\{y \in F_x : (x,y) \in O_N\}$ is dense in F_x. Hence $\bigcap_{n\in\mathbb{N}} B_n \subseteq R$, which means s is a winning strategy for the player α is the $BM(R)$-game. Hence, by Lemma 4.1, R is residual in X. $\square$

Theorem 4.4. *Suppose that Y is a W-space and X is a Baire space. If Y possesses a rich family $\mathcal{F}$ of Baire subspaces then $X \times Y$ is a Baire space. In fact, if Z is any topological space that contains Y as a dense subspace then $X \times Z$ is also a Baire space.*

Proof: Suppose that $\{O_n : n \in \mathbb{N}\}$ are dense open subsets of $X \times Y$ and $U \times V$ is the product of a nonempty open subset U of X with a nonempty open subset V of Y; we will show that $(U \times V) \cap \bigcap_{n\in\mathbb{N}} O_n \neq \varnothing$. To this end, choose $y \in V$ and set $Z := \{y\}$. By the previous theorem there exists a residual subset R of X such that for each $x \in R$ there exists an $F_x \in \mathcal{F}$ such that (i) $y \in F_x$ and (ii) $\{y' \in F_x : (x, y') \in \bigcap_{n\in\mathbb{N}} O_n\}$ is dense in F_x. Choose $x_0 \in U \cap R \neq \varnothing$ and $F_{x_0} \in \mathcal{F}$ such that $y \in F_{x_0}$ and $\{y' \in F_{x_0} : (x_0, y') \in \bigcap_{n\in\mathbb{N}} O_n\}$ is dense in F_{x_0}. In particular, $\{y' \in F_{x_0} : (x_0, y') \in \bigcap_{n\in\mathbb{N}} O_n\} \cap V \neq \varnothing$. Hence, if we choose $y_0 \in \{y' \in F_{x_0} : (x_0, y') \in \bigcap_{n\in\mathbb{N}} O_n\} \cap V$ then $(x_0, y_0) \in (U \times V) \cap \bigcap_{n\in\mathbb{N}} O_n$. This completes the first part of the proof. To see that $X \times Z$ is a Baire space it is sufficient to realise that $X \times Y$ is a dense Baire subspace of $X \times Z$. $\square$

There are many examples of spaces that admit a rich family of Baire spaces that are not hereditarily Baire. For example, if (i) X is a separable Baire space that is not hereditarily Baire; in which case $\mathcal{F} := \{X\}$ is a rich family of Baire spaces, [1] or (ii) Y is a hereditarily Baire W-space such that $Y \times Y$ is not hereditarily Baire, [1], then the family of all nonempty closed separable rectangles gives a rich family of Baire subspaces of $Y \times Y$.

Corollary 4.5. *Suppose that $\{X_s : s \in S\}$ is a nonempty family of W-spaces. If each X_s, $s \in S$, possesses a rich family of Baire subspaces $\mathcal{F}_s$ then for each $a \in \Pi_{s\in S} X_s$, $\Sigma_{s\in S} X_s(a)$ is a W-space with a rich family of Baire subspaces. In particular, $\Sigma_{s\in S} X_s(a)$ is a Baire space.*

Proof: The fact that $\Sigma_{s\in S} X_s(a)$ is a W-space follows directly from Theorem 2.5. Moreover, from Theorem 3.5 we know that $\Sigma_{s\in S}\mathcal{F}_s(a)$ is a rich family, so it remains to show that all the members of $\Sigma_{s\in S}\mathcal{F}_s(a)$ are Baire spaces. To this end, suppose that $E := \Pi_{s\in S} E_s \in \Sigma_{s\in S}\mathcal{F}_s(a)$. Then E is homeomorphic to $\Pi_{s\in C_E} E_s$. However, by [6, Theorem 3.6] E is a separable first countable space. Therefore, by [8, Theorem 3], $\Pi_{s\in C_E} E_s$ is a Baire space. Finally, the fact that $\Sigma_{s\in S} X_s(a)$ is a Baire space now follows from Theorem 3.3. $\square$

Corollary 4.6. *Suppose that $\{X_s : s \in S\}$ is a nonempty family of W-spaces. If each X_s, $s \in S$, possesses a rich family of Baire subspaces $\mathcal{F}_s$ then $\Pi_{s\in S} X_s$ is a Baire space.*

Proof: This follows directly from Corollary 4.5 since for any $a \in \Pi_{s\in S} X_s$, $\Sigma_{s\in S} X_s(a)$ is a dense Baire subspace. $\square$

As a tribute to Professor I. Namioka, let us end this paper with what is essentially a folklore result, apart from the phrasing in terms of rich families, concerning the Namioka property.

Recall that a Baire space X has the *Namioka property* if for each compact Hausdorff space K and continuous mapping $f : X \to C_p(K)$ there exists a dense subset D of X such that f is continuous with respect to the $\|\cdot\|_\infty$-topology on $C(K)$ at each point of D.

Theorem 4.7. *Suppose that X is a topological space with countable tightness (in particular if X is a W-space) that possesses a rich family $\mathcal{F}$ of Baire subspaces then X has the Namioka property.*

Proof: In order to obtain a contradiction let us suppose that X does not have the Namioka property. Then there exists a compact Hausdorff space K and a continuous mapping $f : X \to C_p(K)$ that does not have a dense set of points of continuity with respect to the $\|\cdot\|_\infty$-topology. In particular, since X is a Baire space (by Theorem 3.3), this implies that for some $\varepsilon > 0$ the open set:

$$O_\varepsilon := \bigcup\{U \in 2^X : \ U \text{ is open and } \|\cdot\|_\infty\text{-diam}[f(U)] \leq 2\varepsilon\}$$

is not dense in X. That is, there exists a nonempty open subset W of X such that $W \cap O_\varepsilon = \varnothing$. For each $x \in X$, let $F_x := \{y \in X : \|f(y) - f(x)\|_\infty > \varepsilon\}$. Then $x \in \overline{F_x}$ for each $x \in W$. Moreover, since X has countable tightness, for each $x \in W$, there exists a countable subset C_x of F_x such that $x \in \overline{C_x}$.

Next, we inductively define an increasing sequence of separable subspaces $(F_n : n \in \mathbb{N})$ of X such that:

(i) $W \cap F_1 \neq \varnothing$;
(ii) $\bigcup\{C_x : x \in D_n \cap W\} \cup F_n \subseteq F_{n+1} \in \mathcal{F}$ for all $n \in \mathbb{N}$, where D_n is any countable dense subset of F_n.

Note that since the family $\mathcal{F}$ is rich this construction is possible.

Let $F := \overline{\bigcup_{n\in\mathbb{N}} F_n}$ and $D := \bigcup_{n\in\mathbb{N}} D_n$. Then $\overline{D} = F \in \mathcal{F}$ and $\|\cdot\|_\infty$-diam$[f(U)] \geq \varepsilon$ every nonempty open subset U of $F \cap W$. Therefore, $f|_F$ has no points of continuity in $F \cap W$ with respect to the $\|\cdot\|_\infty$-topology. This however, contradicts [10, Theorem 6] which states the every separable Baire space has the Namioka property. Therefore, the space X must have the Naimoka property. □

This theorem improves upon some results from [4].

References

[1] J. M. Aarts and D. J. Lutzer, The product of totally nonmeagre spaces, *Proc. Amer. Math. Soc.* **38** (1973), 198–200.

[2] J. M. Borwein and W. B. Moors, Separable determination of integrability and minimality of the Clarke subdifferential mapping, *Proc. Amer. Math. Soc.* **128** (2000), 215–221.

[3] J. Cao and W. B. Moors, A survey on topological games and their applications in analysis, *RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **100** (2006), 39–49.

[4] J. Chaber and R. Pol, On hereditarily Baire spaces, σ-fragmentability of mappings and Namioka property, *Topology Appl.* **151** (2005), 132–143.

[5] W. G. Fleissner and K. Kunen, Barely Baire spaces, *Fund. Math.* **101** (1978), 499–504.

[6] G. Gruenhage, Infinite games and generalizations of first-countable spaces, *Topology Appl.* **6** (1976), 339–352.

[7] W. B. Moors, The product of a Baire space with a hereditarily Baire metric space is Baire, *Proc. Amer. Math. Soc.* **134** (2006), 2161–2163.

[8] J. C. Oxtoby, Cartesian products of Baire spaces, *Fund. Math.* **49** (1960/61), 157–166.

[9] J. C. Oxtoby, *The Banach-Mazur game and Banach category theorem, Contributions to the theory of games* Vol III, Annals of Mathematics Studies **39**, Princeton University press, 1957.

[10] J. Saint Raymond, Jeux topologiques et espaces de Namioka, *Proc. Amer. Math. Soc.* **87** (1983), 499–504.

Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland, New Zealand

E-mail address: `pjlin85@yahoo.co.nz, moors@math.auckland.ac.nz`